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On Certain Types of Convergence in Topological Spaces

A thesis

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Contents

List of Symbols

Symbol	Meaning
τ – open	The open set with respect to topology τ .
τ – closed	The closed set with respect to topology τ .
int(A)	The interior of a set A .
cl(A)	The closure of a set A .
S.cl(A)	The semi closure of a set A .
S.int(A)	The semi interior of a set A .
$\subseteq$	Subset .
$\not\subset$	Not subset .
ϕ	The empty set .
$\in$	Belong to .
$\notin$	Not belong to .
X/A	The complement of a set A .
τ– int(A)	The interior of a set A with respect to the topology τ .
μ– int(A)	The interior of a set A with respect to the topology μ .
$\cup$	Union .
$\cap$	Intersection .
FIP	Finite intersection property .

Abstract

The goal of this work is to study certain types of convergence in the topological spaces. We state below some of the main results that are obtained in this work :

1. Let g be a function from a space (X , τ) into a space (Y , μ) , and $x_o \in X$. Then g is :

i) δ – continuous at x_o if and only if whenever a net {fa : a ∈ A} δ – converges to x_o in X . Then the net {g(fa) : a ∈ A} δ – converges to $g(x_o)$.

ii) S. δ – continuous at xo if and only if whenever a net { fa : a ∈A } S.δ – converges to x_o in X . Then the net { g(fa) : a ∈ A } S.δ – coverges to $g(x_o)$.

iii) θ*.δ – continuous at x_o if and only if whenever a net {fa : a ∈ A } θ*.δ – converges to x_o in X . Then the net { g(fa) : a ∈ A } θ*.δ – converges to $g(x_o)$.

iv) S*.δ – continuous at x_o if and only if whenever a net {fa : a ∈ A} S*.δ – converges to x_o in X . Then the net { g(fa) : a ∈ A } S*.δ – converges to $g(x_o)$.

v) δ*.δ – continuous at x_o if and only if whenever a net {fa : a ∈ A} δ*.δ – converges to x_o in X . Then the net { g(fa) : a ∈ A } δ*.δ – converges to $g(x_o)$.

vi) S**.δ – continuous at x_o if and only if whenever a net {fa : a ∈ A} S**.δ – converges to x_o in X . Then the net { g(fa) : a ∈ A } S**.δ – converges to $g(x_o)$.

2. Let (X , τ) be a topological space and let Y ⊆ X . If x_o is a point of X then, x_o is:

i) δ – adherent point of Y , if and only if there exists a net in Y

δ – converging to x_o .

ii) S.δ – adherent point of Y , if and only if there exists a net in Y S.δ – converging to x_o .

iii) θ*.δ – adherent point of Y , If and only if there exists a net in Y θ*.δ – converging to x_o .

iv) S*.δ – adherent point of Y , if and only if there exists a net in Y S*.δ – converging to x_o .

v) δ*.δ – adherent point of Y , if and only if there exists a net in Y δ*.δ – converging to x_o .

vi) S**.δ – adherent point of Y , if and only if there exists a net in Y S**.δ – converging to x_o .

3. Let (X , τ) be a topological and let Y $\subseteq$ X . Then Y consist of all its :

i)δ – accumulation points if and only if no net in Y δ – converges to a point in X/Y .

ii) S.δ – accumulation points if and only if no net in Y S.δ – converges to a point in X/Y .

iii) θ*.δ – accumulation points if and only if no net inY θ*.δ – converges to a point in X/Y .

iv) S*.δ – accumulation points if and only if no net in Y S*.δ – converges to a point in X/Y .

v) δ*.δ – accumulation points if and only if no net in Y δ*.δ – converges to a point in X/Y .

vi) S**.δ – accumulation points if and only if no net in Y S**.δ – converges to a point in X/Y .

4). Let (X , τ) be a :

i) δ – Hausdorff space . Then every δ – convergent net has a unique δ – cluster point and this is the unique δ – limit point of the net .

ii) S.δ – Hausdorff space . Then every δ – convergent net has a unique S.δ – cluster point and this is the unique S.δ – limit point of the net .

iii) θ*.δ – Hausdorff space .Then every θ*.δ – convergent net has a unique θ*.δ – cluster point and this is the unique θ*.δ – limit point of the net .

iv)S*.δ – Hausdorff space .Then every S*.δ – convergent net has a unique S*.δ – cluster point and this is the unique S*.δ – limit point of the net .

v)δ*.δ – Hausdorff space .Then every δ*.δ – convergent net has a unique δ*.δ – cluster point and this is the unique δ*.δ – limit point of the net .

vi)S**.δ – Hausdorff space .Then every S**.δ – convergent net has a unique S**.δ – cluster point and this is the unique S**.δ – limit point of the net .

5.Let (X , τ) be a topological space . A point xo in X is a :

i) δ – cluster point of a net (f , X , A , ≥) if and only if there exists a subnet (g , X , B , ≥*) which δ – converges to x_o .

ii) S.δ – cluster point of a net (f , X , A , ≥) if and only if there exists a subnet (g , X , B , ≥*) which S.δ – converges to x_o.

iii) θ*.δ – cluster point of a net (f , X , A , ≥) if and only if there exists a subnet (g , X , B , ≥*) which θ*.δ – converges to x_o.

iv) S*.δ – cluster point of a net (f , X , A , ≥) if and only if there exists a subnet (g , X , B , ≥*) which S*.δ – converges to x_o.

v) δ*.δ – cluster point of a net (f , X , A , ≥) if and only if there exists a subnet (g , X , A , ≥*) which δ*.δ – converges to x_o .

vi) S**.δ – cluster point of a net (f , X , A , ≥) if and only if there exists a subnet (g , X , A , ≥*) which S**.δ – converges to x_o .

6. A topological space (X , τ) is :

(i) δ – compact , if each net in X has δ – cluster point .

(ii) S.δ – compact , if each net in X has S.δ – cluster point .

(iii) $\theta^*.\delta$ – compact , if each net in X has $\theta^*.\delta$ – cluster point .

(iv) $\delta^*.\delta$ – compact , if each net in X has $\delta^*.\delta$ – cluster point .

INTRODUCTION

The aim of this work is to study the concept of certain types of convergence in the topological spaces . The notation of convergence is one of the basic notions in analysis. Convergence can be described in any topological space by means of nets or filters . In many concrete examples however, convergence is the primitive notion , and the topology , if such exists , is defined only afterwards . From this situation the need has grown to have an axiom system for convergence which makes it possible to recognize whether the convergence is topological .

For net convergence such an axiom system was given in 1937 by Birkhoff [3] , the crucial " topology " axiom being the iterated limit axiom .

In 1963 . Semi open set were introduced and investigated by N. Levine [20].In 1966 , the notations of θ – open subsets , θ – closed subsets and θ – closure were introduced by Veličko[31].Dickman and Proter[8],[9] , Joseph[17] continued the work of Veličko . In 1968 , Veličko introduced the concepts of δ – closure and δ – interior operations. The notion of δ – open sets which are stronger than open sets .Since then , δ – open sets have been widely used in order to introduce new spaces and functions .

In 1990 , L . Al – Swidi [2], study (m ,n) – compactness in terms of multifunction .He give some net characterizations for such spaces .

Transfinite sequences of functions form some special type of nets. J.Ewert in 1993, he investigate the quasi-uniform convergence of transfinite sequences of functions[13].

For a given linear topology τ, on a vector space *E*, the finest linear topology having the same τ ,convergent sequences, and the finest linear topology on *E* having the same τ, precompact sets, are investigated. Also, the sequentially born logical spaces and the sequentially barreled spaces are introduced and some of their propertiesare studied in 1995 ,byA.K.Katsaras and V.Benekas [18] .

V.Tarieladze in 1997 , he is shown that the convergence of convolution products of probability measures on certain non-locally compact topological abelian groups can be verified by means of characteristic functionals. Analogous results are obtained also for almost everywhere convergence of series of independent random elements in the considered groups. A connection with the Sazonov property of the groups is discussed [30].

In 2003, K.Kuwae and T. Shioya[19]. They present a functional analytic framework of some natural topologies on a given family of spectral structures on Hilbert spaces, and study convergence of Riemannian manifolds and their spectral structure induced from the Laplacian . They also consider convergence of Alexandrov spaces, locally finite graphs, and metric spaces with Dirichlet forms. Our study covers convergence of noncompact (or incomplete) spaces whose Laplacian has continuous spectrum.

In 2004, the purpose of D.Maclver's paper[23] is to provide a brief discussion of this general theory of filters , and attempt to demonstrate why they are an interesting and useful way to talk about convergence .

M.E.El–Shafei in 2005 [12] , he introduce and investigate the notion of weakly Hausdorffness in bitopological spaces by using the convergent of nets .

In 2007 M.Sakata present a simplified theory of generalized eigenfunction expansions for a commuting family of bounded operators and with finitely many unbounded operators. He also study the convergence of these expansions, giving an abstract type of uniform convergence result, and illustrate the theory by giving two examples: The Fourier transform on Hecke operators, and the Laplacian operators in hyperbolic spaces [27] .

In chapter one , we shall introduce elementary definitions , theorems , standard operators and relationships on subsets of topological spaces by using semi open , θ – open and δ – open .These operators are those of closure and interior . We see that most properties of them remain true after our alternation .

In chapter two , we have used new definitions and relations

depending on adherent point , accumulation point , cluster point of the net , limit point of the net , Hausdorff space and continuous functions by presenting some properties of the semi open , θ – open and δ – open sets . We make several proofs for some of the continuity theorems which give us great useful in convergence .

The third chapter has two sections. In each one we prove theorems by using the convergence net .In section one , we prove some of theorems depending on convergence with types of Hausdorff spaces.

Section two contains the definition , compact sets and spaces , and we have proved some of the theorems relying on convergence in compact topological spaces.

Chapter One

Elementary and study some types of open sets

Introduction :

In this chapter, we introduce and study three classes of sets: semi open , θ – open and δ – open . We also mention some of elementary definitions, theorems and results which are related to above sets. In addition to, we present notions of interior and closure points and some of relations that are necessary to the work .

Section one : Semi open set

This section deals with the main definitions semi open and semi closed . It also presents a general study of the notions semi interior and semi closure as well as some theorems and properties that are included throughout the work .

Definition 1.1.1 : [20]

Let (X,τ) be a topological space and $A \subseteq X$. A is called semi open in X if and only if there exists $O \in \tau$ such that $O \subseteq A \subseteq cl(O)$. The family of all semi open sets in X is denoted by S.O.(X).

The following remarks give some basic properties of S.O.(X) .

Remarks 1.1.2 : [20]

Let (X,τ) be a topological space , and A be a subset of X . Then :

1) $A \in$ S.O.(X) if and only if $A \subseteq cl[int(A)]$.
2) τ is a sub collection of S.O.(X).
3) A necessary condition for a nonempty set to be in S.O.(X) is that its interior is not empty.

Definition 1.1.3: [6]

Let (X,τ) be a topological space and A be a subset of X . A is called semi closed in X if and only if (X/A) is semi open set in X .If B $\subseteq$ X then the semi closure of B is defined by the intersection of all semi closed sets in X containing B and is denoted by S.cl(B) .

The following results give basic properties of the semi closure :

Theorem 1.1.4 : [6]

Let (X,τ) be a topological space and A $\subseteq$ B $\subseteq$ X . Then :

1) S.cl(A) $\subseteq$ S.cl(B).
2) S.cl(A) is the smallest semi closed set containing A.
3) S.cl(S.cl(A) = S.cl(A).
4) A is semi closed if and only if A =S.cl(A).

Definition 1.1.5 : [6]

Let (X,τ) be a topological space and x $\in$ X . x is called a semi accumulation point of A $\subseteq$X if each semi open set containing x , contains a point of A which is distinct from x .We shall call the set of all semi accumulation points of A by the semi derived set of A and denote it by S.D.(A).

Theorem 1.1.6 : [16]

Let (X,τ) be a topological space and A be a subset of X.Then A is semi open if and only if there exists B $\in$ S.O.(X) such that B $\subseteq$ A $\subseteq$ cl(B).

Definition 1.1.7 : [20]

Let (X.τ) be a topological space and A be a subset of X. A point x $\in$ X is called a semi interior point of A if and only if there exists U $\in$ S.O.(X) such that

$x \in U \subseteq$ A.The set of all semi interior point of A is called the semi interior of A and is denoted by S.int(A).

Theorem 1.1.8: [16]

Let (X,τ) be a topological space . Let A and B be two subsets of X . Then :

1) Int(A) $\subseteq$ S.int(A) and the reverse inclusion is not true .

2)S.int(A) is the largest semi open set contained in A .

3) If A $\subseteq$ B then S.int(A) $\subseteq$ S.int(B) .

4) A is semi open if and only if A = S.int(A) .

5) S.int(S.int(A)) = S.int(A) .

6) S.int($A \cap B$) $\subseteq$ S.int(A) $\cap$ S.int(B) and the reverse inclusion is not true .

Remark 1.1.9 : [26]

Let (X,τ) be a topological space, and A be an open set, then $A \cap D \in$ S.O(X) for each $D \in$ S.O(X) .

Theorem1.1.10 : [16]

Let (X,τ) be a topological space and let $A \subseteq X$ .Then A is semi closed if and only if S.D(A) $\subseteq$ A .

Theorem 1.1.11: [24]

Let (X, τ) be a topological space . If A is semi closed in X , and int(A)$\subseteq B \subseteq A$, then B is semi closed .

Theorem 1.1.12 : [16]

A subset B of a toplogical space (X,τ) is semi closed if and only if there exists a closed set C such that int(C) $\subseteq$ B $\subseteq$ C .

Remark 1.1.13 [20]

Let (X,τ) be a topological space we have :

1) Any union of semi open sets is semi open .

2) Any intersection of semi closed sets is semi closed .

Section two : θ – Open Set

Section two deals with the elementary definitions for θ – open , θ – closed , θ – closure and θ – interior with some theorems and results that are necessary throughout our research .

Definition 1.2.1 : [22]

Let (X,τ) be a topological space and $A \subseteq X$. A is said to be θ – open if and only if for each $x \in A$, there exists an open set U such that $x \in U \subseteq cl(U) \subseteq A$ .It follows that each θ – open set is open .

Definition 1.2.2 : [31]

Let (X,τ) be a topological space . A point $x \in X$ is called a θ – adherent point of a subset A of X if $A \cap cl(N) \neq \phi$ for every open set N containing x .The set of all θ – adherent points of A is called the θ – closure of A , and is denoted by θ.cl(A) .

Remark 1.2.3 : [31]

Let (X,τ) be a topological space . A subset A of X is called θ – closed if A = θ.cl(A) .

Definition 1.2.4 : [4]

Let (X,τ) be a topological space , and let A be a θ – closed set . The complement set of A is called θ – open . The family of all θ – open sets in X is denoted by θ.O.(X).

Lemma 1.2.5 : [11]

If A and B are subsets of a topological space (X,τ) , then

$\theta.cl(A \cup B) = \theta.cl(A) \cup \theta.cl(B)$.

Remark 1.2.6 :

If B is a subset of a topological space (X,τ) , then the θ – closure of B is defined by the intersection of all θ – closed sets in X containing B .

Definition 1.2.7 : [17]

Let (X,τ) be a topological space . A point $x \in X$ is said to be a θ – interior point of A , if there exists an open set U containing x such that $U \subseteq cl(U) \subseteq A$. The set of all θ – interior points of A is said to be the θ – interior of A and is denoted by $\theta.int(A)$.

The following results give basic properties of the θ – interior .

Thoreom 1.2.8 : [4]

Let A and B are subsets of a topological space (X,τ) , the following statements are true :

1) $\theta.int(A)$ is the union of all open sets of X whose closures are contained in A .

2) A is θ.open if and only if $A = \theta.int(A)$.

3) If $A \subseteq B$, then $\theta.int(A) \subseteq \theta.int(B)$.

4) $\theta.int(A) \cup \theta.int(B) \subseteq \theta.int(A \cup B)$.

5) $\theta.int(A) \cap \theta.int(B) = \theta.int(A \cap B)$.

Remark 1.2.9 : [5]

Let (X,τ) be a topological space . The family of all θ – open sets are topology on X which we shall denote by τ_θ .

Section three : δ – Open set

In section three, we give a general study of the δ – open and δ – interior that are necessary with some main notions δ – closed , δ – closure as well as some theorems and results that have close relation to the research .

Definition 1.3.1:

Let (X ,τ) be a topological space and $A \subseteq X$. A is said to be δ – open if and only if for each $x \in A$, there exists an open set U such that $x \in U \subseteq int(cl(U)) \subseteq A$.

It follows that each δ – open set is open set .

Definition 1.3.2 : [31]

Let (X,τ) be a topological space . A point $x \in X$ is called a δ – adherent point of A if and only if $int(clU) \cap A \neq \phi$, for every open set U containing x . The set of all δ – adherent points of A is called the δ – closure of A , and is denoted by δ.cl(A) .

Definition 1.3.3 : [31]

Let (X,τ) be a topological space . If δ.cl(A) = A , then A is said to be δ – closed .

Remark 1.3.4 : [31]

Let A be a subset of a topological space (X,τ) . Then A is called δ – closed if X/A is δ – open .

Note 1.3.5 :

The family of all δ – open sets of a topological space (X,τ) , will be denoted by $\delta.O.(X)$.

Lemma 1.3.6 :[25]

Let (X,τ) be a topological space .The intersection of arbitrary of δ – closed sets in X is δ – closed .

Lemma 1.3.7 : [25]

If A is a subset of a topological space (X,τ) . Then $\delta.cl(A)$ is the intersection of all δ – closed sets in X containing A .

Definition 1.3.8 :

Let (X,τ) be a topological space . A point $x \in X$ is said to be a δ – interior point of A if there exists an open set U containing x such that $x \in U \subseteq int(clU) \subseteq A$. The set of all δ – interior points of A is said to be the δ – interior of A and is denoted by $\delta.int(A)$.

Definition 1.3.9 : [31]

Let (X , τ) be a topological space. A subset A of X is called δ – open if $A = \delta . int(A)$.

Definition 1.3.10 : [29]

A subset A of a topological space (X , τ) is said to be regular open if $A = int(clA)$.

Lemma 1.3.11 : [1]

Let (X , τ) be a topological space and $U \in \tau$ then $int(cl(int(cl(U)))) = int(cl(U))$.

Remark 1.3.12 :

Let (X , τ) be a topological space . A subset A of X is δ – open if and only if each $x \in A$, there exists a regular open set W such that $x \in W \subseteq A$.

Proof :

The " if " part . Let A is δ – open set then, for all $x \in A$ there exists an open Set U containing x such that $x \in U \subseteq int[cl(U)] \subseteq A$. Let $W = int[cl(U)]$, therefore , $int[cl(W)] = int[cl(int[cl(U)])]$.

But $int[cl(U)] = int[cl(int[cl(U)])]$ by Lemma 1.3.12 .Therefore , $int[cl(W)] = int[cl(U)]$, and since $W = int[cl(U)]$ consequently , $W = int[cl(W)]$,that is, W is a regular open set , and so $x \in U \subseteq W \subseteq A$.

The "if only" part . For each $x \in A$, there exists a regular open set W such that $x \in W \subseteq A$. Since W is open set , and $int[cl(W)] = W$, therefore , $x \in W \subseteq int[cl(W)] \subseteq A$.

Hence A is δ – open set by Definition 1.3.1.

Remark 1.3.13:[14]

The family of all δ – open subsets of a topological space (X,τ) is a topology on X which is denoted by τ_δ .

Lemma 1.3.14 :

Let (X,τ) be a topological space and A , B are two subsets of X . If $A \subseteq B$ then $\delta.intA \subseteq \delta.intB$.

Proof :

Let $x \in \delta.intA$ then there exists open set U containing x such that $x \in U \subseteq int(clU) \subseteq A$. Since $A \subseteq B$, we get that $x \in U \subseteq int(clU) \subseteq B$. Hence $x \in \delta.intB$, and consequently, $\delta.intA \subseteq \delta.intB$.

Definition 1.3.15 :

Let (X , τ) be a topological space . A point $x \in X$ is called a δ – accumulation point of a subset A of X , if $int(cl(V)) \cap (A/\{x\}) \neq \phi$ for every open set V containing x . The set of all δ – accumulation points of A is called δ – drived set of A and is denoted by δ.D(A) .

Section four :

Properties and Relationships of Some Types of Open Sets

In the last section, we present some definitions and theorems that are needed throughout the study. We also explain some relationships and graphs for the sets: semi open , θ – open and δ – open .

Remark 1.4.1 : [15]

The following relations hold among modifications of open sets : θ – open , δ – open and semi open .

$$\theta\text{ – open} \rightarrow \delta\text{ – open} \rightarrow \text{open} \rightarrow \text{semi open}$$

Remark 1.4.2 :

Let (X , τ) be a topological space , and $A \subseteq X$ .Then :

1) Every θ – interior point of A is δ – interior point .

2) Every δ – interior point of A is interior point .

Proof :

(1) Let x be a θ – interior point of A , then there exists an open set U such that $x \in U \subseteq cl(U) \subseteq A$ by Definition 1.2.7.

And since $int(cl(U)) \subseteq cl(U)$ therefore , $x \in U \subseteq int[cl(U)] \subseteq A$.

Hence x is δ – interior point of A .

(2) can be proved by the same manner by using Definition 1.3.8.

Lemma 1.4.3 :

Let (X , τ) be a topological space and $A \subseteq X$ then :

1) If A is θ – closed set then , A is δ – closed set ;
2) If A is δ – closed set then , A is closed set ;
3) If A is closed set then , A is semi closed set .

Proof :

(1) Let A be a θ – closed set which implies that (X/A) is θ – open set , then (X/A) is δ – open set by Remark 1.4.1 .Therefore we get that A is δ – closed set .

(2) and (3) can be proved by the same manner .

Note 1.4.4 :

The following relations hold among modifications of closed sets : θ – closed , δ – closed and semi closed .

$$\theta\text{ – closed} \rightarrow \delta\text{ – closed} \rightarrow \text{closed} \rightarrow \text{semi closed}$$

Theorem 1.4.5 : [21]

Let (X,τ) be a topological space and let $A \subseteq X$. Then $S.cl(A) \subseteq cl(A)$.

Remark 1.4.6 : [31]

Let (X,τ) be a topological space, then

1) For any subset $S \subseteq X$, $cl(S) \subseteq \delta.cl(S) \subseteq \theta.cl(S)$.

2) For any $U \in \tau$, $cl(U) = \delta.cl(U) = \theta.cl(U)$.

Note 1.4.7 :

Let A be a subset of a topological space (X , τ) . Then ,

$S.cl(A) \subseteq cl(A) \subseteq \delta.cl(A) \subseteq \theta.cl(A)$

Lemma 1.4.8 :[5]

Let (X,τ) be a topological space then , $\tau_\theta \subseteq \tau_\delta \subseteq \tau$.

Theorem 1.4.9 :

Let (X,τ) be a topological space , and $A \subseteq X$,then :

1) $\delta.int(A) = X/(\delta.cl(X/A))$.
2) $\theta.int(A) = X/(\theta.cl(X/A)$.
3) $S.int(A) = X/(S.cl(X/A)$.

Proof :

(1) $\delta.cl(A) = \bigcap\{F:F$ is δ – closed set and $A\subseteq F\}$. By [Demorgan law], we get that $X/(\delta.cl(A)) = \bigcup\{G:G$ is δ – open set and $G \subseteq X/A\}$, where $G = X/F$.Therefore , $X/(\delta.cl(A)) = \delta.int(X/A)$.

Consequently $X/(\delta.cl(X/A)) = \delta.int(A)$.

Similarly we can prove numbers (2) and (3) .

Theorem 1.4.10: [28]

Let (X,τ) be a topological space , and let $A\subseteq X$. Then , $int(A) = [X/cl(X/A)]$ (i.e. The interior of A is the complement of the closure of the complement of A .

Note 1.4.11:

Let (X, τ) be a topological space , and N be an open set then , $N \subseteq int[cl(N)]$.

Theorem 1.4.12 :

Let (X,τ) be a topological space , and let N be an open set then :

1) $N \subseteq \delta.int[cl(N)]$.
2) $N \subseteq \theta.int[cl(N)]$.
3) $N \subseteq S.int[cl(N)]$
4) $N \subseteq int[\delta.cl(N)]$.

5) $N \subseteq int[\theta.cl(N)]$.

6) $N \subseteq int[S.cl(N)]$.

7) $N \subseteq \delta.int[\delta.cl(N)]$.

8) $N \subseteq \theta.int[\theta.cl(N)]$.

9) $N \subseteq S.int[S.cl(N)]$.

10) $N \subseteq \delta.int[\theta.cl(N)]$.

11) $N \subseteq \theta.int[\delta.cl(N)]$.

Proof :

(1) By Theorem 1.4.9.Part 1 , $\delta.int[cl(N)] = X/[\delta.cl(X/cl(N))]$.

Therefore , $\delta.int[cl(N)] = X/[cl(X/cl(N))]$ by Remark 1.4.6.Part 2 .

$= int[cl(N)]$ by Theorem 1.4.10 ,

and since N is open set , we get that $N \subseteq int[cl(N)]$ by Note 1.4.11, consequently $N \subseteq \delta.int[cl(N)]$.

Similarly we can prove number (2) .

(3) By Theorem 1.1.8.Part 1 , and by assumption N is open , we get that $N = int(N) \subseteq S.int(N)$, but $S.int(N) \subseteq S.int[cl(N)]$.

Consequently $N \subseteq S.int[cl(N)]$.

(4) Since $N \subseteq \delta.cl(N)$, therefore $int(N) \subseteq int[\delta.cl(N)]$, and since N is open set , that is, $N = int(N)$. Hence $N \subseteq int[\delta.cl(N)]$.

(5) and (6) can be proved by the same manner .

(7) From item (1) , $N \subseteq \delta.int[cl(N)]$, and since $cl(N) = \delta.cl(N)$ by Remark 1.4.6.Part 2 , therefore , $N \subseteq \delta.int[\delta.cl(N)]$.

(8) can be proved by the same manner .

(9) Since N $\subseteq$ S.cl(N) , and so S.int(N) $\subseteq$ S.int[S.cl(N)] , also int(N) $\subseteq$ S.int(N) by Theorem 1.1.8.Part 1 , which implies that int(N) $\subseteq$ S.int[S.cl(N)] , but int(N) = N, since N is open set . Consequently N $\subseteq$ S.int[S.cl(N)] .

(10) By remark 1.4.6.Part 2 , and by item (1) , we get that N $\subseteq$ δ.int[θ.cl(N)].

Similarly we can prove number (11) .

Proposition 1.4.13 :

Let (X,τ) be a discrete topological space , and A $\subseteq$ X. Then :

1) A is θ – open set .

2) A is δ – open set .

3) A is semi open set .

Proof :

(1) Since (X,τ) is a discrete topological space , and A $\subseteq$ X then A is open set , and A = int(A), also X/A $\subseteq$ X then X/A is open set therefore, A is closed set, and A = cl(A) .For all x $\in$ A , and since x $\in$ A $\subseteq$ cl(A) $\subseteq$ A . Hence A is θ – open set by Definition 1.2.1.

(2) By proof (1) , we get that A is θ – open set , hence A is δ – open set by Remark 1.4.1.

Similarly we can prove number (3) .

Theorem 1.4.14 :

Let (X,τ) be a discrete topological space , and A $\subseteq$ X then :

1) $A = int[\theta.cl(A)]$.

2) $A = int[\delta.cl(A)]$.

3) $A = int[S.cl(A)]$.

4) $A = \theta.int[cl(A)]$.

5) $A = \delta.int[cl(A)]$.

6) $A = S.int[cl(A)]$.

7) $A = \theta.int[\theta.cl(A)]$.

8) $A = \delta.int[\delta.cl(A)]$.

9) $A = S.int[S.cl(A)]$.

10) $A = \delta.int[\theta.cl(A)]$.

11) $A = \theta.int[\delta.cl(A)]$.

12) $A = int[cl(A)]$.

Proof :

Directly through Proposition 1.4.13 .

Chapter Two

Convergence Net

Introduction :

Chapter two consists of five sections . In section one, we define and study types of adherent points . Section two deals with kinds of accumulation points , while in the third section we give definitions and relationships to the kinds of cluster and limit points of the net . The fourth section deals with elementary definitions of Hausdorff spaces .

In the last section , we present types of continuity and prove theorems about convergence by using continuity.

Section one : Types of Adherent Points

In this section, we define an adherent point by depending on some properties of θ – open set , δ – open set and semi open set .

Let A be a subset of a topological space (X ,τ) and let x ∈ X . Then x is called an adherent point of A , if every open set containing x , contains point of A .[28]

Definition 2.1.1:

Let (X,τ) be a topological space , and let A ⊆ X . A point x ∈ X is called :

1) S.δ – adherent point of A , if int[S.cl(N)] $\cap$ A $\neq \phi$, for every open set N containing x . The set of all S.δ – adherent points of A is called the Semi δ – closure of A , and is denoted by S.δ.cl(A) .

2) θ.δ – adherent point of A , if int[θ.cl(N)] $\cap$ A $\neq \phi$, for every open set N containing x . The set of all θ.δ – adherent points of A is called the θ.δ – closure of A , and is denoted by θ.δ.cl(A) .

3) δ.δ – adherent point of A , if int[δ.cl(N)] $\cap$ A $\neq \phi$, for every open set N containing x . The set of all δ.δ – adherent points of A is called the δ.δ – closure of A , and is denoted by δ.δ.cl(A) .

4) S*.δ – adherent point of A , if S.int[cl(N)] $\cap$ A $\neq \phi$, for every open set N containing x . The set of all S*.δ – adherent points of A is called the S*.δ – closure of A , and is denoted by S*.δ.cl(A) .

5) $\theta^*.\delta$ – adherent point of A , if $\theta.int[cl(N)] \cap A \neq \phi$, for every open set Ncontaining x . The set of all $\theta^*.\delta$ – adherent points of A is called the $\theta^*.\delta$ – closure of A , and is denoted by $\theta^*.\delta.cl(A)$.

6) $\delta^*.\delta$ – adherent point of A , if $\delta.int[cl(N)] \cap A \neq \phi$, for every open set N containing x . The set of all $\delta^*.\delta$ – adherent point of A is called the $\delta^*.\delta$ – closure of A , and is denoted by $\delta^*.\delta.cl(A)$.

7) $S^{**}.\delta$ – adherent point of A , if $S.int[S.cl(N)] \cap A \neq \phi$, for every open set N containing x .The set of all $S^{**}.\delta$ – adherent points of A is called the $S^{**}.\delta$ – closure of A , and is denoted by$S^{**}.\delta.cl(A)$.

8) $\theta^{**}.\delta$ – adherent point of A , if $\theta.int[\theta.cl(N)] \cap A \neq \phi$, for every open set N containing x . The set of all $\theta^{**}.\delta$ – adherent points of A is called the $\theta^{**}.\delta$ – closure of A , and is denoted by $\theta^{**}. \delta.cl(A)$.

9) $\delta^{**}.\delta$ – adherent point of A , if $\delta.int[\delta.cl(N)] \cap A \neq \phi$, for every open set N containing x . The set of all $\delta^{**}.\delta$ – adherent points of A is called the $\delta^{**}.\delta$ – closure of A , and is denoted by $\delta^{**}.\delta.cl(A)$.

10)$\delta^{***}.\delta$ – adherent point of A , if $\theta.int[\delta.cl(N)] \cap A \neq \phi$, for every open set N containing x . The set of all $\delta^{***}.\delta$ – adherent points of A is called the $\delta^{***}.\delta$ – closure of A , and is denoted by $\delta^{***}.\delta.cl(A)$.

11) $\theta^{***}.\delta$ – adherent point of A , if $\delta.int[\theta.cl(N)] \cap A \neq \phi$, for every open set N containing x . The set of all $\theta^{***}.\delta$ – adherent points of A is called the $\theta^{***}.\delta$ – closure of A , and is denoted by $\theta^{***}.\delta.cl(A)$.

The following propositions can be proved by using Definition 1.3.2, Definition 2.1.1 and Remark 1.4.6.Part 2 .

Proposition 2.1.2:

Let (X , τ) be a topological space , and let A $\subseteq$ X . Then for x $\in$ X , the following statement are equivalent :

1) x is δ – adherent point of A.
2) x is θ. δ – adherent point of A .
3) x is $\delta.\delta$ – adherent point of A .

Proposition 2.1.3 :

Let (X , τ) be a topological space , and let A $\subseteq$ X . Then for x $\in$ X , the following statement are equivalent :

1) x is $\theta^*.\delta$ – adherent point of A .
2) x is $\theta^{**}.\ \delta$ – adherent point of A .
3) x is $\delta^{***}.\delta$ – adherent point of A .

Proposition 2.1.4 :

Let (X , τ) be a topological space , and let A $\subseteq$ X . Then for x $\in$ X , the following statement are equivalent :

1) x is $\delta^*.\delta$ – adherent point of A .
2) x is $\delta^{**}.\ \delta$ – adherent point of A .
3) x is $\theta^{***}.\delta$ – adherent point of A .

Lemma 2.1.5 :

Let (X,τ) be a topological space and let A be a subset of X then , every adherent point of A is :

1) S.δ – adherent point.
2) $\theta.\delta$ – adherent point .
3) S*.δ – adherent point .

4) $\theta^*.\delta$ – adherent point .

5) $\delta^*.\delta$ – adherent point .

6) $S^{**}.\delta$ – adherent point .

Proof :

(1) Let x be an adherent point of A then , $N \cap A \neq \phi$ for every open set N containing x .

Since $N \subseteq int[S.cl(N)]$ by Theorem 1.4.12. Part 6 , therefore , $int[S.cl(N)] \cap A \neq \phi$. Hence x is $S.\delta$ – adherent point of A .

Similarly we can prove numbers (2) , (3) , (4) , (5) and (6) .

Example 2.1.6 :

Let X = {a ,b ,c} and $\tau = \{\phi, \{a\}, \{a,b\}, X\}$ and A = {b}. τ–closed = {X, {b ,c} , {c} , ϕ} . Find 1) The adherent points of A .

2) δ – adherent, $S.\delta$ – adherent , $\delta^*.\delta$ – adherent, the $S^*.\delta$ – adherent , $\theta^*.\delta$ – adherent and $S^{**}.\delta$ – adherent points of A .

Solution :

(1) Since {a , b} , X are only open sets containing b and $\{a, b\} \cap A \neq \phi$, and $X \cap A \neq \phi$, therefore b is adherent point of A , also c is adherent point of A. Since X is only open set containing c and $X \cap A \neq \phi$.

But a is not adherent point of A , since {a} is open set containing a and $\{a\} \cap A = \phi$.

(2) Since b and c are adherent points of A , therefore , b and c are δ – adherent, $S.\delta$ – adherent, $\delta^*.\delta$ – adherent, $S^*.\delta$ – adherent A, $\theta^*.\delta$ – adherent and $S^{**}.\delta$ – adherent points of A by Lemma 2.1.5.

Since int[cl(N)] = X , δ.int[cl(N)] = X ,

S.int[cl(N)] = X , θ.int[cl(N)] = X , and since {ϕ, {a} , {a ,b} , X , {a ,c}} the family of all semi open sets in X , therefore , {X , {b , c} , {c} , ϕ, {b}} the family
of all semi closed in X , so int[S.cl(N)] = X and S.int[S.cl(N)] = X , for every open set N containing a , and so X$\cap$ A $\neq \phi$, hence a is δ – adherent A , S.δ – adherent, δ*.δ – adherent, S*.δ – adherent, θ*.δ – adherent and S**.δ – adherent point of A by Definition 2.1.1.

Proposition 2.1.7 :

Let A be a subset of a topological space (X,τ) then every :

1) S. δ – adherent point of A is δ – adherent point .
2) S**.δ – adherent point of A is S*.δ – adherent point .
3) δ – adherent point of A is S*.δ – adherent point .
4) S.δ – adherent point of A is S**.δ – adherent point .
5) δ*.δ – adherent point of A is δ – adherent point .
6) θ*.δ – adherent point of A is δ*.δ – adherent point .
7) θ**.δ – adherent point of A is θ***.δ – adherent point .
8)δ***.δ – adherent point of A is δ**.δ – adherent point .
9) δ**.δ – adherent point of A is θ.δ – adherent point .
10)θ*.δ – adherent point of A is θ***.δ – adherent point .
11)δ***.δ – adherent point of A is δ – adherent point .

Proof :

(1) Let x be a S.δ – adhrenet point of A then , int[S.cl(N)] $\cap$ A $\neq \phi$, for every open set N containing x .Since S.cl(N) $\subseteq$ cl(N) by Theorem 1.4.5, and so int[S.cl(N)] $\subseteq$ int[cl(N)] , therefore , int[cl(N)] $\cap$ A $\neq \phi$ for every open set N containing x . Hence x is δ – adherent point of A .

Similarly we can prove number (2) .

(2) Let x be a δ – adherent point of A then, int[cl(N)] $\cap$ A $\neq \phi$ for every open set N containing x .And every interior point of A is semi interior point by Theorem 1.1.8.Part 1,therefore, S.int[cl(N)] $\cap$ A $\neq \phi$ for every open set N containing x .Hence x is S*.δ – adherent point of A .

(4) , (5) , (6) , (7) and (8) can be proved by adopting the same items .

(9) Let x be a δ**.δ – adherent point of A then ,

δ.int[δ.cl(N)] $\cap$ A $\neq \phi$ for every open set N containing x .

But for any set A δ – interior point of A is interior point by Remark 1.4.2.Part 2,consequently int[δ.cl(N)] $\cap$ A $\neq \phi$, and so δ.cl(N) = θ.cl(N) by Remark 1.4.6.Part 2, therefore ,

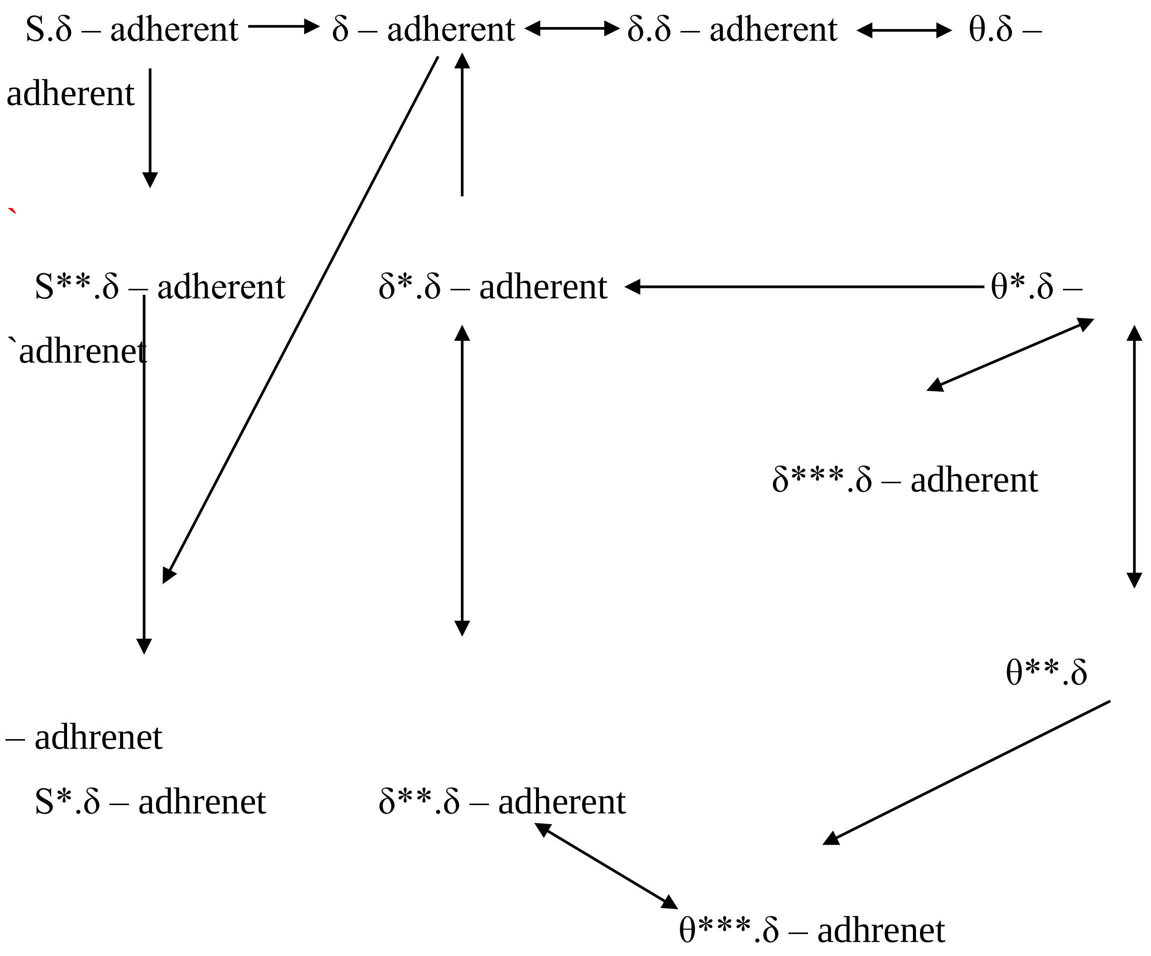
int[θ.cl(N)] $\cap$ A $\neq \phi$ for every open set N containing x .

Hence x is θ.δ – adherent point of A .

Similarly we can prove numbers (10) and (11) .

Note 2.1.8 :

The following diagram is taken from the above proposition and the definition of modification of δ – adherent points stated above .



Section two : Types of Accumulation Points

In this section we define an accumulation point by depending on some properties of θ – open set , δ – open set and semi open set .

Let A be a subset of a topological space (X, τ) and let $x \in X$. Then x is called the accumulation point of A , if every open set containing x , contains point of A other than x .[28]

Definition 2.2.1:

Let A be a subset of a topological space (X, τ) . A point $x \in X$ is called :

1) $S.\delta$ – accumulation point of A , if $int[S.cl(N)] \cap (A/\{x\}) \neq \phi$ for every open set N containing x .

2) $\theta.\delta$ – accumulation point of A , if $int[\theta.cl(N)] \cap (A/\{x\}) \neq \phi$ for every open set N containing x .

3) $\delta.\delta$ – accumulation point of A , if $int[\delta.cl(N)] \cap (A/\{x\}) \neq \phi$ for every open set N containing x .

4) $S^*.\delta$ – accumulation point of A , if $S.int[cl(N)] \cap (A/\{x\}) \neq \phi$ for every open set N containing x .

5) $\theta^*.\delta$ – accumulation point of A , if $\theta.int[cl(N)] \cap (A/\{x\}) \neq \phi$ for every open set N containing x .

6) $\delta^*.\delta$ – accumulation point of A , if $\delta.int[cl(N)] \cap (A/\{x\}) \neq \phi$ for every open set N containing x .

7) $S^{**}.\delta$ – accumulation point of A , if S.int[S.cl(N)] $\cap$ (A/{x}) $\neq \phi$ for every open set N containing x .

8) $\theta^{**}.\delta$ – accumulation point of A , if θ.int[θ.cl(N)] $\cap$ (A/{x}) $\neq \phi$ for every open set N containing x .

9) $\delta^{**}.\delta$ – accumulation point of A , if δ.int[δ.cl(N)] $\cap$ (A/{x}) $\neq \phi$ for every open set N containing x .

10) $\delta^{***}.\delta$ – accumulation point of A , if θ.int[δ.cl(N)] $\cap$ (A/{x}) $\neq \phi$ for every open set N containing x .

11) $\theta^{***}.\delta$ – accumulation point of A , if δ.int[θ.cl(N)] $\cap$ (A/{x}) $\neq \phi$ for every open set N containing x .

The following propositions can be proved by using Definition 1.3.15, Definition 2.2.1 and Remark 1.4.6.Part 2 .

Proposition 2.2.2 :

Let A be a subset of a topological space (X , τ) , and let $x \in X$.
Then , the following statements are equivalents :

1) x is δ – accumulation point of A .
2) x is θ.δ – accumulation point of A .
3) x is δ.δ – accumulation point of A .

Proposition 2.2.3 :

Let A be a subset of a topological space (X, τ), and let $x \in X$. Then , the following statements are equivalents :

1) x is $\theta^*.\delta$ – accumulation point of A .
2) x is $\theta^{**}.\delta$ – accumulation point of A .
3) x is $\delta^{***}.\delta$ – accumulation point of A .

Proposition 2.2.4 :

Let A be a subset of a topological space (X, τ), and let $x \in X$. Then , the following statements are equivalents :

1) x is $\delta^*.\delta$ – accumulation point of A .
2) x is $\delta^{**}.\delta$ – accumulation point of A .
3) x is $\theta^{***}.\delta$ – accumulation point of A .

Lemma 2.2.5 :

Let (X,τ) be a topological space and let A be a subset of X then , every accumulation point of A is :

1) $\theta.\delta$ – accumulation point.
2) $S.\delta$ – accumulation point .
3) $S^*.\delta$ – accumulation point .
4) $\theta^*.\delta$ – accumulation point .
5) $\delta^*.\delta$ – accumulation point .
6) $S^{**}.\delta$ – accumulation point .

Proof :

(1) Let x be an accumulation point of A then ,

$N \cap (A/\{x\}) \neq \phi$ for every open set N containing x .

Since $N \subseteq$ int[θ.cl(N)] by Theorem 1.4.12. Part 5,

therefore , int[θ.cl(N)] $\cap$ (A/{x}) $\neq \phi$ for every open set N containing x . Hence x is θ.δ – accumulation point of A .

(2), (3), (4), (5) and (6) can be proved by following the same procedures applicable to number (1) .

Proposition 2.2.6:

Let A be a subset of a topological space (X,τ) then , every :

1) S.δ – accumulation point of A is δ – accumulation point .
2) S**.δ – accumulation point of A is S*.δ – accumulation point.
3) δ – accumulation point of A is S*.δ – accumulation point .
4) S.δ – accumulation point of A is S**.δ – accumulation point .
5) δ*.δ – accumulation point of A is δ – accumulation point .
6) θ*.δ – accumulation point of A is δ*.δ – accumulation point .
7) θ**.δ – accumulation point of A is θ***.δ – accumulation point .
8) δ***.δ – accumulation point of A is δ**.δ – accumulation point .
9) δ**.δ – accumulation point of A is θ.δ – accumulation point .
10) θ*.δ – accumulation point of A is θ***.δ – accumulation point .
11) δ***.δ – accumulation point of A is δ – accumulation point .

Proof :

(1) Let x be a S.δ – accumulation point of A then ,

int[S.cl(N)] $\cap$(A/{x}) $\neq \phi$ for every open set N containing x .

Since S.cl(N) $\subseteq$ cl(N) by Theorem 1.4.5, and so int[S.cl(N)] $\subseteq$ int[cl(N)] , therefore ,

int[cl(N)] $\cap$ (A/{x}) $\neq \phi$ for every open set N containing x . Hence x is δ – accumulation point of A .

Similarly we can prove number (2) .

(3) Let x be a δ – accumulation point of A then , int[cl(N)] $\cap$ (A/{x}) $\neq \phi$ for every open set N containing x . But every interior point is semi interior point by Theorem 1.1.8.Part 1, therefore , S.int[cl(N)] $\cap$ (A/{x}) $\neq \phi$ for every open set N containing x . Hence x is S*.δ – accumulation point of A . (4), (5), (6), (7) and (8) can be proved by adopting the same items .

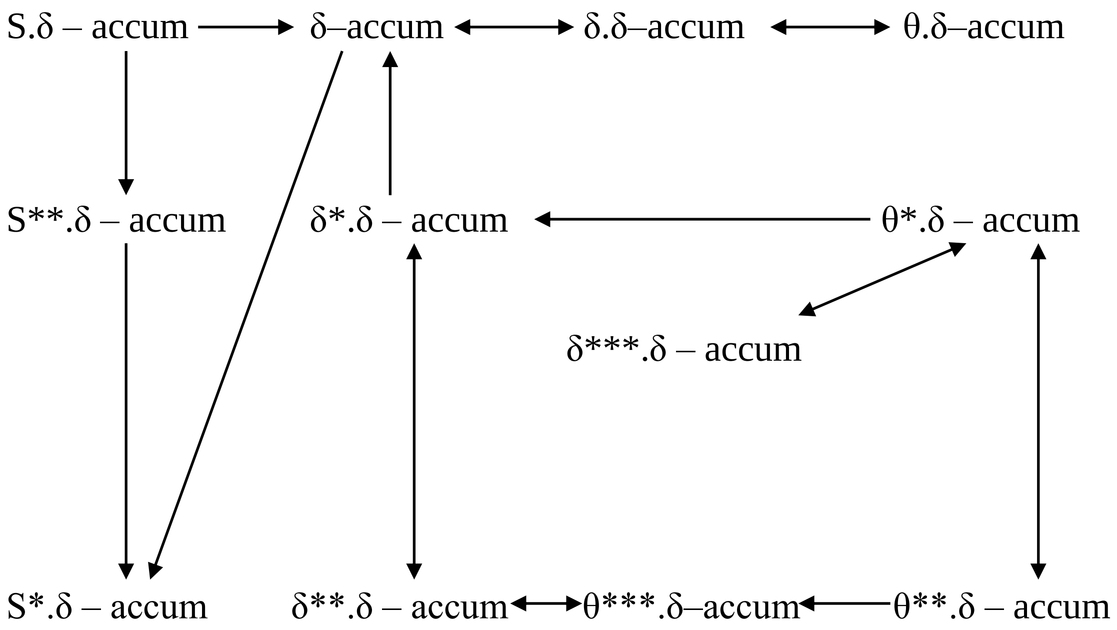
(9) Let x be a δ**.δ – accumulation point of A then , δ.int[δ.cl(N)] $\cap$ (A/{x}) $\neq \phi$ for every open set N containing x .

But every δ – interior point for any set is interior point by Remark 1.4.2.Part 2 , and so δ.cl(N) = θ.cl(N) by Remark 1.4.6.Part 2, therefore , int[θ.cl(N)] $\cap$ (A/{x}) $\neq \phi$ for every open set N containing x . Hence x is θ.δ – accumulation point of A . (10) and (11) can be proved by following the same procedures applicable to the number (9) .

Note 2.2.7 :

The following diagram is taken from the above proposition and the definition of modification of δ – accumulation points stated above .

Let accum = accumulation



Section three : Properties of Convergence Net

Section three deals with convergence by using elementary definitions and results of some properties of semi open , θ – open and δ – open sets .

Definition 2.3.1:[28]

Let A be a non – empty set . We say that a binary relation $\geq$ on A directs the set A , if

[D_1] : $a \in A$ then $a \geq a$.

[D_2] : $a \geq b$ and $b \geq c$ then $a \geq c$ ($a , b , c \in A$) .

[D_3] : given any two members a and b of A , there exists a member $c \in A$ such that $c \geq a$ and $c \geq b$.

We say that a follows b in the order $\geq$ and that b precedes a , if $a \geq b$.

By a directed set , we mean a pair (A , $\geq$) consisting of a non–empty set A and a binary relation $\geq$ defined on A which directs A .

The pair (A, $\geq$) is some times called a directed system .

Definition 2.3.2 :[28]

Let (A , $\geq$) be a directed system and let f be an arbitrary function of A into a set X . Then f is called a net in X and is denoted by (f , X , A , $\geq$) .

We shall denote the image f(a) of $a \in A$ under f by f_a and a net in X will be often denoted by the symbol ($\{f_a : a \in A\}$, $\geq$) or simply by $\{f_a : a \in A\}$.

Definition 2.3.3 : [28]

Let (f , X , A , ≥) be a net and let Y $\subseteq$ X . Then f is said to be frequently in Y , if for each a in A , there exists b ≥ a in A such that $f_b \in$ Y .

Definition 2.3.4 : [28]

Let (X , τ) be a topological space and let (f , X , A , ≥) be a net in X .Then a point $x_o \in$ X is called cluster point of the net f , if f is frequently in every open set N containing x_o , that is , for each a in A , there exists b ≥ a in A such that $f_b \in N$ for every open set N containing x_o .

Definition 2.3.5 : [28]

Let (f , X , A , ≥) be a net and let Y $\subseteq$ X . Then f is said to be eventually in Y , if there exists an element $a_o \in$ A such that for every a ≥ a_o , a $\in$ A then $f_a \in$ Y .

Note 2.3.6 :[28]

Let (f , X , A , ≥) be a net and Y $\subseteq$ X . If the net f is eventually in Y then , f can not be frequently in X/Y .

Definition : 2.3.7 :[28]

Let (f , X , A , ≥) and (g , X , B , ≥*) be two nets . Then g is said to be a subnet of f , if there exists a function

$\Phi : B \to A$ such that

1) $g = f \circ \Phi$ and

2) for each a in A , there exists an element b in B such that $\Phi(x) \geq a$ for every $x \geq^* b$ in B .

Theorem 2.3.8:[28]

Let (X , τ) be a topological space and let (f , X , A , ≥) be a net in X . Let Ω be the collection of subsets of X satisfying the following two conditions :

1) f is frequently in each members of Ω .
2) If S , T are any two members of Ω . Then , there exists a member U of
   Ω such that $U \subseteq S \cap T$. Then there exists a subnet of f which is eventually in each member of Ω .

Definition2.3.9 : [7]

Let (X,τ) be a topological space and let (f , X , A , ≥) be a net in X . We say that f converges to a point $x_o \in X$, if f is eventually in every open set containing x_o , that is , for each open set N containing x_o , there exists an element $a_o \in A$, $a \geq a_o$ then $f_a \in N$, and the point x_o is called limit point of the net f .

Definition 2.3. 10 :

Let (X,τ) be a topological space and let (f , X , A , ≥) be a net in X , and $x_o \in X$. We say that f is :

1) δ – converging to a point x_o , if f is eventually in int[cl(N)] for each open set N containing x_o , that is , for each open set N containing x_o, there exists an element $a_o \in A$, $a \geq a_o$ in A then $f_a \in$ int[cl(N) , and the point x_o is called δ – limit point of the net f .

2) S.δ – converging to a point x_o, if f is eventually in int[S.cl(N)] for each open set N containing x_o , that is , for each open set N containing x_o,

there exists an element $a_o \in A$, $a \geq a_o$ in A then $f_a \in$ int[S.cl(N)] , and the point x_o is called S.δ – limit point of the net f .

3) θ.δ – converging to a point x_o, if f is eventually in int[θ.cl(N)] for each open set N containing x_o , that is , for each open set N containing x_o, there exists an element $a_o \in A$, $a \geq a_o$ in A then $f_a \in$ int[θ.cl(N)] , and the point x_o is called θ.δ – limit point of the net f .

4) δ.δ – converging to a point x_o, if f is eventually in int[δ.cl(N)] for each open set N containing x_o, that is , for each open set N containing x_o, there exists an element $a_o \in A$,$a \geq a_o$ in A then $f_a \in$ int[δ.cl(N)] , and the point x_o is called δ.δ – limit point of the net f .

5) S*.δ – converging to a point x_o, if f is eventuallay in S.int[cl(N)] for each open set N containing x_o , that is , for each open set N containing x_o, there exists an element $a_o \in A$, $a \geq a_o$ in A then $f_a \in$ S.int[cl(N)], and the point x_o is called S.δ – limit point of the net f .

6) θ*.δ – converging to a point x_o, if f is eventually in θ.int[cl(N)] for each open set N containing x_o , that is , for each open set N containing x_o, there exists an element $a_o \in A$, $a \geq a_o$ in A then $f_a \in$ int[cl(N)] , and the point x_o is called θ*.δ – limit point of the net f .

7) δ*.δ – converging to a point x_o, if f is eventually in δ.int [cl(N)] for each open set containing x_o, that is , for each open set N containing x_o, there exists an element $a_o \in A$, $a \geq a_o$ in A then $f_a \in$ δ.int[cl(N)] , and the point x_o is called δ*.δ – limit point of the net f .

8)S**.δ – converging to a point x_o, if is eventually in S.int[S.cl(N)] for each open set N containing x_o , that is , for each open set N containing x_o, there exists an element $a_o \in A$, $a \geq a_o$ in A then $f_a \in$ S.int[S.cl(N)] , and the point x_o is called S**.δ – limit point of the net f .

9)θ**.δ – converging to a point x_o, if f is eventully in θ.int[θ.cl(N)] for each open set N containing x_o , that is , for each open set N containing x_o,there exists an element $a_o \in A$, $a \geq a_o$ in A then $f_a \in$ θ.in[θ.cl(N)] , and the point x_o is called θ**.δ – limit point of the net f

10) δ**.δ – converging to a point x_o, if f is eventually in δ.int[δ.cl(N)] for each open set N containing x_o , that is , for each open set N containing x_o, there exists an element $a_o \in A$,$a \geq a_o$ in A then $f_a \in$ δ.int[δ.cl(N)] , and the point x_o is called δ**.δ – limit point of the net f .

11) δ***.δ – converging to a poin x_o, if f is eventually in θ.int[δ.cl(N)] for each open set N containing x_o , that is , for each open set N containing x_o, there exists an element $a_o \in A$, $a \geq a_o$ in A then $f_a \in$ θ.int[cl(N)] , and the point x_o is called δ***.δ – limit point of the net f .

12) θ***.δ – converging to a point x_o, if f is eventually in δ.int[θ.cl(N)] for each open setN containing x_o, that is , for each open set N containing x_o, there exists an element $a_o \in A$, $a \geq a_o$ in A then $f_a \in$ δ.int[θ.cl(N)] , and the point x_o is called θ***.δ – limit point of the net f .

Example 2.3.11:

Let (X , τ) be an indiscrete topological space .Then , every net in X is :

1) S.δ – converges to every point in X .

2) $\theta.\delta$ – converges to every point in X .

3) $S^*.\delta$ – converges to every point in X .

4) $\theta^*.\delta$ – converges to every point in X .

5) $\delta^*.\delta$ – converges to every point in X .

6) $S^{**}.\delta$ – converges to every point in X .

7) $\theta^{**}.\delta$ –converges to every point in X .

8) $\delta^{**}.\delta$ – converges to every point in X .

10)$\delta^{***}.\delta$ – converges to every point in X .

11)$\theta^{***}.\delta$ – converges to every point in X .

12)δ – converges to every point in X .

Proof :

(1) Let (f , X , A , $\geq$) be a net in X , and let x be an arbitrary point of X .Then , the only open set containing x is X .
Since $f_a \in X$ for all $a \in A$,
and $X \subseteq int[S.cl(X)]$ by Theorem 1.4.12 .Part 6 ,
also $int[S.cl(X)] \subseteq X$, consequently , $X = int[S.cl(X)]$.
Hence $f_a \in int[S.cl(X)]$ for all $a \in A$. It follows that f is $S.\delta$ – converges to x .

(2), (3), (4), (5), (6), (7), (8), (9), (10), (11), and (12) can be proved by the same manner .

Note 2.3.12 :

The above example shows that a net in an indiscrete topological space may be :

1) $S.\delta$ – converges to several different points .

2) $\theta.\delta$ – converges to several different points .

3) $\delta.\delta$ – converges to several different points .

4) $S^*.\delta$ – converges to several different points .

5) $\theta^*.\delta$ – converges to several different points .

6) $\delta^*.\delta$ – converges to several different points .

7) $S^{**}.\delta$ – converges to several different points .

8) $\theta^{**}.\delta$ – converges to several different points .

9) $\delta^{**}.\delta$ – converges to several different points .

10) $\delta^{***}.\delta$ – converges to several different points .

11) $\theta^{***}.\delta$ – converges to several different points.

12) δ – converges to several different points .

Example 2.3.13 :

Let (X,τ) be a discrete topological space and $x_o \in X$ and let $(f , X , A , \geq)$ be a net in X then, the net f is :

1) $S.\delta$ – converges to a point x_o if and only if f is eventually in $\{x_o\}$,that is ,if and only if there exists an element $a_o \in A$ such that $f_a = x_o$ for all $a \geq a_o$ in A .

2) $\theta.\delta$ – converges to a point x_o if and only if f is eventually in $\{x_o\}$, that is , if and only if there exists an element $a_o \in A$ such that $f_a = x_o$ for all $a \geq a_o$ in A .

3) $\delta.\delta$ – converges to a point x_o if and only if f is eventually in $\{x_o\}$, that is ,if and only if there exists an element $a_o \in A$ such that $f_a = x_o$ for all $a \geq a_o$ in A .

4) $S^*.\delta$ – converges to a point x_o if and only if f is eventually in $\{x_o\}$, that is ,if and only if there exists an element $a_o \in A$ such that $f_a = x_o$ for all $a \geq a_o$ in A .

5) $\theta^*.\delta$ – converges to a point x_o, if and only if f is eventually in $\{x_o\}$, that is , if and only if there exists an element $a_o \in A$ such that $fa = x_o$ for all $a \geq a_o$ in A .

6) $\delta^*.\delta$ – converges to a point x_o , if and only if f is eventually in $\{x_o\}$, that is , if and only if there exists an element $a_o \in A$ such that $f_a = x_o$ for all $a \geq a_o$ in A .

7) $S^{**}.\delta$ – converges to a point x_o if and only if f is eventually in $\{x_o\}$, that is , if and only if there exists an element $a_o \in A$ such that $f_a = x_o$ for all $a \geq a_o$ in A .

8) $\theta^{**}.\delta$ – converges to a point x_o if and only if f is eventually in $\{x_o\}$, that is ,if and only if there exists an element $a_o \in A$ such that $f_a = x_o$ for all $a \geq a_o$ in A .

9) $\delta^{**}.\delta$ – converges to a point x_o if and only if f is eventually in $\{x_o\}$, that is , if and only if there exists an element $a_o \in A$ such that $f_a = x_o$ for all $a \geq a_o$ in A .

10) $\delta^{***}.\delta$ – converges to a point x_o if and only if f is eventually in $\{x_o\}$, that is ,if and only if there exists an element $a_o \in A$ such that $f_a = x_o$ for all $a \geq a_o$ in A .

11) $\theta^{***}.\delta$ – converges to a point x_o if and only if f is eventually in $\{x_o\}$, that is ,if and only if there exists an element $a_o \in A$ such that $f_a = x_o$ for all $a \geq a_o$ in A .

12) δ – converges to a point x_o , if and only if f is eventually in $\{x_o\}$, that is , if and only if there exists an element $a_o \in A$ such that $f_a = x_o$ for all $a \geq a_o$ in A .

Proof :

(1) Let f S.δ – converges to x_o and since (X , τ) is a discrete topological space , therefore ,$\{x_o\}$ is an open set containing x_o, and so there exists an element $a_o \in A$, $a \geq a_o$ in A which implies that $f_a \in int[S.cl(\{x_o\})]$.

But by Theorem 1.4.14.Part 3, $\{x_o\} = int[S.cl(\{x_o\})]$, consequently $f_a \in \{x_o\}$, that is , f is eventually in $\{x_o\}$.

Conversely, Let f be eventually in $\{x_o\}$. Since every open set containing x_o contains $\{x_o\}$, therefore f is eventually in every open set N containing x_o, and since N = int[S.cl(N)] by Theorem 1.4.14.Part 3, It follows that f is eventually in int[S.cl(N)] for every open set N containing x_o. Hence f S.δ – converges to x_o .

(2), (3), (4), (5), (6), (7), (8), (9), (10), (11) and (12) can be proved by adopting the same items .

The following Propositions 2.3.14, 2.3.15 and 2.3.16, can be proved by using Definition 2.3.10, and Remark 1.4.6.Part 2 .

proposition 2.3.14:

Let (X, τ) be a topological space, and let (f , X , A , ≥) be a net in X , and $x \in X$. Then the following statements are equivalents :

1) x is δ – limit point of the net f .

2) x is θ.δ – limit point of the net f .

3) x is δ.δ – limit point of the net f .

proposition 2.3.15 :

Let (X, τ) be a topological space, and let (f , X , A , $\geq$) be a net in X , and x $\in$ X . Then the following statements are equivalents :

1) x is θ*.δ – limit point of the net f .

2) x is θ**.δ – limit point of the net f .

3) x is δ***.δ – limit point of the net f .

proposition 2.3.16 :

Let (X, τ) be a topological space, and let (f , X , A , $\geq$) be a net in X , and x $\in$ X . Then the following statements are equivalents :

1) x is δ*.δ – limit point of the net f .

2) x is δ**.δ – limit point of the net f .

3) x is θ***.δ – limit point of the net f .

proposition 2.3.17 :

Let (X , τ) be a topological space and let (f , X , A , $\geq$) be a net in X , then every :

1)S.δ – limit point of the net f is δ – limit point .

2) S**.δ – limit point of the net f is S*.δ – limit point .

3) δ – limit point of the net f is S*.δ – limit point .

4) S.δ – limit point of the net f is S**.δ – limit point .

5) δ*.δ – limit point of the net f is δ – limt point .

6) θ*.δ – limit point of the net f is δ*.δ – limit point .

7) θ**.δ – limit point of the net f is θ***.δ – limit point .

8) δ***.δ – limit point of the net f is δ**.δ – limit point .

9) δ**.δ – limit point of the net f is θ.δ – limit point .

10)θ*.δ – limit point of the net f is θ***.δ – limit point .

11)δ***.δ – limit point of the net f is δ – limit point .

Proof :

(1) Let x_o be a S.δ – limit point of the net f then , for each open set N containing x_o there exists an element $a_o \in A$ such that $a \geq a_o$, $a \in A$ then fa $\in$ int[S.cl(N)] . By Theorem 1.4.5, we get that S.cl(N) $\subseteq$ cl(N) , So int[S.cl(N)] $\subseteq$ int[cl(N)] , therefore , fa $\in$ int (cl(N) , for each open set N containing x_o .

Hence x is δ – limit point of the net f .

Similarly we can prove number (2) .

(3) Let x_o be a δ – limit point of the net f then , for each open set N containing x_o ,there exists an element $a_o \in A$ such that $a \geq a_o$, $a \in A$ then fa $\in$ int[cl(N)] .But every interior point is semi interior point by Theorem 1.1.8.Part 1, therefore , fa $\in$ S.int[cl(N)] for each open set N containing x_o .

Hence x is S*.δ – limit point of the net f .

Similarly we can prove numbers (4) , (5) , (6) , (7) and (8) .

(9) Let x_o be a δ**.δ – limit point of the net f then for every open set N containing x_o, there exists an element $a_o \in A$ such that for each $a \in A$, $a \geq a_o$ then fa $\in$ δ.int[δ.cl(N)] . But every δ – interior point is interior point by Remark 1.4.2.Part 2, and so δ.cl(N) = θ.cl(N) by Remark 1.4.6.Part 2 , therefore , fa $\in$ int[θ.cl(N)] for every open set N containing x_o . Hence x is θ.δ – limit point of the net f .

Similarly we can prove numbers (10) and (11) .

proposition 2.3.17 :

Let (X , τ) be a topological space and let (f , X , A , ≥) be a net in X , then every :

1)S.δ – limit point of the net f is δ – limit point .

2) S**.δ – limit point of the net f is S*.δ – limit point .

3) δ – limit point of the net f is S*.δ – limit point .

4) S.δ – limit point of the net f is S**.δ – limit point .

5) δ*.δ – limit point of the net f is δ – limt point .

6) θ*.δ – limit point of the net f is δ*.δ – limit point .

7) θ**.δ – limit point of the net f is θ***.δ – limit point .

8) δ***.δ – limit point of the net f is δ**.δ – limit point .

9) δ**.δ – limit point of the net f is θ.δ – limit point .

10)θ*.δ – limit point of the net f is θ***.δ – limit point .

11)δ***.δ – limit point of the net f is δ – limit point .

Proof :

(1) Let x_o be a S.δ – limit point of the net f then , for each open set N containing x_o there exists an element $a_o \in A$ such that $a \geq a_o$, $a \in A$ then $fa \in$ int[S.cl(N)] . By Theorem 1.4.5, we get that S.cl(N) $\subseteq$ cl(N) , So int[S.cl(N)] $\subseteq$ int[cl(N)] , therefore , $fa \in$ int (cl(N) , for each open set N containing x_o .

Hence x is δ – limit point of the net f .

Similarly we can prove number (2) .

(3) Let x_o be a δ – limit point of the net f then , for each open set N containing x_o ,there exists an element $a_o \in A$ such that $a \geq a_o$, $a \in A$ then $fa \in$ int[cl(N)] .But every interior point is semi interior point by Theorem 1.1.8.Part 1, therefore , $fa \in$ S.int[cl(N)] for each open set N containing x_o .

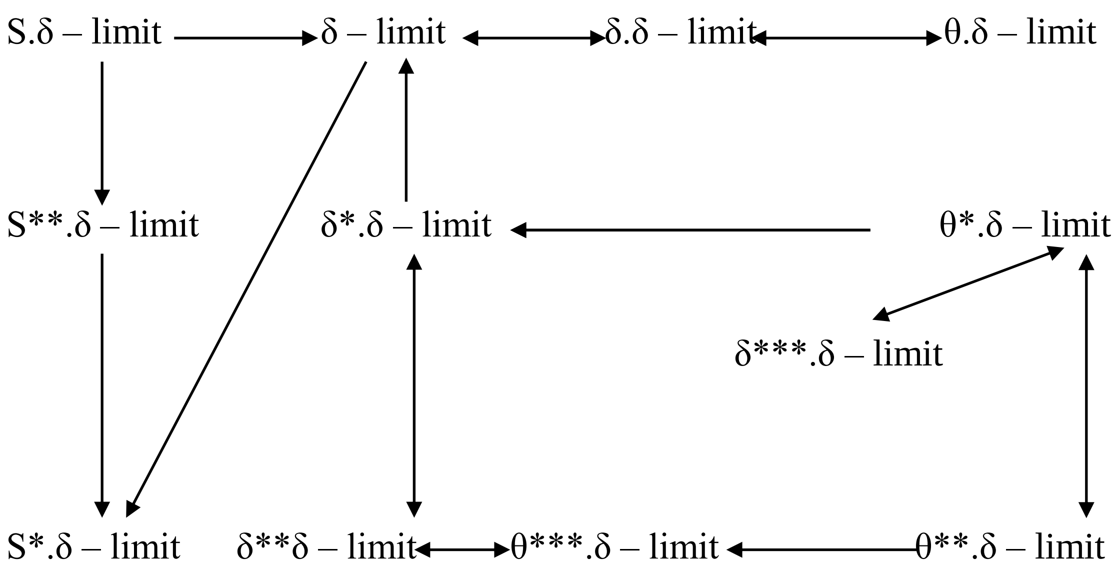
Hence x is S*.δ – limit point of the net f .

Similarly we can prove numbers (4) , (5) , (6) , (7) and (8) .

(9) Let x_o be a δ**.δ – limit point of the net f then for every open set N containing x_o, there exists an element $a_o \in A$ such that for each $a \in A$, $a \geq a_o$ then $fa \in$ δ.int[δ.cl(N)] . But every δ – interior point is interior point by Remark 1.4.2.Part 2, and so δ.cl(N) = θ.cl(N) by Remark 1.4.6.Part 2 , therefore , $fa \in$ int[θ.cl(N)] for every open set N containing x_o . Hence x is θ.δ – limit point of the net f .
Similarly we can prove numbers (10) and (11) .

Note 2.3.18 :

The following diagram is taken from the above proposition and definition of modification of δ – limit points of the net stated above .



Lemma 2.3.19 :

Let (X , τ) be a topological space and let (f , X , A , ≥) be a net in X , converges to a point $x_o \in$ X . Then the net f is :

1) S.δ – converges to x_o .
2) δ – converges to x_o .
3) S*.δ – converges to x_o .
4) θ*.δ – converges to x_o .
5) δ*.δ – converges to x_o .
6) S**.δ – converges to x_o .

Proof :

(1) Suppose (f , X , A , ≥) be a net converging to a point x_o in X then , for every open set N containing x_o , there exists $a_o \in$ A such that for each a $\in$ A , a ≥ a_o then fa $\in$ N . Since N $\subseteq$ int[S.cl(N)] by Theorem 1.4.12.Part 6, therefore ,fa $\in$ int[S.cl(N)] for every open set N containing x_o . Hence f is S.δ – converges net to the point $x_o \in$ X .
(2), (3), (4), (5) and (6) can be proved by following the same procedures applicable to the number (1) .

Definition 2.3.20 :

Let (X , τ) be a topological space and let (f , X , A , ≥) be a net in X then , a point x $\in$ X is called :

1)δ – cluster point of the net f , if for each open set U containing x , f is frequently in int[cl(U)] , that is , for each a in A , there exists b ≥ a in A such that $f_b \in$ int[cl(U)] , for each open set U containing x .

2) S.δ – cluster point of the net f , if for each open set U containing x , f is frequently in int[S.cl(U)] ,that is, for each a in A , there exists $b \geq a$ in A such that $f_b \in$ int[S.cl(U)] , for each open set U containing x.

3) $\theta.\delta$ – cluster point of the net f , if for each open set U containing x , f is frequently in int[(θ.cl(U)] , that is , for each a in A , there exists $b \geq a$ in A such that $f_b \in$ int[θ.cl(U)] , for each open set U containing x .

4) $\delta.\delta$ – cluster point of the net f , if for each open set U containing x , f is frequently in int[δ.cl(U)] ,that is, for each a in A , there exists $b \geq a$ in A such that $f_b \in$ int[δ.cl(U)] , for each open set U containing x .

5) S*.δ – cluster point of the net f , if for each open set U containing x, f is frequently in S.int[cl(U)] , that is , for each a in A , there exists $b \geq a$ inA such that $f_b \in$ S.int[cl(U)] , for each open set U containing x .

6)θ*.δ – cluster point of the net f , if for each open set U containing x , f is frequently in θ.int[cl(U) , that is , for each a inA , there exists $b \geq a$ in A such that $f_b \in$ θ.int[cl(U)] , for each open set U containing x .

7) δ*.δ – cluster point of the net f , if for each open set U containing x , f is frequently in δ.int[cl(U)] , that is , for each a in A , there exists $b \geq a$ in A such that $f_b \in$ δ.int[cl(U)] , for each open set U containing x .

8) S**.δ – cluster point of the net f , if for each open set U containing x , f is frequently in S.int[S.cl(U)] , that is , for each a in A , there exists $b \geq a$ in A such that $f_b \in$ S.int[S.cl(U)] , for each open set U containing x .

9)θ**.δ – cluster point of the net f , if for each open set U containing x , f is frequently in θ.int[θ.cl(U)] , that is , for each a in A , there exists b $\geq$ a inA such that $f_b \in \theta$.int[θ.cl(U)] , for each open set U containing x .

10)δ**.δ – cluster point of the net f , if for each open set U containing x , f is frequently in δ.int[δ.cl(U)] , that is , for each a in A, there exists b $\geq$ a in A such that $f_b \in \delta$.int[δ.cl(U)] , for each open set U containing x .

11)δ***.δ – cluster point of the net f , if for each open set U containing x , f is frequently in θ.int[δ.cl(U)] , that is , for each a in A there exists b $\geq$ a in A such that $f_b \in \theta$.int[δ.cl(U) , for each open set U containing x .

12)θ***.δ – cluster point of the net f , if for each open set containing x , f is frequently in δ.int[θ.cl(U)] , that is , for each a in A there exists b $\geq$ a in A such that $f_b \in \delta$.int[θ.cl(U)] , for each open set U containing x .

Lemma 2.3.21 :

Let (X,τ) be a topological space and let (f , X , A , $\geq$) be a net in X , then every :

1) S. δ – limit point of the net f is S.δ – cluster point .
2) θ.δ – limit point of the net f is θ.δ – cluster point .
3) δ.δ – limit point of the net f is δ.δ – cluster point .
4) S*.δ – limit point of the net f is S*.δ – cluster point .
5) θ*.δ – limit point of the net f is θ*.δ – cluster point .
6) δ*.δ – limit point of the net f is δ*.δ – cluster point .
7) S**.δ – limit point of the net f is S**.δ – cluster point .
8) θ**.δ – limit point of the net f is θ**.δ – cluster point .
9) δ**.δ – limit point of the net f is δ**.δ – cluster point .

10)δ***.δ – limit point of the net f is δ***.δ – cluster point .

11)θ***.δ – cluster point of the net f is θ***.δ – cluster point .

12) δ – limit point of the net f is δ – cluster point .

Proof :

(1) Let x be a S.δ – limit point of f , and let N be an arbitrary open set containing x then , there exists $a_o \in A$ such that for each $a \geq a_o$ in A , then $f_a \in$ int[S.cl(N)]………(1). Since A is a directed set ,we get that for all $c \in A$, and since $a_o \in A$,

there exists $b \in A$ such that $b \geq c$ and $b \geq a_o$. Therefore ,

$f_b \in$ int [S.cl(N)] from (1) .Hence x is S.δ – cluster point of f .

Similarly we can prove numbers (2), (3), (4), (5), (6), (7), (8), (9), (10) , (11) and (12) .

Theorem 2.3.22 :

Let (X , τ) be a topological space . A point x_o in X is a :

1) δ – cluster point of a net (f , X , A , $\geq$) if and only if there exists a subnet (g , X , B , $\geq$*) which δ – converges to x_o .

2) S.δ – cluster point of a net (f , X , A , $\geq$) if and only if there exists a subnet (g , X , B , $\geq$*) which S.δ – converges to x_o.

3) θ*.δ – cluster point of a net (f , X , A , $\geq$) if and only if there exists a subnet (g , X , B , $\geq$*) which θ*.δ – converges to x_o.

4) S*.δ – cluster point of a net (f , X , A , $\geq$) if and only if there exists a subnet (g , X , B , $\geq$*) which S*.δ – converges to x_o.

5) $\delta^*.\delta$ – cluster point of a net (f , X , A , $\geq$) if and only if there exists a subnet (g , X , A , $\geq^*$) which $\delta^*.\delta$ – converges to x_o .

6) $S^{**}.\delta$ – cluster point of a net (f , X , A , $\geq$) if and only if there exists a subnet (g , X , A , $\geq^*$) which $S^{**}.\delta$ – converges to x_o .

Proof :

(1) The " if " part . Assume that f has a subnet g which δ – converges to x_o .To prove that x_o is a δ – cluster point of f . Let N be τ – open set containing x_o and let a_o be any element of A . Since g is a subnet of f , there exists a function

$\Phi : B \rightarrow A$ such that

1) $g = f \circ \Phi$ and

2) For each a in A , there exists an element b in B such that $\Phi(x) \geq a$ for every $x \geq^* b$ in B .

Hence by (2) . Corresponding to $a_o \in A$, there exists an element $b_o \in B$ such that $\Phi(x) \geq a_o$ for every $x \geq^* b_o$. Since g δ – converges to x_o , there exists an element $p \geq^* b_o$ in B such that $g(p) \in int[cl(N)]$. Now let $\Phi(p) = q$. Then $q \in A$ and $q \geq a_o$. Also $f(q) = f(\Phi(p)) = (f\circ\Phi)(p) = g(p) \in int[cl(N)]$. Thus we have Shown that for each element a_o in A , there exists an element $q \geq a_o$ in A such that $f(q) \in int[cl(N)]$. Hence f is frequently in int[cl(N)] . It follows that x_o is a δ – cluster point of f .

The " only if " part . Let x_o be a δ – cluster point of the net (f , X , A , $\geq$) and let $N(x_o)$ be the collection of all open subsets of X containing x_o .

Let $\Omega = \{N_1 : N_1 = int[cl(N)] \text{ and } N \in N(x_o)\}$. If $L_1 = int[cl(L)]$

and M_1 = int[cl(M)] are any two members of Ω , since L ∩ M ∈ N(x_o), then int[cl(L ∩ M)] is also a member of Ω and int[cl(L ∩ M)] ⊆ int[cl(L)] and so int[cl(L ∩ M)] ⊆ int[cl(M)] , also since x_o is a δ – cluster point of f , then f is frequently in each member of Ω . Hence by Theorem 2.3.8, there exists a subnet g of f which is eventually in each member of Ω . This implies that g is δ – converges to x_o.

(2) , (3) , (4) , (5) and (6) can be proved by adopting the same items .

Corollary 2.3.23 :

Let (X , τ) be a topological space . A point x_o in X is (δ – cluster point , S.δ – cluster point , θ*.δ – cluster point , S*.δ – cluster point , δ*.δ – cluster point and S**.δ – cluster point) of a net (f , X , A , ≥) , if there exists a subnet (g , X , B , ≥*) which converges to x_o .

Proof :

By Lemma 2.3.19, and by Theorem 2.3.22.

The following Propositions 2.3.24, 2.3.25and 2.3.26 can be proved by using Definition 2.3.20, and Remark 1.4.6.Part 2 .

proposition 2.3.24:

Let (X, τ) be a topological space, and let (f , X , A , ≥) be a net in X , and x ∈ X . Then the following statements are equivalents :

1) x is δ – cluster point of the net f .

2) x is θ.δ – cluster point of the net f .

3) x is $\delta.\delta$ – cluster point of the net f .

proposition 2.3.25 :

Let (X, τ) be a topological space, and let (f , X , A , $\geq$) be a net in X , and x $\in$ X . Then the following statements are equivalents :

1) x is $\theta^*.\delta$ – cluster point of the net f .

2) x is $\theta^{**}.\delta$ – cluster point of the net f .

3) x is $\delta^{***}.\delta$ – cluster point of the net f .

proposition 2.3.26:

Let (X, τ) be a topological space, and let (f , X , A , $\geq$) be a net in X , and x $\in$ X . Then the following statements are equivalents :

1) x is $\delta^*.\delta$ – cluster point of the net f .

2) x is $\delta^{**}.\delta$ – cluster point of the net f .

3) x is $\theta^{***}.\delta$ – cluster point of the net f .

Lemma 2.3.27 :

Let (X,τ) be a topological space and let (f , X , A , $\geq$) be a net in X then , every cluster point of the net f is :

1) $S.\delta$ – cluster point .
2) δ – cluster point .
3) $\theta^*.\delta$ – cluster point .
4) $S^*.\delta$ – cluster point .
5) $\delta^*.\delta$ – cluster point .

6) S**.δ – cluster point .

Proof :

(1) Suppose $x_o \in X$, and x_o is a cluster point of the net f , then for every open set N containing x_o, and for all $a \in A$ there exists $b \geq a$ in A such that $f_b \in N$.

Since $N \subseteq int[S.cl(N)]$ by Theorem 1.4.12. Part 6, therefore , $f_b \in int[S.cl(N)]$. Hence x_o is S.δ – cluster point of the net f .

(2), (3), (4), (5) and (6) can be proved by the same manner .

Example 2.3.28 :

Let X = {1 , 2 , 3 } and τ = { ϕ, {1} , {1 , 2} , X } , and let f be a net of N into X , where N is the set of natural numbers.

Find :

(1) the limit points , the δ – limit points , the S.δ – limit points, the δ*.δ – limit points, the S*.δ – limit points , the θ*.δ – limit points and the S**.δ – limit points .(2)The cluster points , the δ – cluster points , the S.δ – cluster points , the δ*.δ – cluster points ,the S*.δ– cluster points , the θ*.δ – cluster points and the S**.δ – cluster points of the following nets in X ,

(a) < 1 , 2 , 1 , 2 , 1 , 2 , …….. > . (b) < 1 , 1 , 1 , 1 , …….. > .

Solution :

(a) (1) . This net converges to the point 2 , since the only open sets containing 2 are {1 , 2} and X both of which contain all term of all net . Similarly converges to 3 , since the only open set containing 3 is X which contains all terms of the net , therefore 2 and 3 are limit points of this net ,also the points 2 and 3 are δ – limit points , S.δ – limit points, δ*.δ – limit points , S*.δ – limit points , θ*. δ – limit points and S**. δ – limit points of this net in X by Lemma 2.3.19 . But 1 is not a limit point of this net , since {1} is open set containing 1 such that the net is not eventually in this open set {1} .

Since τ – closed = {X , {2 , 3} , {3} , ϕ}, therefore ,

int[cl(N)] = X , S.int[cl(N)] = X , δ.int[cl(N)] = X ,

θ.int[cl(N)] = X , and since {ϕ, {a} , {a ,b} , X , {a ,c}} the family of all semi open sets in X , therefore , {X , {b , c} , {c} , ϕ, {b}} the family of all semi closed in X , so int[S.cl(N)] = X and S.int[S.cl(N)] = X, for every open set N containing 1 . and so the net is eventually in X , hence 1 is δ – limit point , S.δ – limit point , δ*.δ – limit point , S*.δ – limit point, θ*.δ – limit point and S**.δ – limit point of A , by Definition 2.3.10 .

(2) Since 2 and 3 are limit points of this net they are certainly cluster points . The point 1 is also a cluster point of this net since the only open sets containing 1 are {1} , {1, 2} and X each of which contains an infinite number of terms of net , hence 1, 2 and 3 are the cluster points of this net,and by Lemma 2.3. 21, we get that 1 , 2 and 3 are δ – cluster point , S.δ – cluster point , δ*.δ – cluster point , S*.δ – cluster point , θ*.δ – cluster point and S**.δ – cluster point of A .

(b) (1) The open sets containing 1 are {1} , {1, 2} and X , since the net < 1 , 1 , 1 , 1 , …….. > is eventually in each of these open sets , it follows that 1 is a limit point of this net . Again the open sets containing 2 are {1 , 2} and X in each of which the net eventually lies . Hence 2 is also a limit point of this net .

Similarly 3 is a limit point of this net since the only open set containing c is X in which the net lies . Thus 1 , 2 and 3 are limit points of this net . Also 1 , 2 and 3 are δ – limit point , S.δ – limit point , δ*.δ – limit point , S*.δ – limit point , θ*.δ – limit point and S**.δ – limit point of A) by Lemma 2.3.19 .

(b) (2) Since 1, 2 and 3 are limit points , δ – limit points , S.δ–limit points , δ*.δ – limit points , S*.δ – limit points , θ*.δ – limit points and S**.δ – limit points of A , they are certainly cluster points , δ – cluster points ,S.δ – cluster points , δ*.δ – cluster points , S*.δ – cluster points, θ*.δ – cluster points and S**.δ – cluster points of A , by Lemma 2.3.21 .

proposition 2.3.29 :

Let (X,τ) be a topological space and let (f , X , A , ≥) be a net in X then , every:

1) S.δ – cluster point of the net f is δ – cluster point .

2) S**.δ – cluster point of the net f is S*.δ – cluster point .

3) δ – cluster point of the net f is S*.δ – cluster point .

4) S.δ – cluster point of the net f is S**.δ – cluster point .

5) δ*.δ – cluster point of the net f is δ – cluster point .

6) θ*.δ – cluster point of the net f is δ*.δ – cluster point .

7) θ**.δ – cluster point of the net f is θ***.δ – cluster point .

8)δ***.δ – cluster point of the net f is δ**.δ – cluster point .

9)δ**.δ – cluster point of the net f is θ.δ – cluster point .

10)θ*.δ – cluster point of the net f is θ***.δ – cluster point .

11)δ***.δ – cluster point of the net f is δ – cluster point .

Proof :

(1) Let x be a S.δ – cluster point of the net f then , for each open set U containing x , f is frequently in int[S.cl(U)] , that is , for each open set U containing x and for each a in A , there exists $b \geq a$ in A such that $f_b \in$ int[S.cl(U)] . and since S.cl(U) $\subseteq$ cl(U) by Theorem 1.4.5, so int[S.cl(U)] $\subseteq$ int[cl(U)] , therefore , $f_b \in$ int[cl(U) , for each open set U containing x .
Hence x is δ – cluster point of the net f .
Similarly we can prove number (2) .

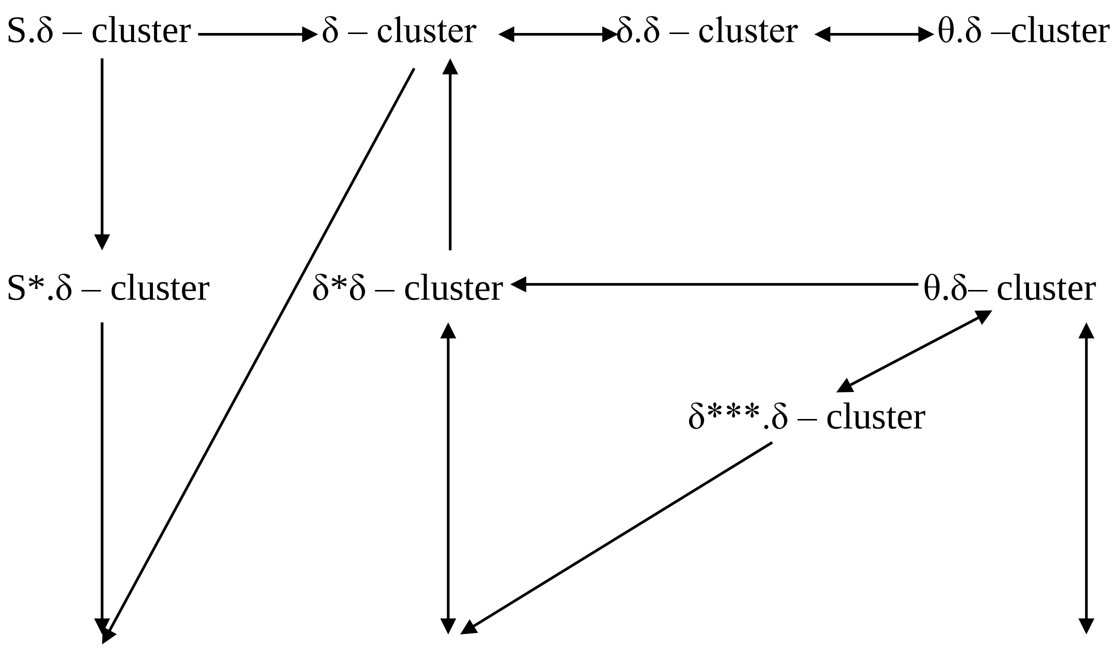
(3) Let x be a δ – cluster point of the net f then , for each open set U containing x , f is frequently in int[cl(U)] , that is , for each open set U containing x and for each a in A , there exists $b \geq a$ in A such that $f_b \in$ int[cl(U)] . But for any set every interior point is semi interior point by Theorem 1.1.8.Part 1 , therefore , we get that $f_b \in$ S.int[cl(U)] for each open set U containing x .
Hence x is S*.δ – cluster point of the net f .
Similarly we can prove numbers (4) , (5) , (6) , (7) and (8) .

9) Let x be δ**.δ – cluster point of the net f then , for each open set U containing x , f is frequently in δ.int[δ.cl(U)] .
But every δ – interior point is interior point by Remark 1.4.2.Part 2, and so δ.cl(U) = θ.cl(U) by Remark 1.4.6.Part 2 .
Therefore , f_b is frequently in int[θ.cl(U)] , for each open set U containing x . Hence x is θ.δ – cluster point of the net f .
(10) and (11) can be proved by following the same procedures applicable to the number (9) .

Note 2.3.30 :

The following diagram is taken from the above proposition and definition

of modification of δ – cluster points of the net stated above .



S*. δ – cluster δ**.δ – cluster ⟷ θ***.δ–cluster ⟵ θ**.δ –cluster

Section four :Types of Hausdorff Spces

In this section, we define and study the kinds of Hausdorff spaces by using some properties of semi open , θ – open and δ – open sets .

A topological space (X , τ) is said to be a Hausdorff space , if for every pair of distinct points x , y of X ,there exist disjoint open sets of x and y , that is , there exist open sets N containing x and M containing y such that $N \cap M = \phi$.[28]

Definition 2.4.1 :

A topological space (X , τ) is said to be a :

1) δ – Hausdorff space , if for all points $x \neq y$ in X ,there exists open sets N containing x and M containing y such that

$$\text{int}[\text{cl}(N)] \cap \text{int}[\text{cl}(M)] = \phi \ .$$

2) S.δ – Hausdorff space , if for all points $x \neq y$ in X , there exists open sets containing x and M containing y , Such that

$$int[S.cl(N)] \cap int[S.cl(M)] = \phi .$$

3) θ.δ – Hausdorff space , if for all points $x \neq y$ in X , there exists open sets N containing x and M containing y , such that

$$int[\theta.cl(N)] \cap int[\theta.cl(M)] = \phi .$$

4) δ.δ – Hausdorff space , if for all points $x \neq y$ in X , there exists open sets N containing x and M containing y , such that

$$int[\delta.cl(N)] \cap int[\delta.cl(M)] = \phi .$$

5) S*. δ – Hausdorff space , if for all $x \neq y$ in X , there exists open sets N containing x and M containing y , such that

$$S.int[cl(N)] \cap S.int[cl(M)] = \phi .$$

6) θ*.δ – Hausdorff space , if for all points $x \neq y$ in X , there exist open sets N containing x and M containing y , such that

$$\theta.int[cl(N)] \cap \theta.int[cl(M)] = \phi .$$

7) δ*.δ – Hausdorff space , if for all $x \neq y$ in X , there exists open sets N containing x and M containing y , such that

$$\delta.int[cl(N)] \cap \delta.int[cl(M)] = \phi .$$

8) S**.δ – Hausdorff space , if for all points $x \neq y$ in X , there exist open sets N containing x and M containing y , such that

$$S.int[S.cl(N)] \cap S.int[S.cl(M)] = \phi .$$

9) $\theta^{**}.\delta$ – Hausdorff space , if for all points $x \neq y$ in X , there exists open sets N containing x and M containing y, such that

$$\theta.int[\theta.cl(N)] \cap \theta.int[\theta.cl(M)] = \phi .$$

10) $\delta^{**}.\delta$ – Hausdorff space , if for all points $x \neq y$ in X ,there exists open sets N containing x and M containing y, such that

$$\delta.int[\delta.cl(N)] \cap \delta.int[\delta.cl(M)] = \phi .$$

11) $\delta^{***}.\delta$ – Hausdorff space , if for all points $x \neq y$ in X , there exists open sets N containing x and M containing y, such that

$$\theta.int[\delta.cl(N)] \cap \theta.int[\delta.cl(M)] = \phi .$$

12) $\theta^{***}.\delta$ – Hausdorff space , if for all points $x \neq y$ in X , there exists open sets N containing x and M containing y , such that

$$\delta.int[\theta.cl(N)] \cap \delta.int[\theta.cl(M)] = \phi .$$

The following propositions can be proved by using Definition 2.4.1, and Remark 1.4.6.Part 2 .

Proposition 2.4.2 :

Let (X , τ) be a topological space , then the following statements are equivalents :

1) (X , τ) is δ – Hausdorff space .
2) (X , τ) is $\theta.\ \delta$ – Hausdorff space .

3) (X , τ) is δ.δ – Hausdorff space .

Proposition 2.4.3 :

Let (X , τ) be a topological space , then the following statements are equivalents :

1) (X , τ) is θ*.δ – Hausdorff topological space .

2) (X , τ) is θ**. δ – Hausdorff topological space .

3) (X , τ) is δ***.δ – Hausdorff topological space .

Proposition 2.4.4 :

Let (X , τ) be a topological space , then the following statements are equivalents :

1) (X , τ) is δ*.δ – Hausdorff space .

2) (X , τ) is δ**. δ – Hausdorff space .

3) (X , τ) is θ***.δ – Hausdorff space .

Lemma 2.4.5 :

Let (X , τ) be a topological space then , Every :

1) S.δ – Hausdorff space is Hausdorff space .

2) θ.δ – Hausdorff space is Hausdorff space .

3) S*.δ – Hausdorff space is Hausdorff space .

4) θ*.δ – Hausdorff space is Hausdorff space .

5) δ*.δ – Hausdorff space is Hausdorff space .

6) S**.δ – Hausdorff space is Hausdorff space .

Proof :

(1) Let (X , τ) be a S.δ – Hausdorff space then , for all points $x \neq y$ in X there exist open sets N containing x and M containing y such that int[S.cl(N)] $\cap$ int[S.cl(M)] = ϕ . But N $\subseteq$ int[S.cl(N)] , and also M $\subseteq$ int[S.cl(M)] by Theorem 1.4. 12.Part 6 , we get that N $\cap$ M = ϕ. Hence , the topological space (X ,τ) is Hausdorff space .

(2) , (3) , (4) , (5) and (6) can be proved by adopting the same items .

Proposition 2.4.6 :

Let (X,τ) be a topological space then , every :

1) δ – Hausdorff space is S.δ – Hausdorff space .
2) S*.δ – Hausdorff space is S**.δ – Hausdorff space .

3) S*.δ – Hausdorff space is δ – Hausdorff space .
4) S**.δ – Hausdorff space is S.δ – Hausdorff space .
5) δ – Hausdorff space is δ*.δ – Hausdorff space .
6) δ*.δ – Hausdorff space is θ*.δ – Hausdorff space .
7) θ***.δ – Hausdorff space is θ**.δ – Hausdorff space .
8) δ**.δ – Hausdorff space is δ***.δ – Hausdorff space .
9) θ.δ – Hausdorff space is δ**δ – Hausdorff space .
10)θ***.δ – Hausdorff space is θ*.δ – Hausdorff space .
11) δ – Hausdorff space is δ***.δ – Hausdorff space .

Proof :

(1) Let (X,τ) be a δ – Hausdorff space then , for all points $x \neq y$ in X , there exist open sets N containing x , and M containing y , such that int[cl(N)] $\cap$ int[cl(M)] = ϕ .
Since S.cl(N) $\subseteq$ cl(N) and S.cl(M) $\subseteq$ cl(M) by Theorem 1.4.5.

And so int[S.cl(N)] $\subseteq$ int[cl(N)] , int[S.cl(M)] $\subseteq$ int[cl(M)] .
This implies that int[S.cl(N)] $\cap$ int[S.cl(M)] = ϕ , for all points x ≠ y in X . Hence (X,τ) is S.δ – Hausdorff space .
Similarly we can prove number (2) .

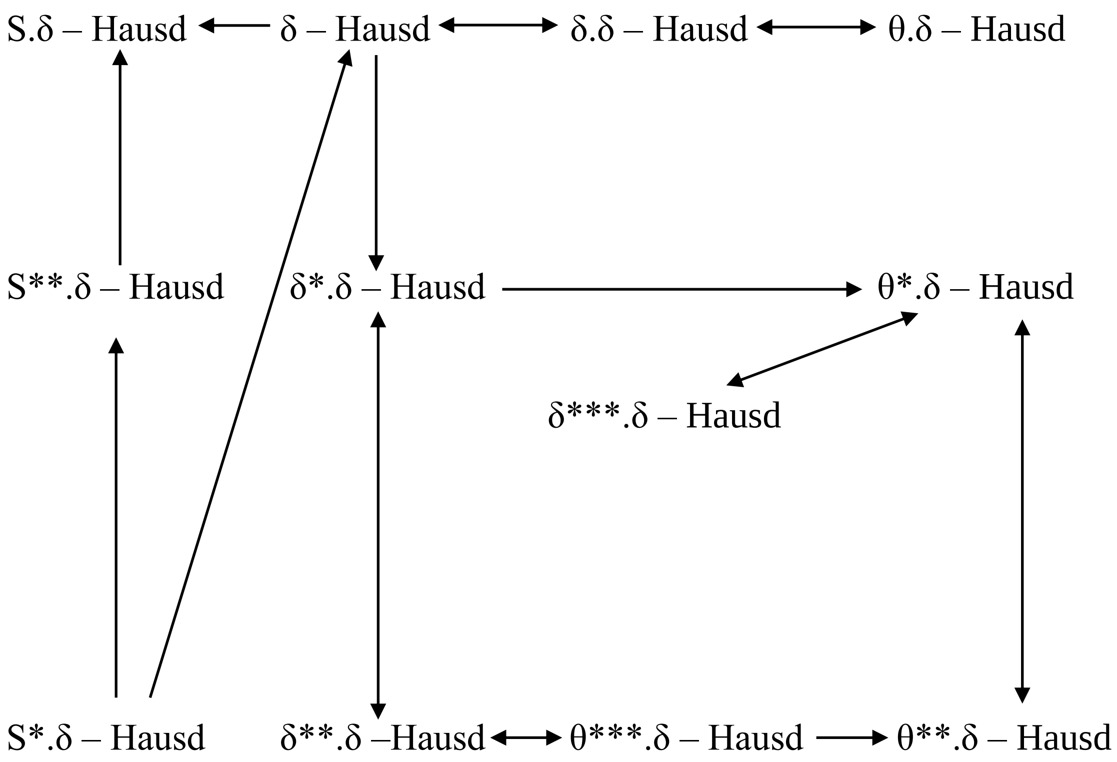
3) Let (X,τ) be a S*.δ – Hausdorff space then , for all points x ≠ y in X ,there exist open sets N containing x and M containing y such that ,S.int[cl(N)] $\cap$ S.int[cl(M)] = ϕ . Since for any set every interior point is semi interior point by Theorem 1.1.8.Part 1, therefore , int[cl(N)] $\cap$ int[cl(M) = ϕ , for all x ≠ y in X .Hence (X,τ) is δ – Hausdorff space .
(4) , (5) , (6) , (7) and (8) can be proved by adopting the same items.

(9) Let (X,τ) be a θ.δ – Hausdorff space then , for all points x ≠ y in X , there exist open sets N containing x and M containing y such that , int[θ.cl(N)] $\cap$ int[θ.cl(M)] = ϕ . Since every δ – interior point is interior point by Remark 1.4.2.Part 2, therefore , δ.int[θ.cl(N)] $\cap$δ.int[θ.cl(M)] = ϕ . And since δ.cl(N) = θ.cl(N), and δ.cl(M) = θ.cl(M) by Remark 1.4.6.Part 2, this implies that δ.int[δ.cl(N)] $\cap$ δ.int[δ.cl(M)] = ϕ , for all x ≠ y in X .
Hence (X,τ) is δ**.δ – Hausdorff space .
Similarly we can prove numbers (10) and (11) .

Note 2.4.7:

The following diagram is taken from the proposition and the definition of modifications of δ – Hausdorff spaces stated above .

Let Hausd = Hausdorff



Section five : Types of Continuous Functions

In this sections , we define the types of continuous function and prove theorems by using convergence .

Let g be a function of a space (X , τ) into a space (Y , μ) , then g is said to be continuous at $x_o \in X$, if for every μ – open set M containing g(x) , there exists a τ – open set N containing x such that $g(N) \subseteq M$.[28]

Definition 2.5.1:

Let g be a function of a space (X , τ) into a space (Y, μ) , and let $x \in X$. Then g is said to be :

1) δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(\tau - int[cl(N)]) \subseteq (\mu - int[cl(M)]) ;$$

2) S.δ – continuous at x , if for all μ –open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(\tau - int[S.cl(N)]) \subseteq (\mu - int[S.cl(M)]);$$

3) θ.δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(\tau - int[\theta.cl(N)]) \subseteq (\mu - int[\theta.cl(M)]) ;$$

4) δ.δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(\tau - int[\delta.cl(N)]) \subseteq (\mu - int[\delta.cl(M)])$$

5) S*.δ – continuous at x, if for all μ – open set M containing g(x) ,

there exists a τ – open set N containing x such that

$$g(\tau - S.int[cl(N)]) \subseteq (\mu - S.int[cl(M)]);$$

6) $\theta^*.\delta$ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(\tau - \theta.int[cl(N)]) \subseteq (\mu - \theta.int[cl(M)]);$$

7) $\delta^*.\delta$ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(\tau - \delta.int[cl(N)]) \subseteq (\mu - \delta.int[cl(N)]);$$

8) $S^{**}.\delta$ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(\tau - S.int[S.cl(N)]) \subseteq (\mu - S.int[S.cl(M)]);$$

9) $\theta^{**}.\delta$ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(\tau - \theta.int[\theta.cl(N)]) \subseteq (\mu - \theta int[\theta.cl(M)]);$$

10) $\delta^{**}.\delta$ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(\tau - \delta.int[\delta.cl(N)] \subseteq (\mu - \delta.int[\delta.cl(M)]);$$

11) $\delta^{***}.\delta$ – continuous at x , if for all μ – open set M containing g(x), there exists a τ – open set N containing x such that

$$g(\tau - \theta.int[\delta.cl(N)]) \subseteq (\mu - \theta.int[\delta.cl(M)]);$$

12) $\theta^{***}.\delta$ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(\tau - \delta.\text{int}[\theta.\text{cl}(N)]) \subseteq (\mu - \delta.\text{int}[\theta.\text{cl}(M)]) .$$

The following propositions can be proved by using Definition 2.5.1, and Remark 1.4.6.Part 2 .

Proposition 2.5.2 :

Let g be a function of a space (X , τ) into a space (Y , μ) , and $x_o \in X$. Then , the following statements are equivalents :

1) g is δ – continuous at x_o.

2) g is $\theta.\delta$ – continuous at x_o .

3) g is $\delta.\delta$ – continuous at x_o .

Proposition 2.5.3 :

Let g be a function of a space (X , τ) into a space (Y , μ) , and $x_o \in X$. Then , the following statements are equivalents :

1) g is $\theta^*.\delta$ – continuous at x_o .

2) g is $\theta^{**}.\delta$ – continuous at x_o .

3) g is $\delta^{***}.\delta$ – continuous at x_o .

Proposition 2.5.4 :

Let g be a function of a space (X , τ) into a space (Y , μ) , and $x_o \in X$. Then , the following statements are equivalents :

1) g is $\delta^*.\delta$ – continuous at x_o .

2) g is $\delta^{**}.\delta$ – continuous at x_o .

3) g is $\theta^{***}.\delta$ – continuous at x_o .

Theorem 2.5.5 :

Let g be a function of a space (X , τ) into a space (Y , μ) , and $x_o \in X$. Then g is :

1) δ – continuous at x_o if and only if whenever a net $\{f_a : a \in A\}$ δ – converges to x_o in X . Then the net $\{g(f_a) : a \in A\}$ δ – converges to $g(x_o)$.

2) S. δ – continuous at x_o if and only if whenever a net $\{f_a : a \in A\}$ S.δ – converges to x_o in X . Then the net $\{ g(f_a) : a \in A \}$ S.δ – coverges to $g(x_o)$.

3) $\theta^*.\delta$ – continuous at x_o if and only if whenever a net $\{ f_a : a \in A \}$ $\theta^*.\delta$ – converges to x_o in X . Then the net $\{ g(f_a) : a \in A \}$ $\theta^*.\delta$ – converges to $g(x_o)$.

4) $S^*.\delta$ – continuous at x_o if and only if whenever a net $\{ f_a : a \in A\}$ $S^*.\delta$ – converges to x_o in X . Then the net $\{ g(f_a) : a \in A \}$ $S^*.\delta$ – converges to $g(x_o)$.

5) $\delta^*.\delta$ – continuous at x_o if and only if whenever a net $\{ f_a : a \in A\}$ $\delta^*.\delta$ – converges to x_o in X . Then the net $\{ g(f_a) : a \in A \}$ $\delta^*.\delta$ – converges to $g(x_o)$.

6) $S^{**}.\delta$ – continuous at x_o if and only if whenever a net $\{ f_a : a \in A\}$ $S^{**}.\delta$ – converges to x_o in X . Then the net $\{ g(f_a) : a \in A \}$ $S^{**}.\delta$ – converges to $g(x_o)$.

Proof :

(1) Suppose g is δ – continuous at x_o and f_a δ – converges to x_o . To show that $g(f_a)$ δ – converges to $g(x_o)$. Let M be any μ – open set containing $g(x_o)$. Since g is δ – continuous at x_o then , there exists τ – open set N containing x_o , such that $g(\text{int}[cl(N)]) \subseteq \mu - \text{int}[cl(M)]$. Since f_a δ – converges to x_o , there exists $a_o \in A$ such that $a \geq a_o$, $a \in A$ then $f_a \in \tau - \text{int}[cl(N)]$ and therefore , $a \geq a_o$ which impilies that $g(f_a) \in g(\tau - \text{int}[cl(N)]) \subseteq \mu - \text{int}[cl(M)]$. Hence $g(f_a)$ δ – converges to $g(x_o)$.

Conversely , let f_a δ – converges to x_o then $g(f_a)$ δ – converges to $g(x_o)$, and suppose , if possible g is not δ – continuous at x_o . Then for some μ – open set M containing $g(x_o)$, $g(\tau - \text{int}[cl(N)]) \not\subset \mu - \text{int}[cl(M)]$, for any τ – open set N containing x_o . Thus for each τ – open set N containing x_o , we can choose x_{N1} where $N_1 = \text{int}[cl(N)]$, and $x_{N1} \in \tau - \text{int}[cl(N)]$ such that $g(x_{N1}) \notin \mu - \text{int}[cl(M)]$, let $N(x_o) = \{N : N \subseteq X$ and N is open set containing $x_o\}$, and put $\mathbf{\Omega} = \{ N_1 : N_1 = \tau - \text{int}[cl(N)]$, and $N \in N(x_o)\}$. We must prove the collection $\mathbf{\Omega}$ is directed set by the inclusion relation $\subseteq$.

1) For all $N_1 \in \mathbf{\Omega}$ then $N_1 \subseteq N_1$.

2) Let N_1 , N_2 and $N_3 \in \mathbf{\Omega}$, such that $N_1 \subseteq N_2$ and $N_2 \subseteq N_3$ then $N_1 \subseteq N_3$.

3) Let N_1 and $M_1 \in \mathbf{\Omega}$ such that $N_1 = \text{int}[cl(N)]$, $M_1 = \text{int}[cl(M)]$, where N and $M \in N(x_o)$, therefore, $N \cap M \in N(x_o)$, and $N \cap M \subseteq N$ and $N \cap M \subseteq M$, also $\text{int}[cl(N \cap M)] \subseteq \text{int}[cl(N)] = N_1$ and

$int[cl(N\cap M)] \subseteq int[cl(M)] = M_1$, and $int[cl(N\cap M)] \in \Omega$.

Consequently the collection Ω is directed by the inclusion

relation $\subseteq$, which implies that $\{x_{N1} : N_1 \in \Omega\}$ is a net in X .

It is easy to see that the net$\{x_{N1} : N_1 \in \Omega\}$ δ – converges to x_o .

For if $U_1 = int[cl(U)]$ and U is open set containing x_o then for every $N_1 = int[cl(N)]$, where N is open set containing x_o and $N_1 \geq U_1$ in Ω,

that is , $N_1 \subseteq U_1$in Ω we have that $x_{N1} \in N_1 \subseteq U_1$. But $\{g(x_{N1}) : N_1 \in \Omega\}$ net in Y which does not δ – converge to $g(x_o)$, for M is a μ – open set containing $g(x_o)$ such that $g(x_{N1}) \notin \mu$ – $int[cl(M)]$ for every $N_1 \in \Omega$.

(2) , (3) , (4) , (5) and (6) can be proved by the same manner .

Corollary 2.5.6 :

Let g be a function of a space (X , τ) into a space (Y , μ) , and $x_o \in X$. If g is :

1) δ – continuous at x_o whenever a net $\{ f_a : a \in A\}$ converges to x_o. Then the net $\{g(f_a) : a \in A\}$ δ – converges to $g(x_o)$.

2) S. δ – continuous at x_o whenever a net $\{ f_a : a \in A \}$ converges to x_o . Then the net $\{ g(f_a) : a \in A \}$ S.δ – coverges to $g(x_o)$.

3)θ*.δ – continuous at x_o whenever a net $\{ f_a : a \in A\}$ converges to x_o . Then the net $\{ g(f_a) : a \in A \}$ θ*.δ – converges to $g(x_o)$.

4) S*.δ – continuous at x_o whenever a net $\{f_a : a \in A\}$ converges to x_o .

Then the net { $g(f_a)$: a $\in$ A } S*.δ – converges to $g(x_o)$.

5) δ*.δ – continuous at x_o whenever a net { f_a : a $\in$ A} converges to x_o .
Then the net { $g(f_a)$: a $\in$ A } δ*.δ – converges to $g(x_o)$.

6) S**.δ – continuous at x_o whenever a net { f_a : a $\in$ A} converges to x_o .
Then the net { $g(f_a)$: a $\in$ A } S**.δ – converges to $g(x_o)$.

Proof :

Directly by Theorem 2.5.5, and Lemma 2.3.19 .

Definition 2.5.7:

Let g be a function of a space (X , τ) into a space (Y , μ) , and let x $\in$ X . Then g is called :

1) almost δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(N) \subseteq (\mu - \text{int}[\text{cl}(M)]) ;$$

2) almost S.δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(N) \subseteq (\mu - \text{int}[S.\text{cl}(M)]);$$

3) almost θ.δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g (N) \subseteq (\mu - \text{int}[\theta.\text{cl}(M)]) ;$$

4) almost δ.δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(N) \subseteq (\mu - \text{int}[\delta.\text{cl}(M)]) ;$$

5) almost S*.δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(N) \subseteq (\mu - S.\text{int}[\text{cl}(M)]);$$

6) almost θ*.δ – containuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(N) \subseteq (\mu - \theta.\text{int}[\text{cl}(M)]) ;$$

7) almost δ*.δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(N) \subseteq (\mu - \delta.\text{int}[\text{cl}(N)]) ;$$

8) almost S**.δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(N) \subseteq (\mu - S.\text{int}[S.\text{cl}(M)]) ;$$

9) almost θ**.δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(N) \subseteq (\mu - \theta\text{int}[\theta.\text{cl}(M)]) ;$$

10)almost δ**.δ – continuous at x , if for all μ – open set M containing g(x), there exists a τ – open set N containing x such that

$$g(N) \subseteq (\mu - \delta.int[\delta.cl(M)]) ;$$

11)almost δ***.δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(N) \subseteq (\mu - \theta.int[\delta.cl(M)]) ;$$

12)almost θ***.δ – continuous at x , if for all μ – open set M containing g(x) , there exists a τ – open set N containing x such that

$$g(N) \subseteq (\mu - \delta.int[\theta.cl(M)]) .$$

The following propositions can be proved by using Definition 2.5.7 and Remark 1.4.6.Part 2 .

Proposition 2.5.8 :

Let g be a function of a space (X , τ) into a space (Y , μ) , and $x_o \in X$. Then , the following statements are equivalents :

1) g is almost δ – continuous at x_o .

2) g is almost $\theta.\delta$ – continuous at x_o .

3) g is almost $\delta.\delta$ – continuous at x_o .

Proposition 2.5.9:

Let g be a function from a space (X , τ) into a space (Y , μ) , and $x_o \in X$. Then , the following statements are equivalents :

1) g is almost θ*.δ – continuous at x_o .

2) g is almost θ**.δ – continuous at x_o .

3) g is almost δ***.δ – continuous at x_o .

Proposition 2.5.10:

Let g be a function of a space (X , τ) into a space (Y , μ) , and $x_o \in X$. Then , the following statements are equivalents :

1) g is almost δ*.δ – continuous at x_o.
2) g is almost δ**.δ – continuous at x_o .
3) g is almost θ***.δ – continuous at x_o .

Lemma 2.5.11:

Let g be a function of a space (X , τ) into a space (Y , μ) , and $x_o \in X$. If g is continuous at x_o then :

1) g is almost δ – continuous at x_o .
2) g is almost S.δ – continuous at x_o .
3) g is almost θ*.δ – continuous at x_o .
4) g is almost S*.δ – continuous at x_o .
5) g is almost δ*.δ – continuous at x_o .
6) g is almost S**.δ – continuous at x_o .

Proof :

(1) Let g be a continuous at x_o . Then for each μ – open set M containing $g(x_o)$, there exist τ – open set N containing x_o such that $g(N) \subseteq M$. But $M \subseteq int[cl(M)]$ by Note 1.4.11 , therefore , for each μ – open set M containing $g(x_o)$, there exist τ – open set N containing x_o such that $g(N) \subseteq int[cl(M)]$. Hence g is almost δ – continuous at x_o .

(2) , (3) , (4) , (5) and (6) can be proved by the same manner .

Proposition 2.5.12:

Let g be a function of a space (X , τ) into a space (Y , μ) , and $x_o \in X$. If g is :

1) almost S.δ – continuous at x_o then g is almost δ – continuous at x_o .

2) almost θ*.δ – continuous at x_o then g is almost θ***.δ – continuous at x_o .

3) almost S**.δ – continuous at x_o then g is almost S*.δ – continuous at x_o .

4) almost δ – continuous at x_o then g is almost S*.δ – continuous at x_o .

5) almost S.δ – continuous at x_o then g is almost S**.δ – continuous at x_o .

6) almost δ*.δ – continuous at x_o then g is almost δ – continuous at x_o .

7) almost θ*.δ – continuous at x_o then g is almost δ*.δ – continuous at x_o .

8)almost θ**.δ – continuous at x_o then g is almost θ***.δ – continuous at x_o .

9) almost δ***. δ – continuous at x_o then g is almost δ.δ – continuous at x_o .

10)almost δ***. δ – continuous at x_o then g is almost δ**.δ – continuous at x_o .

11)almost δ**. δ – continuous at x_o then g is almost θ.δ – continuous at x_o .

Proof :

(1) Let g be a almost S.δ – continuous at x_o . Then for each μ – open set M containing $g(x_o)$, there is τ – open set N containing x_o such that $g(N) \subseteq int[S.cl(M)]$. Since $S.cl(M) \subseteq cl(M)$ by Theorem 1.4.5, and so $int[S.cl(M)] \subseteq int[cl(M)$, which implies that $g(N) \subseteq int[cl(M)]$.
Hence g is almost δ – continuous at x_o .
Similarly we can prove number (2) .

(3) Let g be a almost δ – continuous at x_o . Then for each μ – open set M containing $g(x_o)$, there is τ – open set N containing x_o such that $g(N) \subseteq int[cl(M)]$. Since every interior point is semi interior point by Theorem 1.1.8.Part 1, therefore , $g(N) \subseteq S.int[cl(M)]$ .This implies that g is almost S*.δ – continuous at x_o .
Similarly we can prove numbers (4) , (5) , (6) , (7) and (8) .

(9) Let g be a almost δ**.δ – continuous at x_o . Then for each μ – open set M containing $g(x_o)$, there is τ – open set N containing x_o such that $g(N) \subseteq \delta.int[\delta.cl(M)]$. But every δ – interior point is interior point by Remark 1.4.2.Part 2, and so $\delta.cl(M) = \theta.cl(M)$ by

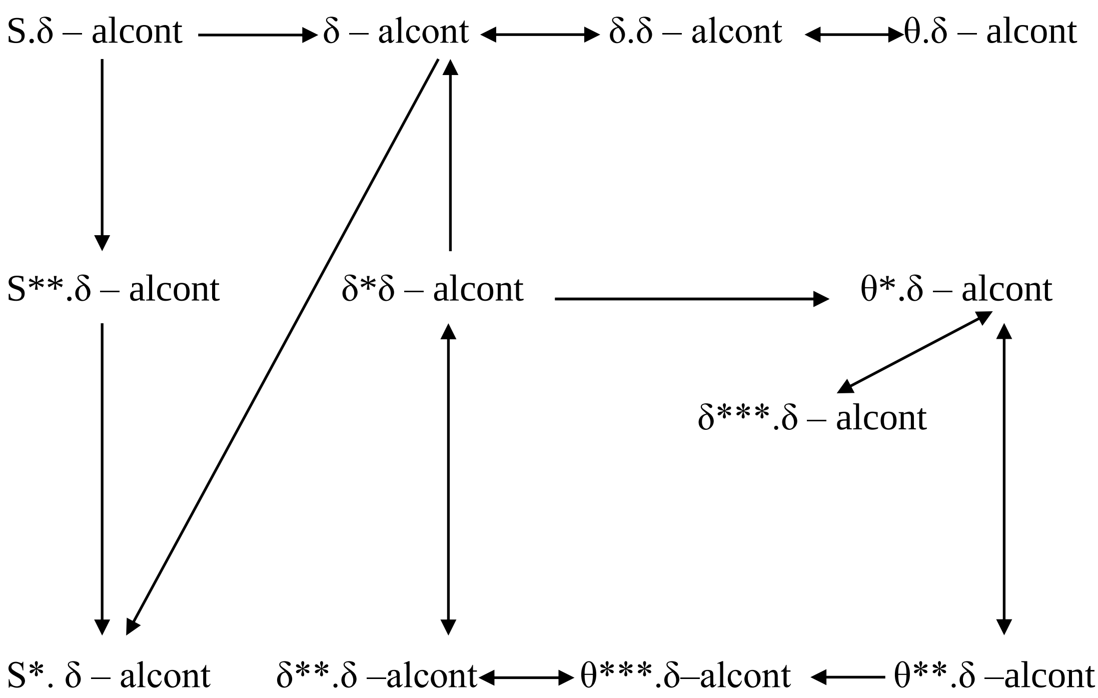
Remark 1.4.6.Part 2, which implies that , $g(N) \subseteq int[\theta.cl(M)]$.

Therefore , g is almost δ – continuous at x_o .

Note 2.5.13 :

The following diagram is taken from the above proposition and definition of modification of almost δ – continuous functions stated above .

Let alcont = almost continuous



Proposition 2.5.14 :

Let g be a function of a space (X , τ) into a space (Y , μ) , and $x_o \in X$. If g is :

1) δ – continuous at x_o then g is almost δ – continuous at x_o .

2) S.δ – continuous at x_o then g is almost S.δ – continuous at x_o .

3) $\theta.\delta$ – continuous at x_o then g is almost $\theta.\delta$ – continuous at x_o .

4) $\delta.\delta$ – continuous at x_o then g is almost $\delta.\delta$ – continuous at x_o.

5) S*. δ – continuous at x_o then g is almost S*.δ – continuous at x_o .

6) δ*.δ – continuous at x_o then g is almost δ*.δ – continuous at x_o .

7) θ*. δ – continuous at x_o then g is almost θ*.δ – continuous at x_o.

8) S**.δ – continuous at x_o then g is almostS**.δ – continuous at x_o .

9) δ**.δ – continuous at x_o then g is almost δ**.δ – continuous at x_o.

10)θ**.δ – continuous at x_o then g is almost θ**.δ – continuous at x_o.

11)δ***.δ – continuous at x_o then g is almost δ***.δ – continuous at x_o.

12)θ***.δ – continuous at x_o then g is almost θ***.δ – continuous at x_o.

Proof :

(1) Let g be a δ – continuous at x_o . Then , for each μ – open set M containing $g(x_o)$, there is τ – open set N containing x_o such that $g(\text{int}[cl(N)]) \subseteq \text{int}[cl(M)]$. But $N \subseteq \text{int}[cl(N)]$ by Note 1.4.11, therefore , $g(N) \subseteq \text{int}[cl(M)]$ which implies that g is almost δ –

continuous at x_o .

(2) , (3) , (4) , (5) , (6) , (7) , (8) , (9) , (10) , (11) and (12)) can be proved by following the same procedures applicable to number (1) .

Corollary 2.5.15:

Let g be a function of a space (X , τ) into a space (Y , μ) , and $x_o \in X$. Then g is :

1) almost δ – continuous at x_o . If every δ – converges net $\{fa : a \in A\}$ to x_o such that the net $\{g(fa) : a \in A\}$ which δ – converges to $g(x_o)$.

2) almost S.δ – continuous at x_o . If every S.δ – converges net $\{fa : a \in A\}$ to x_o such that the net $\{g(fa) : a \in A\}$which S.δ – converges to $g(x_o)$.

3) almost $\theta^*.\delta$ – continuous at x_o. If every$\theta^*.\delta$ – converges net $\{fa : a \in A\}$ to x_o such that the net $\{g(fa) : a \in A\}$which $\theta^*.\delta$ – converges to $g(x_o)$.

4) almost $S^*.\delta$ – continuous at x_o . If every $S^*.\delta$ – converges net $\{fa : a \in A\}$ to x_o such that the net $\{g(fa) : a \in A\}$which $S^*.\delta$ – converges to $g(x_o)$.

5) almost $\delta^*.\delta$ – continuous at x_o . If every $\delta^*.\delta$ – converges net $\{fa : a \in A\}$ to x_o such that the net $\{g(fa) : a \in A\}$which $\delta^*.\delta$ – converges to $g(x_o)$.

6) almost $S^{**}.\delta$ – continuous at x_o . If every $S^{**}.\delta$ – converges net

{fa : a $\in$ A} to x_o such that the net {g(fa) : a $\in$ A} which S**.δ – converges to $g(x_o)$.

Proof :

By Theorem 2.5.5, and by Proposition 2.5.14 . .

Corolary 2.5.16 :

Let g be a function of a space (X , τ) into a space (Y , μ) , and $x_o \in$ X . Then g is :

1) almost δ – continuous at x_o if and only if whenever a net {fa : a $\in$ A} converges to x_o in X . Then the net {g(fa) : a $\in$ A} δ – converges to $g(x_o)$.

2) almost S. δ – continuous at x_o if and only if whenever a net { fa : a $\in$A } converges to x_o in X . Then the net { g(fa) : a $\in$ A } S.δ – coverges to $g(x_o)$.

3) almost θ*.δ – continuous at x_o if and only if whenever a net {fa : a $\in$ A } converges to x_o in X . Then the net { g(fa) : a $\in$ A } θ*.δ – converges to $g(x_o)$.

4) almost S*.δ – continuous at x_o if and only if whenever a net {fa : a $\in$ A} converges to x_o in X . Then the net { g(fa) : a $\in$ A } S*.δ – converges to $g(x_o)$.

5) almost δ*.δ – continuous at x_o if and only if whenever a net

$\{fa : a \in A\}$converges to x_o in X . Then the net $\{g(fa) : a \in A\}$ $\delta^*.\delta$ – converges to $g(x_o)$.

6) almost $S^{**}.\delta$ – continuous at x_o if and only if whenever a net $\{fa : a \in A\}$ converges to x_o in X . Then the net $\{ g(fa) : a \in A \}$ $S^{**}.\delta$ – converges to $g(x_o)$.

Proof :

(1) Suppose g is almost δ – continuous at x_o and f_a converges to x_o . To show that $g(f_a)$ δ – converges to $g(x_o)$. Let M be any μ – open set containing $g(x_o)$. Since g is almost δ – continuous at x_o then , there exists τ – open set N containing x_o , such that $g(N) \subseteq \mu$ – int[cl(M)] . Since f_a converge to x_o , there exist $a_o \in A$ such that $a \geq a_o$, $a \in A$ then $f_a \in (N)$ and therefore , $a \geq a_o$ which implies that $g(f_a) \in g(N) \subseteq \mu$ – int[cl(M)] . Hence $g(f_a)$ δ – converges to $g(x_o)$.

Conversely , let f_a converges to x_o which implies that $g(f_a)$ δ – converges to $g(x_o)$, we get that f_a δ – converges to x_o by Lemma 2.3.19.Part 2 . Therefore , g is almost δ – continuous at x_o by Corollary 2.5.15.Part 2 .

(2) , (3) , (4) , (5) and (6) can be proved by following the same procedures applicable to number (1) .

Theorem 3. 1.4 :

A topological space (X , τ) is :

1) δ – Hausdorff if and only if for every net in X can δ – converge to at most one point .

2) S.δ – Hausdorff if and only if for every net in X can S.δ – converge to at most one point .

3) θ*.δ – Hausdorff if and only if for every net in X can θ.δ – converge to at most one point .

4) S*.δ – Hausdorff if and only if for every net in X can S*.δ – converge to at most one point .

5) δ*.δ – Hausdorff if and only if for every net in X can δ*.δ – converge to at most one point .

6) S**.δ – Hausdorff if and only if for every net in X can S**.δ – converge to at most one point .

Proof :

(1) The " if " part . Assume that every net in X can δ – converge to at most one point and suppose if possible , that the space X is not δ – Hausdorff space . Then there exists two distinct points x and y in X

such that for every open sets N containing x and M containing y , the intersection int[cl(N)] ∩ int[cl(M)] $\neq \phi$.

Let N(x) = {N : N $\subseteq$ X , and N is open set containing x},

Let $\Omega = \{ int[cl(N)] : N \in N(x) \}$, and let $N(y) = \{ M : M \subseteq X$, and M is open set containing y $\}$, and $\Phi = \{ int[cl(M)] : M \in N(y)\}$.
Then both Ω and Φ are directed by the inclusion relation $\subseteq$. put $N_1 = int[cl(N)]$ and put $M_1 = int[cl(M)]$. Consider the collection $\Psi = \{ (N_1 , M_1) : N_1 \in \Omega , M_1 \in \Phi$, and $N_1 \cap M_1 \neq \phi \}$, we define $(N_1, M_1) \geq (U_1, V_1)$, if $N_1 \subseteq U_1$ and $M_1 \subseteq V_1$. We must show that Ψ is directed by the inclusion relation $\subseteq$.

1) For all $(N_1 , M_1) \in \Psi$, $N_1 \subseteq N_1$ and $M_1 \subseteq M_1$ then ,
$(N_1 , M_1) \geq (N_1 , M_1)$.

2)Let (A_1 , B_1) , (C_1 , D_1) and $(E_1 , F_1) \in \Psi$ such that
$(A_1 , B_1) \geq (C_1 , D_1)$ and $(C_1 , D_1) \geq (E_1 , F_1)$. Since $A_1 \subseteq C_1$ and $C_1 \subseteq E_1$ then $A_1 \subseteq E_1$, and also $B_1 \subseteq D_1$ and $D_1 \subseteq F_1$ then $B_1 \subseteq F_1$ therefore $(A_1 , B_1) \geq (E_1 , F_1)$.

3)Let (A_1, B_1) and $(C_1 , D_1) \in \Psi$, such that $A_1 = int[cl(A)]$, $B_1= int[cl(B)]$, $C_1 = int[cl(C)]$ and $D_1= int[cl(D)]$.
A and C $\in N(x)$, B and D $\in N(y)$ .So $A \cap C \in N(x)$, and $int[cl(A\cap C)] \subseteq int[cl(A)]$, and $int[cl(A\cap C)] \subseteq int[cl(C)]$.
Also $B\cap D \in N(y)$, and $int[cl(B\cap D)] \subseteq int[cl(B)]$, and $int[cl(B\cap D)] \subseteq int[cl(D)]$. Let $int[cl(A\cap C)] = U_1$, and $int[cl(B\cap D)] = V_1$. Since $A\cap C \in N(x)$ and $B\cap D \in N(y)$, we get that $U_1 \in \Omega$ and $V_1 \in \Phi$ and so $U_1 \cap V_1 \neq \phi$, therefore , $(U_1 , V_1) \in \Psi$. Since $U_1 \subseteq A_1$ and $V_1 \subseteq B_1$ hence , $(U_1 , V_1) \geq (A_1 , B_1)$, and so $U_1 \subseteq C_1$ and $V_1 \subseteq D_1$ hence $(U_1 , V_1) \geq (C_1 , D_1)$. Consequently $(\Psi , \geq)$

is directed set . Since $N_1 = int[cl(N)]$ for all open set N containing x which intersect $M_1= int[cl(M)]$ for all open sets M containing y , we have $N_1 \cap M_1 \neq \phi$ for all $(N_1 , M_1) \in \Psi$. Now consider the function $f : \Psi \to X$; $f(N_1, M_1) = x(N_1, M_1)$ for all $(N_1, M_1) \in \Psi$. Then f is a net in X δ – converges to both x and y . To see this , let S be any open set containing x , and T be any open set containing y such that $S_1 = int[cl(S)]$, $T_1 = int[cl(T)]$,and let $E_1 = int[cl(E)]$ and $F_1 = int[cl(F)]$ .Then for each $(E_1, F_1) \geq (S_1 , T_1)$, that is , $E_1 \subseteq S_1$ and $F_1 \subseteq T_1$, we have $f(E_1,F_1) = x(E_1,F_1) \in E_1 \cap F_1 \subseteq S_1 \cap T_1$. Hence $f(E_1, F_1) \in$ S1 and $f(E_1, F_1) \in T_1$. It follows that f δ – converges to both x and y . But this is a contradiction . Hence X must be a δ – Hausdorff space .

The“ only if ” part . Let X be a δ – Hausdorff space , and let x and y be two distinct points of X . Then there exist open sets N containing x and M containing y such that $int[cl(N)] \cap int[cl(M) = \phi$. Since a net can not be eventually in each of two disjoint sets . It is evident that no net in X can δ – converge to both x and y . Hence a net in X can δ – converge to at most one point .

(2) , (3) , (4) , (5) and (6) can be proved by adopting the same items.

Theorem : 3.1.5 :

Let (X , τ) be a :

1) δ – Hausdorff space . Then every δ – convergent net has a unique δ – cluster point and this is the unique δ – limit point of the net .

2) S.δ – Hausdorff space . Then every δ – convergent net has a unique S.δ – cluster point and this is the unique S.δ – limit point of the net .

3)θ*.δ – Hausdorff space .Then every θ*.δ – convergent net has a unique θ*.δ – cluster point and this is the unique θ*.δ – limit point of the net .

4)S*.δ – Hausdorff space .Then every S*.δ – convergent net has a unique S*.δ – cluster point and this is the unique S*.δ – limit point of the net .

5)δ*.δ – Hausdorff space .Then every δ*.δ – convergent net has a unique δ*.δ – cluster point and this is the unique δ*.δ – limit point of the net .

6)S**.δ – Hausdorff space .Then every S**.δ – convergent net has a unique S**.δ – cluster point and this is the unique S**.δ – limit point of the net .

Proof :

(1) We know that in a δ – Hausdorff space , every δ – convergent net has a unique δ – limit point by Theorem 3.1.4.Part 1. Let p be the unique δ – limit point of a δ – convergent net f in X .Since every δ – limit point is also δ – cluster point by Lemma 2.3.21.Part 12,we get that p is δ – cluster point of f .

Suppose if possible that , f has another δ – cluster point q distinct from p , since X is δ – Hausdorff , there exist disjoint open sets N and M of p and q respectively , such that $int[cl(N)] \cap int[cl(M)] = \phi$. Since p is δ – limit point of the net f , f is eventually in int[cl(N)] and since $int[cl(N)] \subseteq X/int[cl(M)]$ by Note 2.3.6, then , f can not be

frequently in int[cl(M)] . But this is a contradiction with our supposition that , q is a δ – cluster point of f . Hence f can not have two distinct δ – cluster points accordingly , every δ – convergent net in X has a unique δ – cluster point .

(2) , (3) , (4) , (5) and (6) can be proved by adopting the same items .

Section 3.2 :

The Relationship of Convergence in Compact Spaces

In this section , we prove some of theorems by convergence in topological spaces on kinds of compact spaces.

A subset A of topological space (X , τ) is compact , if every open cover of A has a finite subcover .[28]

Definition 3.2.1 :

A topological space (X , τ) is called :

1) δ – compact space , if for every open cover U of X has a finite subfamily U_o of U such that $X = \bigcup \{ int[cl(V) : V \in U_o \}$.

2) $S.\delta$ – compact space , if for every open cover U of X has a finite subfamily U_o of U such that $X = \bigcup \{ int[S.cl(V)] : V \in U_o \}$.

3) $\theta.\delta$ – compact space , if for every open cover U of X has a finite subfamily U_o of U such that $X = \bigcup \{int[\theta.cl(V)] : V \in U_o \}$.

4) $\delta.\delta$ – compact space , if for every open cover U of X has a finite subfamily U_o of U such that $X = \bigcup \{int[\delta.cl(V)] : V \in U_o \}$.

5) $S^*.\delta$ – compact space , if for every open cover U of X has a finite subfamily U_o of U such that $X = \bigcup \{ S.int[cl(V) : V \in U_o \}$.

6) $\theta^*.\delta$ – compact space , if for every open cover U of X has a finite subfamily U_o of U such that $X = \bigcup \{\theta.int[cl(V) : V \in U_o \}$.

7) $\delta^*.\delta$ – compact space , if for every open cover U of X has a finite subfamily U_o of U such that $X = \bigcup \{\delta.int[cl(V) : V \in U_o\}$.

8) $S^{**}.\delta$ – compact space , if for every open cover U of X has a finite subfamily U_o of U such that $X = \bigcup \{ S.int[S.cl(V)] : V \in U_o \}$.

9) $\theta^{**}.\delta$ – compact space , if for every open cover U of X has a finite subfamily U_o of U such that $X = \bigcup \{\theta.int[\theta.cl(V)] : V \in U_o\}$.

10) $\delta^{**}.\delta$ – compact space, if for every open cover U of X has a finite subfamily U_o of U such that $X = \bigcup \{\delta.int[\delta.cl(V)] : V \in U_o\}$.

11) $\delta^{***}.\delta$ – compact space , if for every open cover U of X has a finite subfamily U_o of U such that $X = \bigcup \{\theta.int[\delta.cl(V)] : V \in U_o\}$.

12) $\theta^{***}.\delta$ – compact space , if for every open cover U of X has a finite subfamily U_o of U such that $X = \bigcup \{\delta.int[\theta.cl(V) : V \in U_o\}$.

The following Propositions 3.2.2, 3.2.3 and 3.2.4 can be proved by using Definition 3.2.1 and Remark 1.4.6.Part 2 .

Proposition 3.2.2 :

Let (X , τ) be a topological space . Then , the following statements are equivalents :

1) (X , τ) is δ – compact topological space .

2) (X , τ) is $\theta.\delta$ – compact topological space .

3) (X , τ) is $\delta.\delta$ – compact topological space .

Proposition 3.2.3 :

Let (X , τ) be a topological space . Then , the following statements are equivalents :

1) (X , τ) is θ*.δ – compact topological space .

2) (X , τ) is θ**.δ – compact topological space .

3) (X , τ) is δ***.δ – compact topological space .

Proposition 3.2.4:

Let (X , τ) be a topological space . Then , the following statements are equivalents :

1) (X , τ) is δ*.δ – compact topological space .

2) (X , τ) is δ**.δ – compact topological space .

3) (X , τ) is θ***.δ – compact topological space .

Proposition 3.2.5 :

Let (X , τ) be a topological space then , every :

1) S.δ – compact space is δ – compact space .
2) S**.δ – compact space is S*.δ – compact space .
3) δ – compact space is S*.δ – compact space .
4) S.δ – compact space is S**.δ – compact space .
5) δ*.δ – compact space is δ – compact space .
6) θ*.δ – compact space is δ*.δ – compact space .
7) θ**.δ – compact space is θ***.δ – compact space .
8) δ***.δ – compact space is δ**.δ – compact space .
9) δ**.δ – compact space is θ.δ – compact space .
10) θ*.δ – compact space is θ***.δ – compact space .
11) δ***.δ – compact space is δ – compact space .

Proof :

(1) Let (X , τ) be a S.δ – compact space then , for every open cover U of X there exists a finite subfamily U_o of U such that $X = \bigcup \{ int[S.cl(V)] : V \in U_o \}$. But $S.cl(V) \subseteq cl(V)$ by Theorem 1.4.5, therefore, $X = \bigcup \{ int[cl(V)] : V \in U_o \}$.

Hence (X , τ) is δ – compact space .

Similarly we can prove number (2) .

(3) Let (X ,τ) be a δ – compact space then , for every open cover U of X has finite subfamily U_o of U such that $X = \bigcup \{ int[cl(V)] : V \in U_o \}$ .But every interior point is semi interior point by Theorem 1.1.8.Part 1,therefore , $X = \bigcup \{ S.int[cl(V)] : V \in U_o \}$. Hence (X , τ) is S.δ – compact space .

(4) , (5) , (6) , (7) and (8) can be proved by the same manner .

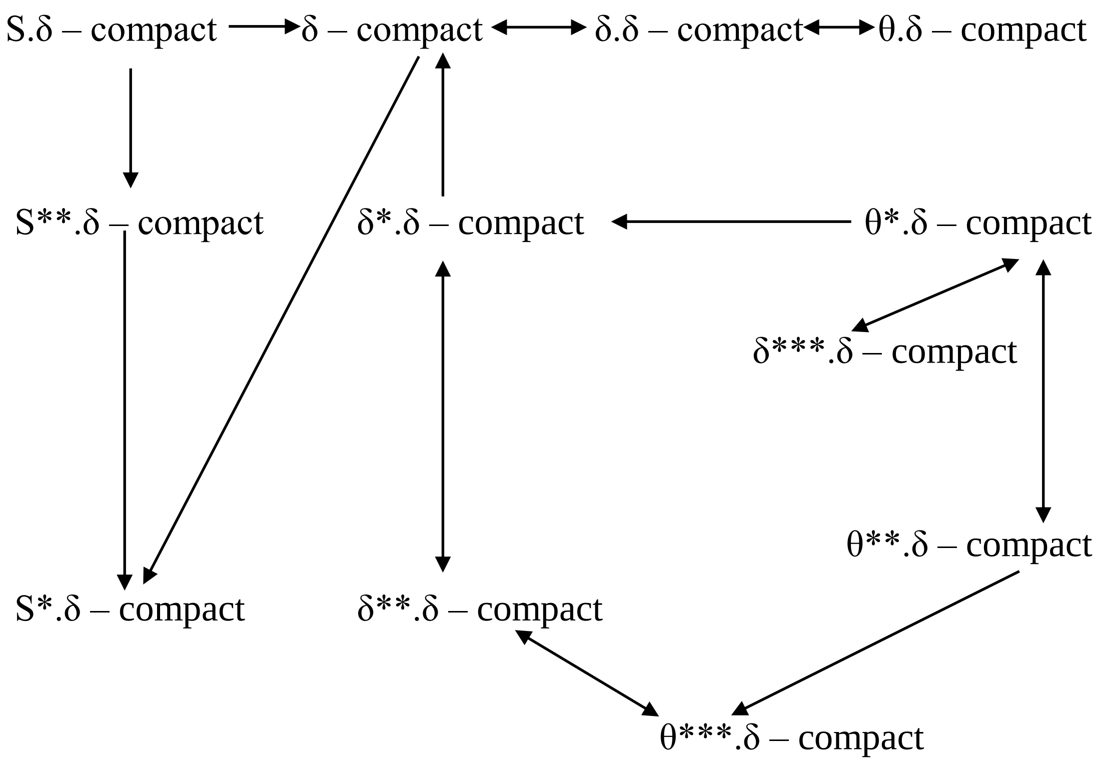
(9) Let (X , τ) be a δ**.δ – compact space then , for every open cover U of X there exists a finite subfamily U_o of U such that $X = \bigcup \{ \delta.int[\delta.cl(V) : V \in U_o \}$. But every δ – interior point is interior point by Remark 1.4.2.Part 2 , and also $\delta.cl(V) = \theta.cl(V)$ by Remark 1.4.6.Part 2 ,therefore , $X = \bigcup \{ int[\theta.cl(V) : V \in U_o \}$.

Hence (X , τ) is θ.δ – compact space .

Similarly we can prove numbers (10) and (11) .

Note 3.2.6 :

The following diagram is taken from the above proposition and definition of modification of δ – compact spaces stated above .



Lemma 3.2.7 :

Every compact topological space (X , τ) is :

1) S.δ – compact topological space .

2) δ – compact topological space .

3) S*.δ – compact topological space .

4) θ*.δ – compact topological space .

5) δ*.δ – compact topological space .

6) S**.δ – compact topological space .

Proof :

(1) Let (X , τ) be a compact topological space then , every open cover U of X has a finite subcover U_o of U such that $X = \cup\{V : V \in U_o , V \in \tau\}$. But $V \subseteq int[S.cl(V)]$ by Theorem 1.4.12.Part.6 , therefore , $X = \cup\{int[S.cl(V)]: V \in U_o \}$. Hence X is S.δ – compact space .

(2) , (3) , (4) , (5) and (6) can be proved by adopting the same items .

Theorem 3.2.8 : [28]

A topological space (X, τ) is compact if and only if each net in X has a cluster point .

Corollary 3.2.9 :

Let (X,τ) be a topological space , if each net in X has a cluster point then (X, τ) is :

1) δ – compact space .
2) S.δ – compact space .
3) $\theta^*.\delta$ – compact space .
4) $S^*.\delta$ – compact space .
5) $\delta^*.\delta$ – compact space .
6) $S^{**}.\delta$ – compact space .

Proof :

(1) By Theorem 3.2.8, we get that (X,τ) is compact space , and so (X,τ) is δ – compact by Lemma 3.2.7.Part 2 .

Similarly we can prove numbers (2) , (3) , (4) , (5) , and (6) .

Definition 3.2.10 :

A subset A of a topological space (X , τ) is called :

1) δ – compact set , if for every open cover U of A there , exists a finite subfamily U_o of U such that

$$A \subseteq \bigcup \{ int[cl(V) : V \in U_o \} .$$

2) S.δ – compact set , if for every open cover U of A there , exists a finite subfamily U_o of U such that

$$A \subseteq \bigcup \{ int[S.cl(V)] : V \in U_o \} .$$

3) $\theta.\delta$ – compact set , if for every open cover U of A there, exists a finite subfamily U_o of U such that

$$A \subseteq \bigcup \{ int[\theta.cl(V)] : V \in U_o \} .$$

4) $\delta.\delta$ – compact set , if for every open cover U of A there, exists a finite subfamily U_o of U such that

$$A \subseteq \bigcup \{ int[\delta.cl(V)] : V \in U_o \} .$$

5) S*.δ – compact set , if for every open cover U of A there, exists a finite subfamily U_o of U such that

$$A \subseteq \bigcup \{ S.int[cl(V)] : V \in U_o \} .$$

6) θ*.δ – compact set , if for every open cover U of A there , exists a finite subfamily U_o of U such that

$$A \subseteq \bigcup \{ \theta.int[cl(V)] : V \in U_o \} .$$

7) $\delta^*.\delta$ – compact set , if for every open cover U of A there, exists a finite subfamily U_o of U such that

$$A \subseteq \bigcup \{ \delta.int[cl(V)] : V \in U_o \} .$$

8) $S^{**}.\delta$ – compact set , if for every open cover U of A there, exists a finite subfamily U_o of U such that

$$A \subseteq \bigcup \{ S.int[S.cl(V)] : V \in U_o \} .$$

9) $\theta^{**}.\delta$ – compact set , if for every open cover U of A there, exists a finite subfamily U_o of U such that

$$A \subseteq \bigcup \{\theta.int[\theta.cl(V)] : V \in U_o \} .$$

10) $\delta^{**}.\delta$ – compact set , if for every open cover U of A there, exists a finite subfamily U_o of U such that

$$A \subseteq \bigcup \{\delta.int[\delta.cl(V)] : V \in U_o \} .$$

11) $\delta^{***}\ \delta$ – compact set , if for every open cover U of A there,exists a finite subfamily U_o of U such that

$$A \subseteq \bigcup \{\theta.int[\delta.cl(V)] : V \in U_o \} .$$

12) $\theta^{***}.\delta$ – compact set , if for every open cover U of A there , exists a finite subfamily U_o of U such that

$$A \subseteq \bigcup \{\delta.int[\theta.cl(V)] : V \in U_o \} .$$

The following Propositions 3.2.11, 3.2.12 and 3.2.13 can be proved by using Definition 3.2.10 and Remark 1.4.6.Part 2 .

Proposition 3.2.11 :

Let (X , τ) be a topological space , and A $\subseteq$ X . Then , the following statements are equivalents :

1) A is δ – compact set .

2) A is $\theta.\delta$ – compact set .

3) A is $\delta.\delta$ – compact set .

Proposition 3.2.12 :

Let (X , τ) be a topological space , and A $\subseteq$ X . Then , the following statements are equivalents :

1) A is $\theta^*.\delta$ – compact set .

2) A is $\theta^{**}.\delta$ – compact set .

3) A is $\delta^{***}.\delta$ – compact set .

Proposition 3.2.13 :

Let (X , τ) be a topological space , and A $\subseteq$ X Then , the following statements are equivalents :

1) A is $\delta^*.\delta$ – compact set .

2) A is $\delta^{**}.\delta$ – compact set .

3) A is $\theta^{***}.\delta$ – compact set .

Proposition 3.2.14 :

Let (X , τ) be a topological space and let A $\subseteq$ X . If A is :

1) $S^{**}.\delta$ – compact set then A is $S^*.\delta$ – compact set .

2) $S.\delta$ – compact set then A is δ – compact set .

3) δ – compact set then A is $S^*.\delta$ – compact set .

4) $S.\delta$ – compact set then A is $S^{**}.\delta$ – compact set .

5) $\delta^*.\delta$ – compact set then A is δ – compact set .

6) $\theta^*.\delta$ – compact set then A is $\delta^*.\delta$ – compact set .

7) $\theta^{**}.\delta$ – compact set then A is $\theta^{***}.\delta$ – compact set .

8) $\delta^{***}.\delta$ – compact set then A is $\delta^{**}.\delta$ – compact set .

9) $\delta^{**}.\delta$ – compact set then A is $\theta.\delta$ – compact set .

10) $\theta^*.\delta$ – compact set then A is $\theta^{***}.\delta$ – compact set .

11) $\delta^{***}.\delta$ – compact set then A is δ – compact set .

Proof :

(1) Let A be a $S^{**}.\delta$ – compact set then , for every open cover U of A there exists a finite subfamily U_o of U such that $A \subseteq \cup\{S.int[S.cl(V)] : V \in U_o\}$. But $S.cl(V) \subseteq cl(V)$ by Theorem 1.4.5, therefore , $A \subseteq \cup\{S.int[cl(V)] : V \in U_o\}$. Hence A is $S^*.\delta$ – compact set .
Similarly we can prove number (2) .

(3) Let A be a δ – compact set then , for every open cover U of A there exists a finite subfamily U_o of U such that $A \subseteq \cup\{int[cl(V) : V \in U_o\}$. But every interior point is semi interior point by Theorem 1.1.8.Part 1, therefore , $A \subseteq \cup\{S.int[cl(V)] : V \in U_o\}$. Hence A is $S^*.\delta$ – compact set .
Similarly we can prove numbers (4) , (5) , (6) , (7) and (8) .

(9) Let A be a $\delta^{**}.\delta$ – compact set then , for every open cover U of A , there exists a finite subfamily U_o of U such that $A \subseteq \cup\{\delta.int[\delta.cl(V) : V \in U_o\}$. But every δ –interior point is interior point , by Remark 1.4.2.Part 2, and also , $\delta.cl(V) = \theta.cl(V)$ by Remark 1.4.6.Part 2 , therefore , $A \subseteq \cup\{int[\theta.cl(V)] : V \in U_o\}$. Hence A is $\theta.\delta$ – compact set .

(10) and (11) can be proved by adopting the same items .

Definition 3.2.15 : [10]

A collection C of sets is said to have the finite intersection property (FIP) or to be finitely common , if the intersection of members of each finite subcollection of C is non – empty .

Definition 3.2.16:

Let (X , τ) be a topological space . A collection $\rho = \{ p_\lambda : \lambda \in \Lambda \}$ of subsets of X is said to have :

1) δ^*.FIP , if for every finite subset Λ_o of Λ the subcollection $\{cl\,[int(p_\lambda)] : \lambda \in \Lambda_o \}$ has non–empty intersection , (i.e .

$$\bigcap_{\lambda \in \Lambda_o} cl[int(p_\lambda)] \neq \phi \ .$$

2) S.δ^*.FIP , if for every finite subset Λ_o of Λ the subcollection

$\{S.cl[int(p_\lambda)] : \lambda \in \Lambda_o \}$ has non–empty intersection , (i.e.

$$\bigcap_{\lambda \in \Lambda_o} S.cl[int(p_\lambda)] \neq \phi \ .$$

3) $\theta^*.\delta^*$.FIP , if for every finite subset Λ_o of Λ the subcollection $\{cl[\theta.int(p_\lambda)] : \lambda \in \Lambda_o \}$ has non–empty intersection , (i.e.

$$\bigcap_{\lambda \in \Lambda_o} cl[\theta.int(p_\lambda)] \neq \phi \ .$$

4) $S^*.\delta^*$.FIP , if for every finite subset Λ_o of Λ the subcollection

$\{ cl[S.int(p_\lambda)] : \lambda \in \Lambda_o \}$ has non–empty intersection , (i.e.

$$\bigcap_{\lambda \in \Lambda o} cl[S.int(p_\lambda)] \neq \phi \;;$$

5) $\delta^*.\delta^*$.FIP , if for every finite subset Λ_o of Λ the subcollection $\{cl[\delta.int(p_\lambda) : \lambda \in \Lambda_o\}$ has non–empty intersection , (i.e.

$$\bigcap_{\lambda \in \Lambda o} cl[\delta.int(p_\lambda)] \neq \phi \;.$$

6) $S^{**}.\delta^*$.FIP , if for every finite subset Λo of Λ the subcollection $\{S.cl[S.int(p_\lambda)] : \lambda \in \Lambda_o\}$ has non–empty intersection , (i.e.

$$\bigcap_{\lambda \in \Lambda o} S.cl[S.int(p_\lambda)] \neq \phi \;.$$

Proposition 3.2.17 :

1)If the collection $\rho = \{p_\lambda : \lambda \in \Lambda\}$ of open subsets of a topological space (X , τ) has FIP . Then ρ has δ^*.FIP .

2)If the collection $\rho = \{p_\lambda : \lambda \in \Lambda\}$ of open subsets of a topological space (X , τ) has FIP . Then ρ has $S.\delta^*$.FIP .

3) If the collection $\rho = \{p_\lambda : \lambda \in \Lambda\}$ of θ – open subsets of a topological space (X , τ) has FIP . Then ρ has $\theta^*.\delta^*$.FIP .

4) If the collection $\rho = \{p_\lambda : \lambda \in \Lambda\}$ of semi open subsets of a topological space (X , τ) has FIP . Then ρ has $S^*.\delta^*$.FIP .

5) If the collection $\rho = \{p_\lambda : \lambda \in \Lambda\}$ of δ – open subsets of a topological

space (X , τ) has FIP . Then ρ has δ*.δ*.FIP .

6) If the collection ρ ={p_λ : λ ∈ Λ} of semi open subsets of a topological space (X , τ) has FIP . Then ρ has S**.δ*.FIP .

Proof :

(1) Since ρ has FIP . So each finite subsets Λ_o of Λ ,

$\bigcap_{\lambda \in \Lambda_o} p_\lambda \neq \phi$. But $p_\lambda = int(p_\lambda) \subseteq cl[int(p_\lambda)]$ for all λ ∈ Λ .Then ,

$\bigcap_{\lambda \in \Lambda_o} cl[int(p_\lambda)] \neq \phi$. Thus ρ has δ*.FIP.

(2) , (3) , (4) , (5) and (6) can be proved by the same manner .

Proposition 3.2.18 :

1) Let ρ = {p_λ : λ ∈ Λ} be a collection of open subsets of a topological space (X , τ) . If {cl(p_λ) : λ ∈ Λ } has the FIP . Then ρ has δ*.FIP.

2) Let ρ = {p_λ : λ ∈ Λ} be a collection of open subsets of a topological space (X , τ) . If {S.cl(p_λ) : λ ∈ Λ} has the FIP .Then ρ has S.δ*.FIP.

3) Let ρ ={p_λ : λ ∈ Λ} be a collection of θ – open subsets of a topological space (X , τ) . If {cl(p_λ) : λ ∈ Λ } has the FIP . Then ρ has θ*.δ*.FIP.

4) Let ρ = {p_λ : λ ∈ Λ} be a collection of semi open subsets of a topo – logical space (X , τ) . If {cl(p_λ) : λ ∈ Λ } has the FIP . Then ρ has S*. δ*.FIP.

5)Let ρ = {p_λ : λ ∈ Λ} be a collection of δ – open subsets of a topological space (X , τ) . If $\{cl(p_\lambda) : \lambda \in \Lambda\}$ has the FIP . Then ρ has δ*.δ*.FIP.

6)Let $\rho = \{p_\lambda : \lambda \in \Lambda\}$ be a collection of semi open subsets of a topolo – gical space (X , τ) . If $\{S.cl(p_\lambda) : \lambda \in \Lambda\}$ has the FIP . Then ρ has S**.δ*.FIP.

Proof :

(1) Suppose $\{cl(p_\lambda) : \lambda \in \Lambda\}$ has FIP , so for each finite subsets Λ_o of Λ , we get that $\cap cl(p_\lambda) \neq \phi$.

Now since for all $\lambda \in \Lambda$, p_λ is open set , which implies that $cl[int(p_\lambda)] = cl(p_\lambda)$ for all $\lambda \in \Lambda$. Hence $\bigcap_{\lambda \in \Lambda o} cl[int(p_\lambda)] \neq \phi$, for each finite subset Λ_o of Λ . Thus ρ has δ*.FIP .

(2) , (3) , (4) , (5) and (6) can be proved by the same manner .

Theorem 3.2.19:

A topological space (X , τ) is :

1) δ – compact if and only if every collection of closed subsets with the δ*.FIP have a non–empty intersection .

2) S.δ – compact if and only if every collection of closed subsets with the S.δ*.FIP have a non–empty intersection .

3) θ*.δ – compact if and only if every collection of closed subsets with the θ*.δ*.FIP have a non–empty intersection .

4) S*.δ – compact if and only if every collection of closed subsets with the S*.δ*.FIP have a non–empty intersection .

5) $\delta^*.\delta$ – compact if and only if every collection of closed subsets with the $\delta^*.\delta^*$.FIP have a non–empty intersection .

6) $S^{**}.\delta$ – compact if and only if every collection of closed subsets with the $S^{**}.\delta^*$.FIP have a non–empty intersection .

Proof :

(1) Suppose that (X , τ) be a δ – compact . Let $F=\{F_\lambda : \lambda \in \Lambda\}$ be a collection of closed subsets of X with the δ^*.FIP and suppose if possible , $\bigcap_{\lambda\in\Lambda} F_\lambda = \phi$. Then $X/[\bigcap_{\lambda\in\Lambda} F_\lambda] = X$, and by (De –Morgan law) we get that $\bigcup_{\lambda\in\Lambda} X/[F_\lambda] = X$. But X is δ – compact space . Hence there exists a finite subset Λ_o of Λ such that $\bigcup_{\lambda\in\Lambda o} int[cl(X/F_\lambda)] = X$ which implies that $\bigcup_{\lambda\in\Lambda o} X/[cl(int(F_\lambda))] = X$, and so $\bigcap_{\lambda\in\Lambda o} cl[int(F_\lambda)] = \phi$, which contradicts with δ^*.FIP of F.

Conversely . Let $K=\{K_\lambda : \lambda \in \Lambda\}$ be any open cover of X , so $\bigcup_{\lambda\in\Lambda} K_\lambda = X$, then $X/[\bigcup_{\lambda\in\Lambda} K_\lambda] = \phi$, which implies that $\bigcap_{\lambda\in\Lambda} X/K_\lambda = \phi$. Hence by hypothesis , there exists a finite subsets Λo of Λ , such that $\bigcap_{\lambda\in\Lambda o} cl\,(int[X/(K_\lambda)]) = \phi$, so $\bigcap_{\lambda\in\Lambda o} X/\,int[cl\,(K_\lambda)] = \phi$.

So by (De – Morgan law) , we get that $\bigcup_{\lambda\in\Lambda o} int[cl(K_\lambda)] = X$ .Thus (X , τ) is δ – compact space .

(2) , (3) , (4) , (5) , and (6) can be proved by adopting the same items .

Theorem 3.2.20 :

Let (X , τ) be a topological space and let Y be a :

1) δ – closed subset of X . If x_o is a δ – cluster point of a net $(f , X , A , \geq)$ which is eventually in Y , then $x_o \in Y$.

2) S.δ – closed subset of X . If x_o is a S.δ – cluster point of a net $(f , X , A , \geq)$ which is eventually in Y , then $x_o \in Y$.

3) $\theta^*.\delta$ – closed subset of X . If x_o is a $\theta^*.\delta$ – cluster point of a net $(f , X , A , \geq)$ which is eventually in Y , then $x_o \in Y$.

4) $S^*.\delta$ – closed subset of X . If x_o is a $S^*.\delta$ – cluster point of a net $(f , X , A , \geq)$ which is eventually in Y , then $x_o \in Y$.

5) $\delta^*.\delta$ – closed subset of X . If x_o is a $\delta^*.\delta$ – cluster point of a net $(f , X , A , \geq)$ which is eventually in Y , then $x_o \in Y$.

6) $S^{**}.\delta$ – closed subset of X . If x_o is a $S^{**}.\delta$ – cluster point of a net $(f , X , A , \geq)$ which is eventually in Y , then $x_o \in Y$.

Proof :

(1) By Theorem 2.3.22.Part 1, there exist a subnet g of the net f which δ – converges to x_o .Since f is eventually in Y, g may be considered as a net in Y which δ – converges to x_o . Hence by Theorem 3.1.1.Part 1, we get that x_o is δ – adherent point of Y , consequently, $x_o \in \delta.cl(Y)$. Since Y is δ – closed , we have $\delta.cl(Y) = Y$. Hence $x_o \in Y$.

(2) , (3) , (4) , (5) and (6) can be proved by the same manner .

Theorem 3.2.21 :

Let (X , τ) be a topological space and let (f , X , A , $\geq$) be a net in X . For each a in A , let Ma = { f(x) : x $\geq$ a in A } . Then a point p of X is a :

1) δ – cluster point of f if and only if p $\in$ δ.cl(Ma) for all a $\in$ A .

2) S.δ – cluster point of f if and only if p $\in$ S.δcl(Ma) for all a $\in$ A .

3) θ*.δ – cluster point of f if and only if p $\in$ θ*.δ.cl(Ma) for all a $\in$ A .

4) S*.δ – cluster point of f if and only if p $\in$ S*.δ.cl(Ma) for all a $\in$ A .

5) δ*.δ – cluster point of f if and only if p $\in$ δ*.δ.cl(Ma) for all a $\in$ A .

6) S**.δ – cluster point of f if and only if p $\in$ S**.δ.cl(Ma) for all a $\in$A .

Proof :

(1) The " if " part . Let p $\in$ δ.cl(Ma) for all a $\in$ A , and suppose if Possible p is not a δ – cluster point of f then , there exists an open set U containing p , and an element a in A such that f(x) $\notin$ int[cl(U)] for all x $\geq$ a in A , this implies that int[cl(U)] $\cap$ Ma = ϕ . It follows that p $\notin$ δ.clMa for this a . But this is a contradiction . Hence p is a δ – cluster point of f .

The " only if " part . Let p be a δ – cluster point of f and let N be any open set containing p .Then f is frequently in int[cl(N)] , that is , for each a $\in$ A , there exist x $\geq$ a in A such that f(x) $\in$ int[cl(N)] , hence Ma $\cap$ int[cl(N)] $\neq$ ϕ for every a $\in$ A . Thus for each open set N containing p , int[cl(N) intersect Ma for all a $\in$ A . It follows that $p \in \delta$.cl(Ma) for all a $\in$ A .

(2) , (3) , (4) , (5) and (6) can be proved by the same manner .

Theorem 3.2.22 :

A topological space (X , τ) is :

(1) δ – compact , if each net in X has δ – cluster point .

(2) S.δ – compact , if each net in X has S.δ – cluster point .

(3) $\theta^*.\delta$ – compact , if each net in X has $\theta^*.\delta$ – cluster point .

(4) $\delta^*.\delta$ – compact , if each net in X has $\delta^*.\delta$ – cluster point .

Proof :

(1) Suppose every net in X have δ – cluster point and let K be a collection of closed subsets of X with δ^*FIP .

Let Ω = { D : D is the δ^*– intersection of a finite sub collection of K }

(i.e. $D = \bigcap_{\lambda \in \Lambda_o} cl\,[int(K_\lambda)]$, where $K_\lambda \in$ K , Λ_o finite index . We must prove Ω is directed set by the inclusion relation $\subseteq$. Since K has δ^*.FIP then , each D in Ω is non – empty .

1) If A $\in \Omega$ then A $\subseteq$ A .
2) Let A , B and C $\in \Omega$ and if A $\subseteq$ B and B $\subseteq$ C then , A $\subseteq$ C .
3) Let A and B $\in \Omega$, since K has δ^*.FIP , which implies that there exist finite indexes Λ_o and Γ_o such that $A = \bigcap_{\lambda \in \Lambda_o} cl[int(K_\lambda)]$,

$B = \bigcap_{\lambda \in \Gamma_o} cl[int(K_\lambda)]$. So $A \cap B = (\bigcap_{\lambda \in \Lambda_o} cl[int(K_\lambda)) \cap (\bigcap_{\lambda \in \Gamma_o} cl[int(K_\lambda)])$ then

$A \cap B = \bigcap_{\lambda \in \Lambda_o \cup \Gamma_o} cl[int(K_\lambda)]$. Since Λ_o and Γ_o are finite sets , so $\Lambda_o \cup \Gamma_o$ is finite set. Thus $A \cap B \in \Omega$, and $A \cap B \subseteq A$ and so $A \cap B \subseteq B$.

Then , $(\Omega , \subseteq)$ is directed set . Since each D is non – empty , by the axiom of choice , we may choose a point x(D) in D .

Now consider the function

$f : \Omega \rightarrow X$; $f(D) = x(D)$ for all $D \in \Omega$.

Then f is a net in X . By hypothesis f must have a δ – cluster point , say x_o . Let E be an arbitrary member of Ω . Then for every $D \geq E$ (i.e. $(D \subseteq E)$ in Ω , we have $f(D) = x(D) \in D \subseteq E$. Hence f is eventually in the δ – closed set E .

It follows from Theorem 3.2.20.Part 1, that $x \in E$. Since E was chosen arbitrary $x_o \in \cap \Omega \subseteq \cap K$. Hence $\cap K \neq \phi$ and consequently, X is δ – compact by Theorem 3.2.19.Part 1.

Similarly we can prove numbers (2) , (3) and (4) .

Corollary 3.2.23 :

A topological space (X , τ) is :

(1) δ – compact , if each net in X has a subnet which δ – converges to some points in X .

(2) S.δ – compact , if each net in X has a subnet which S.δ – converges to some points in X .

(3) θ*.δ – compact , if each net in X has a subnet which θ*.δ – converges to some points in X .

(4) $\delta^*.\delta$ – compact , if each net in X has a subnet which $\delta^*.\delta$ – converges to some points in X .

Proof :

It is an immediate consequence of Theorem 2.3.22, and the Theorem 3.2.22.

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المستخلص

الهدف الرئيسي من هذا العمل هو دراسة أنماط معينه من التقارب في فضاءات التبولوجيا. ادناه بعض النتائج الرئيسية التي تم الحصول عليها:

1. اذا كانت g دالة من الفضاء التبولوجي (X,τ) الى الفضاء التبوليجي(Y,μ) و النقطة$x_o \in X$ فان g تكون :

i) δ - مستمرة عند x_o اذا وفقط اذا عندما الشبكة $\{fa : a A\}\delta$– تقترب الى x_o فان الشبكة $\{ g(fa) : a A \}\ \delta$ – تقترب الى $g(x_o)$.

ii) $S.\delta$ - مستمرة عند x_o اذا وفقط اذا عندما الشبكة $\{fa : a A\}S.\delta$– تقترب الى x_o فان الشبكة $\{ g(fa) : a A \}\ S.\delta$ – تقترب الى $g(x_o)$.

iii) $\theta^*.\delta$ - مستمرة عند x_o اذا وفقط اذا عندما الشبكة $\{fa : a A\}\theta^*.\delta$– تقترب الى x_o فان الشبكة $\{ g(fa) : a A \}\ \theta^*.\delta$ – تقترب الى $g(x_o)$.

iv) $S^*.\delta$ - مستمرة عند x_o اذا وفقط اذا عندما الشبكة $\{fa : a A\}S^*.\delta$– تقترب الى x_o فان الشبكة $\{ g(fa) : a A \}\ S^*.\delta$ – تقترب الى $g(x_o)$

v) $\delta^*.\delta$ - مستمرة عند x_o اذا وفقط اذا عندما الشبكة $\{fa : a A\}\delta^*.\delta$– تقترب الى x_o فان الشبكة $\{ g(fa) : a A \}\ \delta^*.\delta$ – تقترب الى $g(x_o)$.

vi) $S^{**}.\delta$ - مستمرة عند x_o اذا وفقط اذا عندما الشبكة – $\{fa : a A\}S^*.\delta$ تقترب ال x_o فان الشبكة $\{ g(fa) : a A \}\ S^{**}.\delta$ – تقترب الى $g(x_o)$.

2. ليكن (X,τ) فضاء تبولوجي ولتكن $Y \subseteq X$ ،اذا كانت x_o نقطة من نقاط X فان x_o تكون :-

i) نقطة δ– ملتصقة في Y ، اذا وفقط اذا توجد شبكة في $Y\ \delta$– متقاربة الى x_o.

ii) نقطة $S.\delta$– ملتصقة فيY، اذا وفقط اذا توجد شبكة في $S.\delta Y$– متقاربة الى x_o .

iii) نقطة $\delta.*\theta$– ملتصقة في Y ، اذا وفقط اذا توجد شبكة في $\delta Y.*\theta$– متقاربة الى x_o .

iv) نقطة $\delta.*S$– ملتصقة في Y ، اذا وفقط اذا توجد شبكة في $\delta Y.*S$– متقاربة الى x_o .

v) نقطة $\delta.*\delta$– ملتصقة في Y ، اذا وفقط اذا توجد شبكة في $\delta Y.*\delta$– متقاربة الى x_o .

vi) نقطة $\delta.**S$– ملتصقة في Y ، اذا وفقط اذا توجد شبكة في $\delta Y.**S$– متقاربة الى x_o.

3. لتكن(X,τ) فضاء تبولوجي ولتكن $Y \subseteq X$. فان Y تتألف من كل :

i) نقاطها δ– التراكمية اذا وفقط اذا لاتوجد شبكة في δ Y – تقترب الى نقطة في X/Y .

ii) نقاطها $\delta.S$– التراكمية ل Y اذا وفقط اذا لاتوجد شبكة في $\delta Y.S$– تقترب الى نقطة في X/Y .

iii) نقاطها $\delta.*\theta$– التراكمية ل Y اذا وفقط اذا لاتوجد شبكة في Y $\delta.*\theta$– تقترب الى نقطة في X/Y .

iv) نقاطها $\delta.*S$– التراكمية ل Y اذا وفقط اذا لاتوجد شبكة في Y $\delta.*S$ – تقترب الى نقطة في X/Y .

v) نقاطها $\delta.*\delta$– التراكمية ل Y اذا وفقط اذا لاتوجد شبكة في Y $\delta.*\delta$ – تقترب الى نقطة في X/Y .

vi) نقاطها $\delta.**S$– التراكمية ل Y اذا وفقط اذا لاتوجد شبكة في Y $\delta.**S$ – تقترب الى نقطة في X/Y .

4. اذا كان (X,τ) فضاء تبولوجي :

i) δ – هاوزدورف فان كل شبكة δ– متقاربة لها نقطةδ – تجمع وحيدة وهي نقطةδ – نهاية وحيدة للشبكة.

ii) $S.\delta$ – هاوزدورف فان كل شبكة $S.\delta$– متقاربة لها نقطة$S.\delta$ – تجمع وحيدة وهي نقطة$S.\delta$– نهاية وحيدة للشبكة.

iii) $\delta.*\theta$ – هاوزدورف فان كل شبكة $\delta.*\theta$– متقاربة لها نقطة$\delta.*\theta$ – تجمع وحيدة وهي نقطة$\delta.*\theta$ – نهاية وحبدة للشبكة.

iv)$S*.\delta$ – هاوزدورف فان كل شبكة $\delta.*S$– متقاربة لها نقطة$\delta.*S$ – تجمع وحيدة وهي نقطة$\delta.*S$– نهاية وحبدة للشبكة.

v) $\delta.*\delta$ – هاوزدورف فان كل شبكة $\delta.*\delta$– متقاربة لها نقطة$\delta.*\delta$ – تجمع وحيدة وهي نقطة$\delta.*\delta$– نهاية وحبدة للشبكة.

vi) $S**.\delta$ – هاوزدورف فان كل شبكة $\delta.**S$– متقاربة لها نقطة $\delta.**S$– تجمع وحيدة وهي نقطة$\delta.**S$– نهاية وحبدة للشبكة.

5. ليكن (X,τ) فضاء تبولوجي. النقطة x_o في X تكون :

i) نقطة δ – تجمع للشبكة $(f , X , A , \geq)$ اذا وفقط اذا توجد هنالك شبكة فرعية $(g , f , B , \geq^*)$ تكون δ– تقترب الى x_o .

ii) نقطة $S.\delta$ – تجمع للشبكة $(f , X , A , \geq)$ اذا وفقط اذا توجد هنالك شبكة فرعية

$(g , f , B , \geq^*)$ تكون $S.\delta$– تقترب الى x_o .

iii) نقطة $\delta.*\theta$ – تجمع للشبكة $(f , X , A , \geq)$ اذا وفقط اذا توجد هنالك شبكة فرعية

$(g , f , B , \geq^*)$ تكون $\delta.*\theta$– تقترب الى x_o .

iv) نقطة $\delta .*S$– تجمع للشبكة $(f , X , A , \geq)$ اذا وفقط اذا توجد هنالك شبكة فرعية $(g , f , B , \geq^*)$ تكون $\delta.*S$– تقترب الى x_o .

v) نقطة $\delta.*\delta$ – تجمع للشبكة $(f , X , A , \geq)$ اذا وفقط اذا توجد هنالك شبكة فرعية

$(g , f , B , \geq^*)$ تكون $\delta.*\delta$– تقترب الى x_o .

vi) نقطة δ.**S – تجمع للشبكة ($f , X , A , \geq$) اذا وفقط اذا توجد هنالك شبكة فرعية ($g , f , B , \geq^*$) تكون δ.**S– تقترب الى x_o .

6. الفضاء التبولوجي (X,τ) يكون :

i) δ– متراص اذا كانت كل شبكة في X لها نقطة δ– تجمع.

ii) $S.\delta$ – متراص اذا كانت كل شبكة في X لها نقطة $S.\delta$– تجمع .

iii) $\theta^*.\delta$– متراص اذا كانت كل شبكة في X لها نقطة$\theta^*.\delta$ – تجمع.

iv) $\delta^*.\delta$– متراص اذا كانت كل شبكة في X لها نقطة$\delta^*.\delta$ – تجمع .

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل – كلية التربية

قسم الرياضيات

حول أنماط معينة من
التقارب في فضاءات التبولوجيا

رسالة مقدمة الى
قسم الرياضيات – كلية التربية – جامعة بابل
كجزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

من قبل
ضياء محمد مهدي

بإشراف
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