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Some Types of g-Lifting Modules and Their Generalizations

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بسم الله الرحمن الرحيم

اللَّهَ ... عِبَادِهِ الْعُلَمَاءُ إِنَّ اللَّهَ عَزِيزٌ غَفُورٌ ﴿

صدق الله العلي العظيم

سورة فاطر الآية ﴿٢٨﴾

Dedication

I dedicate my dissertation work to my family. A special feeling of gratitude

My loving parents Dad and Mom

whose words of encouragement and push for tenacity ring in my ears.

My brothers and sisters

Dr. Saif, Dr. Zaid, Miss Hala, Miss Adraa. who have always loved me unconditionally and whose good examples have taught me to work hard for the things that I aspire to achieve.

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List of Symbols

The symbols	Meaning
$\boldsymbol{R}$	an associative ring with identity
$\boldsymbol{M_R}$	M is a right R-module
$\boldsymbol{N \leq K}$	N is a submodule of K
$\boldsymbol{N \trianglelefteq K}$	N is an essential submodule of K
$\boldsymbol{N \leq^{max} K}$	N is a maximal submodule of K
$\boldsymbol{N \leq^{\oplus} K}$	N is a direct summand of K
$\boldsymbol{N \ll K}$	N is a small submodule of K
$\boldsymbol{N \ll_{\delta} K}$	N is a δ-small submodule of K
$\boldsymbol{N \ll_{g} K}$	N is a g-small submodule of K
$\boldsymbol{Rad(M)}$	the Jacobson radical of a module M
$\boldsymbol{Rad_g(M)}$	the generalized radical of a module M
$\boldsymbol{Z(M)}$	the singular submodule of a module M
$\boldsymbol{\delta(M)}$	the sum of all δ-small submodules of M
$\boldsymbol{\overline{Z}_g(M)}$	the intersection of all kernels of $g \in Hom(M, N)$, N is g-small module
$\boldsymbol{\oplus_{i \in I} M_i}$	the direct sum of modules M_i, for all $i \in I$
PID	principal ideal domain
$\boldsymbol{P_g(M)}$	the sum of all generalized radical submodules of M
$\boldsymbol{ann_R(M)}$	the right annihilator ideal of M in R
$\boldsymbol{(E :_R M)}$	the residual of E by M in R
$\boldsymbol{\mathcal{L}(N)}$	the set of all elements of R that are not prime N
$\boldsymbol{ACC}$	ascending chain condition of submodules
$\boldsymbol{DCC}$	descending chain condition of submodules

Abstract

The primary objective of this study was to explore the characterizations, features, and relationships associated with seven concepts. Various examples and important consequences were provided, along with illustrations of the connections between these concepts.

The second objective was to examine the relationships between the ideas in this work and other types of modules, such as indecomposable, semisimple, generalized hollow, g-lifting, ⊕-g-supplemented, distributive, uniform, weak duo, projective modules, etc. The attributes, characterizations, and examples of these modules were explored, and direct connections to other notions were identified. Conditions that make the classes of the seven modules closed under direct sums were also provided.

This study presented several concepts in module theory. Firstly, the notion of principally generalized radical-supplemented modules, abbreviated as PG-radical supplemented, was introduced. In this concept, an R-module M was considered PG-radical supplemented if, for any cyclic submodule N of M has PG-radical supplement. The concept of P-($\mathrm{P}_{\mathrm{g}}^{*}$)-modules, introduced as generalization of principally g-lifting modules.

Additionally, the concept of ⊕-PG-radical supplemented modules was introduced, an R-module M was considered ⊕-PG-radical supplemented if, for any cyclic submodule N of M has direct summand PG-radical supplement. The idea of a $\mathrm{Rad}_{\mathrm{g}}$-lifting module was discussed, defining it as an R-module M in which, for any submodule N of M with $Rad_g(M) \subseteq N$, there was a decomposition $M = A \oplus B$ such that $A \le N$ and $N \cap B \ll_g B$. This concept

was extended to a ring R being called Rad_g-lifting if it behaved as an R-module. Furthermore, this idea was applied to cyclic submodules and termed as principally Rad_g-lifting modules. Moreover, the concept of generalized Rad_g-lifting modules, denoted as G-Rad_g-lifting, was introduced, where an R-module M was considered G-Rad_g-lifting if, for every $N \leq M$ with $Rad_g(M) \subseteq N$, there was a decomposition $M=A \oplus B$ such that $A \leq N$ and $N \cap B \subseteq Rad_g(B)$. The case in which a ring R was considered G-Rad_g-lifting was also discussed.

Introduction

Throughout this dissertation all rings are associative with identity unless otherwise stated, and all modules are right unitary.

A proper submodule N of a module M is called small in M, i.e., $N \ll M$ if, for any submodule L of M with $M = N + L$, then $L = M$, equivalently, for any $L \subset M$, $N + L \subset M$ [22]. If M is an R-module, the Jacobson radical of M is denoted by $Rad(M)$, and defined as the intersection of all maximal submodules of M, or the sum of all small submodules of M [22]. If M has no maximal submodule, then we set $Rad(M) = M$. Many years ago, many authors [7] gave a definition of Rad-supplemented module as, if for every submodule N in M, we have that $M = N + K$ and $N \cap K \subseteq Rad(K)$ for some $K \leq M$. Later in 2012 the concept of ⊕-Rad-supplemented module have been introduced in [11] if for every direct summand submodule N in M, we have that $M = N + K$ and $N \cap K \subseteq Rad(K)$ for some $K \leq M$.

A nonzero submodule N of a module M is said to be essential in M, i.e, $N \trianglelefteq M$ if, for any $L \leq M$ with $N \cap L = 0$ implies $L = 0$ [22]. In [42] Zhang and Zhou call submodule K as g-small in M, i.e., $K \ll_g M$ if, for every essential submodule T of M such that $M = K + T$ implies that $T = M$. Also, Zhou and Zhang [42] defined the generalized radical of an R-module M as the intersection of all submodules that are both essential and maximal submodule of M and denoted by $Rad_g(M)$. Equivalently, $Rad_g(M)$ is the sum of all g-small submodules of M. The definition of g-radical supplemented module was presented by [26].

In 2020, Ghawi introduce the definition of ⊕-g-radical supplemented module as, a module M is called ⊕-g-radical supplemented if, every submodule of M has a g-radical supplement that is a direct summand, i.e., for every $N \leq M$, there is a direct summand H of M such that $M = N + H$ and $N \cap H \subseteq Rad(H)$ [14]. In [15] Ghawi introduced the definition of (P_g^*)-modules as, if for $N \leq M$, there exists a direct summand H of M where $H \leq N$ and $N/H \subseteq Rad_g(M/H)$, a module M is said to have (P_g^*) property or (P_g^*)-module.

Lifting modules have been extensively studied by several authors, and various generalizations for these modules were given, for example, g-lifting modules [33], principally lifting modules [19], principally g-lifting modules [13], …etc. The ideas in this work.

In this work, we will introduce three main concepts as generalizations of the concepts of generalized radical supplemented modules, (P_g^*)-modules and ⊕-generalized radical supplemented modules respectively. Then we presented two ideas of Rad_g-lifting modules and G-Rad_g-lifting modules with their generalizations principally Rad_g-lifting modules and principally G-Rad_g-lifting modules respectively.

According to the order of the chapters: This dissertation consists of five chapters. Chapter one is divided into two sections. In section 1, we mentioned all the definitions that we used in this work. In section 2, we represented and explained all the theorems, propositions and corollaries that building this work as lemmas.

In chapter two of this study is entitled "PG-radical supplemented and P-(P_g^*)-property" is divided into six sections. In sections 1 & 4, the notion of PG-radical supplemented and P-(P_g^*)-modules, were defined and examined. We'll go through the characterization of the notions of PG-radical supplemented modules (P-(P_g^*)-modules), with some of their features. Also, we take some examples, remarks and studied the closure under direct summands and factor modules. We started by important propositions then concluded major theorems where we investigate the factors of PG-radical supplemented modules (P-(P_g^*)-modules) in some cases. A direct summand of the PG-radical supplemented modules (P-(P_g^*)-modules), has been investigated in a series of results which begin by several propositions.

Then we gave a condition that makes a PG-radical supplemented modules (P-(P_g^*)-modules), inherited by its direct summands. Also, we discussed the direct sums of them under some cases.

Sections 2 & 5 are about the relation between PG-radical supplemented modules (P-(P_g^*)-modules) and other types of modules such as, generalized hollow, semisimple, …etc. Also, we are proofed.

In sections 3 & 6, we looked at and explored the behavior of PG-radical supplemented modules (P-(P_g^*)-modules) under localization. So, we gave some lemmas and theorems. As well, we investigated the P-(P_g^*) property as a ring, in section 6.

Chapter three is entitled "⊕-PG-radical supplemented modules" and it has three sections. As a more advanced notion of ⊕-PG-radical supplemented modules, we defined and investigated a ⊕-PG-radical supplemented modules, and some basic properties that are shows in section 1 of this chapter. By using the indecomposable modules, we provide many equivalent cases for the notion ⊕-PG-radical supplemented modules.

We describe the direct sums of the notions of ⊕-PG-radical supplemented modules, under certain conditions. Also, in the same section we looked into the direct summands and factors of ⊕-PG-radical supplemented modules. The factor is proved under many conditions as we see in many propositions. We showed that if, M/L is a ⊕-PG-radical supplemented module, then M does not need to be a ⊕-PG-radical supplemented, for a submodule L of a module M by couture example.

Also, we investigated the conditions of direct summands for ⊕-PG-radical supplemented modules. Later we conclude a corollaries which give us a cases that make a direct sum of ⊕-PG-radical supplemented modules is ⊕-PG-radical supplemented if the module is satisfied one of the following cases, weak-duo, duo, weakly distributive or M has SIP.

In section 2, we discussed numerous relations between the idea of ⊕-PG-radical supplemented modules and other types of modules. We start by many theorems which declare these relations with ⊕-PG-radical supplemented modules under special conditions. Then we start to form different equivalence relations under case zero generalized radical. As well, if we have g-V-ring (*e*-noncosingular) we had the same results. Through this section by remarks we show that some cases are necessary in theorems. Also, we give two cases that can make the factor $M/Rad_g(M)$ of a cyclic and ⊕-PG-radical supplemented module M over a PID R is principally semisimple. We ended this section with a five-point equivalents that is true only if, the module has zero generalized radical.

In section 3, we discussed the ⊕-PG-radical supplemented property as a ring. In addition, we investigate the behavior of ⊕-PG-radical supplemented modules under the localization. We have talked about some interesting results.

Chapter four, is entitled "$\mathrm{Rad_g}$-lifting modules and P-$\mathrm{Rad_g}$-lifting modules" is divided into six sections. In sections 1 & 4, we defined the class of modules that is known as $\mathrm{Rad_g}$-lifting and P-$\mathrm{Rad_g}$-lifting respectively. Therefore, we presented the main proposition which gave us seven equivalents assertions to our definition. Also, there were certain properties, examples, characterizations and instances provided. As well we gave a condition that make a direct sum of Rad_g-lifting modules (P-Rad_g-lifting modules) is also Rad_g-lifting modules (P-Rad_g-lifting modules). Basic characteristics and factor modules will be checked. We discuss the factor module of Rad_g-lifting modules (P-Rad_g-lifting modules) in many propositions and corollaries.

In sections 2 & 5, several relations are made between our notions and different types of modules. As well as the classes, semisimple (principally semisimple) modules, g-lifting (principally g-lifting) modules, ⊕-g-supplemented (principally ⊕-g-supplemented) modules, (P_g^*)-modules (principally (P_g^*)-modules), $sgrs^{\oplus}$-modules (principally $sgrs^{\oplus}$-modules) and ⊕-G-Rad_g-supplemented (⊕-PG-Rad_g-supplemented) modules are equivalent only if, the generalized radical of the module is equal to zero. We build another theorem that has the same condition over a ring *R* but with a special group of some types of *R*-modules.

Finally, in sections 3 & 6, we discuss a localization of Rad_g-lifting modules and P-Rad_g-lifting modules. The Rad_g-lifting and P-Rad_g-lifting properties is also studied as a ring in these two sections.

Chapter five is, entitled "G-Rad_g-lifting modules and PG-Rad_g-lifting modules" is divided into six sections. In sections 1 & 4, we defined a class of modules that is known as a G-Rad_g-lifting modules and PG-Rad_g-lifting modules respectively, also, there were certain properties, examples, characterizations, instances provided, direct summands, factor modules, homomorphism images. We collected and remembered several examples of as well as some observations that are useful in the writing of this dissertation as well as clarify some concepts and prove them elsewhere.

In sections 2 & 5 within this chapter, several connections are made between our notions and different types of modules that we mentioned for them previously which they were had direct relation with our definitions and indirect sometimes, which required us to add some cases. We build another theorem that has the same condition over a ring R but with a special group of some types of R-modules.

In two sections 3 & 6, during these sections we exposed the concept of localization, and we studied how it effects on $\mathrm{Rad_g}$-lifting (P-$\mathrm{Rad_g}$-lifting) modules. As well as, we discussed these notions as rings.

Note in the last two chapters we included a case within the basic definition, which is $Rad_g(N) \subseteq M$ for each submodule N of a module M. This required lead us to add conditions sometimes and delete conditions at other times.

Chapter Three is titled "⊕-PG-Radical Supplemented Modules" and is divided into three sections. In this chapter, we explore the concept of ⊕-PG-radical supplemented modules in greater depth. We define and investigate ⊕-PG-radical supplemented modules, presenting their basic properties in the first section. By utilizing indecomposable modules, we establish several equivalent cases for ⊕-PG-radical supplemented modules.

We discuss the direct sums of ⊕-PG-radical supplemented modules under specific conditions in the same section. Additionally, we examine the direct summands and factors of ⊕-PG-radical supplemented modules. We provide proofs for these factors under various conditions, as indicated by multiple propositions. Furthermore, we demonstrate that if M⁄L is a ⊕-PG-radical supplemented module, M itself does not necessarily have to be ⊕-PG-radical supplemented, as exemplified in our corollary.

In this section, we also investigate the conditions for direct summands of ⊕-PG-radical supplemented modules. Towards the end, we derive corollaries that outline cases where the direct sum of ⊕-PG-radical supplemented modules is also ⊕-PG-radical supplemented, assuming that the module satisfies one of the following conditions: weak-duo, duo, weakly distributive, or M has SIP.

Moving on to Section 2 of this chapter, we explore various relationships between ⊕-PG-radical supplemented modules and other module types. We begin by introducing several theorems that establish these relationships under specific conditions. Additionally, we examine different equivalence relations under the context of the zero generalized radical and g-V-ring (g-noncosingular) settings.

By providing remarks, we highlight the necessity of certain cases in the theorems. Furthermore, we present two cases where the factor $M / (Rad_g(M))$ of a cyclic and ⊕-PG-radical supplemented module M over a PID R is principally semisimple. We conclude this section with a five-point equivalence that holds true only if the module has a zero generalized radical.

In Section 3, we shift our focus to the ⊕-PG-radical supplemented property as a ring. We also investigate how ⊕-PG-radical supplemented modules behave under localization, presenting several interesting results.

Chapter Four, titled "$\mathrm{Rad_g}$-Lifting Modules and P-$\mathrm{Rad_g}$-Lifting Modules," is divided into six sections. In Sections 1 and 4, we define two classes of modules: $\mathrm{Rad_g}$-lifting and P-$\mathrm{Rad_g}$-lifting. We present the main proposition, which provides seven equivalent assertions to our definitions. We also explore various properties, examples, characterizations, and instances. Additionally, we discuss conditions under which the direct sum of $\mathrm{Rad_g}$-lifting modules (P-$\mathrm{Rad_g}$-lifting modules) remains $\mathrm{Rad_g}$-lifting modules (P-$\mathrm{Rad_g}$-lifting modules). We delve into basic characteristics and factor modules, examining them through multiple propositions and corollaries.

In Sections 2 and 5 of this chapter, we establish connections between our notions and different types of modules discussed earlier. We demonstrate equivalence among classes, such as semisimple (principally semisimple) modules, g-lifting (principally g-lifting) modules, $\oplus$-g-supplemented (principally $\oplus$-g-supplemented) modules, (P_g^*)-modules (principally (P_g^*)-modules), sgrs$^{\oplus}$-modules (principally sgrs$^{\oplus}$-modules), and $\oplus$-PG-$\mathrm{Rad_g}$-supplemented ($\oplus$-PG-$\mathrm{Rad_g}$-supplemented) modules when the generalized radical of the module equals zero. We introduce another theorem with the same condition applicable to a special group of R-modules over a ring R.

In Sections 3 and 6, we examine the localization of $\mathrm{Rad_g}$-lifting modules and P-$\mathrm{Rad_g}$-lifting modules. We also study these properties as rings.

To conclude, the last two chapters include a case within the basic definition, which is $Rad_g(M) \subseteq N$ for each submodule N of a module M. This occasionally leads us to add or remove conditions as needed.

In this chapter we will review several of important definitions and outcomes. This chapter is divided into two sections. In section 1, we recall many basic definitions which we need in our work. In section 2, we are going to include several theorems, propositions, and previous corollaries as lemmas.

1.1. Basic definitions.

In this section of the chapter, we will state most of the definitions that we need in our work.

Definition 1.1.2. [22, p.19] If 0 and M are the only submodules of an R-module M, then M is called simple.

Definition 1.1.3. [22, p.19] A submodule E of an R-module M is said to be maximal, denoted by $E \leq^{max} M$ if, $E \neq M$ and for any submodule G of M with $E \subset G \leq M$, then $G = M$.

Definition 1.1.4. [22, p.19] A submodule E of an R-module M is said to be minimal, if, for $E \neq M$ and for any submodule G of M with $G \subset E \leq M$, then $G = 0$.

Definition 1.1.5. [22, p.18] A submodule of an R-module M is called cyclic if it is generated by m. So that, a module M is called cyclic if it generated by a subset $X = \{m\}$ contains one element only, that is $M = \langle\{m\}\rangle$ or M is called cyclic if and only if there is an $m \in M$, where $M = mR$.

Definition 1.1.6. [22, p.22] If an R-module M has a finite generating set, it is said to be finitely generated, say X, that is $M = \langle X \rangle$.

Definition 1.1.7. [22, p.31] A submodule E of an R-module M is called a direct summand of M, denoted by $E \leq^{\oplus} M$ if, there exists a submodule C of M such that $E + C = M$ and $E \cap C = 0$.

Definition 1.1.8. [22, p.191] If every submodule of an R-module M is a direct summand, then M is called semisimple.

Definition 1.1.9. [36, p.47] An R-module M is called regular if all its cyclic submodules are direct summands of M. Later it is called principally semisimple [13].

Definition 1.1.10. [22, p.32] An R-module M is called an indecomposable if the only direct summands of M are 0 and M.

Definition 1.1.11. [22, p.106] A submodule E of an R-module M is said to be small in M, denoted by $E \ll M$ if, for any submodule X of M with $M = E + X$, then $X = M$, or, for any $X \subset M$, $E + X \subset M$.

Definition 1.1.12. [22, p.106] A nonzero submodule E is said to be essential in an R-module M, denoted by $E \trianglelefteq M$ if, for any $X \leq M$ and $E \cap X = 0$ implies X= 0, equivalently, for any submodule $X \neq 0$, $E \cap X \neq 0$.

Definition 1.1.13. [16, p.17] let A and B are two submodules of an R-module M, then A is relative complement of B, if A is maximal in $A \cap B = 0$.

Definition 1.1.14. [22, p.212] The Jacobson Radical of an R-module M denoted by $Rad(M)$, and defined as the intersection of all maximals of M, or the sum of all small submodules of M. If M has no maximal submodule, then $Rad(M) = M$.

Definition 1.1.15. [40, p.351] A nonzero R-module M is called hollow if every proper submodule is small in M.

Definition 1.1.16. [20] A nonzero R-module M is called principally hollow (denoted by , P-hollow) if every proper cyclic submodule is small in M.

Definition 1.1.17. [16, p.85] A nonzero R-module is called uniform if all its nonzero submodules are essential.

Definition 1.1.18. [16, p.17] Let M be an R-module, the annihilator of a module M in R denoted by $ann(M)_R$, defined as the set of those $r \in R$ such that $mr = 0$ for all $m \in M$.

Definition 1.1.19. [16, p.31] Let M be an R-module, put $Z(M) = \{m \in M: ann(M)_R \trianglelefteq R\}$. $Z(M)$is called the singular submodule of M. M is called singular if $Z(M) = M$, and M called nonsingular if $Z(M) = 0$.

Definition 1.1.20. [42] A submodule K of an R-module M is called g-small in M, can write $K \ll_g M$ if, for every essential L in M such that $M = K + L$ then $L = M$. The authors Zhou and Zhang call a g-small submodule as an e-small submodule.

Definition 1.1.21. [42] A generalized Radical of an R-module M is the intersection of all essential maximal in M or, the sum of all g-small `submodules, and it is denoted by $Rad_g(M)$. If M has no maximal submodule, then we set $Rad_g(M) = M$.

Definition 1.1.22. [43] A submodule N of an R-module M is called δ-small in M, denoted by $N \ll_\delta M$ if, whenever $M = N + X$ with M/X singular, we have $M = X$.

Definition 1.1.23. [43] Let M be an R-module, define $\delta(M)$ as the following: $\delta(M) = \sum\{N \leq M | N$ is a δ-small submodule of $M\}$. Since, every δ-small submodule is g-small, then $\delta(M) \leq Rad_g(M)$.

Definition 1.1.24. [3, p.357] an R-module M is called to be local if, $Rad(M)$ is maximal in M and $Rad(M) \ll M$.

Definition 1.1.25. [35] An R-module M is called to be g-local if, $Rad_g(M)$ is maximal in M and $Rad_g(M) \ll_g M$.

Definition 1.1.26. [21] Let M be an R-module, the sum of all generalized Radical submodules of M denoted by $P_g(M)$, i.e., $P_g(M) = \sum\{E \leq M \big| Rad_g(E) = E\}$.

Definition 1.1.27. [26] An R-module M is called generalized hollow if each its proper submodule is g-small.

Definition 1.1.28. [14] An R-module M is called principally generalized hollow if each its proper cyclic submodule is g-small.

Definition 1.1.29. [22, p.147] An R-module M is said to be Noetherian (Artinian) if every nonempty set of submodules have a maximal (minimal, respectively) element ordered by inclusion.

Definition 1.1.30. [40, p. 539] An R-module M over ring R is called uniserial if all submodules are totally ordered by inclusion. This simply means that for any two submodules E_1 and E_2 of M, either $E_1 \leq E_2$ or $E_2 \leq E_1$.

Definition 1.1.31. [22, p.23] An R-module M is called free if it is having basis.

Definition 1.1.32.[22, P.124**]** An R-module N is called injective if and only if for any two R-modules A, B and for any monomorphism $f: A \to B$ and any homomorphism $g: A \to N$, there exists a homomorphism $h: B \to N$ such that $h \circ f = g$. If $N \trianglelefteq M$ and M is an injective module, then M is said to be an injective envelop of N, denoted by $E(N) = M$.

Definition 1.1.33. [22, p.117] An R-module P is called projective if and only if for any two R-modules H, G and for any epimorphism $\tau: H \to G$ and for any $g: P \to G$ homomorphism, there is $\sigma: P \to H$ homomorphism with $\tau \circ \sigma = g$.

Definition 1.1.34. [3, p.184] Let E and M be two R-modules. M is called projective with respect to a module E (or E-projective) in case for each $X \leq E$, and for each homomorphism $h: M \rightarrow E/X$ could be lifted a homomorphism $g: M \rightarrow E$, i.e., $\pi \circ g = h$ where $\pi: E \rightarrow E/X$ is a natural map. A module which is projective with respect to itself is said to be a quasi-projective (or self-projective) module [31, p.68].

Definition 1.1.35. [31, p.74] Let $I = \{1,2, \dots, n\}$, where n be a positive integer. The class of modules $\{M_i | \ i \in I\}$ is said to be relatively projective if for M_i is M_j-projective for all $(i \neq j) \in I$.

Definition 1.1.36. [40. p.40] A submodule E of an R-module M is said to be fully invariant if, $f(E) \subseteq E$ for all $f \in End(M)$.

Definition 1.1.37. [33] If all submodules of an R-module M are fully invariant, then M is called a duo module.

Definition 1.1.38. [33] If all direct summands of an R-module M are fully invariant, then M is called a weak duo module.

Definition 1.1.39. [8] A submodule H of a module M is called distributive if $H \cap (G + L) = (H \cap G) + (H \cap L)$ or $H + (G \cap L) = (H + G) \cap (H + L)$ for all submodules G, L of M. A module M is called distributive if all submodules of M are distributive.

Definition 1.1.40. [6] A submodule H of an R-module M is weak distributive if, $H = (H \cap X) + (H \cap Y)$ for all submodules X, Y of M such that $X + Y = M$. A module M is called weakly distributive if every submodule of M is a weak distributive submodule of M.

Definition 1.1.41. [40, p.359] An R-module M is called π-projective if, for any two submodules H, G of M with $M = H + G$ there is an $f \in End_R(M)$ such that $Imf \leq H$ and $Im(1 - f) \leq G$.

Definition 1.1.42. [41, p.50] An R-module M is called refinable if for all submodules U and V of M with $M = U + V$, there is a direct summand $\acute{U}$ of M such that $\acute{U} \subseteq U$ and $M = \acute{U} + V$.

Definition 1.1.43. [39] An R-module M is said to have the summand intersection property, briefly SIP, if the intersection of any two direct summands of M is again a direct summand.

Definition 1.1.44. [2] An R-module M is said to have the summand sum property, briefly SSP, if the sum of any two direct summands of M is also direct summand of M.

Definition 1.1.45. [17] A submodule L of an R-module M is called g-coclosed if whenever $N \leq L$ with $L/N \ll_g M/N$ implies $L = N$.

Definition 1.1.46 [40, p.348] For submodules L, E of an R-module M. Then L is called a supplement of E in M if, L is a minimal element in the set of submodules X of M with $E + X = M$, or equivalently, L is a supplement submodule of E in M if, $E + L = M$ and $E \cap L \ll L$. A module M is called supplemented if every submodule of M has a supplement in M.

Definition 1.1.47. [1] An R-module M is said to be principally supplemented if for all cyclic submodule L of M, there is a submodule X of M with $M = L + X$ and $L \cap X$ is small in X.

Definition 1.1.48. [26] Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $M = U + L$ with $L \trianglelefteq V$ implies $L = V$, then V is called a g-supplement of U in M; or equivalently, V is a g-supplement of U in M if and only if $M = U + V$ and $U \cap V \ll_g V$. If every submodule of M has a g-supplement in M, then M is called a g-supplemented module.

Definition 1.1.49. [32] An R-module M is called principally g-supplemented if every cyclic submodule of M has a principally g-supplement in M, i.e., for $m \in M$, there is $L \leq M$ such that $M = mR + L$ and $mR \cap L \ll_g L$.

Definition 1.1.50. [31, p.95] An R-module M is said to be $\oplus$-supplemented if every submodule of M has a supplement that is a direct summand, i.e., for every $L \leq M$, there is a direct summand N of M with $L + N = M$ and $L \cap N \ll N$.

Definition 1.1.51. [38] An R-module M is called principally $\oplus$-supplemented if for any cyclic submodule L of M, there is a direct summand X of M such that $M = L + X$ and $L \cap X$ is small in X.

Definition 1.1.52. [14] An R-module M is called $\oplus$-g-supplemented if every submodule of M has a g-supplement that is a direct summand of M, i.e., for any $E \leq M$ there is a direct summand C of M with $E + C = M$ and $E \cap C \ll_g C$. For any ring R, in case a module R_R is $\oplus$-g-supplemented, then R is said to be a $\oplus$-g-supplemented ring.

Definition 1.1.53. [32] An R -module M is called principally $\oplus$ -g-supplemented if every cyclic submodule of M has a principally g-supplement that is a direct summand of M, i.e., that is, for each $m \in M$, there exists a submodule L of M such that $M = mR + L = L' \oplus L$ for some $L' \leq M$ with $mR \cap L \ll_g L$.

Definition 1.1.54. [19] An R-module M is called principally δ-supplemented if for all cyclic submodule N of M, there exists a submodule X of M such that $M = N + X$ and $N \cap X \ll_\delta X$.

Definition 1.1.55. [37] An R -module M is said to be principally $\oplus$-δ-supplemented if for all cyclic submodule N of M, there exists a direct summand X of M such that $M = N + X$ and $N \cap X \ll_\delta X$.

Definition 1.1.56. [25] An R-module M is called $\oplus$-G-$\mathrm{Rad_g}$-supplemented if, any submodule N of M with $Rad_g(M) \subseteq N$ there is a direct summand A of M such that $M = N + A$ and $N \cap A \subseteq Rad_g(M)$.

Definition 1.1.57. [21] Let R be a ring. An R-module M is said to be principally $\oplus$-G-$\mathrm{Rad_g}$-supplemented (briefly, $\oplus$-PG-$\mathrm{Rad_g}$-supplemented) if, any cyclic submodule mR of M with $Rad_g(M) \subseteq mR$, there exists a direct summand A of M such that $M = mR + A$ and $mR \cap A \subseteq Rad_g(A)$. A ring R is called $\oplus$-PG-$\mathrm{Rad_g}$-supplemented if, R_R is $\oplus$-PG-$\mathrm{Rad_g}$-supplemented.

Definition 1.1.58. [27] Let M be an R-module, U, $V \leq M$ and $M = U + V$ such that $U \cap V \leq Rad_g(V)$, then V is called a generalized Radical supplement, briefly g-Radical supplement, of U in M. Moreover, if every submodule of M has a generalized Radical supplement in M, then M is called a generalized Radical supplemented, briefly g-Radical supplemented.

Definition 1.1.59. [14] An R-module M is called $\oplus$-g-Radical supplemented if, every submodule of M has a g-Radical supplement that is a direct summand.

Definition 1.1.60. [10] An R-module M is called strongly generalized Radical supplemented or briefly, sgrs-module if, any submodule of M containing $Rad_g(M)$ has a g-supplement in M, i.e., for any $N \leq M$ with $Rad_g(M) \subseteq N$ there exists a submodule K of M such that $N + K = M$ and $N \cap K \ll_g K$. As an example, every semisimple module is a sgrs-module.

Definition 1.1.61. [32] An R-module is called a principally strongly g-Radical supplemented module (briefly principally sgrs-module) if, for $m \in M$ with $Rad_g(M) \subseteq mR$, there exists a submodule X of M such that $M = X + mR$ and $mR \cap X$ is g-small in X.

Definition 1.1.62. [21] An R-module M is called strongly generalized $\oplus$-Radical supplemented (briefly, sgrs$^{\oplus}$-module) if for every submodule L of M with $Rad_g(M) \subseteq L$ has a g-supplement which is a direct summand of M, i.e., for any $L \leq M$ with $Rad_g(M) \subseteq L$, there exists a direct summand X of M such that $M = L + X$ and $L \cap X$ is g-small in X.

Definition 1.1.63. [21] An R-module M is said to be a principally strongly generalized $\oplus$-Radical supplemented (briefly, principally sgrs$^{\oplus}$-module) if, for any cyclic submodule N of M such that $Rad_g(M) \subseteq N$ has a g-supplement that is a direct summand of M, i.e., for any $m \in M$ with $Rad_g(M) \subseteq mR$, there exists a direct summand X of M such that $M = mR + X$ and $mR \cap X \ll_g X$.

Definition 1.1.64. [23] An R-module M is called lifting (or satisfies (D_1)) if, any submodule L of M contains a direct summand N of M with $L/N \ll M/N$, or equivalently, for every $L \leq M$, there exists a decomposition $M = N \oplus X$ such that $N \leq L$ and $L \cap X \ll M$, also $L \cap X \ll X$.

Definition 1.1.65. [20] An R-module M is called principally lifting if, for all cyclic submodule L of M, there exists a decomposition $M = H \oplus G$ such that $H \leq L$ and $L \cap G \ll M$.

Definition 1.1.66. [31, p.57] An R-module M is said have:

($\boldsymbol{D_2}$) Property: if for any $H \leq M$ such that M/H is isomorphic to a direct summand of M, then H is so a direct summand of M.

($\boldsymbol{D_3}$) Property: if for any direct summands H and G of M with $M = H + G$ then $H \cap G$ is also a direct summand of M.

Definition 1.1.67. [34] An R-module M is called g-lifting if, for any $L \leq M$, there is a decomposition $M = H \oplus G$ such that $H \leq L$ and $L \cap G \ll_g M$, also in G.

Definition 1.1.68. [13] An R-module M is called principally g-lifting if, for each $m \in M$, M has a decomposition $M = H \oplus G$ such that $H \leq mR$ and $mR \cap G \ll_g G$.

Definition 1.1.69. [22, p.124] A pair (P, f) is called a cover of an R-module M if, P is a module and $f: P \to M$ is an epimorphism with $kerf$ is small in P. If P is a projective module, (P, f) is said to be a projective cover of M. An R-module M is called semiperfect if and only if every factor module of M has a projective cover.

Definition 1.1.70. [14] A pair (P, f) is called g-cover of M if, P is a module and $f: P \to M$ is an epimorphism with $kerf$ is g-small in P. If P is a projective, (P, f) is said to be a projective g-cover of M. As well, a module M is said to be g-semiperfect if every factor module of M has a projective g-cover.

Definition 1.1.71. [1] An R-module M is called principally semiperfect if every factor module of M by a cyclic submodule has a projective cover.

Definition 1.1.72. [14] An R-module M is called principally g-semiperfect if, all its factor modules by cyclic submodules has a projective g-cover.

Definition 1.1.73. [35] An R-module M is called $\mathcal{T}$-*e*-noncosingular relative to E if, for any nonzero homomorphism $f: M \to E$, Imf is not *e*-small in E. M is called $\mathcal{T}$-*e*-noncosingular if M is $\mathcal{T}$-*e*-noncosingular relative to M.

Definition 1.1.74. [35] An R-module M is called "*e*-small", in this study we called it "g-small", if M is g-small in injective envelope of M. Then a *e*-cosingular, in [13] indicated to be a g-cosingular submodule which denote $\overline{Z}_g(M)$, defined as follows: $\overline{Z}_g(M) = \bigcap\{ker\sigma \mid \sigma: M \to E, E$ is a g-small$\}$.

Definition 1.1.78. [14] An module M is said to be a g-cosingular (g-noncosingular) if, $\overline{Z}_g(M) = 0$ (resp. $\overline{Z}_g(M) = M$).

Definition 1.1.79. [15] An R-module M is said to have (P_g^*)-property or may be called a (P_g^*)-module if for each $L \leq M$, there is a direct summand H of M such that $H \leq L$ and $L/H \subseteq Rad_g(M/L)$.

Definition 1.1.80. [28, p.97] A ring R is called V-ring if, for every simple R-module M, is injective. Moreover, $Rad(M) = 0$. Also a ring R is called g-V-ring if, for every R-module M, $Rad_g(M) = 0$. [21]

Definition 1.1.81. [30, p.42] A multiplicatively closed set (or, a multiplicative set) is a subset S of a ring R with identity satisfying two conditions:

(1) $1 \in S$.

(2) For all x and y in S, the product $xy \in S$.

Definition 1.1.82. [5] An R-module M is said to be multiplication if, for $E \leq M$, $E = IM$ for ideal I of R. Equivalently, M is a multiplication R-module if, for any submodule E of M, $E = M.(E:_R M)$ where $(E:_R M) = \{r \in R: Mr \subseteq E\}$.

Definition 1.1.83. [5] An R-module M is called faithful if, $(0:_R M) = 0$.

Definition 1.1.84. [30, P.61] Let R be a ring and let M be an R-module. Let S be a multiplicatively closed set in R. Let $\mathcal{T}$ be the set of all ordered pairs (a, b) where $a \in R$ and $b \in S$. We have $S^{-1}M$ is an R-module with respect to this operation. The R-module $S^{-1}M$ is said to be "a quotient $S^{-1}R$-module", or a "module of quotient". Note that if $0 \in S$, then $S^{-1}M = 0$.

Als, $S^{-1}R$ is called a quotient ring of R, or a ring of quotient of R.

Definition 1.1.85. [4] Let R be a commutative ring, M an R-module and E an R-submodule of M. An element $r \in R$ is called prime to E if $rm \in E$ ($m \in M$) implies that $m \in E$. Denote by $\mathcal{L}(E)$ the set of all elements of R that are not prime to E, i.e., $\mathcal{L}(E) = \{r \in R | \ rm \in E$ for some $m \in M \setminus E\}$.

1.2. Basic results.

In this section of the chapter, we will mention most of the results that we will rely on in this dissertation for the coming chapters.

Lemma 1.2.1. [40, 19.3] Let C, X, E and M be R-modules.

(1) If $C \leq X \leq M$, then $X \ll M$ if and only if $C \ll M$ and $X/C \ll M/C$.

(2) $C_1, C_2, \dots, C_n$ are small submodules of M if and only if $\sum_{i=1}^{n} C_i$ is small in M.

(3) For $C \ll M$ and $f: M \to E$ a homomorphism, we have $f(C) \ll E$.

(4) If $C \leq X \leq M$ and X a direct summand in M, then $C \ll M$ if and only if $C \ll X$.

Lemma 1.2.2. [3, Proposition 5.20] let M be R-module, and $C_1 \leq M_1 \leq M$, $C_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$; then.

(1) $C_1 \oplus C_2 \ll M_1 \oplus M_2$ if and only if $C_1 \ll M_1$ and $C_2 \ll M_2$.

(2) $C_1 \oplus C_2 \trianglelefteq M_1 \oplus M_2$ if and only if $C_1 \trianglelefteq M_1$ and $C_2 \trianglelefteq M_2$.

Lemma 1.2.3. [12, Theorem 2.5] Let R be a commutative ring with identity and M a non-zero multiplication R-module. Then

(1) every proper submodule of M is contained in a maximal submodule of M.

(2) C is a maximal submodule of M if and only if there exists a maximal ideal P of R such that $C = PM \neq M$.

Lemma 1.2.4. [42, Proposition 2.3] Let E be a submodule of an R-module M. The following are equivalent.

(1) $E \ll_g M$.

(2) If $C + E = M$, then C is a direct summand of M with M/C a semisimple module.

Lemma 1.2.5. [42, Proposition 2.5]

(1) Assume that E, C, X are submodules of an R-module M with $C \subseteq E$.

(i) If $E \ll_g M$, then $C \ll_g M$ and $E/C \ll_g M/C$.

(ii) $E + X \ll_g M$ if and only if $E \ll_g M$ and $X \ll_g M$.

(2) If $C \ll_g M$ and $f: M \to E$ is any homomorphism, then $f(C) \ll_g E$. In particular, if $C \ll_g M \subseteq E$, then $C \ll_g E$.

(3) Assume that $C_1 \subseteq M_1 \subseteq M$, $C_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$, then $C_1 \oplus C_2 \ll_g M_1 \oplus M_2$ if and only if $C_1 \ll_g M_1$ and $C_2 \ll_g M_2$.

Lemma 1.2.6. [18, Proposition 3.2] Let M be an R-module, let $C \leq E \leq M$ be submodules of M. If $C \ll_g M$ and $E \leq^{\oplus} M$, then $C \ll_g E$.

Lemma 1.2.7. [15, Lemma 2.5] Let M be an R-module and $X \leq E \leq^{\oplus} M$. If $X \subseteq Rad_g(M)$ then $X \subseteq Rad_g(E)$.

Lemma 1.2.8. [42, Corollary 2.11]

(1) If $f: M \to \grave{M}$ is an R-homomorphism, then $f(Rad_g(M)) \subseteq Rad_g(\grave{M})$. In particular, $Rad_g(M)$ is a fully invariant submodule of M.

(2) If every proper essential submodule of M is contained in a maximal submodule of M, then $Rad_g(M)$ is the unique largest g-small submodule.

Lemma 1.2.9. [35, Lemma 2.2] Let M be an R-module and $m \in M$. The following are equivalent:

(1) $m \in Rad_g(M)$.

(2) $mR \ll_g M$.

Lemma 1.2.10. [35, Corollary 2.3] Let $M = \oplus_{i \in I} M_i$ be a module, then $Rad_g(M) = \oplus_{i \in I} Rad_g(M_i)$.

Lemma 1.2.11. [18, Proposition 3.7] Let M be an indecomposable module. Then a proper submodule E of M is small if and only if it is g-small.

Lemma 1.2.12. [14, Lemma 5.4] M be finitely generated R-module, then $Rad_g(M) \ll_g M$.

Lemma 1.2.13. [18, Proposition 3.15] Let M be an arbitrary R-module. Then $Rad_g(M) = M$ if and only if all it is finitely generated submodules of M are all g-small.

Lemma 1.2.14. [33, Lemma 2.1] Let an R-module $M = \bigoplus_{i\in I} M_i$ be a direct sum of submodule M_i $(i \in I)$, and let E be a fully invariant submodule of M. Then $E = \bigoplus_{i\in I}(E \cap M_i)$.

Lemma 1.2.15. [34, Lemma 6] Let E, L be submodules of an R-module M such that $E + L$ has a g-supplement X in M and $E \cap (X + L)$ has a g-supplement U in E. Then $X + U$ is a g-supplement of L in M.

Lemma 1.2.16. [34, Lemma 8] Let M_1 and E be submodules of an R-module M and M_1 be a g-supplemented module. If $M_1 + E$ has a g-supplement in M, then so does E.

Lemma 1.2.17. [14, Lemma 2.12] Suppose that M is an R-module with $L \leq E \leq M$ and $X \leq M$. Assume E is a g-supplement of X. Then,

(1) L is g-small in E if and only if L is g-small in M.

(2) $Rad_g(E) = E \cap Rad_g(M)$

Lemma 1.2.18. [13, Lemma 2.3] Let L, X and N be submodules of an R-module M such that L is a g-supplement of X and $N \ll_g M$, then L is a g-supplement of $X + N$ in M.

Lemma 1.2.19. [37, Lemma 3.14] Let M be an R-module and E a fully invariant submodule of M. If $M = M_1 \oplus M_2$ for some submodules M_1 and M_2 of M, then $M/E = [(M_1 + E)/E] \oplus [(M_2 + E)\ /E]$.

Lemma 1.2.20. [34, Theorem 4] The following are equivalent for an R-module M.

(1) $Rad_g(M)$ is Artinian .

(2) Every g-small submodule of M is Artinian .

(3) M satisfies DCC on g-small submodules.

Lemma 1.2.21. [35, Proposition 3.7] The following are equivalent for a ring R.

(1) Every right R-module is τ-g-noncosingular.

(2) Every right R-module is g-noncosingular.

(3) For any right R-module M, $Rad_g(M) = 0$.

Lemma 1.2.22. [Modular Law,22, P.30**]** For submodules A , B and C in an R-module M with $B \leq C$, it follows that $(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B$.

Lemma 1.2.23. [40, 31.3] Let M be a semisimple R-module. Then M is Artinian if and only if M is finitely generated.

Lemma 1.2.24. [31, Lemma 4.6] Let M be an R-module with (D_2). Then M has (D_3).

Lemma 1.2.25. [31, Proposition 4.38] Any quasi-projective R-module has (D_2).

Lemma 1.2.26. [31, Lemma 4.47] let $M = S \oplus T = E + T$ be an R-module where S is T-projective. Then $M = \grave{T} \oplus T$ where $\grave{T} \leq E$.

Lemma 1.2.27. [22, Theorem 5.4.1] A module is projective if and only if it is isomorphic to a direct summand of a free module.

Lemma 1.2.28. [22, Corollary 8.2.2] Let R be a ring, then.

(1) R is semisimple if and only if any right and left R-module is semisimple.

(2) R is semisimple if and only if any right and any left R-module is injective if and only if any right and any left R-module is projective.

(3) R is semisimple if and only if any simple right R-module and any simple left R-module is projective.

Lemma 1.2.29. [24, Lemma 2.3] Let E be a direct summand of an R-module M and let L be a submodule of M such that M/L is projective and $M = E + L$. Then $E \cap L$ is a direct summand of M.

Lemma 1.2.30. [40, 41.14] Let M be a π-projective module. If $M = U + V$ and U a direct summand in M, then there exists $\acute{V} \subset V$ with $M = U \oplus \acute{V}$.

Lemma 1.2.31. [40**,** 41.14] The following cases are equivalent for an R-module $M = M_1 \oplus M_2$.

(1) M_1 is M_2-projective.

(2) For any submodule N of M with $M = E + M_2$, there exists a submodule E_1 of E such that $M = E_1 \oplus M_2$.

Lemma 1.2.32. [23, Theorem 6.1.2] Let M be an R-module and let $\mathrm{N} \leq \mathrm{M}$.

(1) The following are equivalent:

(i) M is Artinian.

(ii) N and M/N are Artinian.

(2) The following are equivalent:

(i) M is Noetherian.

(ii) N and M/N are noetherian.

(iii) Every submodule of is finitely generated.

Lemma 1.2.33. [29, Proposition 2.1] For a submodule $E \subset M$, the following are equivalent.

(1) M/E is semisimple.

(2) for any $X \leq M$ there exists a submodule $L \leq M$ such that $X + L = M$ and $X \cap L \subseteq E$;

(3) there is a decomposition $M = M_1 \oplus M_2$ with M_1 is semisimple, $E \trianglelefteq M_2$ and M_2/E is semisimple.

Lemma 1.2.34. [35, Proposition 2.8] The following cases will be equivalent for a g-local module M.

(1) M is local.

(2) M is indecomposable.

Lemma 1.2.35. [17, Proposition 2.6] Suppose that M is an R-module and $Rad_g(M) \neq M$. Then M is generalized hollow if and only if M is local.

Lemma 1.2.36. [21, Lemma 2.2.13] $P_g(M)$ is a fully invariant submodule for each module M.

Lemma 1.2.37. [21, Lemma 2.2.20] Let M be an R-module such that $E \leq M$ and $Rad_g(M) \subseteq E$. Then,

(1) If E is a direct summand of M, then $Rad_g(M) = Rad_g(E)$.

(2) If $Rad_g(M)$ is a direct summand of M, then $Rad_g(M) = P_g(M)$.

Lemma 1.2.38. [9, 2.17] For an R-module M the following are equivalent.

(1) M is uniserial.

(2) Every factor module of M is uniform.

(3) Every factor module of M has zero or simple socle.

(4) Every submodule of M is hollow.

(5) Every finitely generated submodule of M is local.

(6) Every submodule of M has at most one maximal submodule.

Lemma 1.2.39. [9, 2.18] Let M be a Noetherian and uniserial R-module. Then any submodule of M is fully invariant.

Lemma 1.2.40. [13, Lemma 3.7] Let R be an arbitrary ring and let $f: H \to G$ be a homomorphism of R-modules H, G. Then,

(1) $f(\bar{Z}_g(H)) \leq \bar{Z}_g(G)$.

(2) The class of all g-noncosingular modules is closed under homomorphic images.

Lemma 1.2.41. [27, Lemma 8] Let M be an R-module such that $U, V \leq M$ and $L \leq U$. If V is a g-Radical supplement of U in M. Then $(V + L)/L$ is a g-Radical supplement of U/L in M/L.

Lemma 1.2.42. [13, Lemma 3.20] For an indecomposable R-module M, the following are equivalent:

(1) M is principally generalized hollow.

(2) M is principally g-lifting.

Lemma 1.2.43. [13, Theorem 4.3**]** The following assertions are equivalent for a projective R-module M.

(1) M is principally g-semiperfect.

(2) M is principally g-lifting.

Lemma 1.2.44. [21, Remarks 3.1.2] Every principally sgrs$^{\oplus}$-module is also a principally sgrs-module.

Lemma 1.2.45. [21, Proposition 3.1.10] Let M be a principally sgrs$^{\oplus}$-module has a cyclic generalized Radical. Then, $M = C_1 \oplus C_2$ where C_1 is a module with $Rad_g(C_1)$ is g-small in C_1 and C_2 is a module with $Rad_g(C_2) = C_2$.

Lemma 1.2.46. [21, Proposition 3.3.7] A refinable R-module M is a principally sgrs-module if and only if, it is a principally sgrs$^{\oplus}$-module.

Lemma 1.2.47. [40] Every nonzero uniform R-module is indecomposable.

Lemma 1.2.48. [15, Proposition 3.2] If M is an R-module and $Rad_g(M) = M$, then M is a (P_g^*)-module.

Lemma 1.2.49. [15, Theorem 2.3] Let M be an R-module. Then the following two assertions are equivalent.

(1) M is a (P_g^*)-module.

(2) for any submodule $L \leq M$, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq L$ and $L \cap M_2 \subseteq Rad_g(M_2)$.

Lemma 1.2.50. [32, Proposition 3.1.4] Any principally $\oplus$-g-supplemented module is principally g-supplemented.

Lemma 1.2.51. [32, Proposition 3.1.5] Every principally g-lifting module is principally $\oplus$-g-supplemented.

Lemma 1.2.52. [15, Proposition 3.1] Let M be an R-module, consider the following:

(1) M is semisimple.

(2) M has (P_g^*) property.

(3) Each summand of M is $\oplus$-g-Radical supplemented.

(4) M is $\oplus$-g-Radical supplemented.

(5) M is g-Radical supplemented. Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (5)$. If $Rad_g(M) = 0$, then $(5) \Longrightarrow (1)$.

Lemma 1.2.53. [32, Proposition 3.1.7] Consider the following for an R-module,

(1) $Rad_g(M) = M$;

(2) M is a principally generalized hollow.

(3) M is a principally g-lifting.

(4) M is a principally $\oplus$-g-supplemented.

Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4)$. If M is non-cyclic indecomposable then $(4) \Longrightarrow (1)$.

Lemma 1.2.54. [21, Remark 3.1.3(4)] If M is a cyclic R-module over a PID R, then we know that every submodule of M is also cyclic. Hence, any cyclic principally sgrs$^{\oplus}$-module over a PID R is a sgrs$^{\oplus}$-module.

Lemma 1.2.55. [21, Corollary 3.3.16] Let M be a module over a PID R has the following cases:

(1) M is a distributive (or, projective).

(2) M is a cyclic refinable.

Then $M/Rad_g(M)$ is a principally semisimple module if and only if, M is a principally sgrs$^{\oplus}$-module.

Lemma 1.2.56. [32, Theorem 2.1.10]. If M is a principally g-supplemented and weakly distributive R-module, then any direct summand of M is principally g-supplemented.

Lemma 1.2.57. [32, Corollary 3.1.24] Let M be a distributive R-module. Consider the following conditions:

(1) M is principally $\oplus$-g-supplemented.

(2) M is principally g-supplemented.

(3) $M/Rad_g(M)$ is principally semisimple. Then $(1) \Longrightarrow (2) \Longrightarrow (3)$, if M is refinable and $Rad_g(M) \ll_g M$, $(3) \Longrightarrow (1)$.

Lemma 1.2.58. [21, Theorem 4.3.1] Let M be an R-module, consider the following conditions:

(1) M is principally semisimple.

(2) M is principally g-lifting.

(3) M is principally ⊕-g-supplemented.

(4) M is a principally sgrs$^{\oplus}$-module.

(5) M is ⊕-PG-$\mathrm{Rad_g}$-supplemented.

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$. Also $(5) \Rightarrow (1)$ if, $Rad_g(M) = 0$.

Lemma 1.2.59. [32, Proposition 4.2.1] Let M be an R-module, consider the following conditions:

(1) M is principally semisimple.

(2) M is principally lifting.

(3) M is principally g-lifting.

(4) M is principally ⊕-supplemented.

(5) M is principally ⊕-g-supplemented.

(6) M is principally supplemented.

(7) M is principally g-supplemented.

(8) M is principally sgrs-module.

Then $(1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (6)$ and $(1) \Rightarrow (3) \Rightarrow (5) \Rightarrow (7) \Rightarrow (8)$. Also $(8) \Rightarrow (1)$ if, $Rad_g(M) = 0$.

Lemma 1.2.60. [32, Theorem 3.3.1] Let M be a projective R-module, the following are equivalents:

(1) M is g-semiperfect.

(2) M is principally g-lifting.

(3) M is principally ⊕-g-supplemented.

(4) M is principally g-supplemented.

Lemma 1.2.61. [21, Corollary 4.3.2] Let M be a cyclic module over a PID R with $Rad_g(M) = 0$. The following are equivalent.

(1) M is semisimple.

(2) M is principally semisimple.

(3) M is g-lifting.

(4) M is principally g-lifting.

(5) M is $\oplus$-g-supplemented.

(6) M is principally $\oplus$-g-supplemented.

(7) M is a sgrs$^{\oplus}$-module.

(8) M is a principally sgrs$^{\oplus}$-module.

(9) M is $\oplus$-G-Rad_g-supplemented.

(10) M is $\oplus$-PG-Rad_g-supplemented.

Lemma 1.2.62. [18, Corollary 3.19] Let M be a faithful, finitely generated and multiplication module over a commutative ring R with identity, and $X \subset M$. The following are equivalent:

(1) $X \ll_g M$.

(2) $(X:_R M) \ll_g R$.

(3) $X = IM$ for some $I \ll_g R$.

Lemma 1.2.63. [12, Theorem 1.6] Let R be a commutative ring with identity and M a faithful R-module. Then M is a multiplication module if and only if.

(1) $\bigcap_{\lambda\in\Lambda}(I_\lambda M) = (\bigcap_{\lambda\epsilon\Lambda} I_\lambda)M$ if for each non-empty collection of ideals I_λ $(\lambda \in \Lambda)$ of R, and

(2) for any submodule E of M and ideal I of R such that $E \subset IM$ there exists an ideal G with $G \subset I$ and $E \subseteq GM$.

Lemma 1.2.64. [12, Theorem 3.1] Let R be a commutative ring with identity and M a faithful multiplication R-module. Then the following are equivalent.

(1) M is finitely generated.

(2) If H and G are ideals of R such that $HM \subseteq GM$ then $H \subseteq G$.

(3) For each submodule E of M, there exists a unique ideal I of R such that $E = IM$.

(4) $M \neq HM$ for any proper ideal H of R.

(5) $M \neq HM$ for any maximal ideal P of R.

Lemma 1.2.65. [32, Lemma 2.3.1] let M be a module and S a multiplicative closed subset of R such that $\mathcal{L}(L) \cap S = \emptyset$ for any $L \leq M$. Then $E \subseteq \grave{E}$ in M if and only if $S^{-1}E \subseteq S^{-1}\grave{E}$ in $S^{-1}M$.

Lemma 1.2.66. [32, Lemma 2.3.2] suppose that M is an R-module, $E \leq M$ and S a multiplicative closed subset of R such that $\mathcal{L}(L) \cap S = \emptyset$ for any $L \leq M$. Then E is cyclic in M as R-module if and only if $S^{-1}E$ is cyclic in $S^{-1}M$ as $S^{-1}R$-module.

Lemma 1.2.67. [32, Lemma 2.3.7] Let M be a module and S a multiplicative closed subset of R such that for any proper submodule L of M, $(L:_M b) = L$ for any $b \in S$. Then $E \subseteq \grave{E}$ in M if and only if $S^{-1}E \subseteq S^{-1}\grave{E}$ in $S^{-1}M$.

Lemma 1.2.68. [32, Lemma 2.3.8] Suppose that M is an R-module, $E \subset M$ and S be a multiplicative closed subset in R. If for any proper submodule L of M, $(L:_M b) = L$, for all $b \in S$. Then E is cyclic in M as R-module if and only if $S^{-1}E$ is cyclic in $S^{-1}M$ as $S^{-1}R$-module.

Lemma 1.2.69. [32, Lemma 2.3.3] Let M be an R-module, $E \leq M$ and S a multiplicative closed subset of R such that $\mathcal{L}(L) \cap S = \emptyset$ for $L \leq M$. Then,

(1) $E \trianglelefteq L \leq M$ as R-module if and only if $S^{-1}E \trianglelefteq S^{-1}L \leq S^{-1}M$ as $S^{-1}R$-module.

(2) $E \ll_g L \leq M$ as R-module if and only if $S^{-1}E \ll_g S^{-1}L \leq S^{-1}M$ as $S^{-1}R$-module.

Lemma 1.2.70. [32, Lemma 2.3.9] let M be an R-module, $E \leq M$ and S a multiplicative closed subset of R such that for any proper submodule L of M, $(L:_M b) = L$, for all $b \in S$. Then,

(1) $E \trianglelefteq L \leq M$ as R-module if and only if $S^{-1}E \trianglelefteq S^{-1}L \leq S^{-1}M$ as $S^{-1}R$-module.

(2) $E \ll_g L \leq M$ as R-module if and only if $S^{-1}E \ll_g S^{-1}L \leq S^{-1}M$ as $S^{-1}R$-module.

Lemma 1.2.71. [32, Lemma 1.2.21] Let E and L be two submodules of an R-module M and let S be a multiplicative closed subset in R. Then,

(1) $S^{-1}(E \cap L) = (S^{-1}E) \cap (S^{-1}L)$.

(2) $S^{-1}(E + L) = (S^{-1}E) + (S^{-1}L)$.

Lemma 1.2.72. [12, Theorem 2.13] Let R be a commutative ring with identity and M a faithful multiplication R-module. A submodule E of M is essential if and only if there exists an essential ideal I of R such that $E = IM$.

Lemma 1.2.73. [43, Lemma 1.9**]** Assume that P is a projective module, then $\delta(P) = P\delta(R)$ such that $\delta(P)$ is the intersection of all essential maximal submodules of P.

Lemma 1.2.74. [16, proposition 1.3] Let N, L be two submodules of an R-module M, if N is relative complement for L. Then $N \oplus L \trianglelefteq M$.

This chapter consists of six sections. In sections one and four, we define and study two concepts principally generalized Radical supplemented modules and principally (P_g^*)-module respectively. Several properties, illustrate examples, characterizations, factors, and direct summands of these modules are presented. In sections two and five, we investigated some connections between our definitions and other types of modules. In sections three and six, we study and investigate the behavior of principally generalized Radical supplemented modules and principally (P_g^*)-modules respectively, under localization.

2.1. Principally Generalized Radical supplemented modules

In this part, we will present the definition of principally generalized Radical supplemented modules, as well as a number of their properties and examples about them will be discussed in this section. First, we will suppose that our following main definition:

Definition 2.1.1. An R-module M is said to be principally generalized Radical supplemented, briefly PG-Radical supplemented if, for any $m \in M$, and mR is a submodule of M, there exists a submodule L of M such that $M = mR + L$ and $mR \cap L \subseteq Rad_g(L)$.

Remarks and Examples 2.1.2.

(1) Evidently, we see that any g-Radical supplemented module is PG-Radical supplemented. However, if M is a cyclic module over a PID R, then we know that every submodule of M is also cyclic. Then, a cyclic module M over a PID R is PG-Radical supplemented if and only if it is g-Radical supplemented module.

(2) Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{24}$. As we know that all g-small submodules of M_R are $\{\bar{0}, 2\mathbb{Z}_{24}, 4\mathbb{Z}_{24}, 6\mathbb{Z}_{24}, 8\mathbb{Z}_{24}, 12\mathbb{Z}_{24}\}$, hence $Rad_g(M) = 2\mathbb{Z}_{24}$. If L is any cyclic submodule of M. So, we have three cases:

(a) if $L = M$, then trivially $\bar{0}$ is a g-Radical supplement of L in M.

(b) if $3\mathbb{Z}_{24} \neq L \subset M$, then $L + M = M$ and $L \cap M = L \subseteq 2\mathbb{Z}_{24} = Rad_g(M)$.

(c) if $L = 3\mathbb{Z}_{24}$. Let $Y = 4\mathbb{Z}_{24}$, then $L + Y = M$. It is easily to see that Y is semisimple, then $Rad_g(Y) = Y$. Hence, $L \cap Y = 12\mathbb{Z}_{24} \subseteq 4\mathbb{Z}_{24} = Rad_g(Y)$.

This means that any submodule of M has a g-Radical supplement in M. Hence, $M = \mathbb{Z}_{24}$ is a PG-Radical supplemented as $\mathbb{Z}$-module.

(3) Any principally generalized hollow module is PG-Radical supplemented, while the converse need not be true, in general, as seen by (**2**) $\mathbb{Z}_{24}$ as $\mathbb{Z}$-module is PG-Radical supplemented but not principally generalized hollow.

(4) Suppose that $M = R = \mathbb{Z}$. Let $n, m \in \mathbb{Z}^+$ with $\text{g.c.d}(n, m) = 1$. As a result of this, $n\mathbb{Z} + m\mathbb{Z} = \mathbb{Z}$. As $Rad_g(a\mathbb{Z}) \subseteq Rad_g(\mathbb{Z})$, then $Rad_g(a\mathbb{Z}) = 0$ for any $a \in \mathbb{Z}_+$, we deduce $n\mathbb{Z} \cap m\mathbb{Z} = (nm)\mathbb{Z} \neq 0$, that is $n\mathbb{Z} \cap m\mathbb{Z} \nsubseteq Rad_g(n\mathbb{Z})$ and $n\mathbb{Z} \cap m\mathbb{Z} \nsubseteq Rad_g(m\mathbb{Z})$. Therefore, the $\mathbb{Z}$-module $\mathbb{Z}$ does not be PG-Radical supplemented.

(5) Any principally semisimple is PG-Radical supplemented, the converse need not be true in general, as we see $\mathbb{Q}$ as a $\mathbb{Z}$-module is PG-Radical supplemented [see examples 2.2.3(2)] but not principally semisimple.

Now, we will study the behavior of PG-Radical supplemented summands.

Theorem 2.1.3. Let M be a PG-Radical supplemented duo R-module. Then every direct summand of M is PG-Radical supplemented.

Proof. Let $M = M_1 \oplus M_2$ be a PG-Radical supplemented duo R-module and let $m \in M_1$. There is a submodule T of M such that $M = mR + T$ and $mR \cap T \subseteq Rad_g(T)$. By the modular law, $M_1 = mR + (M_1 \cap T)$. By Lemma 1.2.14, $T = (M_1 \cap T) \oplus (M_2 \cap T)$. We prove that $mR \cap (M_1 \cap T) \subseteq Rad_g(M_1 \cap T)$. If $x \in mR \cap (M_1 \cap T)$ and as $mR \cap (M_1 \cap T) \subseteq mR \cap T$, then $x \in mR \cap T$, so that $x \in Rad_g(T)$. Thus $xR \ll_g T$, by Lemma 1.2.9. As $xR \leq M_1 \cap T \leq^{\oplus} T$, As a result of this, $xR \ll_g M_1 \cap T$, by Lemma 1.2.6, thus $xR \subseteq Rad_g(M_1 \cap T)$, and so $x \in xR \subseteq Rad_g(M_1 \cap T)$ that is what we have to prove. □

Now, we propose and prove the following lemma that present the idea in theorem 2.1.5.

Lemma 2.1.4. Let $M = M_1 \oplus M_2 = W + E$ be an R-module and $W \leq M_1$. If M is weakly distributive and $W \cap E \subseteq Rad_g(E)$, then $W \cap E \subseteq Rad_g(M_1 \cap E)$.

Proof. Let $x \in W \cap E$, then $x \in Rad_g(E)$ and so $xR \ll_g E$, by Lemma 1.2.9. As M is weakly distributive, $E = (M_1 \cap E) \oplus (M_2 \cap E)$. As $xR \leq W \leq M_1$ and $xR \leq E$ then $xR \leq M_1 \cap E \leq^{\oplus} E$, Lemma 1.2.6 implies that $xR \ll_g M_1 \cap E$. Thus, $xR \subseteq Rad_g(M_1 \cap E)$, and $x = x.1 \in xR \subseteq Rad_g(M_1 \cap E)$. This completes the proof. □

Theorem 2.1.5. Let M be a weakly distributive R-module and $M_1 \leq^{\oplus} M$. If M is a PG-Radical supplemented R-module, then M_1 is PG-Radical supplemented.

Proof. Let $M = M_1 \oplus M_2$ be a weakly distributive PG-Radical supplemented R-module and $x \in M_1$, for some $M_2 \leq M$. There exsist a submodule E of M such that $M = xR + E$ and $xR \cap E \subseteq Rad_g(E)$, Since, M is PG-Radical supplemented. By modular law, we have $M_1 = M_1 \cap M = M_1 \cap (xR + E) = xR + (M_1 \cap E)$. Also $M = M_1 \oplus M_2 = xR + E$, $xR \leq M_1$ and $xR \cap E \subseteq Rad_g(E)$, we deduce that $xR \cap E = xR \cap (M_1 \cap E) \subseteq Rad_g(M_1 \cap E)$, by Lemma 2.1.4. This mean that $M_1 \cap E$ is a PG-Radical supplement of xR in M_1, and then M_1 is PG-Radical supplemented. □

Proposition 2.1.6. Let $M = \bigoplus_{i \in I} M_i$ be an infinite direct sum of PG-Radical supplemented R-modules $\{M_i | \ i \in I\}$. If every cyclic submodule of M is fully invariant, then M is PG-Radical supplemented.

Proof. Assume that $M = \bigoplus_{i \in I} M_i$ is an R-module and $m \in M$. According to the hypothesis, mR is fully invariant, then $mR = \bigoplus_{i \in I} (mR \cap M_i)$, by Lemma

1.2.14. Since, $mR \cap M_i$ is a cyclic submodules of M_i and so M_i is a PG-Radical supplemented R-module, for $i \in I$, then there exist a submodule T_i of M_i such that $M_i = (mR \cap M_i) + T_i$ and $(mR \cap M_i) \cap T_i = mR \cap T_i \subseteq Rad_g(T_i)$. So $M = \oplus_{i \in I} M_i = mR + (\oplus_{i \in I} T_i)$. Again by Lemma 1.2.14, we have $mR \cap (\oplus_{i \in I} T_i) = \oplus_{i \in I} (mR \cap T_i) \subseteq \oplus_{i \in I} Rad_g(T_i) = Rad_g(\oplus_{i \in I} T_i)$, as required. □

Proposition 2.1.7. Let $M = M_1 \oplus M_2$ be a direct sum of PG-Radical supplemented modules M_1 and M_2. If any cyclic submodule of M is weak distributive, then M is PG-Radical supplemented.

Proof. As the same argument of Proposition 2.1.6. □

Corollary 2.1.8. Let M be a module, then.

(1) if $M = \oplus_{i \in I} M_i$ is a duo infinite direct sum of modules $\{M_i | \, i \in I\}$. Then M is PG-Radical supplemented if and only if M_i is PG-Radical supplemented, for $i \in I$.

(2) if $M = M_1 \oplus M_2$ is a weakly distributive direct sum of modules M_1 and M_2. Then M is PG-Radical supplemented if and only if M_1 and M_2 are PG-Radical supplemented.

Proof. (1) It follows directly by Theorem 2.1.3 and Proposition 2.1.6.

(2) It follows directly by Theorem 2.1.5 and Proposition 2.1.7. □

Lemma 2.1.9. If $f: M \to E$ is a homomorphism and C is PG-Radical supplement in M with $kerf \leq C$, then $f(C)$ is a PG-Radical supplement in $f(M)$.

Proof. If C is a PG-Radical supplement in M, then there exists a cyclic $W \leq M$ such that $W + C = M$ and $W \cap C \subseteq Rad_g(C)$. Thus, $f(W) + f(C) = f(M)$. As, $kerf \leq C$, so $f(W) \cap f(C) = f(W \cap C) \subseteq f(Rad_g(C)) \subseteq Rad_g(f(C))$, by Lemma 1.2.8(1). Clearly, $f(W)$ is a cyclic submodule of $f(M)$. This implies, $f(C)$ is a PG-Radical supplement of $f(W)$ in $f(M)$. □

Following, we will investigate the factors of PG-Radical supplemented modules under some cases.

Proposition 2.1.10. Let M be a PG-Radical supplemented module and $C \leq M$. If any cyclic submodule of M has a g-Radical supplement contains C, then M/C is PG-Radical supplemented.

Proof. Let $\overline{m}R$ be any cyclic submodule of M/C, then $\overline{m}R = (mR + C)/C$ for some $m \in M$. According to the hypothesis, there exists $W \leq M$ such that $C \leq W$, $mR + W = M$ and $mR \cap W \subseteq Rad_g(W)$. Suppose that a natural map $\pi: M \to M/C$. Since $ker\pi = C \leq W$, so by Lemma 2.1.9, $\pi(W)$ is a g-Radical supplement of $\pi(mR) = (mR + C)/C = \overline{m}R$ in M/C, and this completes the proof. □

The following example shows the converse of Proposition 2.1.10 is not true, in general.

Example 2.1.11. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/24\mathbb{Z}$. We illustrated this by Remarks and Examples 2.1.2(2) that $\mathbb{Z}/24\mathbb{Z}$ is a PG-Radical supplemented as $\mathbb{Z}$-module, while the $\mathbb{Z}$-module $\mathbb{Z}$ does not be PG-Radical supplemented, see Remarks and Examples 2.1.2(4).

Proposition 2.1.12. Let M be a distributive and PG-Radical supplemented module. Then $M/Rad_g(M)$ is principally semisimple.

Proof. Let M be a PG-Radical supplemented module and let $\overline{m}R$ be a cyclic submodule of $M/Rad_g(M)$, so $\overline{m}R = (mR + Rad_g(M))/Rad_g(M)$ for some $m \in M$. Then there is a submodule C of M so that $mR + C = M$ and $mR \cap C \subseteq Rad_g(C)$. So, we get that $M/Rad_g(M) = (mR + Rad_g(M)/Rad_g(M)) + \big(C + Rad_g(M)/Rad_g(M)\big) = \overline{m}R + (C + Rad_g(M))/Rad_g(M)$. Since, M is distributive, hence $\overline{m}R \cap ((C + Rad_g(M))/Rad_g(M)) = [mR + Rad_g(M) \cap \big(C + Rad_g(M)\,\big)\,] \,/\, Rad_g(M)\,] = (\,(\,mR \cap C) + Rad_g(M))/\, Rad_g(M) \subseteq$

$(Rad_g(C) + Rad_g(M))]/Rad_g(M) = Rad_g(M)/Rad_g(M)$. This lead as to $M/Rad_g(M) = \overline{m}R \oplus (C + Rad_g(M)/Rad_g(M))$, and hence $M/Rad_g(M)$ is principally semisimple. □

The following corollary is obtained by Proposition 2.1.12 and Remarks and Examples 2.1.2(5).

Corollary 2.1.13. Let M be a distributive and PG-Radical supplemented module. Then $M/Rad_g(M)$ is PG-Radical supplemented.

Proposition 2.1.14. Let $\alpha: M_1 \to M_2$ be any R-epimorphism from a PG-Radical supplemented module M_1 into module M_2 which any cyclic submodule of M_1 contains $ker\alpha$, then M_2 is PG-Radical supplemented.

Proof. Let $\grave{m}R$ be any cyclic submodule of M_2, then $\grave{m} = \alpha(m)$ for some $m \in M_1$. Since M_1 is a PG-Radical supplemented module and mR a submodule of M_1, then there is a submodule W of M_1 such that $mR + W = M_1$ and $mR \cap W \subseteq Rad_g(W)$. As a result of this, $\grave{m}R + \alpha(W) = M_2$. By assuming we get $ker\alpha \leq mR$. Therefore, $\alpha(mR) \cap \alpha(W) = \alpha(mR \cap W) \subseteq \alpha(Rad_g(W)) \subseteq Rad_g(\alpha(W))$ that is $\grave{m}R \cap \alpha(W) \subseteq Rad_g(\alpha(W))$. So, $\alpha(W)$ is a g-Radical supplement of $\grave{m}R$ in M_2, and then M_2 is PG-Radical supplemented. □

Corollary 2.1.15. Let M be a PG-Radical supplemented R-module and $W \leq M$. If any cyclic submodule of M contains W, then M/W is PG-Radical supplemented.

Proof. Assume the natural epimorphism map $\pi: M \to M/W$. As $ker\pi = W$, then by assuming, every cyclic submodule of M contains $ker\pi$, so that M/W is PG-Radical supplemented, by Proposition 2.1.14. □

Proposition 2.1.16. If a module $M = T \oplus N$. Then N is PG-Radical supplemented if and only if for any cyclic submodule E/T of M/T, there is a submodule W of N such that $M = W + E$ and $W \cap E \subseteq Rad_g(W)$.

Proof. Suppose that N is PG-Radical supplemented. Let E/T be a cyclic submodule of M/T. Then $E/T = (xR + T)/T$ and $x = \alpha_1 + \alpha_2$ where $\alpha_1 \in T$, $\alpha_2 \in N$. Thus $E/T = (\alpha_2 R + T)/T$. Hence there is $W \leq N$ with $N = \alpha_2 R + W$ with $\alpha_2 R \cap W \leq Rad_g(W)$. As a result of this, $E = \alpha_2 R + T$ and $M = T + (\alpha_2 R + W) = E + W$. Therfore we get that $E \cap W = (\alpha_2 R + T) \cap W \leq [\alpha_2 R \cap (T + W)] + [T \cap (\alpha_2 R + W)]$. Since, $T \cap (\alpha_2 R + W) \subseteq T \cap N = 0$, we deduce that $E \cap W \leq \alpha_2 R \cap (T + W) \leq \alpha_2 R$, and so $E \cap W \leq \alpha_2 R \cap W$. Since, $\alpha_2 R \cap W \subseteq Rad_g(W)$, we deduce $E \cap W \subseteq Rad_g(W)$.

Conversely, suppose that $\alpha_2 \in N$ and the cyclic submodule $(\alpha_2 R + T)/T$ of M/T. By our assuming, there is $W \leq N$ with $M = (\alpha_2 R + T) + W$ and $(\alpha_2 R + T) \cap W \leq Rad_g(W)$. By modular law, we deduce that $N = N \cap M = N \cap \big((\alpha_2 R + T) + W\big) = (\alpha_2 R + W) + (N \cap T) = \alpha_2 R + W$. It is enough to show that $W \cap (T + \alpha_2 R) = \alpha_2 R \cap (T + W) = \alpha_2 R \cap W$. Therefore, $\alpha_2 R \cap (T + W) \leq [T \cap (W + \alpha_2 R)] + [W \cap (\alpha_2 R + T)] = W \cap (\alpha_2 R + T) \leq [\alpha_2 R \cap (T + W)] + [T \cap (W + \alpha_2 R)] = \alpha_2 R \cap (T + W)$. Since we have $T \cap (W + \alpha_2 R) \subseteq T \cap N = 0$. It follows that $W \cap (T + \alpha_2 R) = \alpha_2 R \cap (T + W) = \alpha_2 R \cap W$. Hence, $\alpha_2 R \cap W \subseteq Rad_g(W)$. Hence, N is PG-Radical supplemented.

Proposition 2.1.17. let M be a PG-Radical supplemented R-module and $L \leq M$. If $L \cap Rad_g(M) = 0$, then L is principally semisimple.

Proof. Let $x \in L$. As M is a PG-Radical supplemented module, there exists a submodule W of M with $M = xR + W$ and $xR \cap W \subseteq Rad_g(W) \subseteq Rad_g(M)$. By the modular law, $L = L \cap M = L \cap (xR + W) = xR + (L \cap W)$. Since, $xR \cap (L \cap W) \subseteq L \cap Rad_g(M) = 0$, we get $L = xR \oplus (L \cap W)$. Hence, xR is a direct summand of L, and so L is principally semisimple. □

Proposition 2.1.18. Let M be a cyclic and PG-Radical supplemented module over a PID R. Then $M = W \oplus C$, where W is a principally semisimple submodule and C a submodule has essential generalized Radical.

Proof. Since, $Rad_g(M) \leq M$, so by lemma 1.2.74 then there is a submodule W of M so that $W \oplus Rad_g(M) \trianglelefteq M$. As, $W \cap Rad_g(M) = 0$ and M a PG-Radical supplemented R-module, so by Proposition 2.1.17, W is principally semisimple. Since, M is a cyclic module over a PID R, and $W \leq M$, then W is a cyclic submodule. As M is a PG-Radical supplemented module, there is a submodule C of M such that $M = W + C$ and $W \cap C \subseteq Rad_g(C) \subseteq Rad_g(M)$. As, $W \cap Rad_g(M) = 0$, then $W \cap C = 0$. Thus $M = W \oplus C$. It follows that, $Rad_g(M) = Rad_g(W \oplus C) = Rad_g(W) \oplus Rad_g(C)$, by Lemma 1.2.10. So, $W \oplus Rad_g(M) = W \oplus Rad_g(C)$. Hence, $W \oplus Rad_g(C) \trianglelefteq M = W \oplus C$, then by Lemma 1.2.2(2) $Rad_g(C) \trianglelefteq C$. □

2.2. Connections with PG-Radical supplemented modules

We will present, in this section, many relationships between the concept of PG-Radical supplemented module and other types of modules.

Proposition 2.2.1. Let M be a principally g-supplemented modules then M is PG-Radical supplemented.

Proof. Let M be a principally g-supplemented module and $m \in M$. Then there is a submodule E of M such that $mR + E = M$ and $mR \cap E \ll_g E$. Thus, $mR \cap E \subseteq Rad_g(E)$ and hence M is PG-Radical supplemented. □

Corollary 2.2.2. Every principally $\oplus$-g-supplemented and hence every principally g-lifting module is PG-Radical supplemented.

Proof. We have M is a principally g-supplemented module, from Lemmas 1.2.50 and 1.2.51 and by Proposition 2.2.1 M is PG-Radical supplemented. □

Examples 2.2.3.

(1) As application of Corollary 2.2.2, $M = \mathbb{Q} \oplus \mathbb{Z}_2$ as a $\mathbb{Z}$-module is PG-Radical supplemented, as M is principally ⊕-g-supplemented see [32, Example 3.1.6].

(2) The $\mathbb{Z}$-module $\mathbb{Q}$ is PG-Radical supplemented, since $\mathbb{Q}_{\mathbb{Z}}$ is principally g-supplemented, according [32, Examples 2.1.4. (1)]. Therefore, there are submodules of an arbitrary module do not inherit the property of PG-Radical supplemented, as we see that the $\mathbb{Z}$-module $\mathbb{Z} \leq \mathbb{Q}$ is not PG-Radical supplemented, while $\mathbb{Z}$-module $\mathbb{Q}$ is PG-Radical supplemented.

Proposition 2.2.4. Let M be a module with $Rad_g(M) = 0$. Then the following are equivalent.

(1) M is principally semisimple.

(2) M is principally g-lifting.

(3) M is principally ⊕-g-supplemented.

(4) M is principally g-supplemented.

(5) M is PG-Radical supplemented.

Proof. $(\mathbf{1}) \Leftrightarrow (\mathbf{2}) \Leftrightarrow (\mathbf{3}) \Leftrightarrow (\mathbf{4})$ By Lemma 1.2.59.

$(\mathbf{4}) \Longrightarrow (\mathbf{5})$ By Proposition 2.2.1.

$(\mathbf{5}) \Longrightarrow (\mathbf{1})$ If $m \in M$, and as M is PG-Radical supplemented, there $W \leq M$ such that $mR + W = M$ and $mR \cap W \subseteq Rad_g(W) \subseteq Rad_g(M)$, that implies $mR \cap W = 0$. Thus, $mR \leq^{\oplus} M$ and hence (1), holds. □

Corollary 2.2.5. Let M be a module over a g-V-ring R. Then the following are equivalent.

(1) M is principally semisimple.

(2) M is principally g-lifting.

(3) M is principally ⊕-g-supplemented.

(4) M is principally g-supplemented.

(5) M is PG-Radical supplemented.

Proof. Since R is a g-V-ring R, hence $Rad_g(M) = 0$, thus the result follows by Proposition 2.2.4. □

Corollary 2.2.6. Let R be an arbitrary ring whose every right R-module is g-noncosingular. Then the following are equivalents.

(1) M is principally semisimple.

(2) M is principally g-lifting.

(3) M is principally ⊕-g-supplemented.

(4) M is principally g-supplemented.

(5) M is PG-Radical supplemented.

Proof. Since M is an g-noncosingular R-module, thus by lemma 1.2.21 $Rad_g(M) = 0$, hence the result come by Proposition 2.2.4. □

Proposition 2.2.7. If M is a cyclic module over PID R with $Rad_g(M) = 0$, then the following are equivalents.

(1) M is semisimple.

(2) M is principally semisimple.

(3) M is g-lifting.

(4) M is principally g-lifting.

(5) M is ⊕-g-supplemented.

(6) M is principally ⊕-g-supplemented.

(7) M is g-supplemented.

(8) M is principally g supplemented.

(9) M is g-Radical supplemented.

(10) M is PG-Radical supplemented.

Proof. $(\mathbf{2}) \Leftrightarrow (\mathbf{4}) \Leftrightarrow (\mathbf{6}) \Leftrightarrow (\mathbf{8}) \Leftrightarrow (\mathbf{10})$ By Proposition 2.2.4

$(\mathbf{1}) \Leftrightarrow (\mathbf{2}), (\mathbf{3}) \Leftrightarrow (\mathbf{4}), (\mathbf{5}) \Leftrightarrow (\mathbf{6}), (\mathbf{7}) \Leftrightarrow (\mathbf{8})$ and $(\mathbf{9}) \Leftrightarrow (\mathbf{10})$ since M is a cyclic module over PID R, then all its submodules are cyclic. □

Proposition 2.2.8. Let M be an R-module so that $Rad_g(M) \subseteq mR$ for any $m \in M$. Then M is principally g-supplemented if and only if M is PG-Radical supplemented and $Rad_g(C) \ll_g C$ for any g-supplement C of mR in M.

Proof. $\Longrightarrow$) From Proposition 2.2.1, M is a PG-Radical supplemented module. Let C be any g-supplement of mR in M, It follows $M = mR + C$ and $mR \cap C \ll_g C$. As $Rad_g(C) \subseteq Rad_g(M) \cap C \subseteq mR \cap C$ imply $Rad_g(C) \ll_g C$.

$\Longleftarrow$) Let $mR \leq M, m \in M$. By assuming, $M = mR + C$ and $mR \cap C \subseteq Rad_g(C)$ for a $C \leq M$. As $Rad_g(C) \ll_g C$, then $mR \cap C \ll_g C$, as required. □

Proposition 2.2.9. Let M be an R-module whose $Rad_g(M)$ is a minimal submodule of M. Then M is principally semisimple if and only if M is PG-Radical supplemented.

Proof. The necessity by Remarks and Examples 2.1.2(5). Conversely, suppose $m \in M$. According to the hypothesis, there exists a $W \leq M$ with $mR + W = M$ and $mR \cap W \subseteq Rad_g(W)$. Therefore, $mR \cap W \subseteq Rad_g(M)$. As $Rad_g(M)$ is minimal in M, we have $mR \cap W = 0$. Hence $mR \oplus W = M$, as required. □

Corollary 2.2.10. Let M be a module with $Rad_g(M)$ is minimal in M. Then the following are equivalents.

(1) M is principally semisimple.

(2) M is principally g-lifting.

(3) M is principally $\oplus$-g-supplemented.

(4) M is principally g-supplemented.

(5) M is PG-Radical supplemented.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) By Lemma 1.2.59.

$(\mathbf{4}) \Longrightarrow (\mathbf{5})$ By Proposition 2.2.1.

$(\mathbf{5}) \Longrightarrow (\mathbf{1})$ By Proposition 2.2.9. □

Corollary 2.2.11. Consider M is a distributive refinable R-module with $Rad_g(M) \ll_g M$. Then M is PG-Radical supplemented if and only if $M/Rad_g(M)$ is principally semisimple.

Proof. The necessity is clear by Proposition 2.1.12. Conversely, Lemma 1.2.57 implies M is a principally g-supplemented module and hence it is PG-Radical supplemented, by Proposition 2.2.1. □

Corollary 2.2.12. Let M be a distributive, refinable and finitely generated module. Then M is PG-Radical supplemented if and only if $M/Rad_g(M)$ is principally semisimple.

Proof. As M is a finitely generated R-module, Lemma 1.2.12 implies $Rad_g(M) \ll_g M$. Hence the result is obtained by Proposition 2.2.11.

Corollary 2.2.13. Let M be a distributive and refinable module with $Rad_g(M) \ll_g M$. Then the following are equivalents.

(1) M is principally ⨁-g-supplemented.

(2) M is principally g-supplemented.

(3) M is PG-Radical supplemented.

(4) $M/Rad_g(M)$ is principally semisimple.

Proof. $(\mathbf{1}) \Leftrightarrow (\mathbf{2}) \Leftrightarrow (\mathbf{4})$ By Lemma 1.2.57.

$(\mathbf{3}) \Leftrightarrow (\mathbf{4})$ By Proposition 2.2.11. □

Corollary 2.2.14. Let M be a distributive, refinable and finitely generated module. Then the following are equivalents.

(1) M is principally g-supplemented.

(2) M is principally ⨁-g-supplemented.

(3) M is PG-Radical supplemented.

(4) $M/Rad_g(M)$ is principally semisimple.

Proof. By Lemma 1.2.12, $Rad_g(M) \ll_g M$. Therefore, the result follows Corollary 2.2.13. □

Corollary 2.2.15. Let M be a distributive, refinable and noetherian module. Then the following are equivalents:

(1) M is principally g-supplemented.

(2) M is principally ⊕-g-supplemented.

(3) M is PG-Radical supplemented.

(4) $M/Rad_g(M)$ is principally semisimple.

Proof. Since any noetherian module is finitely generated. Hence, the result follows by Corollary 2.2.14. □

2.3. Localization of PG-Radical supplemented modules

We will investigate the behavior of PG-Radical supplemented modules under localization. Before that we need to prove the following two lemmas.

Lemma 2.3.1. Let M be an R-module, $E \leq \grave{E} \leq M$ and S is a multiplicative closed subset of R with $\mathcal{L}(W) \cap S = \emptyset$ for any $W \leq M$. Then E is a maximal submodule of $\grave{E}$ if and only if $S^{-1}E$ is a maximal submodule of $S^{-1}\grave{E}$.

Proof. Suppose that E is a maximal submodule of $\grave{E}$. As E is a proper submodule of $\grave{E}$, so $S^{-1}E$ is a proper submodule of $S^{-1}\grave{E}$. Let $S^{-1}C$ be any submodule of $S^{-1}\grave{E}$ such that $S^{-1}E \subseteq S^{-1}C \subseteq S^{-1}\grave{E}$. From Lemma 1.2.65, we get that $E \subseteq C \subseteq \grave{E}$. As E is maximal in $\grave{E}$, so either $E = C$ or $C = \grave{E}$. Again, by Lemma 1.2.65, $S^{-1}E = S^{-1}C$ or $S^{-1}C = S^{-1}\grave{E}$, and so $S^{-1}E$ is a maximal submodule of $S^{-1}\grave{E}$.

Conversely, suppose that $S^{-1}E$ is a maximal submodule of $S^{-1}\grave{E}$. As $S^{-1}E$ is a proper submodule of $S^{-1}\grave{E}$, so it is easily E is a proper submodule of $\grave{E}$. Let C be a submodule of $\grave{E}$ such that $E \subseteq C \subseteq \grave{E}$. From Lemma 1.2.65, $S^{-1}E \subseteq$

$S^{-1}C \subseteq S^{-1}\grave{E}$. Since $S^{-1}E$ is maximal in $S^{-1}\grave{E}$, so either $S^{-1}E = S^{-1}C$ or $S^{-1}C = S^{-1}\grave{E}$. Again, by Lemma 1.2.65, We get that $E = C$ or $C = \grave{E}$, and hence E is a maximal submodule of $\grave{E}$. □

Lemma 2.3.2. Let M be an R-module, $E \leq M$ and S is a multiplicative closed subset of R such that $\mathcal{L}(W) \cap S = \emptyset$ for any $W \leq M$. Then $Rad_g(S^{-1}E) = S^{-1}(Rad_g(E))$.

Proof. Let $x \in Rad_g(S^{-1}E)$. Then $x = \frac{n}{s}$, for some $n \in E$ and $s \in S$. For any essential maximal submodule C of E, then from Lemma 1.2.69 (1) and Lemma 2.3.1, $S^{-1}C$ is an essential maximal submodule of $S^{-1}E$ and so $x = \frac{n}{s} \in S^{-1}C$, then $\frac{n}{s} = \frac{l}{t}$, for some $l \in C$ and $t \in S$, then $ptn = psl \in C$, for some $p \in S$. If $n \notin C$, then $pt \in \mathcal{L}(C)$ and as $\mathcal{L}(C) \cap S = \emptyset$, so $pt \notin S$, which is a contradiction, and thus $n \in C$, so that $n \in Rad_g(E)$ and so $x = \frac{n}{s} \in S^{-1}(Rad_g(E))$. Hence, $Rad_g(S^{-1}E) \subseteq S^{-1}(Rad_g(E)$.

Conversely, suppose that $a = \frac{x}{s} \in S^{-1}(Rad_g(E))$, for some $x \in Rad_g(E)$ and $s \in S$. For any essential maximal submodule $S^{-1}C$ of $S^{-1}E$, Lemma 1.2.69(1) and Lemma 2.3.1 implies C is an essential maximal submodule of E, then $x \in C$ and so $a = \frac{x}{s} \in S^{-1}C$ hence $a \in Rad_g(S^{-1}E)$. Therefore, $S^{-1}(Rad_g(E)) \subseteq Rad_g(S^{-1}E)$, as required. □

Theorem 2.3.3. Let M be an R-module and S a multiplicative closed subset of R such that $\mathcal{L}(T) \cap S = \emptyset$ for any $T \leq M$. Then M is PG-Radical supplemented as R-module if and only if $S^{-1}M$ is PG-Radical supplemented as $S^{-1}R$-module.

Proof. Let M be PG-Radical supplemented as R-module and let $S^{-1}E$ be any cyclic submodule of $S^{-1}M$ as $S^{-1}R$-module. By Lemma 1.2.66, E is cyclic in M as R-module. Then there is a submodule W of M so that $E + W = M$ and $E \cap W \subseteq Rad_g(W)$. From Lemmas 1.2.71, 1.2.65 and 2.3.2, $S^{-1}E + S^{-1}W =$

$S^{-1}M$ and $S^{-1}E \cap S^{-1}W \subseteq S^{-1}(Rad_g(W)) = Rad_g(S^{-1}W)$. Hence, $S^{-1}M$ is PG-Radical supplemented as $S^{-1}R$-module.

Conversely, suppose that $S^{-1}M$ is PG-Radical supplemented as $S^{-1}R$-module, and let E be any cyclic submodule of M as R-module. From Lemma 1.2.66, $S^{-1}E$ is cyclic in $S^{-1}M$ as $S^{-1}R$-module. Then there exists a submodule $S^{-1}W$ of $S^{-1}M$, $S^{-1}E + S^{-1}W = S^{-1}$and $S^{-1}E \cap S^{-1}W \subseteq Rad_g(S^{-1}W)$. By Lemmas 1.2.71, 1.2.65 and 2.3.2, we have that $E + W = M$ and $E \cap W \subseteq Rad_g(W)$. That is, M is PG-Radical supplemented as R-module. □

Lemma 2.3.4. Let M be an R-module, $m \in M$ and S be a multiplicative closed subset of R. Then $S^{-1}(mR) = \frac{m}{s}.S^{-1}R$ for any $s \in S$.

Proof. Let $\frac{mr}{t} \in S^{-1}(mR)$, for some $r \in R$ and $t \in S$. Then $\frac{mr}{t} = \frac{mrs}{ts} = \frac{m}{s}.\frac{rs}{t} \in \frac{m}{s}.S^{-1}R$, we deduce that $S^{-1}(mR) \subseteq \frac{m}{s}.S^{-1}R$.Conversely, let $\frac{m}{s}.\frac{r}{s_1} \in \frac{m}{s}.S^{-1}R$, for some $r \in R$ and $s_1 \in S$. Put $s_2 = ss_1$, then $\frac{m}{s}.\frac{r}{s_1} = \frac{mr}{ss_1} = \frac{mr}{s_2} \in S^{-1}(mR)$. Thus, $\frac{m}{s}.S^{-1}R \subseteq S^{-1}(mR)$ and hence $S^{-1}(mR) = \frac{m}{s}.S^{-1}R$ for any $s \in S$. □

Lemma 2.3.5. Let M be an R-module, $E \subset M$ and S be a multiplicative closed subset of R. If for every submodule $W \subset M$, $(W:_M s) = W$, for all $s \in S$. Then $Rad_g(S^{-1}E) = S^{-1}(Rad_g(E))$.

Proof. Suppose that $\frac{n}{s} \in Rad_g(S^{-1}E)$, for some $n \in E$ and $s \in S$. Then $\frac{n}{s}.S^{-1}R \ll_g S^{-1}E$, so by Lemma 2.3.4, $S^{-1}(nR) = \frac{n}{s}.S^{-1}R$ and so $nR \ll_g E$ from Lemma 1.2.70(2). Thus $n \in Rad_g(E)$ and then $\frac{n}{s} \in S^{-1}(Rad_g(E))$. Therefore, $Rad_g(S^{-1}E) \subseteq S^{-1}(Rad_g(E))$.

Conversely, suppose $\frac{x}{s} \in S^{-1}(Rad_g(E))$, for some $x \in Rad_g(E)$ and $s \in S$. As a result of this, $xR \ll_g E$, Lemma 1.2.70(2) implies $S^{-1}(xR) \ll_g S^{-1}E$.

From Lemma 2.3.4, $\frac{x}{s}.S^{-1}R = S^{-1}(xR)$ that is; $\frac{x}{s}.S^{-1}R \ll_g S^{-1}E$, and hence $\frac{x}{s} \in Rad_g(S^{-1}E)$. Therefore, $S^{-1}(Rad_g(E)) \subseteq Rad_g(S^{-1}E)$, as required. □

Theorem 2.3.6. Let M be an R-module and S a multiplicative closed subset of R such that for any $W \subset M$, $(W:_M s) = W$, for all $s \in S$. Then M is PG-Radical supplemented as R-module if and only if $S^{-1}M$ is PG-Radical supplemented as $S^{-1}R$-module.

Proof. Suppose that M is PG-Radical supplemented as R-module, and $S^{-1}E$ a cyclic submodule of $S^{-1}M$ as $S^{-1}R$-module. By Lemma 1.2.68, E is a cyclic submodule of M as R-module. Then there exists $W \leq M$ such that $E + W = M$ and $E \cap W \subseteq Rad_g(W)$. From Lemmas 1.2.71, 1.2.67 and 2.3.5, we get $S^{-1}E + S^{-1}W = S^{-1}M$ and so that $S^{-1}E \cap S^{-1}W \subseteq S^{-1}(Rad_g(W)) = Rad_g(S^{-1}W)$. Therefore, $S^{-1}M$ is PG-Radical supplemented as $S^{-1}R$-module.

Conversely, let $S^{-1}M$ be PG-Radical supplemented $S^{-1}R$-module, and let E be a cyclic submodule of M as R-module. By Lemma 1.2.68, $S^{-1}E$ is cyclic in $S^{-1}M$ as $S^{-1}R$-module. Then there exists a $S^{-1}W \leq S^{-1}M$ with $S^{-1}E + S^{-1}W = S^{-1}M$ and $S^{-1}E \cap S^{-1}W \subseteq Rad_g(S^{-1}W)$. By Lemmas 1.2.71, 1.2.67 and 2.3.5, we deduce that $E + W = M$ and $E \cap W \subseteq Rad_g(W)$, as required. □

2.4. Principally (P_g^*)-modules

In this section we will introduce the definition of principally (P_g^*)-modules, as well as a some of their properties and examples about them will be discussed.

Definition 2.4.1. An R-module M is called a principally (P_g^*)-module, briefly P-(P_g^*)-module or have P-(P_g^*) property if, for any $m \in M$, there is a direct summand D of M such that $D \leq mR$ and $mR/D \subseteq Rad_g(M/D)$.

Remarks and Examples 2.4.2.

(1) It is easily to ensure that every principally g-lifting module is P-($P_g{}^*$)-module. As an application example; we have $\mathbb{Q}$ and $\mathbb{Z}_{p^n}$ as $\mathbb{Z}$-modules are principally g-lifting for all prime number p and $n \in \mathbb{Z}^+$, see [13, Examples 3.2], hence they are principally ($P_g{}^*$)-module modules.

(2) It is clear that any ($P_g{}^*$)-module is a P-($P_g{}^*$)-module. However, if M is a cyclic module over a PID R, then M is a P-($P_g{}^*$)-module if and only if it is a ($P_g{}^*$)-module.

(3) The P-($P_g{}^*$)-module not inherited by its submodules, for instance, by **(1)**, $\mathbb{Q}$ is a P-($P_g{}^*$)-$\mathbb{Z}$-module, while $\mathbb{Z}_\mathbb{Z} \leq \mathbb{Q}_\mathbb{Z}$ is not P-($P_g{}^*$)-module, to see this: it is well known that $Rad_g(\mathbb{Z}_\mathbb{Z}) = 0$, then for any $n\mathbb{Z} \leq \mathbb{Z}$ we have $Rad_g(n\mathbb{Z}_\mathbb{Z}) \subseteq Rad_g(\mathbb{Z}_\mathbb{Z}) = 0$ for any $n \in \mathbb{Z}^+$. However, if $n\mathbb{Z} \leq \mathbb{Z}$ and $n > 1$, then there is only decomposition $\mathbb{Z} = (0)\oplus\mathbb{Z}$ so that $(0) \leq n\mathbb{Z}$ while $n\mathbb{Z} \cap \mathbb{Z} = n\mathbb{Z} \nsubseteq 0 = Rad_g(n\mathbb{Z})$.

In the following theorem, we shall propose a characterization for P-($P_g{}^*$)-modules.

Theorem 2.4.3. Let M be an R-module. Then the following are equivalent.

(1) M is a P-($P_g{}^*$)-module.

(2) for any $m \in M$, there is a decomposition $M = E\oplus W$ such that $E \leq mR$ and $mR \cap W \subseteq Rad_g(W)$.

(3) for any $m \in M$, there is a decomposition $M = E\oplus W$ such that $E \leq mR$ and $mR \cap W \subseteq Rad_g(M)$.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Let $m \in M$, so by (1), there is a direct summand E of M such that $E \leq mR$ and $mR/E \subseteq Rad_g(M/E)$. Thus $M = E\oplus W$ for some $W \leq M$. By modular law, $mR = mR \cap M = mR \cap (E\oplus W) = E\oplus(mR \cap W)$. As $M/E \cong W$, so we have that $\alpha: M/E \to W$ is an R-isomorphism. Since,

$mR/E \subseteq Rad_g(M/E)$, we have that $mR \cap W = \alpha\big((E \oplus (mR \cap W))/E\big) = \alpha(mR/E) \subseteq \alpha(Rad_g(M/E)) \subseteq Rad_g(W)$.

$(\mathbf{2}) \Longrightarrow (\mathbf{1})$ If $m \in M$, by **(2)**, then there is a decomposition $M = E \oplus W$ so that $E \leq mR$ and $mR \cap W \subseteq Rad_g(W)$. We have $mR = E \oplus (mR \cap W)$. As, $W \cong M/E$, then there is an R-isomorphism $\varphi: W \to M/E$. As $mR \cap W \subseteq Rad_g(W)$, thus $mR/E = (E \oplus (mR \cap W))/E = \varphi(mR \cap W) \subseteq \varphi(Rad_g(W)) \subseteq Rad_g(M/E)$. Hence, M is a P-(P_g^*)-module.

$(\mathbf{1}) \Longrightarrow (\mathbf{3})$ It is directly followed by Theorem 2.4.3.

$(\mathbf{3}) \Longrightarrow (\mathbf{1})$ Let $m \in M$. By **(2)**, there is a decomposition $M = L \oplus Y$ such that $L \leq mR$ and $mR \cap Y \subseteq Rad_g(M)$. As $mR \cap Y \leq Y \leq^{\oplus} M$, Lemma 1.2.7 implies $mR \cap Y \subseteq Rad_g(Y)$, thus **(1)** holds, by Theorem 2.4.3. □

Proposition 2.4.4. Let M be a P-(P_g^*)-module. If $m \in M$, then there exists a submodule E of M contained in mR such that $M = E \oplus C$ and C is a g-Radical supplement of mR in M.

Proof. Assume $m \in M$. Since, M is a P-(P_g^*)-module, then there exists a submodule E of M in mR such that $M = E \oplus C$ and $mR \cap C \subseteq Rad_g(C)$. It follows that, $M = mR + C$. Hence C is a g-Radical supplement of mR in M. □

Theorem 2.4.5. The following are equivalent for an R-module M and $m \in M$.

(1) M is a P-(P_g^*)-module.

(2) there exists an idempotent $e \in End(M)$, $e(M) \leq mR$ and $(1-e)mR \subseteq Rad_g(1-e)(M)$;

(3) mR has a g-Radical supplement Y so that $mR \cap Y \leq^{\oplus} mR$.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Let $m \in M$. By Theorem 2.4.3, there is a decomposition $M = L \oplus Y$ so that $L \leq mR$ and $mR \cap Y \subseteq Rad_g(Y)$. For a decomposition $M = L \oplus Y$, there is an idempotent $e \in End(M)$ so that $L = e(M)$ and $Y = (1-\mathrm{e})M$. Since $e(M) = L \leq mR$, then we can easily to show that

$(1-e)mR = mR \cap (1-e)M$. As $Y = (1-e)M$ and $mR \cap Y \subseteq Rad_g(Y)$, so $(1-e)mR = mR \cap (1-e)M = mR \cap Y \subseteq Rad_g(Y) = Rad_g(1-e)(M)$.

$(\mathbf{2}) \Longrightarrow (\mathbf{3})$ Putting $L = e(M)$ and $Y = (1-e)M$, then $M = e(M) \oplus (1-e)(M) = L \oplus Y$. As $L = e(M) \leq mR$, we conclude that $M = mR + Y$. Also $mR \cap Y = mR \cap (1-e)M = (1-e)mR \subseteq Rad_g(1-e)(M) = Rad_g(Y)$. So, $Y = (1-e)M$ is a g-Radical supplement of mR in M. On the other hand, as $L \leq mR$, we have $mR = mR \cap M = mR \cap (L \oplus Y) = L \oplus (mR \cap Y)$, by the modular law, that is $mR \cap Y \leq^{\oplus} mR$.

$(\mathbf{3}) \Longrightarrow (\mathbf{1})$ Let $mR = L \oplus (mR \cap Y)$ for some $L \leq mR$. As Y is a g-Radical supplement of mR in M, $M = mR + Y$ and $mR \cap Y \subseteq Rad_g(Y)$. Therefore, $M = mR + Y = L \oplus (mR \cap Y) + Y = L \oplus Y$. Then the result is obtained by Theorem 2.4.3. □

Theorem 2.4.6. The following are equivalent for an R-module M.

(1) M is a P-($P_g{}^*$)-module;

(2) for each $m \in M$, then mR can be written as $mR = T \oplus S$, where T is a direct summand of M and $S \subseteq Rad_g(M)$;

(3) for each $m \in M$, there exists two ideals I and J of R so that $mR = mI \oplus mJ$, where mI is a direct summand of M and $mJ \subseteq Rad_g(M)$.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ If $m \in M$, so by a P-($P_g{}^*$)-module for M, there exists a submodule T of M contained in mR such that $M = T \oplus E$ and $mR \cap E \subseteq Rad_g(E)$. Put $S = mR \cap E$, it follows that $mR = T \oplus S$ and $S \subseteq Rad_g(M)$.

$(\mathbf{2}) \Longrightarrow (\mathbf{1})$ Let $m \in M$, by **(2)**, then there exists a direct summand T of M, $mR = T \oplus S$ and $S \subseteq Rad_g(M)$. Thus $M = T \oplus \acute{T}$ for some $\acute{T} \leq M$. As a result of this, $mR = mR \cap M = T \oplus (mR \cap \acute{T})$. Suppose that a projection $\acute{\rho}: T \oplus \acute{T} \longrightarrow \acute{T}$. We deduce $S = \acute{\rho}(T \oplus S) = \acute{\rho}(mR) = mR \cap \acute{T}$, and so $mR \cap \acute{T} \subseteq Rad_g(M)$. Since $mR \cap \acute{T} \leq \acute{T} \leq^{\oplus} M$, Lemma 1.2.7, implies $mR \cap \acute{T} \subseteq Rad_g(\acute{T})$.

(1) ⟹ (3) If $m \in M$, by Theorem 2.4.5, there is an $e = e^2 \in End(M)$, $e(M) \leq mR$ and $(1-e)mR \subseteq Rad_g(1-e)(M)$. Consider, $M = eM \oplus (1-e)M$. Let $r \in R$ so that $mr = (1-e)\grave{m}$ for some $\grave{m} \in M$, then $\grave{m} = em + mr \in mR$ since $e(M) \leq mR$, and so $mR \cap (1-e)M \leq mR \cap (1-e)mR$. As a result of this, $mR \cap (1-e)M = (1-e)mR$. Therefore, we have $mR = eM \oplus (1-e)mR$. Putting $I = \{s \in R | ms \in eM\}$ and $J = \{r \in R | \, mr \in (1-e)mR\}$. Therefore, $mR = mI \oplus mJ$, where $mI = eM \leq^{\oplus} M$ and $mJ = (1-e)mR \subseteq Rad_g(M)$.

(3) ⟹ (1) By **(3)**, if $m \in M$, then there exists two ideals I and J of R, such that $mR = mI \oplus mJ$, where $mI \leq^{\oplus} M$ and $mJ \subseteq Rad_g(M)$. So that, $M = mI \oplus S$ for some $S \leq M$. Hence $mR = mI \oplus (mR \cap S)$, and so $mR \cap S \cong mJ$, imply $mR \cap S \subseteq Rad_g(M)$. As $mR \cap S \leq S \leq^{\oplus} M$, Lemma 1.2.7 implies that $mR \cap S \subseteq Rad_g(S)$. Hence M is a principally (P_g^*)-module. □

We will discuss some properties of a P-(P_g^*)-module according to their submodules, direct summands, and factor modules.

Proposition 2.4.7. Let M be a P-(P_g^*)-module. Then for $m \in M$, there exists $E \leq^{\oplus} M$, $(E + mR)/E \subseteq Rad_g(M/E)$ and $(E + mR)/mR \subseteq Rad_g(M/mR)$.

Proof. Suppose that M is a P-(P_g^*)-module. If $m \in M$, then there is a direct summand E of M such that $E \leq mR$ and $mR/E \subseteq Rad_g(M/E)$. It follows that, $(E + mR)/E \subseteq Rad_g(M/E)$. As $(E + mR)/mR \ll_g M/mR$, we deduce that $(E + mR)/mR \subseteq Rad_g(M/mR)$. □

Proposition 2.4.8. Let M be a P-(P_g^*)-module and $m \in M$. Then there is a direct summand L of M and a submodule $C \leq M$ such that $L \leq mR$, $mR = L + C$ and $C \subseteq Rad_g(M)$.

Proof. Suppose that M is a P-(P_g^*)-module and let $m \in M$. Then there exists a decomposition $M = L \oplus Y$, $L \leq mR$ and $mR/L \subseteq Rad_g(M/L)$. As a result of this, $mR = L + (mR \cap Y)$. Put $C = mR \cap Y$, we get $mR = L + C$. Since, $M/L \cong Y$, we deduce that $\varphi: M/L \to Y$ is an R-isomorphism. As $mR/L \subseteq Rad_g(M/L)$, implies $mR \cap Y = \varphi(mR/L) \subseteq \varphi(Rad_g(M/L)) \subseteq Rad_g(Y)$, and hence $C \subseteq Rad_g(M)$. □

Proposition 2.4.9. Let M be a P-(P_g^*)-module and $C \leq M$. If M/C is projective, then C is a P-(P_g^*)-module.

Proof. Let $x \in C$, then there exists a decomposition $M = L \oplus Y$ with $L \leq xR$ and $xR \cap Y \subseteq Rad_g(M)$. Thus, $M = C + Y$, and so $C \cap Y$ is a direct summand of M, by Lemma 1.2.29. Also, we have $C = L \oplus (C \cap Y)$ and $xR \cap (C \cap Y) = xR \cap Y \subseteq Rad_g(C \cap Y)$, by Lemma 1.2.7. Therefore, C is a P-(P_g^*)-module. □

Proposition 2.4.10. Let M be a module and $N \leq M$. If any cyclic submodule of M has a (P_g^*) property with a direct summand of M which contains N, then M/N has a P-(P_g^*) property.

Proof. Suppose that $(mR + N)/N$ is a submodule of M/N, where $m \in M$. By assuming, there is a submodule C of M in mR so that $M = C \oplus L$, $N \leq L$ and $mR \cap L \subseteq Rad_g(M)$. Then we have $M/N = ((C + N)/N) \oplus (L/N)$, and $(C + N)/N \leq (mR + N)/N$. Let π indicate the canonical map from a module M into M/N. Since, $ker\pi = N \subseteq L$, then $\pi(mR \cap L) = \pi(mR) \cap \pi(L) = ((mR + N)/N) \cap (L/N)$. So that, $((mR + N)/N) \cap (L/N) \subseteq Rad_g(M/N)$ and hence $((mR + N)/N) \cap (L/N) \subseteq Rad_g(L/N)$, by Lemma 1.2.7. □

Now, we will investigate the quotients of P-(P_g^*)-modules.

Proposition 2.4.11. Let M be a P-(P_g^*)-module, then.

(1) If $C \leq M$ is fully invariant submodule of M, then M/C is so a principally (P_g^*)-module.

(2) If $C \leq M$ is weak distributive submodule of M, then M/C is so a principally (P_g^*)-module.

Proof. (1) Let C be a fully invariant submodule of M and $(mR+C)/C \leq M/C$, where $m \in M$. As $mR \leq M$, then there is a decomposition $M = L \oplus Y$ such that $L \leq mR$ and $mR \cap Y \subseteq Rad_g(Y)$. From Lemma 1.2.19, we get $M/C = [(L+C)/C] \oplus [(Y+C)/C]$. As $L \leq mR$, then $(L+C)/C \leq mR/C$. Now, define a natural map $\pi: Y \to (Y+C)/C$. Since $mR \cap Y \subseteq Rad_g(Y)$, so $\pi(mR \cap Y) \subseteq \pi\left(Rad_g(Y)\right) \subseteq Rad_g((Y+C)/C)$, but we have $(mR/C) \cap ((Y+C)/C) = \pi(mR \cap Y)$, thus we deduce that $(mR/C) \cap ((Y+C)/C) \subseteq Rad_g((Y+C)/C)$. Therefore, M/C is a P-(P_g^*)-module.

(2) Let C be a weak distributive submodule of M and $(mR+C)/C \leq M/C$, where $m \in M$. Then there exists a decomposition $M = L \oplus Y$, $L \leq mR$ and $mR \cap Y \subseteq Rad_g(Y)$. We get that $C = (C \cap L) + (C \cap Y)$. Also, $M/C = (L+C)/C + (Y+C)/C$. Hence, $(L+C) \cap (Y+C) = \left(L + (C \cap Y)\right) \cap (Y+C) = (L \cap Y) + (C \cap Y) + C = C$ imply $(L+C)/C \cap (Y+C)/C = 0$. Therefore, $((L+C)/C) \oplus ((Y+C)/C) = M/C$. Then we go through the same steps to proof **(1)**. □

Corollary 2.4.12. Let M be a duo (or, a weakly distributive) module. If M is a P-(P_g^*)-module, then any factor module of M is also a P-(P_g^*)-module.

Proof. Clear. □

Corollary 2.4.13. If M is a P-(P_g^*)-module, then so is $M/Rad_g(M)$.

Proof. From Lemma 1.2.8 we know that $Rad_g(M)$ is fully invariant, so that the result is obtained by Proposition 2.4.11(1). □

As we see in Remarks and Examples 2.4.2(3), a P-(P_g^*)-module is not inherited by their submodules in general, so that we will give some cases to make it inherited.

Proposition 2.4.14. Each direct summand of a P-(P_g^*)-module is also a P-(P_g^*)-module.

Proof. Let $F \leq^{\oplus} M$ and let M be a P-(P_g^*)-module. If $a \in F$, then there exist two submodules L and Y of M so that $L \leq aR$ and $aR \cap Y \subseteq Rad_g(Y)$, where $M = L \oplus Y$. We deduce that $F = L \oplus (F \cap Y)$. It's obvious to see that $F \cap Y \leq^{\oplus} Y$. By $aR \cap (F \cap Y) \leq aR \cap Y \subseteq Rad_g(Y)$, Lemma 1.2.7 implies $aR \cap (F \cap Y) \subseteq Rad_g(F \cap Y)$, as required. □

Proposition 2.4.15. Let $M = \oplus_{i\in I} M_i$ be a module whose any cyclic submodule is fully invariant. If M_i is a P-(P_g^*)-module for all $i \in I$, then M is a P-(P_g^*)-module.

Proof. Let M_i be a P-(P_g^*)-module for all $i \in I$, and let mR be a submodule of $M = \oplus_{i\in I} M_i$, where $m \in M$. As mR is fully invariant, then by Lemma 1.2.14 $mR = \oplus_{i\in I}(mR \cap M_i)$. As $mR \cap M_i \leq M_i$ for $i \in I$, there is decompositions $M_i = G_i \oplus \grave{G}_i$ so that $G_i \leq mR \cap M_i$ and $(mR \cap M_i) \cap \grave{G}_i = mR \cap \grave{G}_i \subseteq Rad_g(\grave{G}_i)$. We get $M = (\oplus_{i\in I} G_i) \oplus (\oplus_{i\in I} \grave{G}_i)$, $\oplus_{i\in I} G_i \leq \oplus_{i\in I}(mR \cap M_i) = mR$ and then $mR \cap \left(\oplus_{i\in I} \grave{G}_i\right) = \oplus_{i\in I}(mR \cap \grave{G}_i) \subseteq \oplus_{i\in I}(Rad_g(\grave{G}_i)) = Rad_g(\oplus_{i\in I} \grave{G}_i)$, from Lemma 1.2.10. Thus, M is a P-(P_g^*)-module. □

Corollary 2.4.16. If $M = \oplus_{i\in I} M_i$ is a duo module, then M_i is a P-(P_g^*)-module for all $i \in I$, if and only if M is a P-(P_g^*)-module.

Proof. The result follows directly from Propositions 2.4.14 and 2.4.15. □

Proposition 2.4.17. Let $M = M_1 \oplus M_2$ be a module. If M_2 is a P-(P_g^*)-module, then for any cyclic submodule mR/M_1 of M/M_1, mR has g-Radical supplement $D \leq M_2$ that is a direct summand of M, for some $m \in M$.

Proof. Suppose M_2 is a P-(P_g^*)-module. Let mR/M_1 be a cyclic submodule of M/M_1, where $m \in M$. Since, mR is a cyclic submodule of $M = M_1 \oplus M_2$, so it's easy to see that $mR \cap M_2$ is cyclic in M_2. Then there is a decomposition

$M_2 = W \oplus \grave{W}$ such that $W \leq mR \cap M_2$ and $(mR \cap M_2) \cap \grave{W} = mR \cap \grave{W} \subseteq Rad_g(\grave{W})$. Thus, $M_2 = (mR \cap M_2) + \grave{W}$. Also, we get that $M = M_1 \oplus M_2 = mR + M_2 = mR + (mR \cap M_2) + \grave{W} = mR + \grave{W}$. Thus, $\grave{W} \leq M_2$ is a g-Radical supplement of mR so that $\grave{W} \leq^{\oplus} M$. □

Proposition 2.4.18. Let $M = S + T$ be a P-(P_g^*)-module with $T \leq^{\oplus} M$. If $S \cap T$ is cyclic in M, then T contains a g-Radical supplement of S in M.

Proof. Since, P-(P_g^*)-module for M, and as $S \cap T$ is cyclic in M, Theorem 2.4.6 implies $S \cap T = A \oplus N$, where A is a direct summand of M, so in T, and $N \subseteq Rad_g(M)$. Put, $T = A \oplus E$ for some $E \leq T$. As a result of this, $S \cap T = A \oplus (S \cap E)$. Suppose that a projection map $\rho: T = A \oplus E \longrightarrow E$. Since, $N \leq T \leq^{\oplus} M$ and $N \subseteq Rad_g(M)$, Lemma 1.2.7 implies $N \subseteq Rad_g(T)$, and hence $\rho(N) \subseteq \rho(Rad_g(T)) \subseteq Rad_g(E)$. Also, we deduce that $S \cap E = \rho(A \oplus (S \cap E) = \rho(S \cap T) = \rho(A \oplus N) = \rho(N)$, then $S \cap E \subseteq Rad_g(E)$ and $M = S + T = S + A + E = S + E$. Hence, T contains E as a g-Radical supplement of S in M. □

Theorem 2.4.19. Let $M = L_1 \oplus L_2$, where L_1 be a semisimple and L_2 is a P-(P_g^*)-module which are relatively projective, then is a P-(P_g^*)-module.

Proof. Let $m \in M$. Suppose that $(0 \neq)\, mR \leq M$, and let $G = L_1 \cap (mR + L_2)$, so we have two cases:

Case (i) if $G \neq 0$. Since $G \leq L_1$, there exists a submodule G_1 of L_1 such that $L_1 = G \oplus G_1$, and so $M = G \oplus G_1 \oplus L_2 = mR + (L_2 \oplus G_1)$. Thus G is $L_2 \oplus G_1$-projective. By Lemma 1.2.26, there is $C \leq mR$ so that $M = C \oplus (L_2 \oplus G_1)$. We may suppose that $mR \cap (L_2 \oplus G_1) \neq 0$. It is easy to see that $mR \cap (H + G_1) = H \cap (mR + G_1)$ for $H \leq L_2$. In particular, $mR \cap (L_2 + G_1) = L_2 \cap (mR + G_1)$. As a result of this, $mR = C \oplus (mR \cap (L_2 \oplus G_1)) = C \oplus (L_2 \cap (mR + G_1))$. As L_2 is a principally (P_g^*)-module, there is a decomposition $L_2 = S_1 \oplus S_2$ with $S_1 \leq L_2 \cap (mR + G_1)$ and $S_2 \cap (mR + G_1) \subseteq Rad_g(S_2)$. We conclude that $M =$

$(C \oplus S_1) \oplus (S_2 \oplus G_1)$. Also, We get that $C \oplus S_1 \leq mR$ and $mR \cap (S_2 \oplus G_1) = S_2 \cap (mR + G_1) \subseteq Rad_g(M)$. From $S_2 \oplus G_1 \leq^{\oplus} M$, we deduce that $mR \cap (S_2 \oplus G_1) \subseteq Rad_g(S_2 \oplus G_1)$ by Lemma 1.2.7.

Case (ii) if $G = 0$, so $mR \leq L_2$. Since, L_2 is a principally (P_g^*)-module there exists a submodule $S_1 \leq mR$, $L_2 = S_1 \oplus S_2$ and $mR \cap S_2 \subseteq Rad_g(S_2)$ for some submodule $S_2 \leq L_2$. Therefore, $M = S_1 \oplus (L_1 \oplus S_2)$ and $mR \cap (L_1 \oplus S_2) = mR \cap S_2 \subseteq Rad_g(M)$. Again, by Lemma 1.2.7, we deduce that $mR \cap (L_1 \oplus S_2) \subseteq Rad_g(L_1 \oplus S_2)$, and the proof is now complete. □

Proposition 2.4.20. If M is a P-(P_g^*)-module and T a submodule of M. Then T is principally semisimple, whenever $T \cap Rad_g(M) = 0$.

Proof. Suppose that $a \in T$. Since, M is a P-(P_g^*)-module, then there exists a decomposition $M = M_1 \oplus M_2$ so that $M_1 \leq aR$ and $aR \cap M_2 \subseteq Rad_g(M_2)$, and so $aR \cap M_2 \subseteq Rad_g(M)$. We conclude that $aR = aR \cap (M_1 \oplus M_2) = M_1 \oplus (aR \cap M_2)$. Since, $aR \cap M_2 \subseteq T \cap Rad_g(M) = 0$, we get $aR = M_1$, and $aR \leq^{\oplus} M$. Thus, $aR \leq^{\oplus} T$ and T is principally semisimple. □

Theorem 2.4.21. Let M be a cyclic module over a PID R. If M is a P-(P_g^*)-module, then $M = T \oplus N$, where T is principally semisimple and N a module with $Rad_g(N) \trianglelefteq N$.

Proof. As $Rad_g(M) \leq M$, by lemma 1.2.74 there is a $T \leq M$ with $T \oplus Rad_g(M) \trianglelefteq M$. As $T \cap Rad_g(M) = 0$, Proposition 2.4.20 implies T is principally semisimple. Since, M is a cyclic module over a PID R, so T is a cyclic submodule of M. Since, M has P-(P_g^*) property, there is a decomposition $M = M_1 \oplus M_2$ so that $M_1 \leq T$ and $T \cap M_2 \subseteq Rad_g(M_2)$. Thus $T + M_2 = M$. As $T \cap M_2 \subseteq T \cap Rad_g(M) = 0$, so that $M = T \oplus M_2$. Notice that, $Rad_g(T) \subseteq T \cap Rad_g(M) = Rad_g(M) = Rad_g(T \oplus M_2) = Rad_g(T) \oplus Rad_g(M_2) = Rad_g(M_2)$, implies $T \oplus Rad_g(M_2) \trianglelefteq T \oplus M_2$, and so $Rad_g(M_2) \trianglelefteq M_2$. □

2.5. Connections with P-(P_g^*)-modules

This section will discuss several relations between the principally g-Radical supplemented module concept and other types of modules.

As a special case if we have an indecomposable module, we present some interesting results.

Proposition 2.5.1. Any cyclic indecomposable submodule of a P-(P_g^*)-module M is either contained in $Rad_g(M)$ or a direct summand of M.

Proof. Let M be a P-(P_g^*)-module. If $m \in M$ such that mR is an indecomposable submodule of M, then It follows form Theorem 2.4.6, that $mR = T \oplus S$ where $T \leq^{\oplus} M$ and $S \subseteq Rad_g(M)$. As mR is indecomposable, we get either $mR = S$ or $mR = T$, and this end the proof. □

Proposition 2.5.2. Let M be an indecomposable (non-cyclic) module. Then M is a P-(P_g^*)-module if and only if $Rad_g(M) = M$.

Proof. ⟹) Let $m \in M$. Since M is a P-(P_g^*)-module, then there exists a decomposition $M = M_1 \oplus M_2$, $M_1 \leq mR$ and $mR \cap M_2 \subseteq Rad_g(M_2)$. So, either $M_1 = M$ or $M_1 = 0$. If $M_1 = M$, then $M = mR$, a contradiction. Thus, $M_1 = 0$ and $M_2 = M$. We deduce $m \in mR \subseteq Rad_g(M)$. Thus $Rad_g(M) = M$.
⟸) Lemma 1.2.48 implies M is a (P_g^*)-module, Therefore, is a P-(P_g^*)-module, by Remarks and Examples 2.4.2(2). □

As an application example of Proposition 2.5.2, in fact $\mathbb{Q}$ is a P-(P_g^*)-module as $\mathbb{Z}$-module, and it is indecomposable and non-cyclic and $Rad_g(\mathbb{Q}) = \mathbb{Q}$.

Proposition 2.5.3. Let M be a non-cyclic indecomposable R-module. Then the following are equivalents.

(1) $Rad_g(M) = M$.

(2) M is principally g-lifting.

(3) M is principally generalized hollow.

(4) M is a P-(P_g^*)-module.

(5) M is a principally ⊕-g-supplemented.

Proof. $(\mathbf{1}) \Leftrightarrow (\mathbf{2}) \Leftrightarrow (\mathbf{3}) \Leftrightarrow (\mathbf{5})$ By Lemma 1.2.53.

$(\mathbf{1}) \Leftrightarrow (\mathbf{4})$ By Proposition 2.5.2. □

Lemma 2.5.4. Let M be a module whose any proper submodule is contained in a maximal essential submodule, then $Rad_g(M) \ll_g M$.

Proof. Let $E \trianglelefteq M$ such that $Rad_g(M) + E = M$. If $E \neq M$, by assumption, there exists a maximal essential submodule H of M with $E \leq H$. Thus, $Rad_g(M) + H = M$. Since, $Rad_g(M) \subseteq H$, we get $H = M$, a contradiction with the maximality for H. So, $E = M$ and then $Rad_g(M) \ll_g M$. □

Proposition 2.5.5. Let M be a module and $Rad_g(M) \ll_g M$. If M is P-(P_g^*)-module then M is principally g-lifting.

Proof. Suppose that $m \in M$. Since, M is a P-(P_g^*)-module, then by Theorem 2.4.3, there is a decomposition $M = D \oplus H$ such that $D \leq mR$ and $mR \cap H \subseteq Rad_g(H)$. As $Rad_g(H) \subseteq Rad_g(M)$ and $Rad_g(M) \ll_g M$, then $mR \cap H \ll_g M$. Since, $mR \cap H \leq H$ and H is a direct summand of M, Lemma 1.2.6 implies $mR \cap H \ll_g H$. This completes the proof. □

By using Lemma 2.5.4, Proposition 2.5.5 and Remarks and Examples 2.4.2(1), we have the following.

Corollary 2.5.6. Let M be an R-module whose any proper submodule is contained in a maximal essential submodule. Then M is a P-(P_g^*)-module if and only if it is a principally g-lifting.

The following consequence is satisfying by Remarks and Examples 2.4.2(1) and Proposition 2.5.5.

Corollary 2.5.7. Let M be a module with $Rad_g(M) \ll_g M$. Then M is P-(P_g^*)-module if and only if M is principally g-lifting module.

Corollary 2.5.8. Let M be a finitely generated R-module. Then M is P-(P_g^*)-module if and only if M is principally g-lifting module.

Proof. Since M is finitely generated, hence by Lemma 1.2.12 M has g-small generalized Radical. By Corollary 2.5.7, the result is follow. □

Corollary 2.5.9. Let R be a ring. Then R is a principally g-lifting R-module if and only if R is a P-(P_g^*)-R-module.

Proof. Since $R = \langle 1 \rangle$, then R is finitely generated. From Corollary 2.5.8, the result is followed. □

Corollary 2.5.10. Let M be a module with $Rad_g(M) = 0$. M is a P- (P_g^*)-module if and only if M is a principally g-lifting module.

Proof. As $Rad_g(M) = 0 \ll_g M$. Thus, the result follows by Corollary 2.5.7. □

Proposition 2.5.11. A principally generalized hollow module is a P-(P_g^*)-module.

Proof. Let $m \in M$ and M a principally generalized hollow module. Suppose, that $mR = M$, the proof is clear. Let $mR \subset M$. We have the decomposition $M = 0 \oplus M$, where $0 \leq mR$ and $mR \cap M = mR \ll_g M$, Therefore, $mR \cap M \subseteq Rad_g(M)$ and M is a principally (P_g^*)-module. □

The converse of Proposition 2.5.11 is not true, in general, as see in the following example.

Example 2.5.12. Let $M = \mathbb{Z}_{24}$ as $\mathbb{Z}$-module. Then M is not a principally generalized hollow, since $3\mathbb{Z}_{24}$ is not g-small in $\mathbb{Z}_{24}$. Now, we will prove that

$M = \mathbb{Z}_{24}$ is a P-(P_g^*)-module. Consider noting that any proper submodule E of $M = \mathbb{Z}_{24}$ with $E \neq 3\mathbb{Z}_{24}$ is g-small, so there is a trivial decomposition $M = \bar{0} \oplus M$ such that $\bar{0} \leq E$ and $E \cap M = E \ll_g M$, so $E \cap M \subseteq Rad_g(M)$. Also, if $E = 3\mathbb{Z}_{24}$ or $E = \mathbb{Z}_{24}$ then $E \leq^{\oplus} M$ (Since, $M = 3\mathbb{Z}_{24} \oplus 8\mathbb{Z}_{24}$ and $M = \mathbb{Z}_{24} \oplus \bar{0}$) then the proof is clear. So, M is a principally (P_g^*)-$\mathbb{Z}$-module.

Proposition 2.5.13. Every principally semisimple is a P-(P_g^*)-module.

Proof. Let $m \in M$ and M a principally semisimple. Hence, $M = mR \oplus W$ for some $W \leq M$. Therefore, $mR \leq mR$ and $mR \cap W = 0 \subseteq Rad_g(W)$, thus M is a P-(P_g^*)-module, by Theorem 2.4.3. □

The converse of Proposition 2.5.13 is not true, in general, as in Example 2.5.12 the $\mathbb{Z}$-module $\mathbb{Z}_{24}$ is P-(P_g^*)-module but not principally semisimple, in fact, $2\mathbb{Z}_{24}$ is not direct summand of $\mathbb{Z}_{24}$.

Proposition 2.5.14. Let M be a module with zero generalized Radical. Then any P-(P_g^*)-module is principally semisimple.

Proof. Let $m \in M$ and M a P-(P_g^*)-module. Then there is a decomposition $M = T \oplus W$ such that $T \leq mR$ and $mR \cap W \subseteq Rad_g(W)$. As a result of this, $M = T + W = mR + W$ and $mR \cap W \subseteq Rad_g(M) = 0$. Hence, $mR \leq^{\oplus} M$ and M is principally semisimple. □

The following corollary is direct by Propositions 2.5.13 and 2.5.14.

Corollary 2.5.15. Let M be a module with zero generalized Radical. Then M is a P-(P_g^*)-module if and only if M is principally semisimple.

Theorem 2.5.16. For an R-module M consider the following statements:

(1) M is a P-(P_g^*)-module.

(2) M is PG-Radical supplemented.

Then (1) ⟹ (2). If M is a π-projective module whose every g-Radical supplement submodule of M is a direct summand, then (2) ⟹ (1).

Proof. (**1**) ⟹ (**2**) Obvious by definitions.

(**2**) ⟹ (**1**) Let $x \in M$. According **(2)**, M has a submodule E, $M = xR + E$ and $xR \cap E \subseteq Rad_g(E)$. According to the hypothesis, E is a direct summand of M, and so $M = T \oplus E$ for some $T \leq M$. Since $M = T \oplus E = xR + E$ is π-projective, Lemma 1.2.30 implies $M = \grave{E} \oplus E$ such that $\grave{E} \leq xR$, and thus **(1)** holds. □

Proposition 2.5.17. Let M be a projective module whose every g-Radical supplement submodule is a direct summand of M. If $Rad_g(M) \ll_g M$, then the following are equivalent.

(1) M is principally g-semiperfect.

(2) M is principally g-lifting.

(3) M is a P-(P_g^*)-module.

(4) M is PG-Radical supplemented.

Proof. (**1**) ⟺ (**2**) By Lemma 1.2.43.

(**2**) ⟺ (**3**) By Corollary 2.5.7.

(**3**) ⟺ (**4**) By Theorem 2.5.16. □

The following corollary comes by Lemma 1.2.12 and Proposition 2.5.17.

Corollary 2.5.18. Let M be a finitely generated and projective module whose each g-Radical supplement submodule is a direct summand of M. Then the following are equivalent.

(1) M is principally g-semiperfect

(2) M is principally g-lifting.

(3) M is a principally (P_g^*)-module.

(4) M is PG-Radical supplemented.

Corollary 2.5.19. Let R be a ring whose any g-Radical supplement R-submodule is a direct summand of R. Then the following are equivalent.

(1) R is principally g-semiperfect

(2) R is principally g-lifting.

(3) R is a principally (P_g^*)- R-module.

(4) R is PG-Radical supplemented.

Proof. Follows by Corollary 2.5.18, since $R = \langle 1 \rangle$ is a finitely generated and projective R-module . □

2.6. Rings and localization for P-(P_g^*)-Property

In this section we discuss P-(P_g^*) property as a ring. Also, under localization, we are going to study the behavior of PG-Radical supplemented modules. Before this, we must first prove the following lemma.

Lemma 2.6.1. Let M be a faithful multiplication module over a commutative ring R with identity. Then $Rad_g(M) = Rad_g(R).M$.

Proof. We have,

$Rad_g(M) = \cap_\lambda\{C_\lambda \leq M \mid C_\lambda$ is maximal essential in $M\}$

$= \cap_\lambda\{C_\lambda = E_\lambda M \mid E_\lambda$ is maximal essential in $R\}$, by Lemmas 1.2.3 and 1.2.72,

$= \cap_\lambda\{E_\lambda \mid E_\lambda$ is maximal essential in $R\}.M$, by Lemma 1.2.63.

$= Rad_g(R).M$. □

Theorem 2.6.2. Let M be a faithful, finitely generated and multiplication module over a commutative ring R with identity. Then M have P-(P_g^*) property if and only if R have P-(P_g^*) property.

Proof. Suppose M have P-(P_g^*) property. Let I be a cyclic ideal of R. As M is a finitely generated multiplication R-module, so IM is a cyclic submodule of M. According to the hypothesis, there exists a decomposition $M = E \oplus W$ so that $E \leq IM$ and $IM \cap W \subseteq Rad_g(W)$, so in $Rad_g(M)$. We get that $E = JM$ and $W = EM$ for some ideals J and E of R. By Lemma 2.6.1, $Rad_g(M) = Rad_g(R).M$. As a result of this, $M = JM \oplus EM$ so that $JM \leq IM$ and $IM \cap$

$EM \subseteq Rad_g(R).M$. Thus, $M = (J \oplus E)M$ so that $JM \leq IM$ and $(I \cap E)M \subseteq Rad_g(R).M$, by Lemma 1.2.63. By Lemma 1.2.64 $R = J \oplus E, J \leq I$ and $I \cap E \subseteq Rad_g(R)$. Since $I \cap E \subseteq E \leq^{\oplus} R$ and $I \cap E \subseteq Rad_g(R)$, hence $I \cap E \subseteq Rad_g(E)$, by Lemma 1.2.7. Therefore, R have P-($P_g{}^*$) property.

Conversely, suppose that R have P-($P_g{}^*$) property. Let $C = IM$ be a cyclic submodule of M, for some ideal I of R. Since, M is a finitely generated multiplication R-module, then I is a cyclic ideal of R. According to the hypothesis, there exists a decomposition $R = J \oplus E$ so that $J \leq I$ and $I \cap E \subseteq Rad_g(E)$, also in $Rad_g(R)$. As a result of this, $M = (J \oplus E)M$ so that $JM \leq IM$ and $(I \cap E)M \subseteq Rad_g(R).M$. Thus, $M = JM \oplus EM$ such that $JM \leq C$ and $C \cap EM \subseteq Rad_g(M)$, by Lemma 1.2.63 and Lemma 2.6.1. Since $C \cap EM \subseteq EM \leq^{\oplus} M$ and $C \cap EM \subseteq Rad_g(M)$, hence $C \cap EM \subseteq Rad_g(EM)$, by Lemma 1.2.7. Therefore, M have P-($P_g{}^*$) property. □

Theorem 2.6.3. Let M be an R-module and S a multiplicative closed subset of R such that $\mathcal{L}(T) \cap S = \emptyset$ for any $T \leq M$. Then M is a P-($P_g{}^*$)-R-module if and only if $S^{-1}M$ is a P-($P_g{}^*$)- $S^{-1}R$-module.

Proof. Let M be a P-($P_g{}^*$)- R-module and let $S^{-1}E$ be any cyclic submodule of $S^{-1}M$ as $S^{-1}R$-module. From Lemma 1.2.66, E is cyclic in M as R-module. So there is a decomposition $M = W \oplus \grave{W}$, $W \leq E$ and $E \cap \grave{W} \subseteq Rad_g(\grave{W})$. From Lemmas 1.2.71, 1.2.65 and 2.3.2, $S^{-1}M = S^{-1}W \oplus S^{-1}\grave{W}$, $S^{-1}W \leq S^{-1}E$ and $S^{-1}E \cap S^{-1}\grave{W} \subseteq S^{-1}(Rad_g(\grave{W})) = Rad_g(S^{-1}W)$. Hence, $S^{-1}M$ is a principally ($P_g{}^*$)- $S^{-1}R$-module.

Conversely, suppose that $S^{-1}M$ is a P-($P_g{}^*$)-$S^{-1}R$-module, and let H be a cyclic submodule of M as R-module. By Lemma 1.2.66, $S^{-1}H$ is cyclic in $S^{-1}M$ as $S^{-1}R$-module, so there is a decomposition $S^{-1}M = S^{-1}W \oplus S^{-1}\grave{W}$, $S^{-1}W \leq S^{-1}H$ and $S^{-1}H \cap S^{-1}\grave{W} \subseteq Rad_g(S^{-1}W)$. By Lemmas 1.2.71, 1.2.65 and 2.3.2, we deduce that $M = W\bigoplus\grave{W}$, $W \leq H$ and $H \cap \grave{W} \subseteq Rad_g(\grave{W})$. Hence, M is P-(P_g^*)-R-module. □

Theorem 2.6.4. Let M be an R-module and S a multiplicative closed subset of R such that any proper submodule W of M, $(W:_M s) = W$, for all $s \in S$. Then M is a P-(P_g^*)-R-module if and only if $S^{-1}M$ is a P-(P_g^*)- $S^{-1}R$-module.

Proof. Suppose that M is a P-(P_g^*)- R-module. Let $S^{-1}L$ be a cyclic submodule of $S^{-1}M$ as $S^{-1}R$-module. By Lemma 1.2.68, L is cyclic in M as R-module. Then there is a decomposition $M = T\bigoplus N$, $T \leq L$ and $L \cap N \subseteq Rad_g(N)$. By Lemmas 1.2.71, 1.2.67 and 2.3.5, we get $S^{-1}M = S^{-1}T\bigoplus S^{-1}N$, $S^{-1}T \leq S^{-1}L$ and $S^{-1}L \cap S^{-1}N \subseteq S^{-1}(Rad_g(N)) = Rad_g(S^{-1}N)$. So, $S^{-1}M$ is a P-(P_g^*)- $S^{-1}R$-module.

Conversely, let $S^{-1}M$ be a P-(P_g^*)- $S^{-1}R$-module. Let L be any cyclic in M as R-module. By Lemma 1.2.68, $S^{-1}L$ is cyclic in $S^{-1}M$ as $S^{-1}R$-module. Then there is a decomposition $S^{-1}M = S^{-1}D\bigoplus S^{-1}\grave{D}$ so that $S^{-1}D \leq S^{-1}L$ and $S^{-1}L \cap S^{-1}\grave{D} \subseteq Rad_g(S^{-1}D)$. From Lemmas 1.2.71, 1.2.67 and 2.3.5, we get $M = D\bigoplus\grave{D}$, $D \leq L$ and $L \cap \grave{D} \subseteq Rad_g(\grave{D})$. Thus, M is a P-(P_g^*)- R-module. □

This chapter consists of three sections. In section one, we define and study the idea of ⊕-PG-Radical supplemented modules, properties, characterizations and illustrate examples factors and direct summands of these modules are presented. In section two, we investigate some relations between our definition and other kinds of modules. In section three, we study and investigate the behavior of ⊕-PG-Radical supplemented modules under localization.

3.1. ⊕-PG-Radical supplemented modules

We will present definition of ⊕-PG-Radical supplemented modules. This section will go over some of their properties and provide examples of them.

Now we are going to analyze our major definition as follows:

Definition 3.1.1. An R-module M is called principally ⊕-generalized Radical supplemented, briefly ⊕-PG-Radical supplemented if, for any cyclic submodule E of M, there is a direct summand L of M such that $M = E + L$ and $E \cap L \subseteq Rad_g(L)$.

Remarks and Examples 3.1.2.

(1) It is easy to see that any ⊕-g-Radical supplemented module is ⊕-PG-Radical supplemented. As an application, $\mathbb{Q}$ as $\mathbb{Z}$-module is ⊕-g-Radical supplemented and hence ⊕-PG-Radical supplemented.

(2) If M is a cyclic module over a PID R, then the two notions "⊕-PG-Radical supplemented" and "⊕-g-Radical supplemented modules" are identical.

(3) It is evident to see that any ⊕-PG-Radical supplemented module is PG-Radical supplemented. As an application, $\mathbb{Z}$ as $\mathbb{Z}$-module is not PG-Radical supplemented and hence it is not ⊕-PG-Radical supplemented, see Remarks and Examples 2.1.2(4).

(4) Clearly, any principally (P_g^*)-module and hence any principally g-lifting module is ⊕-PG-Radical supplemented. As an application, the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ for prime number p and $n \in \mathbb{Z}^+$ is principally g-lifting, see [13, Examples 3.2(2)] and so it ⊕-PG-Radical supplemented.

(5) Every principally semisimple is ⊕-PG-Radical supplemented.

Now we will prove analogous characterization for ⊕-PG-Radical supplemented modules.

Proposition 3.1.3. Let M be a ⊕-PG-Radical supplemented R-module and $C \leq M$. If every cyclic submodule of M has a ⊕-g-Radical supplement containing C, then M/C is ⊕-PG-Radical supplemented.

Proof. Let $\bar{m}R$ be a submodule of M/C, such that $\bar{m}R = (mR + C)/C$, where $m \in M$. Then there exists a direct summand E of M with $C \leq E$, $M = mR + E$ and $mR \cap E \subseteq Rad_g(E)$. So $M = E \oplus W$ for some submodule W of M. Let $\pi: E \to E/C$ be a canonical map. It is easy to see that $M/C = E/C \oplus (W + C)/C = E/C + \bar{m}R$. By modular law and Lemma 1.2.8, we conclude that $(E/C) \cap \bar{m}R = E/C \cap (mR + C)/C = (E \cap (mR + C))/C = (C + (mR \cap E))/C = \pi(mR \cap E) \subseteq \pi(Rad_g(E)) \subseteq Rad_g(E/C)$. Hence, E/C is a g-Radical supplement of $\bar{m}R$ of M/C, and hence M/C is ⊕-PG-Radical supplemented. □

Proposition 3.1.4. Let M be a ⊕-PG-Radical supplemented R-module and C a fully invariant submodule of M. So M/C is ⊕-PG-Radical supplemented.

Proof. Let mR/C be any cyclic submodule of M/C, where $m \in M$. Then there is a direct summand E of M such that $M = mR + E$ and $mR \cap E \subseteq Rad_g(E)$. Thus $M = E \oplus W$ for some submodule W of M. By Lemma 1.2.19, we have $M/C = ((C + E)/C) \oplus ((C + W)/C)$. Also, $M/C = (mR/C) \oplus ((C + E)/C)$. Let $\pi: E \longrightarrow (C + E)/C$ be a canonical map. By modular low and Lemma 1.2.8, we have $(mR/C) \cap ((C + E)/C) = (mR \cap (C + E))/C = (C + (mR \cap E))/C = \pi(mR \cap E) \subseteq \pi(Rad_g(E)) \subseteq Rad_g((C + E)/C)$. Therefore, $(C + E)/C$ is a ⊕-

PG-Radical supplement of mR/C in M/C, and then M/C is ⊕-PG-Radical supplemented. □

Corollary 3.1.5. The factor R-module of a ⊕-PG-Radical supplemented duo module is ⊕-PG-Radical supplemented.

Proof. Direct from Proposition 3.1.4.

Corollary 3.1.6. If M is a ⊕-PG-Radical supplemented R-module, then $\mathrm{M}/\mathrm{Rad_g}(\mathrm{M})$ is ⊕-PG-Radical supplemented.

Proof. It follows directly from Lemma 1.2.8(1) and Proposition 3.1.4. □

Previously, we note that ⊕-PG-Radical supplemented property is not inherited by their submodules. Now we will give condition to be inherited.

Theorem 3.1.14. Let M be a ⊕-PG-Radical supplemented R-module, $T \leq M$ and M/T is a projective R-module, then T is ⊕-PG-Radical supplemented.

Proof. Let $a \in T$. There is a direct summand N of M such that $M = aR + N$ and $aR \cap N \subseteq Rad_g(N)$. Thus, $M = T + N$ and so $T \cap N \leq^{\oplus} M$, by Lemma 1.2.29. Since, $M = (T \cap N) \oplus F$ for some $F \leq M$. By modular law, we get that $T = T \cap ((T \cap N) \oplus F) = (T \cap N) \oplus (T \cap F)$, this implies $T \cap N \leq^{\oplus} T$. Also, by modular law, $T = aR + (T \cap N)$, also $aR \cap (T \cap N) = aR \cap N \subseteq Rad_g(M)$. $aR \cap (T \cap N) \subseteq T \cap N$ and $T \cap N \leq^{\oplus} M$ implies $aR \cap (T \cap N) \subseteq Rad_g(T \cap N)$ by Lemma 1.2.7, as required. □

Now, we will study the behavior of ⊕-PG-Radical supplemented summands.

Proposition 3.1.7. Suppose $M = M_1 \oplus M_2$ is a ⊕-PG-Radical supplemented R-module, where M_1 is a fully invariant submodule of M. Then M_1 and M_2 are ⊕-PG-Radical supplemented.

Proof. By Proposition 3.1.4, and since $M/M_1 \cong M_2$, then M_2 is ⊕-PG-Radical supplemented. Let $m_1 \in M_1$. As M is a ⊕-PG-Radical supplemented module, then there exists a submodule E of M such that $M = m_1R + E = E \oplus W$ and $m_1R \cap E \subseteq Rad_g(E)$ for some submodule W of M. As M_1 is a fully invariant submodule of M, we have that $M_1 = (M_1 \cap E) \oplus (M_1 \cap W)$. We deduce that $M_1 = M_1 \cap (m_1R + E) = m_1R + (M_1 \cap E)$. Also, $m_1R \cap (M_1 \cap E) = m_1R \cap E \subseteq Rad_g(M)$, and as $m_1R \cap (M_1 \cap E) \leq M_1 \cap E \leq^{\oplus} M$, Lemma 1.2.7 implies $mR \cap (M_1 \cap E) \subseteq Rad_g(M_1 \cap E)$. Thus, M_1 is ⊕-PG-Radical supplemented. □

Corollary 3.1.8. If M is a weak duo and ⊕-PG-Radical supplemented R-module, then every direct summand of M is also ⊕-PG-Radical supplemented.

Proof. Let E is direct summand of M and M is a ⊕-PG-Radical supplemented module, $M = E \oplus W$ for some $W \leq M$. As M is weak duo, then E is a fully invariant submodule, thus E is a ⊕-PG-Radical supplemented R-module from Proposition 3.1.7. □

Corollary 3.1.9. Let M be a duo ⊕-PG-Radical supplemented module. Then every direct summand of M is ⊕-PG-Radical supplemented.

Proof. Clear. □

The proofs of the three following consequences are exactly analogous to the proofs of Theorem 2.1.5, Propositions 2.1.6 and 2.1.7, respectively.

Theorem 3.1.10. Let M be a weakly distributive ⊕-PG-Radical supplemented R-module, then any direct summand of M is ⊕-PG-Radical supplemented.

Proposition 3.1.11. Let $M = \oplus_{i \in I} M_i$ be an infinite direct sum of ⊕-PG-Radical supplemented R-modules $\{M_i | \; i \in I\}$. If every cyclic submodule of M is fully invariant, then M is ⊕-PG-Radical supplemented.

Proposition 3.1.12. Let M_1 and M_2 be ⊕-PG-Radical supplemented R-modules and let $M = M_1 \oplus M_2$. If any cyclic submodule of M is weak distributive, then M is ⊕-PG-Radical supplemented.

However, we have the following corollary.

Corollary 3.1.13. Let M be an R-module, then

(1) if $M = \oplus_{i \in I} M_i$ is a duo infinite direct sum of R-modules $\{M_i | \ i \in I\}$. Then M is ⊕-PG-Radical supplemented if and only if, for $i \in I$, M_i is ⊕-PG-Radical supplemented, for $i \in I$.

(2) if $M = M_1 \oplus M_2$ is a weakly distributive direct sum of R-modules M_1 and M_2. Then M is ⊕-PG-Radical supplemented if and only if M_1 and M_2 are ⊕-PG-Radical supplemented.

Proof. (1) It follows directly by Corollary 3.1.9 and Proposition 3.1.11.

(2) It follows directly by Theorem 3.1.10 and Proposition 3.1.12. □

Proposition 3.1.14. Let M be a ⊕-PG-Radical supplemented R-module satisfies (D_3), then any direct summand of M is ⊕-PG-Radical supplemented.

Proof. Suppose that $T \leq^{\oplus} M$ and $a \in T$. As M is a ⊕-PG-Radical supplemented module and $a \in M$, $M = aR + N$ and $aR \cap N \subseteq Rad_g(N)$ for some direct summand N of M. By modular law, $T = aR + (T \cap N)$. From $M = T + N$ and by property (D_3), $T \cap N$ is direct summand of M. Since, $aR \cap (T \cap N) = aR \cap N \subseteq Rad_g(M)$ and $T \cap N \leq^{\oplus} M$, we have $aR \cap (T \cap N) \subseteq Rad_g(T \cap N)$, by Lemma 1.2.57. Therefore, T is ⊕-PG-Radical supplemented. □

Corollary 3.1.15. Let M be an R-module has the SIP. Then M is ⊕-PG-Radical supplemented if and only if every direct summand of M is ⊕-PG-Radical supplemented.

Proof. By Proposition 3.1.14 since the SIP implies (D_3) property. □

Corollary 3.1.16. Suppose that M is a quasi-projective an R-module. Then M is ⊕-PG-Radical supplemented if and only if every direct summand of M is ⊕-PG-Radical supplemented.

Proof. By Lemmas 1.2.24 and 1.2.25, M has (D_3). Thus, the result is follows directly by Proposition 3.1.14. □

3.2. Connections with ⊕-PG-Radical supplemented modules

This section will look at several connections between the ⊕-PG-Radical supplemented module concept and various kinds of modules.

Proposition 3.2.1. Let M be a principally ⊕-g-supplemented R-module, then M is ⊕-PG-Radical supplemented. However, the converse is true whenever $Rad_g(M) \ll_g M$.

Proof. ⟹) Suppose that M is a principally ⊕-g-supplemented R-module and $m \in M$. Then there exists a direct summand E of M such that $mR + E = M$ and $mR \cap E \ll_g E$. Thus, $mR \cap E \subseteq Rad_g(E)$. Hence, M is ⊕-PG-Radical supplemented.

⟸) Suppose that M is a ⊕-PG-Radical supplemented R-module and $m \in M$. Then there exists a direct summand E of M such that $mR + E = M$ and $mR \cap E \subseteq Rad_g(E) \subseteq Rad_g(M) \ll_g M$. As $mR \cap E \leq E \leq^{\oplus} M$, it follows that $mR \cap E \ll_g E$, by Lemma 1.2.6. So, M is a principally ⊕-g-supplemented R-module.

Example 3.2.2. As an application of Proposition 3.1.15, we have that $\mathbb{Q}\oplus\mathbb{Z}_2$ principally ⊕-g-supplemented as $\mathbb{Z}$-module, see [32, Examples 3.1.6(2)]. Hence, it is will be ⊕-PG-Radical supplemented,

Corollary 3.2.3. Let M be a finitely generated R-module. Then M is ⊕-PG-Radical supplemented if and only if M is principally ⊕-g-supplemented.

Proof. Since M is a finitely generated R-module, Lemma 1.2.12 implies that $Rad_g(M) \ll_g M$. Hence the result is obtained by Proposition 3.2.1. □

Corollary 3.2.4. Let M be a noetherian R-module. Then M is ⊕-PG-Radical supplemented if and only if M is principally ⊕-g-supplemented.

Proof. Since any noetherian R-module is finitely generated, Corollary 3.2.3 implies the result. □

Corollary 3.2.5. Let R be any ring. Then R is ⊕-PG-Radical supplemented if and only if R is principally ⊕-g-supplemented.

Proof. Since $R = \langle 1 \rangle$ is finitely generated, then the result is follows by Corollary 3.2.3. □

Proposition 3.2.6. Let M be a distributive R-module. Consider the following conditions:

(1) M is ⊕-PG-Radical-supplemented.

(2) $M/Rad_g(M)$ is principally semisimple.

Then $(1) \Longrightarrow (2)$. If M is a refinable R-module with $Rad_g(M) \ll_g M$, then $(2) \Longrightarrow (1)$.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ By Remarks and Examples 3.1.2(3) and Proposition 2.1.12.

$(\mathbf{2}) \Longrightarrow (\mathbf{1})$ By Lemma 1.2.57, M is a principally ⊕-g-supplemented R-module and so it is ⊕-PG-Radical supplemented, by Proposition 3.2.1. □

Proposition 3.2.7. A π-projective R-module M is ⊕-PG-Radical supplemented if and only if M has the property P-($P_g{}^*$).

Proof. ⟹) Suppose that $m \in M$. Then $M = mR + C$ and $mR \cap C \subseteq Rad_g(C)$ for a direct summand C of M, since M is ⊕-PG-Radical supplemented. From π-projectivity for M, there exists $W \leq mR$ such that $M = W \oplus C$, from Lemma 1.2.30. Thus, for $m \in M$, there exists a decomposition $M = W \oplus C$ such that $W \leq mR$ and $mR \cap C \subseteq Rad_g(C)$. This implies M has the property P-($P_g{}^*$).

⟸) By Remarks and Examples 3.1.2(4). □

Corollary 3.2.8. Let M be a π-projective R-module. Then M is ⊕-PG-Radical supplemented if and only if every direct summand of M is ⊕-PG-Radical supplemented.

Proof. Suppose M is a ⊕-PG-Radical supplemented R-module. By Proposition 3.2.7, M is a P-(P_g^*)-module. From Proposition 2.4.14, we have that every direct summand of M is a principally (P_g^*)-module, so it is ⊕-PG-Radical supplemented. The converse is clear. □

Corollary 3.2.9. Suppose that M is an R-module with $Rad_g(M) = 0$, then M is ⊕-PG-Radical supplemented if and only if M is principally ⊕-g-supplemented.
Proof. Clear by Proposition 3.2.1. □

The Corollaries 3.2.10 and 3.2.11 come directly from Corollary 3.2.9.

Corollary 3.2.10. Let M be a module over a g-V-ring R. Then M is ⊕-PG-Radical supplemented if and only if M is principally ⊕-g-supplemented.

Corollary 3.2.11. Suppose that M is an g-noncosingular R-module. Then M is ⊕-PG-Radical supplemented if and only if M is principally ⊕-g-supplemented.

Proposition 3.2.12. If M is an R-module such that $Rad_g(M) = M$, then M is ⊕-PG-Radical supplemented.
Proof. If $m \in M$, then $mR \subseteq Rad_g(M)$, by hypothesis. Thus, $mR + M = M$ and $mR \cap M = mR \subseteq Rad_g(M)$. Hence, M is ⊕-PG-Radical supplemented. □

Proposition 3.2.13. Let M be a non-cyclic indecomposable R-module. If M is a ⊕-PG-Radical supplemented R-module then $Rad_g(M) = M$.
Proof. Suppose that M is a ⊕-PG-Radical supplemented R-module and $m \in M$. Then there is a direct summand E of M such that $M = mR + E = E \oplus W$ and $mR \cap E \subseteq Rad_g(E)$ for some $W \leq M$. Hence, either $M = E$ or $E = 0$. If,

$E = 0$, then $M = mR$, which is a contradiction. Thus, $M = E$. We deduce that $m \in mR \subseteq Rad_g(M)$. Hence $Rad_g(M) = M$. □

Corollary 3.2.14. Let M be a non-cyclic indecomposable R-module. Then M is ⊕-PG-Radical supplemented if and only if $Rad_g(M) = M$.

Proof. By Propositions 3.2.12 and 3.2.13. □

In the following theorem we present our main result.

Theorem 3.2.15. Consider that the following conditions for an R-module M:

(1) M is a P-(P_g^*)-module.

(2) Every direct summand of M is ⊕-PG-Radical supplemented.

(3) M is ⊕-PG-Radical supplemented.

(4) M is PG-Radical supplemented.

Then (1) ⟹ (2) ⟹ (3) ⟹ (4). If M is a projective R-module whose any g-Radical supplement submodule of M is a direct summand, then (4) ⟹ (1).

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Let E be any direct summand of M and $n \in E$. Since, M has the property P-(P_g^*), then by Theorem 2.4.3, there exists a decomposition $M = L \oplus \grave{L}$ such that $L \leq nR$ and $nR \cap \grave{L} \subseteq Rad_g(\grave{L})$. By modular law, we can write $E = L \oplus (E \cap \grave{L})$. This implies $E \cap \grave{L}$ is a direct summand of E. It follows that, $E = nR + (E \cap \grave{L})$. Also, we will prove that $nR \cap (E \cap \grave{L}) = nR \cap \grave{L} \subseteq Rad_g(E \cap \grave{L})$. Suppose that $x \in nR \cap \grave{L}$, so $x \in Rad_g(\grave{L})$, since $xR \ll_g \grave{L}$ (also $xR \ll_g M$) by Lemma 1.2.9. As $xR \leq E \cap \grave{L} \leq^{\oplus} E \leq^{\oplus} M$, Lemma 1.2.6 implies $xR \ll_g E \cap \grave{L}$, therefore $x \in xR \subseteq Rad_g(E \cap \grave{L})$. That is $E \cap \grave{L} \leq^{\oplus} E$ and it is a g-Radical supplement for nR in E. Hence, E is ⊕-PG-Radical supplemented.

$(\mathbf{2}) \Longrightarrow (\mathbf{3})$ Clear.

$(\mathbf{3}) \Longrightarrow (\mathbf{4})$ By Remarks and Examples 3.1.2(3).

$(\mathbf{4}) \Longrightarrow (\mathbf{1})$ by Theorem 2.5.17. □

Proposition 3.2.16. If M is a refinable R-module, consider the following cases:

(1) M is a P-(P_g^*)-module.

(2) M is ⊕-PG-Radical supplemented.

(3) M is PG-Radical supplemented.

Then (1) ⇒ (2) ⇔ (3). If M satisfies (D_3) property, then (2) ⇒ (1).

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2}) \Rightarrow (\mathbf{3})$ Follows by Remarks and Examples 3.1.2(3 and 4).

$(\mathbf{3}) \Rightarrow (\mathbf{2})$ Let $m \in M$. Then there exists a submodule E of M such that $E + mR = M$ and $E \cap mR \subseteq Rad_g(E)$. As M is refinable, so there is a direct summand U of M with $U \subseteq E$ and $M = U + mR = U \oplus \grave{U}$ for a submodule $\grave{U}$ of M. To conform that $U \cap mR \subseteq Rad_g(U)$, we deduce $U \cap mR \subseteq Rad_g(M)$. By $U \cap mR \leq U$ and $U \leq^{\oplus} M$, Lemma 1.2.7 imply $U \cap mR \subseteq Rad_g(U)$. Hence, M is ⊕-PG-Radical supplemented.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$ If $m \in M$, then there exists a direct summand E of M such that $E + mR = M$ and $E \cap mR \subseteq Rad_g(E)$. Since M is refinable, then there is $U_1 \leq^{\oplus} M$ such that $U_1 \subseteq E$ and $M = U_1 + mR = U_1 \oplus \grave{U}_1$ for some $\grave{U}_1 \leq M$. As $U_1 \cap mR \subseteq Rad_g(M)$ and $U_1 \leq^{\oplus} M$, Lemma 1.2.7 implies $U_1 \cap mR \subseteq Rad_g(U_1)$. Again, by hypothesis, there exists $U_2 \leq^{\oplus} M$ such that $U_2 \subseteq mR$ and $M = U_1 + U_2 = U_2 \oplus \grave{U}_2$. By property (D_3), $U_1 \cap U_2 \leq^{\oplus} M$, and $M = (U_1 \cap U_2) \oplus W$ for some submodule W of M. By modular law, $U_1 = (U_1 \cap U_2) \oplus (U_1 \cap W)$ and so $M = U_1 + U_2 = U_2 \oplus (U_1 \cap W)$. It is easily to see that $mR \cap (U_1 \cap W) \subseteq Rad_g(U_1)$. Since $mR \cap (U_1 \cap W) \leq U_1 \cap W$ and $U_1 \cap W \leq^{\oplus} U_1$, we get that $mR \cap (U_1 \cap W) \subseteq Rad_g(U_1 \cap W)$ by Lemma 1.2.7, and the proof now is complete. □

Corollary 3.2.17. Let M be a refinable R-module has SIP. Then the following are equivalent.

(1) M is a P-(P_g^*)-module.

(2) M is ⊕-PG-Radical supplemented.

(3) M is PG-Radical supplemented.

Proof. By Proposition 3.2.16, since the SIP implies (D_3) property. □

Proposition 3.2.18. The following are equivalent for a nonzero indecomposable R-module M with $Rad_g(M) \ll_g M$.

(1) M is principally generalized hollow.

(2) M is ⊕-PG-Radical supplemented.

(3) Every direct summand of M is ⊕-PG-Radical supplemented.

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$ Obvious.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$ Let M be an indecomposable module and let $W \leq^{\oplus} M$, so either $W = M$ or $W = 0$. By **(2),** then the result is follows.

$(\mathbf{3}) \Rightarrow (\mathbf{1})$ Suppose that mR is any proper cyclic submodule of M and $m \in M$. Since M is a ⊕-PG-Radical supplemented module, then there exist $T \leq N$ of M such that $mR \cap T \subseteq Rad_g(T)$ and $M = T \oplus N = mR + T$. If $T = 0$ then $mR = M$, which is a contradiction. Thus, $T = M$ and $N = 0$. So, $mR = mR \cap T \subseteq Rad_g(M)$, and then $mR \ll_g M$. Hence, M is principally generalized hollow. □

Corollary 3.2.19. The following cases are equivalent for a finitely generated indecomposable R-module M.

(1) M is principally generalized hollow.

(2) M is ⊕-PG-Radical supplemented.

(3) Every direct summand of M is ⊕-PG-Radical supplemented.

Proof. From Lemma 1.2.12 and Proposition 3.2.18. □

Corollary 3.2.20. The following are equivalent for an indecomposable ring R.

(1) R is principally generalized hollow.

(2) R is ⊕-PG-Radical supplemented.

(3) Every direct summand of R is ⊕-PG-Radical supplemented.

Proof. By Corollary 3.2.19, Since $R = \langle 1 \rangle$ is finitely generated. □

In the following, we will investigate some terms that make the ⊕-PG-Radical supplemented modules to be P-(P_g^*)-modules.

Theorem 3.2.21. Let M be a ⊕-PG-Radical supplemented R-module and satisfy any one of the following:

(1) M is weakly distributive.

(2) M is duo.

(3) M is refinable and have the SIP.

Then M is a P-(P_g^*)-module.

Proof. (1) Let $m \in M$. As M is a ⊕-PG-Radical supplemented R-module, then $M = mR + C$ and $mR \cap C \subseteq Rad_g(C)$ for some a $C \leq^{\oplus} M$. Hence, $M = C \oplus W$ for some $W \leq M$. Since mR is weak distributive in M, then $mR = (mR \cap C) + (mR \cap W)$, also $(mR \cap C) \cap (mR \cap W) \subseteq C \cap W = 0$, this lead as to $mR = (mR \cap C) \oplus (mR \cap W)$ and hence $M = (mR \cap W) \oplus C$ where $mR \cap W \leq mR$ and $mR \cap C \subseteq Rad_g(C)$. So, M is a P-(P_g^*)-module.

(2) Similar to proof **(1)**.

(3) Suppose that M is a ⊕-PG-Radical supplemented R-module and $m \in M$, then $M = mR + C$ and $mR \cap C \subseteq Rad_g(C)$ for some $C \leq^{\oplus} M$. As M is refinable, then there is $W \leq^{\oplus} M$ such that $W \leq mR$ and $M = W + C$. Thus $C \cap W \leq^{\oplus} M$, Since M have the SIP. Thus $M = (C \cap W) \oplus E$ for some $E \leq M$. By modular law, we conclude $C = (C \cap W) \oplus (C \cap E)$, and so $M = W + C = W \oplus (C \cap E)$. Clearly, $mR \cap (C \cap E) \subseteq Rad_g(C \cap E)$ and this end of proof. □

Proposition 3.2.22. Let M be a nonzero indecomposable R-module such that $Rad_g(M) \ll_g M$. Then the following are equivalent.

(1) M is principally ⊕-supplemented.

(2) M is principally ⊕-δ-supplemented.

(3) M is principally ⊕-g-supplemented.

(4) M is ⊕-PG-Radical supplemented.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2}) \Longrightarrow (\mathbf{3})$ Evident.

$(\mathbf{3}) \Longrightarrow (\mathbf{4})$ By Proposition 3.2.1.

$(\mathbf{4}) \Longrightarrow (\mathbf{1})$ Let $m \in M$. Then there is $E \leq^{\oplus} M$ such that $mR + E = M$ and $mR \cap E \subseteq Rad_g(E)$. If $mR \cap E = M$, so $mR = M$ has trivially a principally ⊕-supplement 0 in M. Let $mR \cap E \subset M$. Since, $mR \cap E \subseteq Rad_g(M)$ and $Rad_g(M) \ll_g M$, we get that $mR \cap E \ll_g M$. As M is an indecomposable R-module, Lemma 1.2.11 implies $mR \cap E \ll M$. Thus, $mR \cap E \ll E$ by Lemma 1.2.1(4), and hence **(1)**, holds. □

Corollary 3.2.23. The following cases are equivalent for an indecomposable R-module $M \neq 0$ with $Rad_g(M) \ll_g M$.

(1) M is principally g-lifting.

(2) M is a P-(P_g^*)-module.

(3) M is principally ⊕-supplemented.

(4) M is principally ⊕-δ-supplemented.

(5) M is principally ⊕-g-supplemented.

(6) M is ⊕-PG-Radical supplemented.

(7) Every direct summand of M is principally ⊕-g-supplemented.

(8) Every direct summand of M is ⊕-PG-Radical supplemented.

(9) M is principally generalized hollow.

Proof. (1) ⟺ (2) By Corollary 2.5.8.

(3) ⟺ (4) ⟺ (5) ⟺ (6) By Proposition 3.2.22.

(7) ⟺ (8) by the same way of proof Proposition 3.2.1

$(\mathbf{6}) \Leftrightarrow (\mathbf{8}) \Leftrightarrow (\mathbf{9})$ By Proposition 3.2.18.

$(\mathbf{1}) \Leftrightarrow (\mathbf{9})$ By Lemma 1.2.42. □

Corollary 3.2.24. The following are equivalent for an indecomposable finitely generated R-module M.

(1) M is principally g-lifting.

(2) M is a P-(P_g^*)-module.

(3) M is principally ⊕-supplemented.

(4) M is principally ⊕-δ-supplemented.

(5) M is principally ⊕-g-supplemented.

(6) M is ⊕-PG-Radical supplemented.

(7) Every direct summand of M is principally ⊕-g-supplemented.

(8) Every direct summand of M is ⊕-PG-Radical supplemented.

(9) M is principally generalized hollow.

Proof. From Lemma 1.2.12 and Proposition 3.2.23. □

Corollary 3.2.25. The following are equivalent for a uniform R-module M with $Rad_g(M) \ll_g M$.

(1) M is principally g-lifting.

(2) M is a P-(P_g^*)-module.

(3) M is principally ⊕-supplemented.

(4) M is principally ⊕-δ-supplemented.

(5) M is principally ⊕-g-supplemented.

(6) M is ⊕-PG-Radical supplemented.

(7) Every direct summand of M is principally ⊕-g-supplemented.

(8) Every direct summand of M is ⊕-PG-Radical supplemented.

(9) M is principally generalized hollow.

Proof. Since, every uniform R-module is indecomposable, Corollary 3.2.24 implies the result. □

Corollary 3.2.26. The following are equivalent for an indecomposable ring R.

(1) R is principally g-lifting.

(2) R is a P-(P_g^*)-R-module.

(3) R is principally ⊕-supplemented.

(4) R is principally ⊕-δ-supplemented.

(5) R is principally ⊕-g-supplemented.

(6) R is ⊕-PG-Radical supplemented.

(7) Every direct summand of R is principally ⊕-g-supplemented.

(8) Every direct summand of R is ⊕-PG-Radical supplemented.

(9) R is principally generalized hollow.

Proof. It is evidently by Corollary 3.2.24, as $R = \langle 1 \rangle$ is finitely generated. □

Proposition 3.2.27. Let M be an R-module, consider the following:

(1) M is principally semisimple.

(2) M is a P-(P_g^*)-module.

(3) Each direct summand of M is ⊕-PG-Radical supplemented.

(4) M is ⊕-PG-Radical supplemented.

(5) M is PG-Radical supplemented.

Then $(\mathbf{1}) \Longrightarrow (\mathbf{2}) \Longrightarrow (\mathbf{3}) \Longrightarrow (\mathbf{4}) \Longrightarrow (\mathbf{5})$. If $Rad_g(M) = 0$, then $(\mathbf{5}) \Longrightarrow (\mathbf{1})$.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ and $(\mathbf{3}) \Longrightarrow (\mathbf{4}) \Longrightarrow (\mathbf{5})$ Clear.

$(\mathbf{2}) \Longrightarrow (\mathbf{3})$ By Proposition 2.4.13, any direct summand of M is also a P-(P_g^*)-module, hence any direct summand of M is ⊕-PG-Radical supplemented, according to Remarks and Examples 3.1.2(4).

$(\mathbf{5}) \Longrightarrow (\mathbf{1})$ By Proposition 2.2.4. □

3.3. Rings and localization for ⊕-PG-Radical supplemented

In this part, we will study the ⊕-PG-Radical supplemented property as a ring. Also, we discuss the behavior of ⊕-PG-Radical supplemented modules under localization.

Theorem 3.3.1. Let M be a faithful, finitely generated and multiplication module over a commutative ring R with $1 \neq 0$. Then M is ⊕-PG-Radical supplemented if and only if R is ⊕-PG-Radical supplemented.

Proof. Suppose that M is a ⊕-PG-Radical supplemented R-module. Let I be any cyclic ideal of R. Since M is a finitely generated multiplication R-module, then IM is a cyclic submodule of M. By hypothesis, there exists a direct summand W of M such that $M = IM + W$ and $IM \cap W \subseteq Rad_g(W)$, so in $Rad_g(M)$. We have $W = EM$ for some ideal E of R. By Lemma 2.6.1, $Rad_g(M) = Rad_g(R).M$. It follows that $E \leq^{\oplus} R$ such that $R = I + E$ and $I \cap E \subseteq Rad_g(R)$, by Lemma 1.2.63 and 1.2.64. Since, $I \cap E \subseteq E \leq^{\oplus} R$ and $I \cap E \subseteq Rad_g(R)$, then $I \cap E \subseteq Rad_g(E)$, by Lemma 1.2.7. Therefore, R is ⊕-PG-Radical supplemented.

Conversely, suppose that R is ⊕-PG-Radical supplemented. Let $C = IM$ be a cyclic submodule of M, for some ideal I of R. As M is a finitely generated multiplication R-module, then I is a cyclic ideal of R. According to the hypothesis, there exists a direct summand E of R such that $R = I + E$ and $I \cap E \subseteq Rad_g(E)$, also in $Rad_g(R)$. Since, $EM \leq^{\oplus} M$ such that $M = C + EM$ and $(I \cap E)M \subseteq Rad_g(R).M$, such that $C \cap EM \subseteq Rad_g(M)$ by Lemma 1.2.63 and Lemma 2.6.1. As $C \cap EM \subseteq EM \leq^{\oplus} M$ and $C \cap EM \subseteq Rad_g(M)$, then $C \cap EM \subseteq Rad_g(EM)$, from Lemma 1.2.7. Therefore, M is ⊕-PG-Radical supplemented. □

Theorem 3.3.2. Let M be an R-module and S a multiplicative closed subset of R with $\mathcal{L}(T) \cap S = \emptyset$ for all $T \leq M$. Then M is ⊕-PG-Radical supplemented as R-module if and only if $S^{-1}M$ is ⊕-PG-Radical supplemented as $S^{-1}R$-module.

Proof. Let M be ⊕-PG-Radical supplemented as R-module and let $S^{-1}E$ be any cyclic submodule of $S^{-1}M$ as $S^{-1}R$-module. By Lemma 1.2.66, E is cyclic in M as R-module. Then there exists a submodule W of M such that $E + W = C \oplus W = M$ and $E \cap W \subseteq Rad_g(W)$. By Lemmas 1.2.71, 1.2.65, and 2.3.2, we deduce that $S^{-1}E + S^{-1}W = S^{-1}C \oplus S^{-1}W = S^{-1}M$ and $S^{-1}E \cap S^{-1}W \subseteq Rad_g(S^{-1}W)$. Hence, $S^{-1}M$ is ⊕-PG-Radical supplemented as $S^{-1}R$-module.

Conversely, suppose that $S^{-1}M$ is ⊕-PG-Radical supplemented as $S^{-1}R$-module, and let E be a cyclic submodule of M as R-module. By lemma

1.2.66, $S^{-1}E$ is cyclic in $S^{-1}M$ as $S^{-1}R$-module. Then there is a submodule $S^{-1}W$ of $S^{-1}M$ such that $S^{-1}E + S^{-1}W = S^{-1}H \oplus S^{-1}W = S^{-1}M$ and so $S^{-1}E \cap S^{-1}W \subseteq Rad_g(S^{-1}W)$. From Lemmas 1.2.71, 1.2.65, and 2.3.2, we have $E + W = H \oplus W = M$ and $E \cap W \subseteq Rad_g(W)$. Implies M is ⊕-PG-Radical supplemented as R-module. □

Theorem 3.3.3. Let M be an R-module and let S be a multiplicative closed subset of R such that for any $W \subset M$, $(W:_M s) = W$, for all $s \in S$. Then M is a ⊕-PG-Radical supplemented as R-module if and only if $S^{-1}M$ is ⊕-PG-Radical supplemented as $S^{-1}R$-module.

Proof. Suppose that M is ⊕-PG-Radical supplemented as R-module. Let $S^{-1}E$ be cyclic in $S^{-1}M$ as $S^{-1}R$-module. From Lemma 1.2.68, E is cyclic in M as R-module. Then there exists a $W \leq M$, $E + W = C \oplus W = M$ and $E \cap W \subseteq Rad_g(W)$. From Lemmas 1.2.71, 1.2.67 and 2.3.2, we get that $S^{-1}E + S^{-1}W = S^{-1}C \oplus S^{-1}W = S^{-1}M$ and $S^{-1}E \cap S^{-1}W \subseteq Rad_g(S^{-1}W)$. Therefore, $S^{-1}M$ is ⊕-PG-Radical supplemented as $S^{-1}R$-module.

Conversely, let $S^{-1}M$ be ⊕-PG-Radical supplemented as $S^{-1}R$-module. If C is a cyclic submodule of M as R-module. By Lemma 1.2.68, $S^{-1}C$ is cyclic in $S^{-1}M$ as $S^{-1}R$-module. Then there is a $S^{-1}H \leq S^{-1}M$, $S^{-1}C + S^{-1}H = S^{-1}D \oplus S^{-1}H = S^{-1}M$ and $S^{-1}C \cap S^{-1}H \subseteq Rad_g(S^{-1}H)$. From Lemmas 1.2.71, 1.2.67 and 2.3.5, we get $C + H = D \oplus H = M$ and $C \cap H \ll_g Rad_g(H)$. Hence, M is ⊕-PG-Radical supplemented as R-module. □

Diagrams of Some Important Implications and Their Reverses for Chapters Two & Three

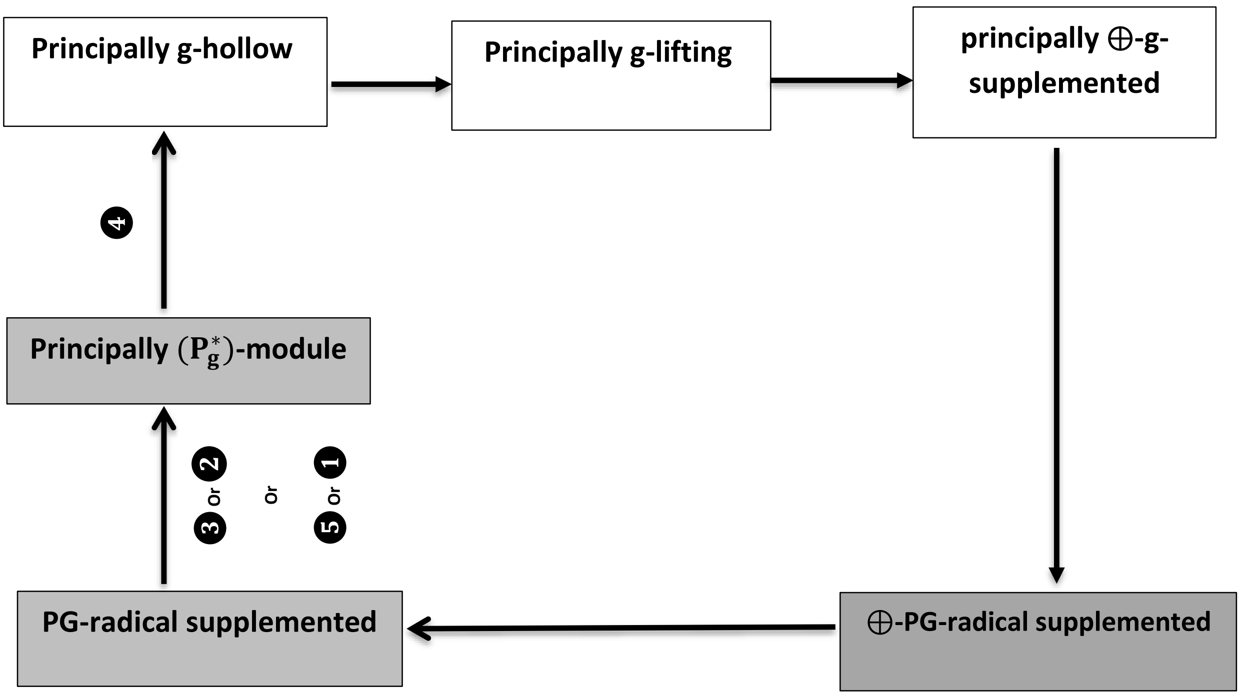


Diagram (1)

Symbol	Condition	Related results
❶	If M is a refinable module & has SIP	Corollary 3.2.18
❷	If M is a projective module & every g-radical supplement submodule is a direct summand of M. & $Rad_g(M) \ll_g M$	Proposition 2.5.17 & 2.5.16
❸	If M is a π-projective module & any g-radical supplement submodule of M is direct summand	Theorem 2.5.16
❹	If M is a non-cyclic indecomposable module	Proposition 2.3.3
❺	If M is a refinable module & satisfies (D_3) property	Proposition 3.2.17

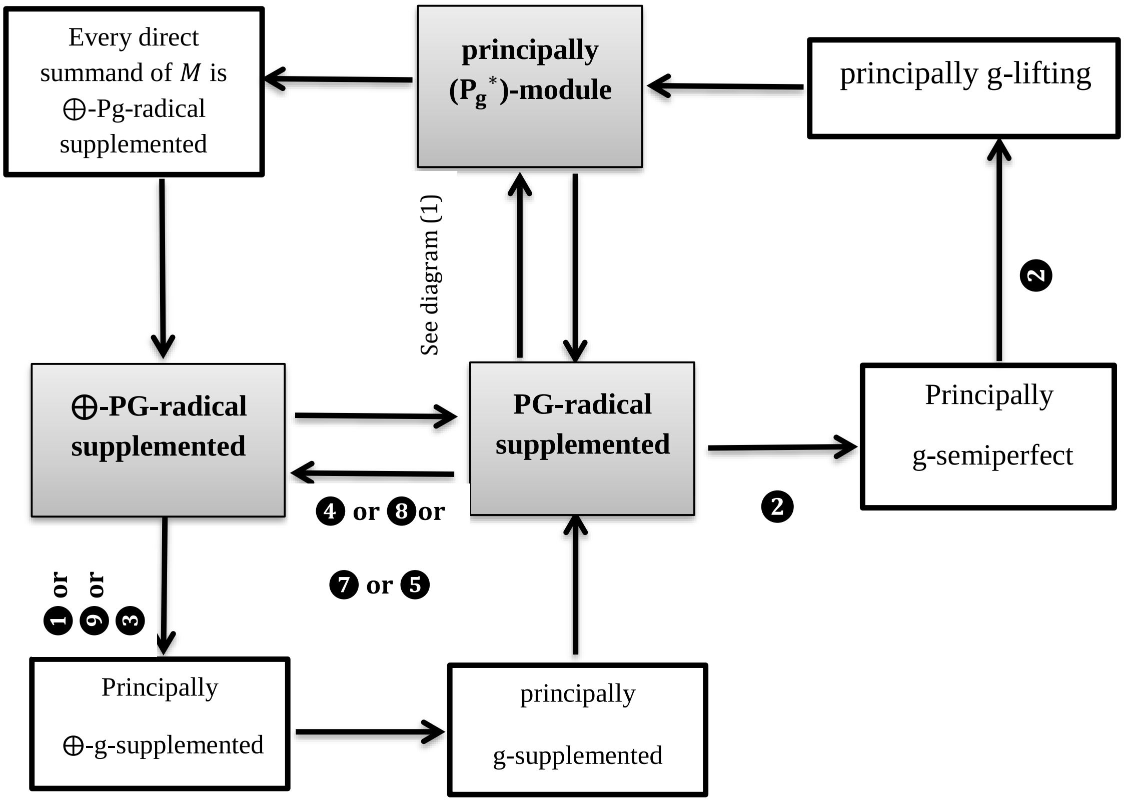


Diagram (2)

Symbol	Condition	Related results
❶	If $Rad_g(M) \ll_g M$.	Proposition 3.2.1
❷	If M is a projective module & any g-radical supplement submodule is direct summand of M & $Rad_g(M) \ll_g M$	Proposition 2.5.17
❸	If $Rad_g(M) = 0$ or, R is g-V-ring or, M is e-noncosingular	Corollaries 3.2.9 & 3.2.10 & 3.2.11
❹	If M is a projective module & any g-radical supplement submodule of M is direct summand	Theorem 3.2.16
❺	If $Rad_g(M) = 0$	Proposition 3.2.28
❼	If M is refinable & has SIP	Corollary 3.2.18
❽	If M is refinable	Proposition 3.2.17
❾	If M be a finitely generated or noetherian R-module.	Corollaries 3.2.3 & 3.2.4

This chapter consists of six sections. In sections one and four, we define and investigate two concepts, Rad_g-lifting modules and P-Rad_g-lifting modules, respectively. Properties, characterizations and illustrate examples factors and direct summands of these modules are presented. In sections two and five, we discussed some relations between our definitions and other types of modules. In sections three and six, we study and investigate the behavior of Rad_g-lifting modules and principally Rad_g-lifting under localization and rings.

4.1. Rad_g-lifting modules

We will present the definition of Rad_g-lifting modules, as well as a number of their properties and examples about them will be discussed in this section.

Definition 4.1.1. An R-module M is called Rad_g-lifting if, every submodule E of M with $Rad_g(M) \subseteq E$, there is a decomposition $M = T \oplus N$ such that $T \leq E$ and $E \cap N \ll_g N$. Thus, a ring R is also called Rad_g-lifting if it is Rad_g-lifting as R-module.

Proposition 4.1.2. Let M be an arbitrary module such that $Rad_g(M) = M$, then M is a Rad_g-lifting module.

Proof. Let E be any submodule of M such that $Rad_g(M) \subseteq E$. By hypothesis, we have that $M \leq E$ this implies $M = E$. Trivially, $M = M \oplus (0)$ where $M \leq E$ and $E \cap (0) = (0) \ll_g (0)$, as required. □

Corollary 4.1.3. If M is an arbitrary module containing no maximal submodule, then M is Rad_g-lifting.

Proof. Since, M containing no maximal, so $Rad(M) = M$. As $Rad(M) \subseteq Rad_g(M)$, so we get $Rad_g(M) = M$. By Proposition 4.1.2, the result is follows. □

Corollary 4.1.4. Let M be a module such that all its finitely generated submodules are g-small, then M is $\mathrm{Rad_g}$-lifting.

Proof. By Proposition 1.2.13 we have $Rad_g(M) = M$, hence by Proposition 4.1.2, M is $\mathrm{Rad_g}$-lifting. □

As an application of above corollary, for a prime number p and $n \in \mathbb{Z}^+$, the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ is $\mathrm{Rad_g}$-lifting, in fact, all its finitely generated submodules are g-small.

Corollary 4.1.5. Let M be any module. Then, $P_g(M)$ is $\mathrm{Rad_g}$-lifting.

Proof. It is enough to prove that, $Rad_g(P_g(M)) = P_g(M)$.

We have, $P_g(M) = \sum\{L \leq M \big| Rad_g(L) = L\} = \sum_{L\leq M} Rad_g(L)$

$$= Rad_g(\sum\{L \leq M \big| Rad_g(L) = L\}) = Rad_g(P_g(M))$$

By Proposition 4.1.2, $P_g(M)$ is a $\mathrm{Rad_g}$-lifting module. □

Remarks and Examples 4.1.6.

(1) By the definitions, every g-lifting module and hence every lifting module is $\mathrm{Rad_g}$-lifting. The converse is not true, in general, for instance: the $\mathbb{Z}$-module $\mathbb{Q}$ is $\mathrm{Rad_g}$-lifting, in fact, $Rad_g(\mathbb{Q}) = \mathbb{Q}$ (see Proposition 4.1.2), while $\mathbb{Z}$-module $\mathbb{Q}$ not g-lifting.

(2) It is clear that every semisimple module is $\mathrm{Rad_g}$-lifting. But the converse need not be true, in general, for instance: It's easy to see that $\mathbb{Z}_8$ as $\mathbb{Z}$-module is hollow, so it is lifting hence by **(1)** is $\mathrm{Rad_g}$-lifting. While the $\mathbb{Z}$-module $\mathbb{Z}_8$ not semisimple.

Proposition 4.1.7. The following are equivalent for a module M.

(1) M is a $\mathrm{Rad_g}$-lifting module.

(2) for every submodule E of M with $Rad_g(M) \subseteq E$, there is a decomposition $M = T \oplus N$ such that $T \leq E$ and $E \cap N \ll_g M$;

(3) for every submodule E of M with $Rad_g(M) \subseteq E$ can be written as $E = H \oplus G$, where H is a direct summand of M and $G \ll_g M$;

(4) for every submodule E of M with $Rad_g(M) \subseteq E$, there exists a direct summand C of M such that $C \leq E$ and $E/C \ll_g M/C$;

(5) for every submodule E of M such that $Rad_g(M) \subseteq E$, and E containing a g-supplement C in M such that $E \cap C$ is a direct summand of E;

(6) for every submodule E of M with $Rad_g(M) \subseteq E$, there is an $e = e^2 \in End(M)$ with $eM \leq E$ and $(1-e)E \ll_g (1-e)M$;

(7) for every submodule E of M with $Rad_g(M) \subseteq E$, there exists a direct summand T of M and $W \ll_g M$ such that $T \leq E$ and $E = T + W$;

(8) for every submodule E of M with $Rad_g(M) \subseteq E$, there is a submodule H of M inside E such that $M = H \oplus W$ and W a g-supplement of E in M.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Clear.

$(\mathbf{2}) \Longrightarrow (\mathbf{1})$ Let $E \leq M$ with $Rad_g(M) \subseteq E$. By **(2)**, there is a decomposition $M = T \oplus N$ such that $T \leq E$ and $E \cap N \ll_g M$. As $E \cap N \leq N \leq^{\oplus} M$, Lemma 1.2.17(1) implies $E \cap N \ll_g N$. Hence, M is $\mathrm{Rad_g}$-lifting.

$(\mathbf{1}) \Longrightarrow (\mathbf{3})$ Suppose that $E \leq M$ such that $Rad_g(M) \subseteq E$. By (1), there exists a decomposition $M = H \oplus S$ such that $H \leq E$ and $E \cap S \ll_g S$. By modular law, $E = H \oplus (E \cap S)$. Putting $E \cap S = G$. Thus $E = H \oplus G$ such that $H \leq^{\oplus} M$ and $G \ll_g M$.

$(\mathbf{3}) \Longrightarrow (\mathbf{4})$ Suppose $E \leq M$ with $Rad_g(M) \subseteq E$. We have a decomposition $E = C \oplus W$ such that $C \leq^{\oplus} M$ and $W \ll_g M$, by **(3)**. Suppose a natural map $\pi: M \longrightarrow M/C$. Since, $W \ll_g M$, we conclude that $\pi(W) \ll_g M/C$, i.e., $(C+W)/C = E/C \ll_g M/C$.

$(\mathbf{4}) \Longrightarrow (\mathbf{1})$ Let $E \leq M$ with $Rad_g(M) \subseteq E$. Then there is a decomposition $M = C \oplus W$ such that $C \leq E$ and $E/C \ll_g M/C$, by **(3)**. By Modular law, we have $E = C \oplus (E \cap W)$. Thus, $E/C \cong E \cap W$ and $M/C \cong W$. It follows that $E \cap W \ll_g W$. Hence, **(1)** holds.

$(\mathbf{1}) \Longrightarrow (\mathbf{5})$ Let $E \leq M$ with $Rad_g(M) \subseteq E$. By **(1)**, there is a decomposition $M = T \oplus C$ such that $T \leq E$ and $E \cap C \ll_g C$. So, $M = E + C$ and by modular low we get $E = T \oplus (C \cap E)$. Hence, E containing a g-supplement C in M such that $E \cap C$ is a direct summand of E.

$(\mathbf{5}) \Longrightarrow (\mathbf{6})$ Let $E \leq M$ with $Rad_g(M) \subseteq E$. By hypothesis, if we suppose E containing a g-supplement C in M with $E \cap C$ is a direct summand of E, $M = E + C$ and $E \cap C \ll_g C$. Also, $E = (E \cap C) \oplus W$ for some $W \leq E$. It follows that $M = W \oplus C$. Suppose that $e: M \to W$; $e(k + l) = k$ and $(1 - e): M \longrightarrow C$; $(1 - e)(k + l) = l$ are projection maps for all $k + l \in M = W \oplus C$. It is easily to confirm that $e = e^2 \in End(M)$. Since, $eM = W \leq E$ and $(1 - e)E = E \cap (1 - e)M = E \cap C \ll_g C = (1 - e)M$.

$(\mathbf{6}) \Longrightarrow (\mathbf{3})$ Suppose that $E \leq M$ such that $Rad_g(M) \subseteq E$. By **(6)**, there is an $e = e^2 \in End(M)$ such that $eM \leq E$ and $(1 - e)E \ll_g (1 - e)M$. Consider noting that $M = eM \oplus (1 - e)M$. By modular law, $E = E \cap (eM \oplus (1 - e)M) = eM \oplus (E \cap (1 - e)M) = eM \oplus (1 - e)E$, such that eM is a direct summand of M and $(1 - e)E \ll_g (1 - e)M$, so in M.

$(\mathbf{1}) \Longrightarrow (\mathbf{7})$ Let $E \leq M$ with $Rad_g(M) \subseteq E$. By **(1)**, there is a submodule T of M containing in E such that $M = T \oplus N$ and $E \cap N \ll_g N$. Put $W = E \cap N$, we deduce that $E = T + W$ and $W \ll_g N$, so in M.

$(\mathbf{7}) \Longrightarrow (\mathbf{1})$ Suppose that E is a submodule of M such that $Rad_g(M) \subseteq E$, so by **(7)**, there exists a direct summand T of M and $W \ll_g M$ such that $T \leq E$ and $E = T + W$. Thus, $M = T \oplus N$ for some $N \leq M$. Since, N is a g-supplement of T in M, so Lemma 1.2.18 implies N is a g-supplement of $E = T + W$ in M, thus $E \cap N \ll_g N$.

(1) ⟹ (8) Let E be any submodule of M such that $Rad_g(M) \subseteq E$. Since, M is a Rad_g-lifting, there exists $H \leq E$ with $M = H \oplus W$ and $E \cap W \ll_g W$. Hence, $M = E + W$ this implies W is a g-supplement of E in M.

(8) ⟹ (1) Clear. □

The Rad_g-lifting property is not inherited by their submodules for instance, the $\mathbb{Z}$-module $\mathbb{Z} \leq \mathbb{Q}$ is not Rad_g-lifting while $\mathbb{Q}$ as $\mathbb{Z}$ -module is Rad_g-lifting, see [Remarks and Examples 4.1.6 (1)].

Now we list some propositions that make Rad_g-lifting property inherited by their submodules under special conditions.

Proposition 4.1.8. Let M be a module. Then M is Rad_g-lifting if and only if, every direct summand of M contains $Rad_g(M)$ is Rad_g-lifting.

Proof. ⟹) Let $E \leq^{\oplus} M$ and $Rad_g(M) \subseteq E$. If U is a submodule of E such that $Rad_g(E) \subseteq U$. By Lemma 1.2.37(1) we have $Rad_g(M) = Rad_g(E)$. Thus, $Rad_g(M) \subseteq U \leq M$. As M is a Rad_g-lifting, there exists a decomposition $M = V \oplus \acute{V}$ such that $V \leq U$ and $U \cap \acute{V} \ll_g \acute{V}$. By modular law, we have that $E = E \cap \left(V \oplus \acute{V}\right) = V \oplus (E \cap \acute{V})$. Also, we have $U \cap \left(E \cap \acute{V}\right) = U \cap \acute{V} \ll_g M$. Since, $U \cap (E \cap \acute{V}) \leq E \cap \acute{V} \leq^{\oplus} M$, from Lemma 1.2.6 we get $U \cap \left(E \cap \acute{V}\right) \ll_g E \cap \acute{V}$. Hence E is Rad_g-lifting.

⟸) Since, trivially $M \leq^{\oplus} M$ and $Rad_g(M) \subseteq M$, Therefore, by hypothesis M is a Rad_g-lifting module. □

Corollary 4.1.9. Suppose that M is a module and $Rad_g(M) \leq^{\oplus} M$. If M is a Rad_g-lifting module, then $Rad_g(M)$ is Rad_g-lifting.

Proof. Clear by Proposition 4.1.8. □

Proposition 4.1.10. Let M is Rad$_g$-lifting and let E be any submodule of M such that $Rad_g(M) \subseteq E$. If E is g-coclosed, then E is a direct summand of M.

Proof. Let E be a g-coclosed submodule of M with $Rad_g(M) \subseteq E$. Since, M is a Rad$_g$-lifting module, by Proposition 4.1.7, there exists a direct summand L of M such that $L \leq E$ and $E/L \ll_g M/L$. Since, E is g-coclosed, $L = E$, this implies $E \leq^{\oplus} M$. □

Corollary 4.1.11. Let M be a Rad$_g$-lifting, and E is a g-coclosed submodule of M containing $Rad_g(M)$, then E is Rad$_g$-lifting.

Proof. Direct from Propositions 4.1.10 and 4.1.8. □

Proposition 4.1.12. If M is a Rad$_g$-lifting module and E an indecomposable submodule of M containing $Rad_g(M)$, then either E is a direct summand or g-small.

Proof. Assume that M is a Rad$_g$-lifting module. Let E be an indecomposable submodule E of M with $Rad_g(M) \subseteq E$. From Proposition 4.1.7, E can be written as $E = H \oplus G$, where H is a direct summand of M and $G \ll_g M$. It follows that either $E = H$ or $E = G$. This completes the proof. □

We now give a condition that make the class of Rad$_g$-lifting is closed under finite direct sums.

Theorem 4.1.13. Let $\{M_i\}_{i=1}^{n}$ be a family of submodules of an R-module M. Such that each M_i is Rad$_g$-lifting R-modules. If $M = \oplus_{i=1}^{n} M_i$ and M is duo, then M is Rad$_g$-lifting.

Proof. Suppose that $\{M_i\}_{i=1}^{n}$ is a family of Rad$_g$-lifting R-modules. To prove that M is Rad$_g$-lifting, it is enough to prove the case when $n = 2$. Let U be a submodule of $M = M_1 \oplus M_2$ and $Rad_g(M) \subseteq U$. Since, U is a fully invariant submodule of M, Lemma 1.2.14 implies $U = (M_1 \cap U) \oplus (M_2 \cap U)$. We deduce that $Rad_g(M_i) \subseteq M_i \cap U$ for $i = 1,2$. Since, M_i is $\mathrm{Rad_g}$-lifting, for $i = 1,2$, then there are two decompositions $M_1 = V_1 \oplus V_2$ and $M_2 = \acute{V}_1 \oplus \acute{V}_2$ with $V_1 \leq M_1 \cap U$ and $(M_1 \cap U) \cap V_2 = U \cap V_2 \ll_{\mathrm{g}} V_2$, also $\acute{V}_1 \leq M_2 \cap U$ and $(M_2 \cap U) \cap \acute{V}_2 = U \cap \acute{V}_2 \ll_{\mathrm{g}} \acute{V}_2$. Thus, $M = M_1 \oplus M_2 = (V_1 \oplus \acute{V}_1) \oplus (V_2 \oplus \acute{V}_2)$ with $V_1 \oplus \acute{V}_1 \leq (M_1 \cap U) \oplus (M_2 \cap U) = U$ and $U \cap (V_2 \oplus \acute{V}_2) = (U \cap V_2) \oplus \big(U \cap \acute{V}_2\big)$ $\ll_g V_2 \oplus \acute{V}_2$, by Lemma 1.2.5(3). So, by mathematical induction, M is $\mathrm{Rad_g}$-lifting. □

Theorem 4.1.14. Let $\{M_i\}_{i=1}^{n}$ be a family of submodules of R-module M. Such that each M_i is $\mathrm{Rad_g}$-lifting R-modules. If $M = \bigoplus_{i=1}^{n} M_i$ and M is distributive, then M is $\mathrm{Rad_g}$-lifting.

Proof. Analogous to proof Theorem 4.1.13. □

Theorem 4.1.15. If $M = M_1 \oplus M_2$ is a direct sum of $\mathrm{Rad_g}$-lifting modules M_1 and M_2 with M_1 is quasi-projective and M_2-projective, then M is $\mathrm{Rad_g}$-lifting.

Proof. Let E be a submodule of $M = M_1 \oplus M_2$ with $Rad_g(M) \subseteq E$. Since, $M_1 \cap (E + M_2) \leq M_1$ and $Rad_g(M_1) \subseteq M_1 \cap (E + M_2)$. Since, M_1 is $\mathrm{Rad_g}$-lifting, then there is a decomposition $M_1 = V_1 \oplus V_2$, $V_1 \leq M_1 \cap (E + M_2)$ and $V_2 \cap (E + M_2) \ll_{\mathrm{g}} V_2$. Since, we have $V_1 \oplus V_2 \oplus M_2 \subseteq (E + M_2) \oplus V_2 \oplus M_2 = E + (V_2 \oplus M_2)$ then $M = E + (V_2 \oplus M_2)$. Since, M_1 is quasi-projective and M_2-projective, then V_1 is $V_2 \oplus M_2$-projective. By Lemma 1.2.31, there is a submodule E_1 of E such that $M = E_1 \oplus (V_2 \oplus M_2)$. It is easy to confirm that $E \cap (L + V_2) = L \cap (E + V_2)$ for any $L \leq M_2$, to see this: as $E \cap (L + V_2) \leq L \cap (E + V_2) + V_2 \cap (E + L)$ and $V_2 \cap (E + L) = 0$, then $(L + V_2) \leq L \cap (E + V_2)$. In the same way, we get $L \cap (E + V_2) \leq E \cap (L + V_2)$. Therefore, $L \cap (E + V_2) = E \cap (C + V_2)$. Also, we conclude that $M_2 \cap (E + V_2) \leq M_2$ with $Rad_g(M_2) \subseteq M_2 \cap (E + V_2)$ and M_2 is a $\mathrm{Rad_g}$-lifting, for then there is a decomposition $M_2 = \acute{V}_1 \oplus \acute{V}_2$, $\acute{V}_1 \leq M_2 \cap (E + V_2) = E \cap (M_2 + V_2) \leq E$ and

$\acute{V}_2 \cap (E + V_2) \ll_g \acute{V}_2$. Therefore, $M = E_1 \oplus (V_2 \oplus M_2) = (E_1 \oplus \acute{V}_1) \oplus (V_2 \oplus \acute{V}_2)$, $E_1 \oplus \acute{V}_1 \leq E$ and $E \cap \left(V_2 \oplus \acute{V}_2\right) = \acute{V}_2 \cap (E + V_2) \ll_g V_2 \oplus \acute{V}_2$. Therefore, M is a Rad_g-lifting. □

Proposition 4.1.16. Let M be a Rad_g-lifting with $Rad_g(M) \neq M$. Then there is a decomposition $M = H \oplus W$ such that W is a g-supplement of $Rad_g(M)$ in M, $Rad_g(W) \ll_g W$ and H is a g-Radical. More specifically, if $P_g(M) = 0$, then $Rad_g(M) \ll_g M$.

Proof. Suppose that $Rad_g(M) \neq M$. Since, M is Rad_g-lifting and $Rad_g(M) \subseteq Rad_g(M)$, so by Proposition 4.1.7(8), there exists a submodule H of M in $Rad_g(M)$ such that $M = H \oplus W$ and W a g-supplement of $Rad_g(M)$ in M, i.e., $M = Rad_g(M) + W$ and $Rad_g(M) \cap W \ll_g W$. Since, W is a direct summand of M, so it is a g-supplement and hence $Rad_g(M) \cap W = Rad_g(W)$, by Lemma 1.2.17, this implies $Rad_g(W) \ll_g W$. Form Lemma 1.2.10, $M = Rad_g(H) \oplus W$. By modular law, we get that $H = H \cap (W \oplus Rad_g(H)) = Rad_g(H) \oplus (W \cap H) = Rad_g(H)$. Hence, H is g-Radical.

Now, suppose that $P_g(M) = 0$. Then, $H = 0$ which implies $M = W$, hence $Rad_g(M) \ll_g M$. □

The converse of Proposition 4.1.16 need not be true, in general for example, if $M = R = \mathbb{Z}$, then $Rad_g(M) = 0 \neq M$. Also, we have $\mathbb{Z} = \mathbb{Z} \oplus (0)$ is the only decomposition of $\mathbb{Z}$-module $\mathbb{Z}$ such that $\mathbb{Z}$ is a g-supplement of $Rad_g(M) = 0$, $Rad_g(\mathbb{Z}) = 0 \ll_g \mathbb{Z}$ and (0) is g-Radical, while $M = \mathbb{Z}$ is not Rad_g-lifting $\mathbb{Z}$-module.

We will discuss the factor module of a Rad_g-lifting modules.

Theorem 4.1.17. Let M be a Rad_g-lifting and $E \leq M$ that satisfy one of the following:

(1) If for any direct summand C of M, $(C+E)/E$ is a direct summand of M/E.

(2) If E is a distributive submodule of M.

(3) If E is a fully invariant submodule of M.

(4) If $M = E \oplus Y$ such that Y containing $Rad_g(M)$.

Then M/E is a Rad_g-lifting module.

Proof. (1) Suppose that $E \leq L \leq M$ with $Rad_g(M/E) \subseteq L/E$. Consider, the natural map $\pi: M \to M/E$. From $Rad_g(M) \subseteq M$, we deduce that $\pi(Rad_g(M)) \subseteq Rad_g(M/E)$, hence $(Rad_g(M)+E)/E \subseteq Rad_g(M/E)$. Thus, $(Rad_g(M)+E)/E \subseteq L/E$, and hence $Rad_g(M) \subseteq L$. Since, M is Rad_g-lifting, then there exists a direct summand C of M such that $C \leq L$ and $L/C \ll_g M/C$, by Proposition 4.1.7(3). By **(1)**, $(C+E)/E$ is a direct summand of M/E. Clearly, $(E+C)/E \leq L/E$. Suppose, a projection map $\rho: \frac{M}{C} \to \frac{M/C}{(E+C)/C}$. Since, $L/C \ll_g M/C$ then $\frac{L}{E+C} \ll_g \frac{M}{E+C}$. Therefore, M/E is Rad_g-lifting.

(2) Suppose that $M = C \oplus \grave{C}$ for some $\grave{C} \leq M$. By **(1)**, it is enough to conform that $(C+E)/E$ is also a direct summand of M/E. It's easy to see that $M/E = ((C+E)/E) + ((\grave{C}+E)/E)$. Now, as E is a distributive submodule of M, then $(C+E) \cap (\grave{C}+E) = (C \cap \grave{C}) + E = E$. So, $((C+E)/E) \cap ((\grave{C}+E)/E) = 0$, as required.

(3) Let H be a direct summand of M, then $M = H \oplus \grave{H}$ for some $\grave{H} \leq M$. Since, E is a fully invariant submodule, so $M/E = ((H+E)/E) \oplus ((H'+E)/E)$, by Lemma 1.2.19, i.e., $(C+E)/E$ is a direct summand of M/E. Hence, M/E is Rad_g-lifting, by **(1)**.

(4) Since, Y is a direct summand of M containing $Rad_g(M)$, thus by Proposition 4.1.8, Y is a Rad_g-lifting module, as $M/E \cong Y$, hence M/E is a Rad_g-lifting module.

Corollary 4.1.18. If M is Rad_g-lifting, then $M/P_g(M)$ is Rad_g-lifting.

Proof. By Lemma 1.2.36, $P_g(M)$ is a fully invariant submodule of M, and so by Theorem 4.1.17(3), we get the result. □

Corollary 4.1.19. The factor module of a distributive (or, duo) Rad_g-lifting module is also Rad_g-lifting.

Proof. Clear. □

Corollary 4.1.20. The homomorphic image of a distributive (or, duo) Rad_g-lifting module is also Rad_g-lifting.

Proof. Suppose M is a distributive Rad_g-lifting module and let $f: M \to \grave{M}$ be any homomorphism. By 1st isomorphism theorem, there exists a submodule U of M such that $M/U \cong f(M)$. By Corollary 4.1.19, M/U is Rad_g-lifting. Hence, $f(M)$ is Rad_g-lifting. □

Corollary 4.1.21. Suppose M be a distributive module, and $L \leq^{\oplus} M$. Then M is a Rad_g-lifting module if and only if L and M/L are both Rad_g-lifting modules.

Proof. $\Longrightarrow$) Suppose that M is a distributive module and L a direct summand of M, so $M = L \oplus N$ for a submodule N of M. By Corollary 4.1.19, M/L is Rad_g-lifting. However, $L \cong M/N$, again by Corollary 4.1.19, we deduce L is Rad_g-lifting.

$\Longleftarrow$) Since, $M \cong L \oplus (M/L)$, the result is included by Theorem 4.1.14. □

Corollary 4.1.22. Let M be a weak duo module, $L \leq^{\oplus} M$. If M is a Rad_g-lifting, then L and M/L are Rad_g-lifting modules.

Proof. Let M be a weak duo and L a direct summand of M, then $M = L \oplus C$ for some $C \leq M$, and L, C are fully invariant submodules. By Theorem 4.1.17(3), M/L is $\mathrm{Rad_g}$-lifting, also $L \cong M/C$ is $\mathrm{Rad_g}$-lifting. □

Corollary 4.1.23. Let M be a duo, and $L \leq^{\oplus} M$. Then M is a $\mathrm{Rad_g}$-lifting module if and only if L and M/L are $\mathrm{Rad_g}$-lifting modules.

Proof. ⟹) Since, every duo module is a weak duo, then the result is follows by Corollary 4.1.22.

⟸) Since, $M \cong L \oplus (M/L)$, we deduce the result by Theorem 4.1.13. □

Corollary 4.1.24. Let $M = \oplus_{i=1}^{n} M_i$ be a duo module. Then M_i is a $\mathrm{Rad_g}$-lifting module for $i \in \{1,2,\dots,n\}$, if and only if M is a $\mathrm{Rad_g}$-lifting module.

Proof. It comes directly by Theorem 4.1.13 and Corollary 4.1.23. □

Corollary 4.1.25. Let $f: M \to \grave{M}$ be a homomorphism from a $\mathrm{Rad_g}$-lifting module M into a module $\grave{M}$. If $kerf$ is a distributive (or, fully invariant) submodule of M, then Imf is $\mathrm{Rad_g}$-lifting.

Proof. Since, M is a $\mathrm{Rad_g}$-lifting module, Theorem 4.1.17(2,3) imply $M/kerf$ is $\mathrm{Rad_g}$-lifting. By 1st theorem isomorphism, $M/kerf \cong Imf$ and so Imf is $\mathrm{Rad_g}$-lifting. □

The following result is consequence from Corollary 4.1.25.

Corollary 4.1.26. Let $f: M \to \grave{M}$ be an epimorphism of R-modules M, $\grave{M}$ such that $kerf$ is a distributive (or, fully invariant) submodule of M. Then $\grave{M}$ is $\mathrm{Rad_g}$-lifting whenever M is $\mathrm{Rad_g}$-lifting.

4.2. Connections with Rad_g-lifting modules

This section will highlight several relations between this concept of Rad_g-lifting and other forms of modules.

Proposition 4.2.1. Every generalized hollow module is Rad_g-lifting.

Proof. Let M be a generalized hollow module and let $C \leq M$ such that $Rad_g(M) \subseteq C$. If $C = M$, trivially, $M = M \oplus (0)$ such that $M \leq C$ and $C \cap (0) \ll_g (0)$. Suppose $C \subset M$, then $C \ll_g M$. It follows that $M = (0) \oplus M$ such that $(0) \leq C$ and $C \cap M = C \ll_g M$, so in C. Hence, M is Rad_g-lifting. □

Example 4.2.2. As an application on Proposition 4.2.1, it is easy to see that $\mathbb{Z}_8$ as $\mathbb{Z}$-module is generalized hollow, so it is Rad_g-lifting.

The reverse of Proposition 4.2.1 need not be true, in general, for instance: $\mathbb{Z}_{24}$ as $\mathbb{Z}$-module is Rad_g-lifting, to see this: it is easy to show that $Rad_g(\mathbb{Z}_{24}) = 2\mathbb{Z}_{24}$. Since, $2\mathbb{Z}_{24}$ is the only proper submodule of $\mathbb{Z}_{24}$ that contains $Rad_g(\mathbb{Z}_{24})$. Thus, $\mathbb{Z}_{24}$ have trivial decomposition $\mathbb{Z}_{24} = (\bar{0}) \oplus \mathbb{Z}_{24}$ such that $(\bar{0}) \leq 2\mathbb{Z}_{24}$ and $2\mathbb{Z}_{24} \cap \mathbb{Z}_{24} = 2\mathbb{Z}_{24} \ll_g \mathbb{Z}_{24}$. While $\mathbb{Z}_{24}$ as $\mathbb{Z}$-module is not generalized hollow, in fact, $3\mathbb{Z}_{24} \ll_g \mathbb{Z}_{24}$.

Proposition 4.2.3. If M is a g-local module, then M is Rad_g-lifting.

Proof. Suppose M is a g-local module. Thus, we have $Rad_g(M)$ is a maximal and g-small submodule of M. Assume $T \leq M$ with $Rad_g(M) \subseteq T$. If $Rad_g(M) = T$, then $T \ll_g M$, trivially, $M = (0) \oplus M$ with $(0) \leq T$ and $T \cap M = T \ll_g M$. Let $Rad_g(M) \subset T$. As, $Rad_g(M)$ is maximal of M, Therefore, $T = M$, trivially, $M = M \oplus (0)$ with $M \leq T$ and $T \cap (0) \ll_g (0)$. Therefore, M is Rad_g-lifting. □

The converse of Proposition 4.2.3 is not true, in general, as seen in the following example.

Example 4.2.4. Assume $R = \mathbb{Z}$ and $M = \mathbb{Q}$. We have that $\mathbb{Q}$ as $\mathbb{Z}$-module is Rad_g-lifting, see Remarks and Examples 4.1.6(1). But $Rad_g(\mathbb{Q})$ dose not be a maximal submodule of $\mathbb{Q}$ and hence, $\mathbb{Q}$ not g-local as $\mathbb{Z}$-module.

Proposition 4.2.5. Every Rad_g-lifting module is a sgrs$^{\oplus}$-module, and hence it is a sgrs-module.

Proof. Suppose M is a Rad_g-lifting module, $E \leq M$ such that $Rad_g(M) \subseteq E$. Then there is a decomposition $M = T \oplus N$ such that $T \leq E$ and $E \cap N \ll_g N$. Thus, $M = E + N$ such that $N \leq^{\oplus} M$ and $E \cap N \ll_g N$. Therefore, M is a sgrs$^{\oplus}$-module. □

As an application of above consequence, by [21, Examples 2.7(3)] $\mathbb{Z}$-module $\mathbb{Z}$ is not an sgrs$^{\oplus}$-module, so it is not Rad_g-lifting. Since, this example and Remarks and Examples 4.1.6(1), it can be said that any submodule of a Rad_g-lifting module may not be Rad_g-lifting.

Proposition 4.2.6. Let M be an indecomposable. If M is an sgrs$^{\oplus}$-module, then M is Rad_g-lifting.

Proof. Suppose that M is an sgrs$^{\oplus}$-module and $E \leq M$ with $Rad_g(M) \subseteq E$. If $E = M$, trivially, there is a decomposition $M = M \oplus 0$ such that $M \leq E$ and $E \cap 0 \ll_g 0$. Suppose $E \neq M$. Then there exists a submodule N of M such that $M = T \oplus N = E + N$ and $E \cap N \ll_g N$. As M is indecomposable, either $N = 0$ or $N = M$. If $N = 0$, then $E = M$, a contradiction. Thus, $T = 0$ and $N = M$. Thus, $T \leq E$ and $E \cap N \ll_g N$. Hence, M is a Rad_g-lifting module.□

The following result follows straightaway from Propositions 4.2.5 and 4.2.6.

Corollary 4.2.7. Let M be indecomposable. Then M is an sgrs$^{\oplus}$-module if and only if M is $\mathrm{Rad_g}$-lifting.

Proposition 4.2.8. Let M be a refinable with SIP. If M is an sgrs$^{\oplus}$-module, then M is $\mathrm{Rad_g}$-lifting.

Proof. Suppose that M be an sgrs$^{\oplus}$-module with the SIP. Let $E \leq M$ such that $Rad_g(M) \subseteq E$. Then there is a direct summand T of M such that $M = E + T$ and $E \cap T \ll_g T$. Since, M is refinable, there exists a direct summand U of M such that U is containing in E and $M = U + T$. By SIP of M, $U \cap T$ is a direct summand of M. Therefore, $M = (U \cap T) \oplus W$ for some submodule W of M. By Modular law, $T = (U \cap T) \oplus (W \cap T)$, and so $M = U + T = U \oplus (W \cap T)$. On the other hand, $E \cap (W \cap T) \ll_g T$. From $E \cap (W \cap T) \leq W \cap T \leq^{\oplus} T$, then $E \cap (W \cap T) \ll_g W \cap T$, by Lemma 1.2.6. □

Proposition 4.2.9. Let M be a uniserial module. Then M is $\mathrm{Rad_g}$-lifting, and hence M/E is $\mathrm{Rad_g}$-lifting whenever $E \leq M$ is fully invariant.

Proof. Let M be a uniserial module, so by Lemma 1.2.38, any submodule of M is a hollow module, hence, M is hollow and so generalized hollow. Thus, by Proposition 4.2.1, M is $\mathrm{Rad_g}$-lifting. Now, if $E \leq M$ is fully invariant, Theorem 4.1.17 implies that M/E is $\mathrm{Rad_g}$-lifting. □

An easy applying example, for a prime number p and a positive integer n, the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ is uniserial, so it is $\mathrm{Rad_g}$-lifting. While $\mathbb{Z}_6$ as $\mathbb{Z}$-module is $\mathrm{Rad_g}$-lifting not uniserial.

Corollary 4.2.10. Let M be a Noetherian uniserial module. Then any factor module of M is $\mathrm{Rad_g}$-lifting.

Proof. Assume M is a Noetherian uniserial module, then every submodule of M is fully invariant by Lemma 1.2.39. Therefore, each factor module of M is Rad_g-lifting, by Proposition 4.2.9. □

Corollary 4.2.11. The factor module of uniserial duo module is also Rad_g-lifting.

Proof. By definition of a duo module and Proposition 4.2.9. □

The following example shows the converse of above results need not be true, in general.

Example 4.2.12. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_p$, where p is a prime number. It is clear that $\mathbb{Z}/p\mathbb{Z}$ is Rad_g-lifting as $\mathbb{Z}$-module because it is simple as $\mathbb{Z}$-module. While the $\mathbb{Z}$-module $\mathbb{Z}$ does not Rad_g-lifting.

Proposition 4.2.13. Let M be a module, consider the following cases:

(1) M is a semisimple module.

(2) M is a g-lifting module.

(3) M is a Rad_g-lifting module.

(4) M is a $(\text{P}_g{}^*)$-module.

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). If $Rad_g(M) = 0$, then (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2}) \Longrightarrow (\mathbf{3})$ Clear.

$(\mathbf{3}) \Longrightarrow (\mathbf{4})$ Let L be submodule of M. As $Rad_g(M) = 0 \subseteq E$, by **(3)** there is a decomposition $M = I_1 \oplus I_2$ such that $I_1 \leq E$ and $E \cap I_2 \ll_g I_2$, so in M, this implies $E \cap I_2 \subseteq Rad_g(M)$. Hence, M is a $(P_g{}^*)$-module, by Lemma 1.2.49.

$(\mathbf{4}) \Longrightarrow (\mathbf{1})$ Let $T \leq M$ with $Rad_g(M) = 0$. By **(4)**, there is a decomposition $M = C \oplus W$ such that $C \leq T$ and $T \cap W \subseteq Rad_g(M)$. Thus, $M = T + W$ and $T \cap W = 0$. Thus, $M = T \oplus W$, and $(\mathbf{1})$ holds. □

By involve the preceding result, we can immediately confirm a previous example, we realize that the $\mathbb{Z}$-module $\mathbb{Z}$ is not Rad_g-lifting, actually, $\mathbb{Z}$-module $\mathbb{Z}$ neither semisimple nor g-lifting, and $Rad_g(\mathbb{Z}) = 0$.

Corollary 4.2.14. If M is a finitely generated Rad_g-lifting module such that $Rad_g(M) = 0$. Then M is an Artinian module.

Proof. From Proposition 4.2.13 and Lemma 1.2.23. □

Corollary 4.2.15. Let M be a module and E a nonzero Rad_g-lifting submodule of M with $E \cap Rad_g(M) = 0$, then E is semisimple.

Proof. Since, $Rad_g(E) \subseteq E \cap Rad_g(M)$, then $Rad_g(E) = 0$ and hence E is semisimple by Proposition 4.2.13. □

Corollary 4.2.16. Let M be a module with $Rad_g(M) = 0$. If, E is a nonzero Rad_g-lifting submodule of M, then E is semisimple.

Proof. Clear by Corollary 4.2.15. □

Corollary 4.2.17. Let $M= M_1 \oplus M_2$ such that M_1 is a semisimple and M_2 is Rad_g-lifting, and they are relatively projective with M_1, then M is Rad_g-lifting.

Proof. From Proposition 4.2.13 we have M_1 is Rad_g-lifting and so by Theorem 4.1.15, M is a Rad_g-lifting. □

Proposition 4.2.18. If M is a Rad_g-lifting, then $M/Rad_g(M)$ is semisimple.

Proof. Suppose $Rad_g(M) \leq H \leq M$. According to the hypothesis, there is a decomposition $M = C \oplus \grave{C}$ for some $C \leq H$ and $H \cap \acute{C} \ll_g \acute{C}$. Therefore, $M = H + \grave{C}$. Since, $\frac{M}{Rad_g(M)} = \frac{H}{Rad_g(M)} + \frac{(\grave{C}+Rad_g(M))}{Rad_g(M)}$, and $\left(\frac{H}{Rad_g(M)}\right) \cap \left(\frac{\grave{C}+Rad_g(M)}{Rad_g(M)}\right) = \frac{H \cap (\grave{C}+Rad_g(M))}{Rad_g(M)} = \frac{Rad_g(M)+\left(H \cap \grave{C}\right)}{Rad_g(M)} = \frac{Rad_g(M)}{Rad_g(M)}$, i.e., $H/Rad_g(M)$ is a direct summand of $M/Rad_g(M)$. Hence, $M/Rad_g(M)$ is a semisimple. □

Corollary 4.2.19. Let M be a Rad_g-lifting such that $Rad_g(M)$ is a direct summand of M then $M/P_g(M)$ is semisimple.

Proof. From Lemma 1.2.37(2), $P_g(M) = Rad_g(M)$. By Proposition 4.2.18, $M/P_g(M)$ is semisimple. □

Corollary 4.2.20. If M is a Rad_g-lifting module, then also $M/Rad_g(M)$.

Proof. From Propositions 4.2.18 and 4.2.13. □

Proposition 4.2.21. Let M be a finitely generated module satisfies DCC on g-small submodules. If M is a Rad_g-lifting, then M is Artinian.

Proof. Suppose that M is Rad_g-lifting. By Proposition 4.2.18, $M/Rad_g(M)$ is semisimple. Since, M is a finitely generated module, then $M/Rad_g(M)$ is so finitely generated, and then $M/Rad_g(M)$ is artinian, see Lemma 1.2.23. Also, M satisfies DCC on g-small submodules implies that $Rad_g(M)$ is Artinian, according to Lemma 1.2.20. Therefore, by Lemma 1.2.22, M is Artinian. □

Corollary 4.2.22. Let M be a cyclic module satisfies DCC on g-small submodules. If M is a Rad_g-lifting, then M is Artinian.

Proof. Clear. □

Proposition 4.2.23. If M is a Rad_g-lifting with $Rad_g(M) \neq M$, then there is a decomposition $M = C_1 \oplus C_2$ such that C_1 is semisimple, $Rad_g(M) \trianglelefteq C_2$ and $C_2/Rad_g(M)$ is semisimple.

Proof. Since, M is a Rad_g-lifting module, Proposition 4.2.18 implies $M/Rad_g(M)$ is a semisimple module. Since, $Rad_g(M) \subset M$ and by Lemma 1.2.33, there is a decomposition $M = C_1 \oplus C_2$ such that C_1 is semisimple, $Rad_g(M) \trianglelefteq C_2$ and $C_2/Rad_g(M)$ is semisimple. □

Proposition 4.2.24. If M is a Rad_g-lifting such that $Rad_g(M) = 0$, then M is refinable.

Proof. Let E, $C \leq M$ with $M = E + C$. Since, M is a Rad$_g$-lifting and $Rad_g(M) = 0 \subseteq E$, then there is a decomposition $M = L \oplus Y$ such that $L \leq E$ and $E \cap Y \ll_g M$. By Modular law $E = L \oplus (E \cap Y)$. Now, since, $E \cap Y \subseteq Rad_g(M) = 0$, so $E = L$ imply $M = L + C$, and $L \leq^{\oplus} M$. Therefore, M is refinable. □

Remark 4.2.25. The term "$Rad_g(M) = 0$" for a module M in Proposition 4.2.24, is necessary, for instance: it is well know that by Remarks and Examples 4.1.6(1), $\mathbb{Q}$ as $\mathbb{Z}$-module is Rad$_g$-lifting, but $\mathbb{Q}_{\mathbb{Z}}$ not refinable, indeed, $Rad_g(\mathbb{Q}_{\mathbb{Z}}) = \mathbb{Q}_{\mathbb{Z}}$, this implies $Rad_g(\mathbb{Q}_{\mathbb{Z}}) \neq 0$.

Theorem 4.2.26. The following are equivalent for a projective R-module M.

(1) For $E \leq M$ with $Rad_g(M) \subseteq E$, the factor M/E containing a projective g-cover.

(2) M is a Rad$_g$-lifting module.

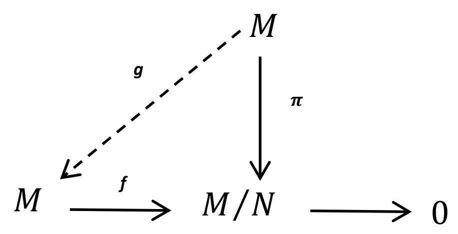
(3) M is a sgrs$^{\oplus}$-module.

(4) M is a sgrs-module.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Suppose that $E \leq M$ such that $Rad_g(M) \subseteq E$. By **(1)**, there exists a projective R-module P and an R-epimorphism $\alpha: P \to M/E$ such that $ker\alpha \ll_g P$. Suppose $\pi: M \to M/E$ is a canonical epimorphism, then there is a homomorphism $\beta: M \to P$ such that $\alpha\beta = \pi$. Thus $P = \beta(M) + ker\alpha$. Since, $ker\alpha \ll_g P$, Lemma 1.2.4 implies $P = \beta(M) \oplus Y$ for some semisimple submodule Y of $ker\alpha$, hence $\beta(M)$ is projective. So $M = ker\beta \oplus W$ for some submodule W of M. It is easy to confirm that $ker\beta \leq ker\pi = E$. To prove that, $E \cap W \ll_g W$. As $ker\alpha \ll_g P$, $ker\alpha \cap \beta(W) = \beta(E \cap W) \ll_g P$. By Lemma 1.2.17(1), $\beta(E \cap W) \ll_g \beta(M) = \beta(W)$. Hence, $E \cap W \ll_g W$, as β is an isomorphism from W into $\beta(W)$. Therefore, M is a Rad$_g$-lifting module.

$(\mathbf{2}) \Longrightarrow (\mathbf{3}) \Longrightarrow (\mathbf{4})$ By Proposition 4.2.5.

$(\mathbf{4}) \Longrightarrow (\mathbf{1})$ Let $E \leq M$ with $Rad_g(M) \subseteq E$. By (4), there exists a submodule T of M such that $M = E + T$ and $E \cap T \ll_g T$. Suppose that $f: M \to M/E$ is a homomorphism defined by $f(x) = a + E$, where $x = n + a \in M, n \in E$ and $a \in T$. Let $\pi: M \to M/E$ be the natural epimorphism. Since, M is projective, then there is a homomorphism $g: M \to M$ such that $fg = \pi$.



Then $f(g(M)) = \pi(M)$, so that $f^{-1}(f(g(M))) = f^{-1}(M/E)$, it follows that $M = g(M) + Kerf = g(M) + E \cap T$. Since, $E \cap T \ll_g T$, and hence it is g-small in M. By Lemma 1.2.4, there exists a semisimple submodule Y of $E \cap T$ such that $M = g(M) \oplus Y$ and such that $g(M)$ is a projective R-module. Hence, $g(M) \cong M/kerg$ implies $kerg$ is a direct summand of $g(M)$, so in M, this implies $M = kerg \oplus N$ for some submodule N of M. Since, N is projective. Let $(fg)|_N$ indicate the restriction of fg on N, that is $(fg)|_N: N \to M/E$. Then $ker((fg)|_N) \leq E \cap T$. Therefore, $ker((fg)|_N) \ll_g M$. Since, $ker((fg)|_N) \subseteq N$ and N is a direct summand of M, Lemma 1.2.17(1) implies $ker((fg)|_N) \ll_g N$. Hence, N is a projective g-cover of M/E. This completes the proof. □

Corollary 4.2.27. The following are equivalent for a ring R.

(1) For $I \leq R$ with $Rad_g(R) \subseteq I$, the factor R/I containing a projective g-cover

(2) R is a Rad_g-lifting ring.

(3) R is a $\text{sgrs}^{\oplus}$-ring.

(4) R is a sgrs-ring.

Proof. Since, $R = \langle 1 \rangle$ is a free R-module, so it is projective. Then the result is obtained by Theorem 4.2.26. □

Corollary 4.2.28. Let R be any semisimple ring, and M be an R-module. Then the following are equivalent.

(1) For $E \leq M$ with $Rad_g(M) \subseteq E$, the factor M/E containing a projective g-cover.

(2) M is a Rad_g-lifting ring.

(3) M is a sgrs$^{\oplus}$-ring.

(4) M is a sgrs-ring.

Proof. Since, R is a semisimple ring, so by Lemma 1.2.28(2), an R-module M is projective. Then the result is obtained by Theorem 4.2.26. □

Proposition 4.2.29. Let P be a projective module containing g-small generalized Radical (i.e., $Rad_g(P) \ll_g P$). Then the following are equivalent.

(1) P is a Rad_g-lifting module.

(2) $P/Rad_g(P)$ is semisimple and for every submodule E of P containing $Rad_g(P)$ and $\bar{E} = E/Rad_g(P)$, there exists a direct summand C of P such that $\bar{E} = \bar{C}$.

Proof. (1) $\Longrightarrow$ (2) Let E be a submodule of P such that $Rad_g(P) \subseteq E$. Since, P is a Rad_g-lifting module, thus by Proposition 4.2.18, $P/Rad_g(P)$ is semisimple. Put $\bar{E} = E/Rad_g(P)$. By Proposition 4.1.7, there exists a decomposition $E = C \oplus W$ such that C is a direct summand of P and $W \ll_g P$. So $W \subseteq Rad_g(P)$. It follows that $C + Rad_g(P) \subseteq E$. On the other hand, we have $E = C + W \subseteq C + Rad_g(P)$, thus $E = C + Rad_g(P)$. Therefore, $E/Rad_g(P) = (C + Rad_g(P))/Rad_g(P)$, and hence $\bar{E} = \bar{C}$.

(2) $\Longrightarrow$ (1) Let $E \leq P$ such that $Rad_g(P) \subseteq E$. As $P/Rad_g(P)$ is semisimple, we have $P/Rad_g(P) = E/Rad_g(P) \oplus W/Rad_g(P)$ for some submodule W of M. By (2), there exists a direct summand T of P such that $P = T \oplus N$ for some submodule N of P, and $\bar{E} = \bar{T}$, such that $W = N + Rad_g(P)$. Thus, $P = T + N + Rad_g(P) = E + N$. Since, $P = E + N$ is projective, so from

Lemma 1.2.32 we get $P = E' \oplus N$ with $E' \subseteq E$. Also, $E \cap N \leq E \cap W = Rad_g(P) \ll_g P$, Therefore, P is a Rad_g-lifting module, by Proposition 4.1.7. □

Proposition 4.2.30. Let M be an indecomposable with $Rad_g(M) \neq M$. If M is Rad_g-lifting, then $Rad_g(M) \ll_g M$. Furthermore, if $Rad_g(M)$ is a maximal submodule of M then the converse is true.

Proof. Suppose that M is an indecomposable and Rad_g-lifting. Since, $Rad_g(M\,) \subseteq Rad_g(M)$, so by Proposition 4.1.16, there is a unique decomposition $M = M \oplus 0$ such that M is a g-supplement of $Rad_g(M)$ and 0 is a g-Radical. Hence, $Rad_g(M) = Rad_g(M) \cap M \ll_g M$.

To prove the converse of this claim, let $Rad_g(M)$ be a maximal submodule of M and $Rad_g(M) \ll_g M$, implies M is g-local. As M is an indecomposable, Lemma 1.2.34 implies M is local. By Lemma 1.2.35, M is generalized hollow and thus by Proposition 4.2.1, M is Rad_g-lifting. □

The converse of Proposition 4.2.30 need not be true, in general, for example, it is well know that the $\mathbb{Z}$-module $\mathbb{Z}$ is not Rad_g-lifting, while $\mathbb{Z}_{\mathbb{Z}}$ is an indecomposable, $\mathbb{Z} \neq Rad_g(\mathbb{Z})$ and $Rad_g(\mathbb{Z}) \ll_g \mathbb{Z}$.

Proposition 4.2.31. Let M be indecomposable. Then M is Rad_g-lifting if and only if, either M is g-Radical or $Rad_g(M)$ is an essential maximal and g-small submodule of M, and hence M is g-local.

Proof. ⟹) Let M be an indecomposable Rad_g-lifting module. Suppose, that $Rad_g(M) \neq M$. By Proposition 4.2.30, we have $Rad_g(M) \ll_g M$. Also, M containing an essential maximal submodule, say E, such that $Rad_g(M) \subseteq E$. By Proposition 4.1.7(8), there exists a submodule H of M inside E such that $M = H \oplus W$ and W is a g-supplement of E in M. Thus, $M = E + W$ and $E \cap W \ll_g W$. As M is an indecomposable module, so either $W = 0$ or $W = M$.

If $W = 0$ then $E = M$, that is a contradiction. Hence $W = M$ and so $E \ll_g M$. Then $E \subseteq Rad_g(M)$. Thus, $Rad_g(M) = E$. This means $Rad_g(M)$ is a maximal submodule of M and $Rad_g(M) \ll_g M$, Therefore, M is g-local.

$\Leftarrow$) If $Rad_g(M) = M$, Proposition 4.1.2 implies M is Rad_g-lifting. Now, if M is g-local, Proposition 4.2.3 implies M is Rad_g-lifting.□

Corollary 4.2.32. Let M be an indecomposable with $Rad_g(M) \neq M$. If M is Rad_g-lifting, then M is local.

Proof. By Proposition 4.2.31, M is a g-local. as stated by Lemma 1.2.34, M is a local. □

Corollary 4.2.33. Let M be an indecomposable with $Rad_g(M) \neq M$. If M is Rad_g-lifting, then M is hollow, and hence generalized hollow.

Proof. Direct from Corollary 4.2.32. □

Proposition 4.2.34. Suppose that M is a module containing SSP and $Rad_g(M)$ is a ⊕-g-supplemented that is a direct summand of M. If M is a Rad_g-lifting, then M is ⊕-g-supplemented.

Proof. Let U be any submodule of M. Since, $Rad_g(M) \subseteq Rad_g(M) + U$, so by Proposition 4.1.7(8), $Rad_g(M) + U$ containing a g-supplement, say L, that is a direct summand of M. Now, as $Rad_g(M) \cap (L + U) \leq Rad_g(M)$ and Since, $Rad_g(M)$ is ⊕-g-supplemented, then $Rad_g(M) \cap (L + U)$ containing a g-supplement, say Y, that is a direct summand of $Rad_g(M)$. Since, $Rad_g(M) \leq^{\oplus} M$, Y is a direct summand of M. As M containing SSP, we have $L + Y$ is a direct summand of M. By Lemma 1.2.15, $L + Y$ is a g-supplement of U in M. Therefore, M is ⊕-g-supplemented. □

4.3. Rings and localization of Rad_g-lifting

In this part, we will investigate the behavior of Rad_g-lifting modules under localization and rings.

Theorem 4.3.1. Let M be a faithful, finitely generated and multiplication module over a commutative ring R with identity. Then M is a Rad_g-lifting R-module if and only if R is a Rad_g-lifting ring.

Proof. Suppose M is a Rad_g-lifting R-module. Let I be an ideal of R such that $Rad_g(R) \subseteq I$. Hence, $Rad_g(R)M \subseteq IM$. Thus, by Lemma 2.6.1 $Rad_g(M) \subseteq IM$, and by the hypothesis, there is a decomposition $M = E \oplus W$ such that $E \leq IM$ and $IM \cap W \ll_g W$. We have access to $E = JM$ and $W = EM$ for some ideals J and E of R. Since, $M = JM \oplus EM$ such that $JM \leq IM$ and $IM \cap EM \ll_g EM$. From Lemmas 1.2.63, 1.2.64 and 1.2.62, we conclude that $R = J \oplus E, J \leq I$ and $I \cap E \ll_g E$. Therefore, R is a Rad_g-lifting ring.

Conversely, let R be a Rad_g-lifting ring. If $C = IM$ is any submodule of M with $Rad_g(M) \subseteq IM$, for some ideal I of R. By Lemma 2.6.1, $Rad_g(R)M \subseteq IM$, hence, $Rad_g(R) \subseteq I$ by Lemma 1.2.64. According to the hypothesis, there exists a decomposition $R = J \oplus E$ such that $J \leq I$ and $I \cap E \ll_g E$. Thus, $M = (J \oplus E)M = JM \oplus EM$ such that $JM \leq IM = C$. By Lemma 1.2.63 and 1.2.62, we have that $C \cap EM = IM \cap EM = (I \cap E)M \ll_g EM$. Hence, M is a Rad_g-lifting R-module. □

Theorem 4.3.2. Let M be an R-module and S a multiplicative closed subset of R such that $\mathcal{L}(T) \cap S = \emptyset$ for any $T \leq M$. Then M is a Rad_g-lifting as R-module if and only if $S^{-1}M$ is a Rad_g-lifting as $S^{-1}R$-module.

Proof. Suppose M is a Rad_g-lifting as R-module and let $S^{-1}E$ be a submodule of $S^{-1}M$ as $S^{-1}R$-module such that $Rad_g(S^{-1}M) \subseteq S^{-1}E$. From Lemma 2.3.2, $S^{-1}(Rad_g(M)) \subseteq S^{-1}E$, thus $Rad_g(M) \subseteq E$ by Lemma 1.2.67. Then there exists a submodule W of M such that $M = H \oplus W$ and $E \cap W \ll_g W$ where $H \leq E$. By Lemmas 1.2.71 and1.2.69 (2), we have $(S^{-1}H) \oplus (S^{-1}W) = S^{-1}M$ and $(S^{-1}E) \cap (S^{-1}W) \ll_g S^{-1}W$ and $S^{-1}H \leq S^{-1}E$. Thus, $S^{-1}M$ is a Rad_g-lifting as $S^{-1}R$-module.

Conversely, Suppose that $S^{-1}M$ is a Rad_g-lifting as $S^{-1}R$-module, and let E be a submodule of M as R-module such that $Rad_g(M) \subseteq E$, thus by Lemma 2.3.2 we get $Rad_g(S^{-1}M) = S^{-1}(Rad_g(M)) \subseteq S^{-1}E$. So, there exists a submodule $S^{-1}W$ of $S^{-1}M$ such that $(S^{-1}H) \oplus (S^{-1}W) = S^{-1}M$ and $(S^{-1}E) \cap (S^{-1}W) \ll_g S^{-1}W$ where $S^{-1}H \leq S^{-1}E$. By Lemmas 1.2.71, 1.2.65 and 1.2.69, we have $H \oplus W = M$ and $E \cap W \ll_g W$ and $H \leq E$, this implies M is a Rad_g-lifting as R-module. □

Theorem 4.3.3. Let M be an R-module and S a multiplicative closed subset of R such that for any $W \subset M$, $(W :_M s) = W$, for all $s \in S$. Then M is a Rad_g-lifting as R-module if and only if $S^{-1}M$ is a Rad_g-lifting as $S^{-1}R$-module.

Proof. Let M be a Rad_g-lifting as R-module, and $S^{-1}E$ be any submodule of $S^{-1}M$ as $S^{-1}R$-module such that $Rad_g(S^{-1}M) \subseteq S^{-1}E$, thus by Lemma 2.3.5, $S^{-1}(Rad_g(M)) \subseteq S^{-1}E$ and so $Rad_g(M) \subseteq E$ by Lemma 1.2.67. Then there exists $W \leq M$ such that $M = H \oplus W$ and $E \cap W \ll_g W$, where $H \leq E$. By Lemmas 1.2.71 and 1.2.70, $S^{-1}M = (S^{-1}H) \oplus (S^{-1}W)$ with $S^{-1}H \leq S^{-1}E$ and $(S^{-1}E) \cap (S^{-1}W) \ll_g S^{-1}W$. Hence $S^{-1}M$ is a Rad_g-lifting as $S^{-1}R$-module.

Conversely, suppose that $S^{-1}M$ be a Rad_g-lifting as $S^{-1}R$-module, and let E be a submodule of M as R-module such that $Rad_g(M) \subseteq E$. By Lemma 2.3.5, $Rad_g(S^{-1}M) = S^{-1}(Rad_g(M)) \subseteq S^{-1}E$. Then there exists a $S^{-1}W \leq S^{-1}M$ such that $(S^{-1}H) \oplus (S^{-1}W) = S^{-1}M$ and $(S^{-1}E) \cap (S^{-1}W) \ll_g S^{-1}W$ where

$S^{-1}H \leq S^{-1}E$. By Lemmas 1.2.71, 1.2.67 and 1.2.70 we deduce that $M = H \oplus W$ such that $H \leq E$ and $E \cap W \ll_g W$. Hence M is a Rad_g-lifting as R-module. □

4.4. Principally Rad_g-lifting modules

In this section we will introduce the definition of principally Rad_g-lifting modules, as well as several of their properties and examples about them will be discussed.

Definition 4.4.1. An R-module M is called principally Rad_g-lifting, briefly P-Rad_g-lifting, if for any cyclic submodule E of M contains $Rad_g(M)$, there is a decomposition $M = T \oplus N$ such that $T \leq E$ and $E \cap N \ll_g N$. In other words, for any $m \in M$ such that $mR \leq M$ with $Rad_g(M) \subseteq mR$, there are submodules T, N of M with $M = T \oplus N$, $T \leq mR$ and $mR \cap N \ll_g N$.

Remarks and Examples 4.4.2.

(1) Every principally semisimple module is P-Rad_g-lifting, such that for any semisimple ring R, all right R-modules are P-Rad_g-lifting, and each submodule of any principally semisimple module so is P-Rad_g-lifting.

(2) For any prime number p and any positive integer n**,** the $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z}$ is P-Rad_g-lifting, while $\mathbb{Z}$ as $\mathbb{Z}$-module is not P-Rad_g-lifting.

(3) Obviously, Rad_g-lifting modules are P-Rad_g-lifting. As application example, by Remarks and Examples 4.1.6, $\mathbb{Q}$ and $\mathbb{Z}_{24}$ as $\mathbb{Z}$-modules are Rad_g-lifting, and so they are P-Rad_g-lifting $\mathbb{Z}$-modules.

(4) If M is a principally g-lifting module then M is P-Rad_g-lifting. Moreover, if $Rad_g(M) = 0$, clearly, the converse is true.

(5) If M is a cyclic module over a PID, then we know that every submodule of M is also cyclic. Hence, any cyclic P-Rad_g-lifting module over a PID is Rad_g-lifting.

(6) If M is a module such that $Rad_g(M)$ is a non-cyclic maximal submodule of M, then M is a P-Rad_g-lifting module.

Proof. Suppose that $m \in M$ such that $Rad_g(M) \subseteq mR$. Since, $Rad_g(M)$ is a non-cyclic submodule, then $Rad_g(M) \neq mR$, this implies $Rad_g(M) \subset mR$. Since, $Rad_g(M)$ is maximal of M, thus $mR = M$, trivially, $M = M \oplus (0)$ such that $M \leq mR$ and $mR \cap (0) \ll_g (0)$. Therefore, M is P-Rad_g-lifting. □

Proposition 4.4.3. The following are equivalent for an R-module M.

(1) M is a P-Rad_g-lifting module.

(2) for any cyclic submodule E of M such that $Rad_g(M) \subseteq E$, there is a decomposition $M = T \oplus N$ such that $T \leq E$ and $E \cap N \ll_g M$;

(3) for any cyclic submodule E of M containing $Rad_g(M)$ can be written as $E = H \oplus G$, where H is a direct summand of M and $G \ll_g M$;

(4) for any cyclic submodule E of M containing $Rad_g(M)$, there exists a direct summand C of M such that $C \leq E$ and $E/C \ll_g M/C$;

(5) for any cyclic submodule E of M containing $Rad_g(M)$, E containing a g-supplement W in M such that $E \cap W$ is a direct summand of E;

(6) for any cyclic submodule E of M containing $Rad_g(M)$, there exists an $e = e^2 \in End(M)$ with $eM \leq E$ and $(1 - e)E \ll_g (1 - e)M$;

(7) for any cyclic submodule E of M containing $Rad_g(M)$, there exists a direct summand T of M and a g-small submodule W of M such that $T \leq E$ and $E = T + W$;

(8) for any cyclic submodule E of M containing $Rad_g(M)$, there is a submodule H of M inside E such that $M = H \oplus W$ and W a g-supplement of E in M;

(9) for each $m \in M$ with $Rad_g(M) \subseteq mR$, there are principal ideals I and J of R such that $mR = mI \oplus mJ$, where $mI \leq^{\oplus} M$ and $mJ \ll_g M$.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Clear.

$(\mathbf{2}) \Longrightarrow (\mathbf{3})$ Let E be a cyclic submodule of M with $Rad_g(M) \subseteq E$. By **(2)**, there is a decomposition $M = T \oplus N$ such that $T \leq E$ and $E \cap N \ll_g M$. By Modular law, $E = T \oplus (E \cap N)$ such that $T \leq^{\oplus} M$ and $E \cap N \ll_g M$.

$(\mathbf{3}) \Longrightarrow (\mathbf{4})$ Suppose E is any cyclic submodule of M such that $Rad_g(M) \subseteq E$. By **(3)**, $E = C \oplus G$, where $C \leq^{\oplus} M$ and $G \ll_g M$. Define, a natural map $\pi: M \longrightarrow M/C$. Since, $G \ll_g M$, we deduce $\pi(G) \ll_g M/C$, i.e., $(C + G)/C = E/C \ll_g M/C$.

$(\mathbf{4}) \Longrightarrow (\mathbf{5})$ Suppose that E is a cyclic submodule of M with $Rad_g(M) \subseteq E$. By **(4)**, there is a direct summand C of M such that $C \leq E$ and $E/C \ll_g M/C$, where $M = C \oplus W$ for some $W \leq M$. Therefore, $M = E + W$. By modular law, $E = C \oplus (E \cap W)$. So, $E/C \cong E \cap W$ and $M/C \cong W$. Thus, $E \cap W \ll_g W$. Hence E containing a g-supplement W in M and $E \cap W$ is a direct summand of E.

$(\mathbf{5}) \Longrightarrow (\mathbf{6})$ Let E be a cyclic submodule of M with $Rad_g(M) \subseteq E$. According to the hypothesis, if we suppose E containing a g-supplement W in M such that $E \cap W$ is a direct summand of E, then $M = E + W$ and $E \cap W \ll_g W$. Also, $E = (E \cap W) \oplus H$ for some $H \leq E$. Thus, $M = H \oplus W$. Suppose, that $e: M \to H$; $e(h + k) = h$ and $(1 - e): M \longrightarrow W$; $(1 - e)(h + k) = k$ are projection maps for all $h + k \in M$. Obviously, $e = e^2$ in $End(M)$. Therefore, $eM \leq E$ and $(1 - e)E = E \cap W \ll_g W = (1 - e)M$.

$(\mathbf{6}) \Longrightarrow (\mathbf{7})$ Suppose E is a cyclic submodule of M such that $Rad_g(M) \subseteq E$. By **(6)**, there is an $e = e^2 \in End(M)$ such that $eM \leq E$ and $(1 - e)E \ll_g (1 - e)M$. We know that $M = eM \oplus (1 - e)M$. We deduce that $E = E \cap M = E \cap (eM \oplus (1 - e)M) = eM \oplus (E \cap (1 - e)M) = eM \oplus (1 - e)E$. If we put

$T = eM$, $W = (1-e)E$ and $N = (1-e)M$. Thus, $T \leq E$ and $E = T + W$ where $W \ll_g N$ (so in M), and T is a direct summand of M.

$(\mathbf{7}) \Longrightarrow (\mathbf{8})$ Suppose E is a cyclic submodule of M such that $Rad_g(M) \subseteq E$. By **(7)**, there exists a direct summand T of M and $W \ll_g M$ such that $T \leq E$ and $E = T + W$. Thus, $M = T \oplus N$ for some $N \leq M$. Thus, N is a g-supplement of T in M, so Lemma 1.2.18 implies N is a g-supplement of $E = T + W$ in M.

$(\mathbf{8}) \Longrightarrow (\mathbf{1})$ Clear.

$(\mathbf{6}) \Longrightarrow (\mathbf{9})$ Suppose that $m \in M$ with $Rad_g(M) \subseteq mR$. By **(6)**, there is an $e = e^2 \in End(M)$ such that $eM \leq mR$ and $(1-e)mR \ll_g (1-e)M$. Notice that $M = eM \oplus (1-e)M$. Let $r \in R$ such that $mr = (1-e)m'$ for some $m' \in M$, then $m' = em' + mr \in mR$, since, $eM \leq mR$, and so $mR \cap (1-e)M \leq (1-e)mR$. Hence, $mR \cap (1-e)M = (1-e)mR$. By modular law, we have that $mR = mR \cap (eM \oplus (1-e)M) = eM \oplus (mR \cap (1-e)M) = eM \oplus (1-e)mR$. Put $I = \{s \in R : ms \in eM\}$ and $J = \{r \in R : mr \in (1-e)mR\}$. It follows that $mR = mI \oplus mJ$, where $mI = eM \leq^{\oplus} M$ and $mJ = (1-e) \ll_g mR(1-e)M$, hence in M.

$(\mathbf{9}) \Longrightarrow (\mathbf{1})$ By **(9)**, for any cyclic submodule E of M containing $Rad_g(M)$, there exists two ideals I and J such that $E = I \oplus J$, where I is direct summand of M and $J \ll_g M$. Thus, $M = I \oplus S$ for some $S \leq M$. Hence $E = I \oplus (E \cap S)$, by modular law, and so $E \cap S \cong J \ll_g M$. Since, $E \cap S \leq S \leq^{\oplus} M$, then $E \cap S \ll_g S$, from Lemma 1.2.17. Therefore, M is a P-Rad_g-lifting module. □

Corollary 4.4.4. Let M be a P-Rad_g-lifting R-module. Then for any indecomposable cyclic submodule E of M containing $Rad_g(M)$, either E is a direct summand or g-small.

Proof. Since, E is a cyclic submodule of M such that $Rad_g(M) \subseteq E$, so by Proposition 4.4.3, $E = H \oplus G$ where H is a direct summand of M and $G \ll_g M$.

Since, E is indecomposable, thus, either $E = H$ or $E = G$. This completes the proof. □

Corollary 4.4.5. Let M be any module over a local ring R. If M is P-$\mathrm{Rad_g}$-lifting, then any cyclic submodule of M containing $Rad_g(M)$ is, either a direct summand of M or g-small.

Proof. Let E be a cyclic submodule of M. Since, any cyclic module over a local ring is a local module, then E is a local R-submodule, and such that it is hollow, thus E is an indecomposable R-submodule. Thus, the result follows by Corollary 4.4.4. □

A P-$\mathrm{Rad_g}$-lifting is not inherited by their submodules in general, as we see, $\mathbb{Q}$ as a $\mathbb{Z}$-module is P-$\mathrm{Rad_g}$-lifting, while $\mathbb{Z} \leq \mathbb{Q}$ is not P-$\mathrm{Rad_g}$-lifting. In following, gives some conditions for a P-$\mathrm{Rad_g}$-lifting module to be inherited by their submodules.

Proposition 4.4.6. Let M be a P-$\mathrm{Rad_g}$-lifting R-module. Then, a submodule E of M with $Rad_g(M) \subseteq E$ is P-$\mathrm{Rad_g}$-lifting if, one of the following cases is hold.

(1) E is a direct summand of M.

(2) E is a cyclic g-coclosed submodule of M.

Proof. (**1**) Let $x \in E$ with $Rad_g(E) \subseteq xR$, where E is a direct summand of M. By Lemma 1.2.37, we have $Rad_g(M) = Rad_g(E)$. So, $Rad_g(M) \subseteq xR \leq M$. As, M is a P-$\mathrm{Rad_g}$-lifting, there is a decomposition $M = V \oplus \acute{V}$ with $V \leq xR$ and $xR \cap \acute{V} \ll_g \acute{V}$. As $V \leq E$, so by modular law, $E = E \cap \left(V \oplus \acute{V}\right) = V \oplus (E \cap \acute{V})$. Also $xR \cap \left(E \cap \acute{V}\right) = xR \cap \acute{V} \ll_g M$. As $xR \cap (E \cap \acute{V}) \leq E \cap \acute{V} \leq^{\oplus} M$, Lemma 1.2.17(1) implies $xR \cap \left(E \cap \acute{V}\right) \ll_g E \cap \acute{V}$. Hence E is P-$\mathrm{Rad_g}$-lifting.

(2) Let E be a cyclic g-coclosed submodule of M with $Rad_g(M) \subseteq E$. Since, M is a P-$\mathrm{Rad_g}$-lifting, so by Proposition 4.4.3(4), there is a direct summand L

of M such that $L \leq E$ and $E/L \ll_g M/L$. Since, E is a g-coclosed submodule of M, $L = E$, this implies $E \leq^{\oplus} M$. By **(1)**, E is P-Rad_g-lifting. □

Corollary 4.4.7. If M is a P-Rad_g-lifting R-module such that $Rad_g(M)$ is a direct summand of M, then $Rad_g(M)$ is P-Rad_g-lifting.

Proof. As $Rad_g(M) \leq^{\oplus} M$ and $Rad_g(M) \subseteq Rad_g(M)$, Proposition 4.4.6(1) imply $Rad_g(M)$ is P-Rad_g-lifting. □

Proposition 4.4.8. Let $M = C + W$ be a P-Rad_g-lifting R-module such that $C \leq M$ and $W \leq^{\oplus} M$. If $C \cap W$ is a cyclic submodule of M such that $Rad_g(M) \subseteq C \cap W$, then W containing a g-supplement of C in M.

Proof. Let $C \cap W$ be a cyclic submodule of M and $Rad_g(M) \subseteq C \cap W$. Since, M is a P-Rad_g-lifting module, we deduce by Proposition 4.4.3(3), $C \cap W = T \oplus N$ where T is a direct summand of M (therefore, in W) and $N \ll_g M$. Write $W = T \oplus E$, for some $E \leq W$. Thus, $C \cap W = T \oplus (C \cap E)$. Suppose that $\pi: W \longrightarrow E$ is the natural projection. As W is a direct summand of M and $N \ll_g M$, we have that $N \ll_g W$ and hence $\pi(N) \ll_g E$. But, $C \cap E = \pi\big(T \oplus (C \cap E)\big) = \pi(C \cap W) = \pi(T \oplus N) = \pi(N)$, so $C \cap E \ll_g E$. Moreover, $M = C + W = C + T + E = C + E$. Thu, W contains E as a g-supplement of C in M. □

Corollary 4.4.9. Let $M = C + mR$ be a P-Rad_g-lifting module over a PID R such that $C \trianglelefteq M$ and $m \in M$. If $Rad_g(M) \subseteq C \cap D$ for each $D \leq^{\oplus} mR$, then mR containing a g-supplement of C in M.

Proof. Assume $M = C + mR$ such that $C \trianglelefteq M$ and $m \in M$. Since, M is P-Rad_g-lifting, so by Proposition 4.4.3(3), we can write $mR = E \oplus S$, where $E \leq^{\oplus} M$ and $S \ll_g M$. So $M = C + mR = C + E + S$, and as $C \trianglelefteq M$ implies $C + E \trianglelefteq M$, and hence $M = C + E$ (since, $S \ll_g M$), where E is a cyclic direct summand of M, thus $C \cap E$ is a cyclic submodule of M containing $Rad_g(M)$ according to the hypothesis, and so by applying Proposition 4.4.8, E (so that, mR) containing a g-supplement of C in M. □

In general, we expect that the sum of two P-$\mathrm{Rad_g}$-lifting modules is not P-$\mathrm{Rad_g}$-lifting, but we could not find an example to confirm this. However, we now give a condition that make the class of P-$\mathrm{Rad_g}$-lifting modules is closed under finite direct sums.

Theorem 4.4.10. Let M be a duo (or, distributive) R-module and $M = \bigoplus_{i=1}^{n} M_i$, where $\{M_i |\ i = 1,2, \dots, n\}$ a finite family of P-$\mathrm{Rad_g}$-lifting modules. Then M is a P-$\mathrm{Rad_g}$-lifting R-module.

Proof. We will prove this in the case when $n = 2$. Let U be any cyclic submodule of a duo R-module $M = M_1 \oplus M_2$ and $Rad_g(M) \subseteq U$. Since, U is a fully invariant submodule of M, Lemma 1.2.14 implies $U = (M_1 \cap U) \oplus (M_2 \cap U)$. We have that $Rad_g(M_i) \subseteq M_i \cap U$ and $M_i \cap U$ is a cyclic submodule of M_i for $i = 1,2$. Since, M_i is P-$\mathrm{Rad_g}$-lifting, for $i = 1,2$, then there are decompositions $M_i = V_i \oplus W_i$ such that $V_i \leq M_i \cap U$ and $(M_i \cap U) \cap W_i = U \cap W_i \ll_g W_i$. Thus, $M = (V_1 \oplus V_2) \oplus (W_1 \oplus W_1)$, $V_1 \oplus V_2 \leq (M_1 \cap U) \oplus (M_2 \cap U) = U$ and $U \cap (W_1 \oplus W_1) = (U \cap W_1) \oplus (U \cap W_2) \ll_g W_1 \oplus W_1$, by Lemma 1.2.5(3). So by using mathematical induction, M is P-$\mathrm{Rad_g}$-lifting. Similarly, when M is a distributive R-module. □

Theorem 4.4.11. Let $M = M_1 \oplus M_2$ be a cyclic module over a PID R such that M_1 and M_2 are P-$\mathrm{Rad_g}$-lifting R-modules and M_1 is quasi-projective and M_2-projective, then M is P-$\mathrm{Rad_g}$-lifting.

Proof. Same as the proof of Theorem 4.1.15.

Corollary 4.4.12. Let $M = M_1 \oplus M_2$ be a cyclic module over a PID R such that M_1 is a principally semisimple R-module and M_2 a principally Rad_g-lifting R-module and they are relatively projective with M_1, then M is a principally Rad_g-lifting module.

Proof. From Proposition 4.4.2(1), we have that M_1 is P-Rad_g-lifting and so by Theorem 4.4.11, $M = M_1 \oplus M_2$ is P-Rad_g-lifting.

Theorem 4.4.13 Let M be a P-Rad_g-lifting R-module and suppose $E \leq M$. If for every direct summand C of M, $(C + E)/E$ is a direct summand of M/E. Then M/E is P-Rad_g-lifting.

Proof. Let $E \leq xR \leq M$ with $x \in M$ and $Rad_g(M/E) \subseteq xR/E$. Consider the natural map $\pi: M \to M/E$. By $Rad_g(M) \subseteq M$, we have that $\pi(Rad_g(M)) \subseteq Rad_g(M/E)$, i.e., $(Rad_g(M) + E)/E \subseteq Rad_g(M/E)$,

thus $(Rad_g(M) + E)/E \subseteq xR/E$, and hence $Rad_g(M) \subseteq xR$. Since, M is a P-Rad_g-lifting module, then by Proposition 4.4.3(4), there is a direct summand H of M such that $H \leq xR$ and $xR/H \ll_g M/H$. By hypothesis, $(H + E)/E$ is a direct summand of M/E. Clearly, $(H + E)/E \leq xR/E$. Suppose a projection map $\rho: \frac{M}{H} \to \frac{M/H}{(H+E)/H}$. Since, $xR/H \ll_g M/H$ then $\frac{xR}{H+E} \ll_g \frac{M}{H+E}$, this implies $\frac{xR/E}{(H+E)/E} \ll_g \frac{M/E}{(H+E)/E}$. Hence, M/E is a P-Rad_g-lifting module. □

Theorem 4.4.14. Let M be a P-Rad_g-lifting R-module and $E \leq M$ that satisfy one of the following:

(1) If E is a distributive submodule of M.

(2) If E is a fully invariant submodule of M.

(3) If Y is a submodule of M containing $Rad_g(M)$ such that $M = E \oplus Y$.

Then M/E is a P-Rad_g-lifting module.

Proof. (1) Assume $M = C \oplus \grave{C}$ for some $\grave{C} \leq M$. From Theorem 4.4.13, it is enough to conform that $(C + E)/E$ is a direct summand of M/E. It is easy to see that $M/E = ((C + E)/E) + ((\grave{C} + E)/E)$. Now, as E is a distributive sub module of M, $(C + E) \cap (\grave{C} + E) = (C \cap \grave{C}) + E = E$. Thus $((C + E)/E) \cap ((\grave{C} + E)/E) = 0$, therefore, M/E is a P-Rad_g-lifting module.

(2) Let H be a direct summand of M, then $M = H \oplus \grave{H}$ for some $\grave{H} \leq M$. As E is a fully invariant submodule of M, then $M/E = ((H + E)/E) \oplus ((H' + E)/E)$, by Lemma 1.2.19, i.e., $(H + E)/E$ is a direct summand of M/E. Hence, M/E is a P-Rad_g-lifting module, by Theorem 4.4.13.

(3) By Proposition 4.4.6(1), Y is a P-Rad_g-lifting module. Thus, $M/E \cong Y$, and then M/E is a P-Rad_g-lifting module. □

Corollary 4.4.15. Let M be a P-Rad_g-lifting R-module, then:

(1) If M is a distributive (or, duo) module, then every factor module of M is also P-Rad_g-lifting.

(2) If $f: M \to \grave{M}$ is a homomorphism containing distributive (or, fully invariant) kernel, then $f(M)$ is P-Rad_g-lifting. Moreover, if f is an epimorphism, then $\grave{M}$ is P-Rad_g-lifting.

Proof. (1) Clear from Theorem 4.4.14(1) and (2), respectively.

(2) Suppose $f: M \to \grave{M}$ is a homomorphism. By 1st isomorphism theorem, we have that $M/kerf \cong f(M)$. From Theorem 4.4.14(1) or (2), $M/kerf$ is P-Rad_g-lifting. Hence, $f(M)$ is P-Rad_g-lifting. □

Proposition 4.4.16. Let M be a weak duo R-module and L a direct summand of M. If M is a P-Rad_g-lifting module, then L and M/L are both P-Rad_g-lifting modules.

Proof. Suppose that M is a weak duo module and L a direct summand of M, then $M = L \oplus C$ where L, C are fully invariant submodules of M. By Theorem 4.4.14(2), M/L and $L \cong M/C$ are P-Rad_g-lifting modules. □

Proposition 4.4.17. Let M be an R-module and L a direct summand of M. Then M is P-Rad_g-lifting if and only if L and M/L are both P-Rad_g-lifting if, one of the following conditions hold:

(1) M is a distributive R-module.

(2) M is a duo R-module.

Proof. (1) Suppose that M is a distributive module and L a direct summand of M, so $M = L \oplus N$ for a submodule N of M. By Corollary 4.4.15(1), we have M/L is P-Rad_g-lifting. However, $L \cong M/N$, again by Corollary 4.4.15(1), L is P-Rad_g-lifting. Conversely, because $M \cong L \oplus (M/L)$, the result is included by Theorem 4.4.10.

(2) Since, every duo module is weak duo, then the result is follows by Proposition 4.4.16 and Theorem 4.4.10. □

Corollary 4.4.18. Assume M is a P-Rad_g-lifting module, then $M/P_g(M)$ is a P-Rad_g-lifting module.

Proof. From Lemma 1.2.36, $P_g(M)$ is a fully invariant submodule of M. Then by Theorem 4.4.14(2), $M/P_g(M)$ is a P-Rad_g-lifting module. □

Corollary 4.4.19. Let $M = \bigoplus_{i=1}^{n} M_i$ be a duo R-module. Then, for any $i = 1,2,\dots,n$, M_i is P-Rad_g-lifting if and only if M is a P-Rad_g-lifting.

Proof. It comes directly from Theorem 4.4.10 and Proposition 4.4.17(2). □

Theorem 4.4.20. Let M be a P-Rad_g-lifting R-module. Then for any cyclic submodule E of M containing $Rad_g(M)$, there is a direct summand C of M with $(C+E)/E \ll_g M/E$ and $(C+E)/C \ll_g M/C$. The converse hold, when M is g-noncosingular.

Proof. Suppose that M is a P-Rad_g-lifting module. If E is a cyclic submodule of M such that $Rad_g(M) \subseteq E$. Then there is a submodule C in E such that $M = C \oplus W$ and $E \cap W \ll_g W$. Since, $(E+C)/E = E/E \ll_g M/E$ and by

Modular law $E = C\oplus(E\cap W)$, so by 2nd isomorphism theorem, $E/C \cong E\cap W$ and $M/C \cong W$, therefore, $E/C \ll_g M/C$, this yield $(C + E)/C \ll_g M/C$.

Conversely, let E be a cyclic submodule of M with $Rad_g(M) \subseteq E$. From a condition, there is a $C \leq^{\oplus} M$ of M such that $(C + E)/E \ll_g M/E$ and $(C + E)/C \ll_g M/C$. Thus, $(C + E)/E$ is g-cosingular. By g-noncosingular for M, then $C + E$ is g-noncosingular, and so by Lemma 1.2.40 $(C + E)/E$ is g-noncosingular. Thus $C + E = E$. Therefore, $C \leq E$ and $E/C \ll_g M/C$. By Proposition 4.4.3(3), M is P-Rad_g-lifting. □

If the factor module of a module M is P-Rad_g-lifting, then it is not necessary for M to be a P-Rad_g-lifting, as seen in the following example.

Example 4.4.21. Suppose $R = \mathbb{Z}$ and $M = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$, where p is a prime number. It is clear that $\mathbb{Z}/p\mathbb{Z}$ is a P-Rad_g-lifting as a $\mathbb{Z}$-module since, it is simple. While the $\mathbb{Z}$-module $\mathbb{Z}$ is not a P-Rad_g-lifting module.

4.5. Connections with P-Rad_g-lifting modules

This section will discuss many relations between the principally Rad_g-lifting module concept and other types of modules.

First we will start with the following proposition.

Proposition 4.5.1. A principally generalized hollow module is P-Rad_g-lifting.

Proof. Suppose M is a principally generalized hollow module, and E a cyclic submodule of M with $Rad_g(M) \subseteq E$. If $E = M$, then there is a decomposition $M = M\oplus(0)$ such that $M \leq E$ and $E \cap (0) \ll_g (0)$. Let $E \subset M$, According to the hypothesis, $E \ll_g M$, then there is a decomposition $M = (0)\oplus M$ such that $(0) \leq E$ and $E \cap M = E \ll_g M$. Hence, M is a P-Rad_g-lifting module. □

The converse of Proposition 4.5.1 is not true, in general, as seen in the following example: The $\mathbb{Z}$-module $\mathbb{Z}_{24}$ is P-Rad_g-lifting, while $\mathbb{Z}_{24}$ is not principally generalized hollow as $\mathbb{Z}$-module, in fact, $3\mathbb{Z}_{24}$ is a proper cyclic submodule which not g-small in $\mathbb{Z}_{24}$.

Also, as application example of Proposition 4.5.1; Since, every finitely submodule in $\mathbb{Q}$ as $\mathbb{Z}$-module is small, then all cyclic submodules are g-small in $\mathbb{Q}$ as $\mathbb{Z}$-module, that is $\mathbb{Q}$ as $\mathbb{Z}$-module is principally generalized hollow, such that it is P-Rad_g-lifting.

Proposition 4.5.2. Let M be an R-module, suppose the following assertions:

(1) M is a P-Rad_g-lifting module.

(2) M is a principally sgrs$^{\oplus}$-module.

(3) M is a principally sgrs-module.

Then $(1) \Longrightarrow (2) \Longrightarrow (3)$. Moreover, if M is indecomposable, then $(2) \Longrightarrow (1)$. If M is refinable, $(3) \Longrightarrow (2)$. The three assertions are equivalent whenever M is an indecomposable refinable module.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Suppose M is a P-Rad_g-lifting module and let $x \in M$ such that $Rad_g(M) \subseteq xR$. By **(1)**, there is a decomposition $M = T \oplus N$ such that $T \leq xR$ and $xR \cap N \ll_g N$. Thus, $M = xR + N$ and $xR \cap N \ll_g N$ for some $N \leq^{\oplus} M$. Hence M is a principally sgrs$^{\oplus}$-module.

$(\mathbf{2}) \Longrightarrow (\mathbf{3})$ By Lemma 1.2.44.

$(\mathbf{2}) \Longrightarrow (\mathbf{1})$ Suppose that M is a principally sgrs$^{\oplus}$-module and let $x \in M$ such that $Rad_g(M) \subseteq xR$. If $xR = M$, trivially, there exists a decomposition $M = M \oplus (0)$ such that $M \leq xR$ and $xR \cap (0) \ll_g (0)$. Let $xR \neq M$, then there is a submodule N of M such that $M = T \oplus N = xR + N$ and $xR \cap N \ll_g N$. As M is an indecomposable module, either $N = 0$ or $N = M$. If $N = 0$, then $xR = M$, which is a contradiction. Thus, $T = 0$ and $N = M$. So, $T \leq xR$ and $xR \cap N \ll_g N$, hence $(\mathbf{1})$ holds.

$(\mathbf{3}) \Longrightarrow (\mathbf{2})$ By Lemma 1.2.46. □

Proposition 4.5.3. Let M be a P-Rad_g-lifting R-module and $0 \neq L \leq M$. If $L \cap Rad_g(M) = 0$, then L is principally semisimple.

Proof. Let $a \in L$. As M is a P-Rad_g-lifting R-module, there is a decomposition $M = T \oplus N$ such that $T \leq aR$ and $aR \cap N \ll_g N$, so in M. Since, $aR \cap N \subseteq Rad_g(M)$. By modular law, $L = L \cap M = L \cap (aR + N) = aR + (L \cap N)$. As $aR \cap (L \cap N) \subseteq L \cap Rad_g(M) = 0$, we get $L = aR \oplus (L \cap N)$. Therefore, $aR \leq^{\oplus} L$ and so L is principally semisimple. □

Corollary 4.5.4. If M is a P-Rad_g-lifting module, then M containing a principally semisimple submodule W such that $Rad_g(M) \oplus W \trianglelefteq M$.

Proof. Since, $Rad_g(M) \leq M$, by Zorn's lemma we may find a submodule W of M such that $Rad_g(M) \oplus W$ is an essential submodule of M. Also, we have $Rad_g(M) \cap W = 0$, then by Proposition 4.5.3, W is principally semisimple. □

Proposition 4.5.5. Let M be an R-module, suppose the following cases:

(1) M is a principally semisimple R-module.

(2) M is a principally g-lifting R-module.

(3) M is a P-Rad_g-lifting R-module.

Then $(\mathbf{1}) \Longrightarrow (\mathbf{2}) \Longrightarrow (\mathbf{3})$**.** If $Rad_g(M) = 0$, then $(\mathbf{3}) \Longrightarrow (\mathbf{1})$**.**

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2}) \Longrightarrow (\mathbf{3})$ Clear.

$(\mathbf{3}) \Longrightarrow (\mathbf{1})$ If $Rad_g(M) = 0$, then $M \cap Rad_g(M) = 0$ and so M is a principally semisimple R-module, by Proposition 4.5.3. □

Corollary 4.5.6. Let M be a P-Rad_g-lifting R-module with $Rad_g(M) = 0$, then every nonzero submodule of M is principally semisimple.

Proof. It comes directly by Proposition 4.5.5. □

Proposition 4.5.7. Let M be a P-Rad_g-lifting R-module with $Rad_g(M) = 0$. Then for each $m,n \in M$ with $M = mR + nR$, there is $H \leq^{\oplus} M$ such that $H \subseteq mR$ and $M = H + nR$.

Proof. Let $m, n \in M$ with $M = mR + nR$. As M is a P-Rad_g-lifting R-module and $Rad_g(M) = 0 \subseteq mR$, then there exists two submodules H and T of M such that $M = H \oplus T$, $H \leq mR$ and $mR \cap T \ll_g M$, by Proposition 4.4.3 By modular law, we deduce that $mR = H \oplus (mR \cap T)$. Moreover, as $mR \cap T \subseteq Rad_g(M) = 0$, so $mR = H$ which implies $M = H + nR$ and $H \leq^{\oplus} M$, as required. □

Proposition 4.5.8. If M is a P-Rad_g-lifting R-module, then $M/Rad_g(M)$ is principally semisimple.

Proof. Let $x \in M$ and $Rad_g(M) \subseteq xR$. According to the hypothesis, there is a decomposition $M = C \oplus \grave{C}$ for some $C \leq xR$ and $xR \cap \acute{C} \ll_g \acute{C}$ (also in M). Therefore, $M = xR + \grave{C}$ and $xR \cap \acute{C} \subseteq Rad_g(M)$. Since, $\frac{M}{Rad_g(M)} = \frac{xR}{Rad_g(M)} + \frac{\grave{C}+Rad_g(M)}{Rad_g(M)}$, and thus $\left(\frac{xR}{Rad_g(M)}\right) \cap \left(\frac{\grave{C}+Rad_g(M)}{Rad_g(M)}\right) = \frac{xR\cap(\grave{C}+Rad_g(M))}{Rad_g(M)} = \frac{Rad_g(M)+(xR\cap\grave{C})}{Rad_g(M)} = 0$, i.e., $xR/Rad_g(M)$ is a direct summand of $M/Rad_g(M)$. Hence, $M/Rad_g(M)$ is a principally semisimple module. □

Corollary 4.5.9. Let M be a P-Rad_g-lifting module then, $M/Rad_g(M)$ is a P-Rad_g-lifting module.

Proof. From Propositions 4.5.5 and 4.5.8. □

Proposition 4.5.10. Suppose M is a cyclic module over a PID R. If M is a P-Rad_g-lifting module satisfies DCC on g-small submodules. Then M is Artinian.

Proof. Let M be a P-Rad_g-lifting module. By Proposition 4.5.8, $M/Rad_g(M)$ is principally semisimple. Since, M is a cyclic R-module, then M is finitely generated and so $M/Rad_g(M)$ is finitely generated. Since, M is a cyclic module over a PID R, then all its submodules are cyclic, Therefore, M is principally semisimple imply M is semisimple, such that by Lemma 1.2.23, $M/Rad_g(M)$ is Artinian. Moreover, M satisfies DCC on g-small submodules implies that $Rad_g(M)$ is Artinian, according to Lemma 1.2.20. Hence, by Lemma 1.2.32, M is Artinian. □

Theorem 4.5.11. Suppose the following cases for a projective R-module M.

(1) For $x \in M$ with $Rad_g(M) \subseteq xR$, the factor M/xR containing a projective g-cover.

(2) M is a P-Rad_g-lifting module.

(3) M is a principally $\text{sgrs}^{\oplus}$-module.

(4) M is a principally sgrs-module.

(5) $M/Rad_g(M)$ is principally semisimple and for every cyclic submodule E of M containing $Rad_g(M)$ and $\bar{E} = E/Rad_g(M)$, there exists a direct summand C of M such that $\bar{E} = \bar{C}$.

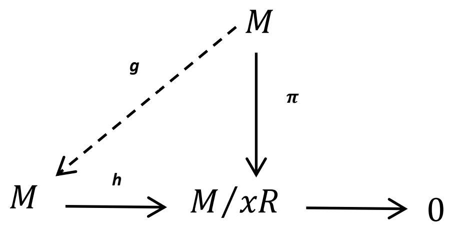
Then $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$ and $(2) \Rightarrow (5)$. If $Rad_g(M) \ll_g M$, then $(5) \Rightarrow (2)$.

Proof.$(\mathbf{1}) \Rightarrow (\mathbf{2})$ Let $x \in M$ such that $Rad_g(M) \subseteq xR$. From **(1)**, there is a projective R-module P and an R-epimorphism $\alpha: P \to M/xR$ such that $ker\alpha \ll_g P$. Suppose, $\pi: M \to M/xR$ is a natural epimorphism, then there is a homomorphism $\beta: M \to P$ such that $\alpha\beta = \pi$. Thus, $\alpha\big(\beta(M)\big) = \pi(M) = M/xR$. By taking the inverse of α, we deduce $P = \beta(M) + ker\alpha$. Since, $ker\alpha \ll_g P$, Lemma 1.2.4 implies $P = \beta(M) \oplus Y$ for some semisimple submodule Y of $ker\alpha$, this implies $\beta(M)$ is projective. Thus, $M = ker\beta \oplus W$ for some $W \leq M$. If $x \in ker\beta$, then $\beta(x) = 0$ and so $0 = \alpha\big(\beta(x)\big) = \pi(x)$, thus $x \in ker\pi$. Thus, $ker\beta \leq ker\pi = xR$. To conform that $xR \cap W \ll_g W$. As $ker\alpha \ll_g P$, $ker\alpha \cap \beta(W) = \beta(xR \cap W) \ll_g P$. By Lemma 1.2.17(1),

$\beta(xR \cap W) \ll_g \beta(M) = \beta(W)$. Therefore, $xR \cap W \ll_g W$, as β is an isomorphism from W into $\beta(W)$. Therefore, M is a P-Rad_g-lifting module.

(2) $\Rightarrow$ **(3)** $\Rightarrow$ **(4)** By Proposition 4.5.2.

(4) $\Rightarrow$ **(1)** Let $x \in M$ such that $Rad_g(M) \subseteq xR$. From (4), there exists a submodule T of M with $M = xR + T$ and $xR \cap T \ll_g T$. Let $h: M \to M/xR$ be a homomorphism defined by $h(m) = a + xR$, where $m = xr + a \in M$, for some $r \in R$ and $a \in T$. Let $\pi: M \to M/xR$ be the natural epimorphism. Since, M is projective, then there is a homomorphism $g: M \to M$ such that $hg = \pi$.



Hence, $h(g(M)) = \pi(M)$, such that $h^{-1}(h(g(M))) = h^{-1}(M/xR)$, Since, $M = g(M) + kerh = g(M) + (xR \cap T)$. Since, $xR \cap T \ll_g T$, so in M, then by Lemma 1.2.4, there exists a semisimple submodule Y of $xR \cap T$ such that $M = g(M) \oplus Y$ and such that $g(M)$ is a projective module. Hence, $g(M) \cong M/kerg$ implies $kerg$ is a direct summand of $g(M)$, so in M, this implies $M = kerg \oplus N$ for some submodule N of M. Since, N is projective. Let $(hg)|_N$ indicate the restriction of hg on N, that is $(hg)|_N: N \to M/xR$. Then $ker((hg)|_N) \leq xR \cap T$. Since, $ker((hg)|_{\boldsymbol{N}}) \ll_g M$. Since, $ker((hg)|_N) \subseteq N$ and N is a direct summand of M, Lemma 1.2.17(1) implies $ker((hg)|_N) \ll_g N$. Hence, N is a projective g-cover of M/xR. This completes the proof.

(2) $\Rightarrow$ **(5)** By Proposition 4.5.9, $M/Rad_g(M)$ is principally semisimple. Let, E be any cyclic submodule of M where $Rad_g(M) \subseteq E$. Put, $\bar{E} = E/Rad_g(M)$. By Proposition 4.4.3(3), there is a decomposition $E = C \oplus W$ such that C is a direct summand of M and $W \ll_g M$, thus $W \subseteq Rad_g(M)$, and hence

$C + Rad_g(M) = E$. Thus, $E/Rad_g(M) = (C + Rad_g(M))/Rad_g(M)$, and hence $\bar{E} = \bar{C}$.

$(\mathbf{5}) \Longrightarrow (\mathbf{2})$ Let E be a cyclic submodule of M such that $Rad_g(M) \subseteq E$. Since, $M/Rad_g(M)$ is semisimple, thus we conclude that $M/Rad_g(M) = \left(E/Rad_g(M)\right) \oplus \left(W/Rad_g(M)\right) = \bar{E} \oplus \left(W/Rad_g(M)\right)$ where $Rad_g(M) \leq W$ for some submodule W of M. By **(2)**, there exists a direct summand T of M such that $M = T \oplus N$ for some submodule N of M, and $\bar{E} = \bar{T}$. Thus $M/Rad_g(M) = \bar{T} \oplus \left((N + Rad_g(M))/Rad_g(M)\right)$, since $W = N + Rad_g(M)$. Hence, $M = T + N + Rad_g(M) = E + N$. Since, $M = E + N$ is projective, Lemma 1.2.32 implies that $M = E' \oplus N$ with $E' \subseteq E$. Also, $E \cap N \leq E \cap W = Rad_g(M) \ll_g M$, hence M is a P-Rad_g-lifting module. □

Corollary 4.5.12. The following are equivalent for a ring R.

(1) For each principal ideal $I \leq R$ with $Rad_g(R) \subseteq I$, the factor R/I containing a projective g-cover.

(2) R is a P-Rad_g-lifting ring.

(3) R is a principally sgrs$^{\oplus}$-ring.

(4) R is a principally sgrs-ring.

(5) $R/Rad_g(R)$ is principally semisimple and for every cyclic ideal I of R containing $Rad_g(R)$ and $\bar{I} = I/Rad_g(R)$, there is a direct summand J of R with $\bar{I} = \bar{J}$.

Proof. Since, $R = \langle 1 \rangle$ is a free R-module, so it is finitely generated projective. By Lemma 1.2.12, $Rad_g(R) \ll_g R$. Thus, we obtained the result by Theorem 4.5.11. □

Corollary 4.5.13. Let R be a semisimple ring and M an R-module. Then the following are equivalent.

(1) For $x \in M$ with $Rad_g(M) \subseteq xR$, the factor M/xR containing a projective g-cover

(2) M is a P-Rad_g-lifting ring.

(3) M is a principally sgrs$^{\oplus}$-ring.

(4) M is a principally sgrs-ring.

Proof. Since, every module over a semisimple ring is projective. Then the result is obtained by Theorem 4.5.11. □

Proposition 4.5.14. Let M be a P-Rad_g-lifting R-module with $Rad_g(M) \neq M$ is cyclic. Then there is a decomposition $M = H \oplus W$ with W is a g-supplement of $Rad_g(M)$ in M, $Rad_g(W) \ll_g W$ and H is a g-Radical. Moreover, if $P_g(M) = 0$, then $Rad_g(M) \ll_g M$.

Proof. Suppose that $Rad_g(M) \neq M$ is a cyclic submodule. Since, M is a P-Rad_g-lifting module and $Rad_g(M) \subseteq Rad_g(M)$, so by Proposition 4.4.3(8), there is a submodule H of M in $Rad_g(M)$ such that $M = H \oplus W$ and W a g-supplement of $Rad_g(M)$ in M, i.e., $M = Rad_g(M) + W$ and $Rad_g(M) \cap W \ll_g W$. Since, W is a direct summand of M, so it is a g-supplement and so $Rad_g(M) \cap W = Rad_g(W)$, by Lemma 1.2.17, implies $Rad_g(W) \ll_g W$. By Lemma 1.2.10, $M = Rad_g(M) + W = Rad_g(H) \oplus W$. By Modular law, we get $H = H \cap (Rad_g(H) \oplus W) = Rad_g(H) \oplus (W \cap H) = Rad_g(H)$. Thus, H is a g-Radical. Now, assume $P_g(M) = 0$. Therefore, $H = 0$, which conclude that $M = W$, hence $Rad_g(M) \ll_g M$. □

The converse of Proposition 4.5.14 need not be true, in general, for example, for the $\mathbb{Z}$-module $\mathbb{Z}$, we have $Rad_g(\mathbb{Z}) = 0 \neq \mathbb{Z}$ is cyclic and a decomposition $\mathbb{Z} = \mathbb{Z} \oplus (0)$ such that $\mathbb{Z}$ is a g-supplement of $Rad_g(\mathbb{Z}) = 0$, $Rad_g(\mathbb{Z}) = 0 \ll_g \mathbb{Z}$ and (0) is a g-Radical, while $\mathbb{Z}$ is not P-Rad_g-lifting $\mathbb{Z}$-module.

Proposition 4.5.15. Let M be an indecomposable module with $Rad_g(M) \neq M$ is cyclic. If M is a P-$\mathrm{Rad_g}$-lifting module, then $Rad_g(M) \ll_g M$.

Proof. Suppose that M is an indecomposable P-$\mathrm{Rad_g}$-lifting module. Since, $Rad_g(M) \subseteq Rad_g(M)$ is cyclic, then by Proposition 4.5.14, there is a unique decomposition $M = M \oplus 0$ such that M is a g-supplement of $Rad_g(M)$ and 0 is a g-Radical. So, $Rad_g(M) = Rad_g(M) \cap M \ll_g M$, as required. □

Corollary 4.5.16. Let M be an indecomposable R-module such that $Rad_g(M)$ is cyclic. If M is a principally $\mathrm{Rad_g}$-lifting module, then either.

(1) M is a cyclic module, or

(2) $Rad_g(M)$ is a g-small submodule of M.

Proof. Suppose M is an indecomposable and principally $\mathrm{Rad_g}$-lifting module such that $Rad_g(M) = aR$ for some $a \in M$. If M is not cyclic, therefore $M \neq aR$ and so $Rad_g(M) \neq M$. This implies $Rad_g(M)$ is a proper cyclic submodule of M, so by Proposition 4.5.15, $Rad_g(M) \ll_g M$. □

Proposition 4.5.17. Let M be an indecomposable R-module containing a cyclic generalized Radical. Then M is P-$\mathrm{Rad_g}$-lifting if and only if, either M is g-Radical or $Rad_g(M)$ is an essential maximal and g-small submodule of M.

Proof. Same as the proof of Theorem 4.2.31. □

Corollary 4.5.18. Let M be an indecomposable module with $Rad_g(M) \neq M$ is cyclic. If M is a P-$\mathrm{Rad_g}$-lifting module, then M is local.

Proof. By Proposition 4.5.17, M is a g-local module. Therefore, M is a local module, by Lemma 1.2.34. □

Proposition 4.5.19. Let M be an R-module, consider the following cases:

(1) M is principally semisimple.

(2) M is principally g-lifting.

(3) M is a P-$\mathrm{Rad_g}$-lifting module

(4) M is a principally sgrs$^{\oplus}$-module.

(5) M is a principally sgrs-module.

Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (5)$. If $Rad_g(M) = 0$, then $(5) \Longrightarrow (1)$.

Proof. (1) ⟹ (2) Clear.

(2) ⟹ (3) By Remarks and Examples 4.4.2(4).

(3) ⟹ (4) ⟹ (5) By Proposition 4.5.2.

$(\mathbf{5}) \Longrightarrow (\mathbf{1})$ By Lemma 1.2.59. □

The following consequence is immediately from Lemma 1.2.21 and Proposition 4.5.19.

Corollary 4.5.20. The following are equivalent for an g-noncosingular R-module M.

(1) M is principally semisimple.

(2) M is principally g-lifting.

(3) M is a P-$\mathrm{Rad_g}$-lifting module

(4) M is a principally sgrs$^{\oplus}$-module.

(5) M is a principally sgrs-module.

Proposition 4.5.21. The following are equivalent for an indecomposable R-module M.

$(\mathbf{1})$ M is a P-$\mathrm{Rad_g}$-lifting module.

$(\mathbf{2})$ For $m \in M$ with $Rad_g(M) \subseteq mR \neq M$, $mR \ll_g M$.

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$ Let $Rad_g(M) \subseteq mR \neq M$ and $m \in M$. As M is a P-$\mathrm{Rad_g}$-lifting module, by Proposition 4.4.3(8), there are submodules H, G of M such that $G \leq mR$, $M = mR + H = G \oplus H$ and $mR \cap H \ll_g H$. Now, if $H = 0$ then $mR = M$ which is a contradiction. By hypothesis, $H = M$ and $G = 0$. Therefore, $mR \ll_g M$.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$ Let $m \in M$ such that $Rad_g(M) \subseteq mR$. If $mR = M$, trivially, there is a decomposition $M = M \oplus (0)$ with $M \leq mR$ and $mR \cap (0) = (0) \ll_g (0)$. If we let $mR \neq M$, according to the hypothesis, we get $mR \ll_g M$. Trivially, $M = (0) \oplus M$ such that $(0) \leq mR$ and $mR \cap M = mR \ll_g M$. Hence, by two cases, **(1)** is hold. □

Corollary 4.5.22. The following are equivalent for a uniform R-module M.

(1) M is a P-Rad_g-lifting module.

(2) For $m \in M$ with $Rad_g(M) \subseteq mR \neq M$, $mR \ll_g M$.

Proof. From Lemma 1.2.47 every uniform module is indecomposable. Then the result is followed by Corollary 4.5.21. □

Proposition 4.5.23. Let M be a P-Rad_g-lifting module containing a cyclic generalized Radical. Then $M = L \oplus K$ such that $Rad_g(L) \ll_g L$ and $Rad_g(K) = K$.

Proof. It comes by Proposition 4.5.2 and Lemma 1.2.45. □

Proposition 4.5.24. Let M be an e-noncosingular R-module. Then M is P-Rad_g-lifting if and only if $M/Rad_g(M)$ is P-Rad_g-lifting.

Proof. By Lemma 1.2.21, $Rad_g(M) = 0$, such that $M/Rad_g(M) \cong M$, as required. □

Theorem 4.5.25. Let R be any ring with $Rad_g(R) = 0$. The following are equivalent.

(1) Every projective R-module is a P-Rad_g-lifting module.

(2) Every free R-module is a P-Rad_g-lifting module.

(3) Every projective R-module is principally semisimple.

(4) Every free R-module is principally semisimple.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Clear.

$(\mathbf{2}) \Longrightarrow (\mathbf{1})$ Let H be a projective R-module. By Lemma 1.2.27, there exists a free R-module F such that $H \leq^{\oplus} F$. By **(2)**, F is a P-Rad_g-lifting module. Since, F is projective, by [42] every g-small submodule is δ-small, thus $Rad_g(F) = \delta(F) = F.\delta(R) \leq F.Rad_g(R) = 0$, by Lemma 1.2.73. Hence, Proposition 4.5.5 implies F is a principally semisimple R-module and then H is principally semisimple, so again by Proposition 4.5.5, H is P-Rad_g-lifting.

$(\mathbf{1}) \Longrightarrow (\mathbf{3})$ Let H be a projective R-module. By **(1)**, H is a P-Rad_g-lifting module. We deduce $Rad_g(H) = 0$, Since, $Rad_g(R) = 0$, Therefore, by Proposition 4.5.5, H is a principally semisimple R-module.

$(\mathbf{3}) \Longrightarrow (\mathbf{1})$ By Proposition 4.5.5.

$(\mathbf{2}) \Longrightarrow (\mathbf{4})$ Similar to proof $(\mathbf{1}) \Longrightarrow (\mathbf{3})$.

$(\mathbf{4}) \Longrightarrow (\mathbf{2})$ By Proposition 4.5.5. □

4.6. Rings and localization of P-Rad_g-lifting

Firstly, we will deal and study the P-Rad_g-lifting property as a ring. Also, we are going to study the behavior of P-Rad_g-lifting modules under localization.

Theorem 4.6.1. Let M be a faithful, finitely generated and multiplication module over a commutative ring R. Then M is a P-Rad_g-lifting R-module if and only if R is a P-Rad_g-lifting ring.

Proof. Suppose M is a P-Rad_g-lifting R-module. Let I be a cyclic ideal of R such that $Rad_g(R) \subseteq I$. As M is a finitely generated multiplication R-module, then IM is a cyclic submodule of M. By Lemma 2.6.1, $Rad_g(M) = Rad_g(R)M \subseteq IM$. So, by the hypothesis, there exists a decomposition $M = E \oplus W$ such that $E \leq IM$ and $IM \cap W \ll_g W$. We have $E = JM$ and $W = EM$ for some ideals J and E of R. Since, $M = JM \oplus EM$ with $JM \leq IM$

and $IM \cap EM \ll_g EM$. By Lemmas 1.2.63, 1.2.64 and 1.2.62, we deduce $R = J \oplus E, J \leq I$ and $I \cap E \ll_g E$. Therefore, R is a P-$\mathrm{Rad_g}$-lifting ring.

Conversely, assume R be a P-$\mathrm{Rad_g}$-lifting ring. If $C = IM$ is a cyclic submodule of M such that $Rad_g(M) \subseteq IM$ for some ideal I of R. Since, M is a finitely generated multiplication R-module, then I is cyclic in R. By Lemma 2.6.1, $Rad_g(R)M \subseteq IM$, hence, $Rad_g(R) \subseteq I$ by Lemma 1.2.64. According to the hypothesis, there is a decomposition $R = J \oplus E$ such that $J \leq I$ and $I \cap E \ll_g E$. Thus, $M = (J \oplus E)M = JM \oplus EM$ such that $JM \leq IM = C$. By Lemmas 1.2.63 and 1.2.62, $C \cap EM = IM \cap EM = (I \cap E)M \ll_g EM$. Thus, M is a P-$\mathrm{Rad_g}$-lifting R-module. □

Theorem 4.6.2. Let M be an R-module and S a multiplicative closed subset of R such that $\mathcal{L}(T) \cap S = \emptyset$ for any $T \leq M$. Then M is a P-$\mathrm{Rad_g}$-lifting as R-module if and only if $S^{-1}M$ is a P-$\mathrm{Rad_g}$-lifting as $S^{-1}R$-module.

Proof. Suppose that M is a P-$\mathrm{Rad_g}$-lifting as R-module and let $S^{-1}E$ be a cyclic submodule of $S^{-1}M$ as $S^{-1}R$-module such that $Rad_g(S^{-1}M) \subseteq S^{-1}E$. By Lemma 1.2.66, E is cyclic in M as R-module. From Lemma 2.3.2, $S^{-1}(Rad_g(M)) \subseteq S^{-1}E$, thus $Rad_g(M) \subseteq E$ by Lemma 1.2.65. Then there exists a submodule W of M such that $M = H \oplus W$ and $E \cap W \ll_g W$ where $H \leq E$. By Lemmas 1.2.71, 1.2.66 and 1.2.69(2), we have that $(S^{-1}H) \oplus (S^{-1}W) = S^{-1}M$ and $(S^{-1}E) \cap (S^{-1}W) \ll_g S^{-1}W$ and $S^{-1}H \leq S^{-1}E$. Hence $S^{-1}M$ is a P-$\mathrm{Rad_g}$-lifting as $S^{-1}R$-module.

Conversely, Suppose that $S^{-1}M$ is a P-$\mathrm{Rad_g}$-lifting as $S^{-1}R$-module, and let E be a cyclic submodule of M as R-module such that $Rad_g(M) \subseteq E$, Thus by Lemma 2.3.2 and Lemma 1.2.65 $Rad_g(S^{-1}M) = S^{-1}(Rad_g(M)) \subseteq S^{-1}E$. From Lemma 1.2.66, $S^{-1}E$ is cyclic in $S^{-1}M$ as $S^{-1}R$-module. From the hypothesis, there is a decomposition $(S^{-1}H) \oplus (S^{-1}W) = S^{-1}M$ and $(S^{-1}E) \cap$

$(S^{-1}W) \ll_g S^{-1}W$ where $S^{-1}H \leq S^{-1}E$. By Lemmas 1.2.71, 1.2.65 and 1.2.69(2), we deduce $H \oplus W = M$ and $E \cap W \ll_g W$ and $H \leq E$, that is M is a P-Rad_g-lifting as R-module. □

Theorem 4.6.3. Suppose that M be an R-module and S a multiplicative closed subset of R such that for any $W \subset M$, $(W :_M s) = W$, for all $s \in S$. Then M is a P-Rad_g-lifting as R-module if and only if $S^{-1}M$ is a P-Rad_g-lifting as $S^{-1}R$-module.

Proof. Let M be a P-Rad_g-lifting as R-module, and $S^{-1}E$ be a cyclic submodule of $S^{-1}M$ as $S^{-1}R$-module such that $Rad_g(S^{-1}M) \subseteq S^{-1}E$. By Lemma 1.2.68, E is cyclic in M as R-module. Also, by Lemma 2.3.2, $S^{-1}(Rad_g(M)) \subseteq S^{-1}E$ and so $Rad_g(M) \subseteq E$ by Lemma 1.2.67. By the hypothesis, there exists a decomposition $M = H \oplus W$ such that $H \leq E$ and $E \cap W \ll_g W$. From Lemmas 1.2.71, 1.2.67 and 1.2.70(2), we deduce that $S^{-1}M = (S^{-1}H) \oplus (S^{-1}W)$, $S^{-1}H \leq S^{-1}E$ and $(S^{-1}E) \cap (S^{-1}W) \ll_g S^{-1}W$. Hence, $S^{-1}M$ is a P-Rad_g-lifting as $S^{-1}R$-module.

Conversely, suppose that $S^{-1}M$ be a P-Rad_g-lifting as $S^{-1}R$-module, and let E be a cyclic submodule of M as R-module such that $Rad_g(M) \subseteq E$. From Lemma 1.2.68, $S^{-1}E$ is cyclic in $S^{-1}M$ as $S^{-1}R$-module, also by Lemmas 2.2.7 and 1.2.67, $Rad_g(S^{-1}M) = S^{-1}(Rad_g(M)) \subseteq S^{-1}E$. According to the hypothesis, then there is a $S^{-1}W \leq S^{-1}M$ such that $(S^{-1}H) \oplus (S^{-1}W) = S^{-1}M$ and $(S^{-1}E) \cap (S^{-1}W) \ll_g S^{-1}W$ where $S^{-1}H \leq S^{-1}E$. From Lemmas 1.2.71, 1.2.67 and 1.2.70(2) we deduce that $M = H \oplus W$ such that $H \leq E$ and $E \cap W \ll_g W$. Hence, M is a P-Rad_g-lifting as R-module. □

This chapter consists of six sections. In sections one and four, we define and study two notions, G-$\mathrm{Rad_g}$-lifting modules and PG-$\mathrm{Rad_g}$-lifting respectively. Features, characterizations and illustrated examples, factors and direct summands of these modules are presented. In sections two and five, we investigate some relations between our definitions and other kinds of modules. In sections three and six, we discuss the behavior of G-$\mathrm{Rad_g}$-lifting modules and PG-$\mathrm{Rad_g}$-lifting as rings. In addition, the localization of these ideas is studied.

5.1. Generalized $\mathrm{Rad_g}$-lifting modules

We will introduce the notion of G-$\mathrm{Rad_g}$-lifting modules, as well as several of their properties and examples about them will be discussed in this section.

Definition 5.1.1. A module M is said to be generalized $\mathrm{Rad_g}$-lifting, briefly G-$\mathrm{Rad_g}$-lifting if, every $E \leq M$ with $Rad_g(M) \subseteq E$ there is a decomposition $M = T \oplus N$ such that $T \leq E$ and $E \cap N \subseteq Rad_g(N)$. A ring R is said to be G-$\mathrm{Rad_g}$-lifting if, R_R is G-$\mathrm{Rad_g}$-lifting.

Beginning, we will give some characterizations of G-$\mathrm{Rad_g}$-lifting modules and prove the equivalency among them.

Proposition 5.1.2. The following are equivalent for a module M.

(1) M is a G-$\mathrm{Rad_g}$-lifting module.

(2) for every submodule E of M such that $Rad_g(M) \subseteq E$ can be written as $E = H \oplus G$, where H is a direct summand of M and $G \subseteq Rad_g(M)$;

(3) for every submodule E of M such that $Rad_g(M) \subseteq E$, there is a direct summand C. of M such that $C \leq E$ and $E/C \subseteq Rad_g(M/C)$;

(4) for every submodule E of M such that $Rad_g(M) \subseteq E$, E containing a g-Radical supplement C in M such that $E \cap C$ is a direct summand of E;

(5) for every submodule E of M such that $Rad_g(M) \subseteq E$, there exists a submodule T of M inside E such that $M = T \oplus C$ and C is a g-Radical supplement of E in M.

(6) for every submodule E of M such that $Rad_g(M) \subseteq E$, there is an $e = e^2 \in End(M)$ with $eM \leq E$ and $(1-e)E \subseteq Rad_g(1-e)M$

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Let $E \leq M$ with $Rad_g(M) \subseteq E$. By **(1)**, there is a decomposition $M = H \oplus S$ such that $H \leq E$ and $E \cap S \subseteq Rad_g(S)$. By the modular law, we have $E = H \oplus (E \cap S)$. Putting $E \cap S = G$. Thus $E = H \oplus G$ such that $H \leq^{\oplus} M$ and $G \subseteq Rad_g(S) \subseteq Rad_g(M)$.

$(\mathbf{2}) \Longrightarrow (\mathbf{3})$ Suppose that $E \leq M$ such that $Rad_g(M) \subseteq E$. We have from **(2)** a decomposition $E = C \oplus W$ such that $C \leq^{\oplus} M$ and $W \subseteq Rad_g(M)$. Suppose, the natural map $\pi: M \longrightarrow M/C$. Since, $W \subseteq Rad_g(M)$, we deduce that $\pi(W) \subseteq Rad_g(M/C)$, i.e., $(C+W)/C = E/C \subseteq Rad_g(M/C)$.

$(\mathbf{3}) \Longrightarrow (\mathbf{1})$ Let $E \leq M$ with $Rad_g(M) \subseteq E$. By **(3)**, there is a decomposition $M = C \oplus W$ such that $C \leq E$ and $E/C \subseteq Rad_g(M/C)$. By the modular law, we have $E = C \oplus (E \cap W)$. Since, $E/C \cong E \cap W$ and $M/C \cong W$. As a result of this, $E \cap W \subseteq Rad_g(W)$. Hence, (1) holds.

$(\mathbf{3}) \Longrightarrow (\mathbf{4})$ Suppose that E is a submodule of M with $Rad_g(M) \subseteq E$. By **(3)**, there exists a direct summand C of M such that $C \leq E$ and $E/C \subseteq Rad_g(M/C)$, where $M = C \oplus W$ for some $W \leq M$. Since, $M = E + W$. By modular law, $E = C \oplus (E \cap W)$. So, $E/C \cong E \cap W$ and $M/C \cong W$. Thus, $E \cap W \subseteq Rad_g(W)$. Hence, E containing a g-Radical supplement W in M and $E \cap W \leq^{\oplus} E$.

$(\mathbf{4}) \Longrightarrow (\mathbf{6})$ Let E be a submodule of M with $Rad_g(M) \subseteq E$. According to the hypothesis, E containing a g-Radical supplement W in M such that $E \cap W$ is a direct summand of E, then $M = W + E$, $E \cap W \subseteq Rad_g(W)$ and $E = (E \cap W) \oplus H$ for some $H \leq E$. Thus, $M = W + (E \cap W) \oplus H = W \oplus H$. let $e: M \to H$; $e(k + h) = h$ and $(1 - e): M \longrightarrow W$; $(1 - e)(k + h) = k$ are projection maps for all $k + h \in M$. Obviously, $e = e^2$ in $End(M)$. Since, $eM = H \leq E$ and so $(1 - e)E = E \cap (1 - e)M = E \cap W \subseteq Rad_g(W) = Rad_g(1 - e)M$.

$(\mathbf{6}) \Longrightarrow (\mathbf{3})$ Let E be any submodule of M with $Rad_g(M) \subseteq E$. By assuming, there is an $e = e^2 \in End(M)$ with $eM \leq E$, and $(1 - e)E \subseteq Rad_g(1 - e)M$. Put $T = eM$ and $N = (1 - e)M$. As $e \in End(M)$ is an idempotent, so it is clear that $M = T \oplus N$. So $M = E + N$. Also, $E \cap N = E \cap (1 - e)M = (1 - e)E \subseteq Rad_g(1 - e)M = Rad_g(N)$, hence N is g-Radical supplement of E. Since, by Lemma 1.2.41 $(T + N)/T$ is g-Radical supplement of E/T in M/T. Hence $E/T = E/T \cap M/T = E/T \cap (T + N)/T \subseteq Rad_g(M/T)$.

$(\mathbf{1}) \Longrightarrow (\mathbf{5})$ Let $E \leq M$ with $Rad_g(M) \subseteq E$. As M is a G-$\mathrm{Rad_g}$-lifting module, so there is a decomposition $M = T \oplus C$ such that $T \leq E$ and $E \cap C \subseteq Rad_g(C)$. Thus, $M = E + C$. Hence, C is a g-Radical supplement of E in M.

$(\mathbf{5}) \Longrightarrow (\mathbf{1})$ Clear. □

Remarks and Examples 5.1.3.

(1) It is obvious to ensure that every $\mathrm{Rad_g}$-lifting module is G-$\mathrm{Rad_g}$-lifting.

(2) The G-$\mathrm{Rad_g}$-lifting module not inherited by its submodules, for instance $\mathbb{Q}$ as $\mathbb{Z}$-module is G-$\mathrm{Rad_g}$-lifting since $\mathbb{Q}$ is a $\mathrm{Rad_g}$-lifting $\mathbb{Z}$-module, while $\mathbb{Z} \leq \mathbb{Q}$ as $\mathbb{Z}$-module is not G-$\mathrm{Rad_g}$-lifting, Since $Rad_g(\mathbb{Z}) = 0 \subseteq n\mathbb{Z}$ for any $n\mathbb{Z} \leq \mathbb{Z}$ where $n \in \mathbb{N} - \{0,1\}$. Since $\mathbb{Z} = \mathbb{Z} \oplus 0$ is the only decomposition of $\mathbb{Z}$-module $\mathbb{Z}$, then $0 \leq n\mathbb{Z}$ but $n\mathbb{Z} \cap \mathbb{Z} = n\mathbb{Z} \nsubseteq 0 = Rad_g(\mathbb{Z})$.

(3) For all prime number p and $n \in \mathbb{Z}^+$, $\mathbb{Z}_{p^n}$ as $\mathbb{Z}$-module is generalized hollow, hence its $\mathrm{Rad_g}$-lifting by Proposition 4.2.1, and so it is G-$\mathrm{Rad_g}$-lifting by **(1)**.

(4) It is clear that any semisimple module is G-Rad$_g$-lifting, in fact, semisimple modules are generalized hollow, however, the converse is not true, in general, in a $\mathbb{Z}$-module $\mathbb{Q}$ is G-Rad$_g$-lifting but not semisimple.

Proposition 5.1.4. Let M be a module. Then M is G-Rad$_g$-lifting if and only if any direct summand E of M containing $Rad_g(M)$ is G-Rad$_g$-lifting.
Proof. $\Longrightarrow$) Suppose that $E \leq^{\oplus} M$ such that $Rad_g(M) \subseteq E$. Let $W \leq E$ with $Rad_g(E) \subseteq W$, then from Lemma 1.2.37 we have that $Rad_g(M) = Rad_g(E)$. Hence $Rad_g(M) \subseteq W$. Since M is a G-Rad$_g$-lifting module, Proposition 5.1.2(2) implies $W = T \oplus N$ such that T is direct summand of M and $N \subseteq Rad_g(M)$. Thus $M = T \oplus D$ for some $D \leq M$. Hence, by modular law $E = T \oplus (E \cap D)$, i.e., $W = T \oplus N$ such that T is direct summand of E and $N \subseteq Rad_g(M) = Rad_g(E)$. Thus by Proposition 5.1.2(2), E is G-Rad$_g$-lifting.

$\Longleftarrow$) Trivially, $M \leq^{\oplus} M$ such that $Rad_g(M) \subseteq M$. Since, by assuming M is a G-Rad$_g$-lifting module. □

Corollary 5.1.5. Let M be a G-Rad$_g$-lifting module such that $Rad_g(M)$ is a direct summand of M, then $Rad_g(M)$ is G-Rad$_g$-lifting.
Proof. Clear by 5.1.4. □

Proposition 5.1.6. Let M be a G-Rad$_g$-lifting module. If $Rad_g(M) = 0$ then M is semisimple.
Proof. Suppose that E be any submodule of M. As $Rad_g(M) = 0 \subseteq E$ and M a G-Rad$_g$-lifting module, then there is a decomposition $M = T \oplus N$ with $T \leq E$ and $E \cap N \subseteq Rad_g(N)$. Thus $M = E + N$. Since, $Rad_g(N) \subseteq Rad_g(M)$, we have that $E \cap N = 0$. Thus $E \leq^{\oplus} M$ and M is semisimple. □

Corollary 5.1.7. Suppose that M be a module with $Rad_g(M) = 0$. Then M is G-$\mathrm{Rad_g}$-lifting if and only if every submodule of M is G-$\mathrm{Rad_g}$-lifting.

Proof. ⟹) By Proposition 5.1.6, M is a semisimple module, that means any submodule of M is a direct summand. Also, notice that every submodule of M contains $Rad_g(M) = 0$. From Proposition 5.1.4, every submodule of M is G-$\mathrm{Rad_g}$-lifting.

⟸) Clear. □

Corollary 5.1.8. Let M be an R-module. Then M is G-$\mathrm{Rad_g}$-lifting if and only if any submodule of M is G-$\mathrm{Rad_g}$-lifting if, one of the following cases hold:

(1) M is g-noncosingular.

(2) R is a g-V-ring.

Proof. (1) By Lemma 1.2.21 and Corollary 5.1.7.

(2) Since R is a g-V-ring, then $Rad_g(M) = 0$. From Corollary 5.1.7, the consequence is obtained. □

Proposition 5.1.9. Assume M is a module whose all its submodules are indecomposable contains $Rad_g(M)$. If M is a G-$\mathrm{Rad_g}$-lifting module, for all $C \leq M$. Then either $C \leq^{\oplus} M$ or, $Rad_g(M) = C$.

Proof. Let M be a G-$\mathrm{Rad_g}$-lifting R-module. If $C \leq M$ is indecomposable such that $Rad_g(M) \subseteq C$, so it comes from Proposition 5.1.2(2), that $C = T \oplus S$ where $T \leq^{\oplus} M$ and $S \subseteq Rad_g(M)$. Since C is indecomposable, then either $C = T$ or $C = S \subseteq Rad_g(M)$, Since $C = T$ or $Rad_g(M) = C$, as required. □

Now, we will discuss an infinite direct sum of G-$\mathrm{Rad_g}$-lifting modules.

Theorem 5.1.10. Let $\{M_i | \, i \in I\}$ be any infinite family submodules of an R-module M, such that each of M_i is G-$\mathrm{Rad_g}$-lifting module. If $M = \oplus_{i \in I} M_i$ and M is a duo module, then M is G-$\mathrm{Rad_g}$-lifting.

Proof. Suppose that $\{M_i|\ i \in I\}$ is a family of G-Rad$_g$-lifting modules such that $M = \bigoplus_{i\in I} M_i$ is a duo module. To conform that $M = \bigoplus_{i\in I} M_i$ is a G-Rad$_g$-lifting module. Let U be a submodule of $M = \bigoplus_{i\in I} M_i$ and $Rad_g(M) \subseteq U$. Since U is a fully invariant submodule of M, Lemma 1.2.14 implies that $U = \bigoplus_{i\in I} (M_i \cap U)$. We deduce that $Rad_g(M_i) \subseteq M_i \cap U$ for $i \in I$. Since, M_i is G-Rad$_g$-lifting, for $i \in I$, then there are decompositions $M_i = V_i \oplus W_i$ such that $V_i \leq M_i \cap U$ and $(M_i \cap U) \cap W_i = U \cap W_i \subseteq Rad_g(W_i)$. Put $V = \bigoplus_{i\in I} V_i$ and $W = \bigoplus_{i\in I} W_i$. As a result of this, $M = V \oplus W$ such that $V = \bigoplus_{i\in I} V_i \leq \bigoplus_{i\in I} (M_i \cap U) = U$ and thus $U \cap W = \bigoplus_{i\in I} (U \cap W_i) \subseteq \bigoplus_{i\in I} Rad_g(W_i) = Rad_g(\bigoplus_{i\in I} W_i) = Rad_g(W)$, by Lemma 1.2.10(1,2). Hence, M is G-Rad$_g$-lifting. □

Theorem 5.1.11. Let $\{M_i|\ i \in I\}$ be any infinite family submodules of an R-module M, such that each of M_i is G-Rad$_g$-lifting module. If $M = \bigoplus_{i\in I} M_i$ and M is a distributive module, then M is G-Rad$_g$-lifting.

Proof. Analogous to proof Theorem 5.1.10. □

Theorem 5.1.12. If $M = M_1 \oplus M_2$ is a direct sum of G-Rad$_g$-lifting modules such that M_1 is quasi-projective and M_2-projective, then M is G-Rad$_g$-lifting.

Proof. Let E be a submodule of $M = M_1 \oplus M_2$ with $Rad_g(M) \subseteq E$. As a result of this, $M_1 \cap (E + M_2) \leq M_1$ and $Rad_g(M_1) \subseteq M_1 \cap (E + M_2)$. Since, M_1 is a G-Rad$_g$-lifting module, then there is a decomposition $M_1 = V_1 \oplus V_2$, with $V_1 \leq M_1 \cap (E + M_2)$ and $V_2 \cap (E + M_2) \subseteq Rad_g(V_2)$. Since $V_1 \oplus V_2 \oplus M_2 \subseteq (E + M_2) \oplus V_2 \oplus M_2 = E + (V_2 \oplus M_2)$ then $M = E + (V_2 \oplus M_2)$. Since, M_1 is a quasi-projective module and M_2-projective, then V_1 is $V_2 \oplus M_2$-projective. From Lemma 1.2.32, there is a submodule E_1 of E with $M = E_1 \oplus (V_2 \oplus M_2)$. It is easy to confirm that $E \cap (L + V_2) = L \cap (E + V_2)$ for any $L \leq M_2$, to see this: as $E \cap (L + V_2) \leq L \cap (E + V_2) + V_2 \cap (E + L)$ and $V_2 \cap (E + L) = 0$, then $E \cap (L + V_2) \leq L \cap (E + V_2)$. By similarly, we have $L \cap (E + V_2) \leq E \cap (L + V_2)$. It follows that $L \cap (E + V_2) = E \cap (L + V_2)$ for any $L \leq M_2$. Moreover, we have $M_2 \cap (E + V_2) \leq M_2$ with $Rad_g(M_2) \subseteq M_2 \cap (E + V_2)$ and M_2 is a G-Rad$_g$-lifting module, so there is a decomposition $M_2 = \acute{V}_1 \oplus \acute{V}_2$, $\acute{V}_1 \leq M_2 \cap (E + V_2) = E \cap (M_2 + V_2) \leq E$ and $\acute{V}_2 \cap (E + V_2) \subseteq Rad_g(\acute{V}_2)$. Since, $M = E_1 \oplus (V_2 \oplus M_2) = (E_1 \oplus \acute{V}_1) \oplus (V_2 \oplus \acute{V}_2)$, $E_1 \oplus \acute{V}_1 \leq E$ and then $E \cap \left(V_2 \oplus \acute{V}_2\right) = \acute{V}_2 \cap (E + V_2) \subseteq Rad_g(V_2 \oplus \acute{V}_2)$, thus M is G-Rad$_g$-lifting. □

Corollary 5.1.13. Let $M = M_1 \oplus M_2$ such that M_1 is a semisimple module and M_2 is a G-Rad$_g$-lifting module, and they are relatively projective with M_1, then M is a G-Rad$_g$-lifting module.

Proof. Since M_1 is a semisimple module, by Remarks and Examples 5.1.3(4), M_1 G-Rad$_g$-lifting and so by Theorem 5.1.12, M is G-Rad$_g$-lifting.□

Theorem 5.1.14. Let M be a G-Rad$_g$-lifting module and $E \leq M$ that satisfy one of the following:

(1) $(C + E)/E \leq^{\oplus} M/E$ for any $C \leq^{\oplus} M$.

(2) If E is a distributive submodule of M.

(3) If E is a fully invariant submodule of M.

(4) If $M = E \oplus G$ and $Rad_g(M) \subseteq G$.

Then M/E is a G-Rad$_g$-lifting module.

Proof. (1) Let $L/E \leq M/E$ such that $E \leq L \leq M$ with $Rad_g(M/E) \subseteq L/E$. Suppose π_E is the natural map. As $Rad_g(M) \subseteq M$, we get $\pi_E(Rad_g(M)) \subseteq Rad_g(M/E) \subseteq L/E$, and then $Rad_g(M) \subseteq L$. Since M is G-Rad$_g$-lifting, then there exists a direct summand C of M such that $C \leq L$ and $L/C \subseteq Rad_g(M/C)$, by Proposition 5.1.2(3). By **(1)**, $(C + E)/E \leq^{\oplus} M/E$. Clearly, $(C + E)/E \leq L/E$. Define $\eta: \frac{M}{C} \rightarrow \frac{M/C}{(E+C)/C}$ as the projection map. Since,

$L/C \subseteq Rad_g(M/C)$, so $\eta(L/C) \subseteq \eta\big(Rad_g(M/C)\big)$, i.e., $\frac{L}{E+C} \subseteq Rad_g(\frac{M}{E+C})$. Thus, M/E is a G-Rad$_g$-lifting module.

(2) We will use part **(1)** to prove this case. Let $M = T \oplus N$ for some $N \leq M$. It is clear that $M/E = ((T+E)/E) + ((N+E)/E)$. As E is distributive, then $(T+E) \cap (N+E) = (T \cap N) + E = E$. So $((T+E)/E) \cap ((N+E)/E) = 0$, that is $(T+E)/E \leq^{\oplus} M/E$.

(3) Let $L \leq^{\oplus} M$, then $M = L \oplus Y$ for some $Y \leq M$. Since, E is fully invariant, Lemma 1.2.19 implies $M/E = ((L+E)/E) \oplus ((Y+E)/E)$, i.e., $(L+E)/E$ is a direct summand of M/E. Hence M/E is a G-Rad$_g$-lifting module, by **(1)**.

(4) Let $M = E \oplus G$ for some $G, E \leq M$. As $Rad_g(M) \subseteq G$, so by Proposition 5.1.4, G is a G-Rad$_g$-lifting module. Since $M/E \cong G$, therefore M/E is a G-Rad$_g$-lifting module. □

Corollary 5.1.15. Let M be a G-Rad$_g$-lifting and f an R-homomorphism from an R-module M into R-module $\grave{M}$. Then Imf is a G-Rad$_g$-lifting module if, $kerf$ is a distributive, or fully invariant submodule of M.

Proof. Theorem 5.1.14(2,3) implies that $M/kerf$ is a G-Rad$_g$-lifting module. By 1st isomorphism theorem, $M/kerf \cong Imf$ and so Imf is a G-Rad$_g$-lifting module. □

Corollary 5.1.16. Let $M = kerf \oplus N$ be a G-Rad$_g$-lifting module such that $f: M \to L$ is an epimorphism and $Rad_g(M) \subseteq N$. Then L is G-Rad$_g$-lifting.

Proof. Clear by Theorem 5.1.14(4), $M/kerf \cong L$ is G-Rad$_g$-lifting. □

Corollary 5.1.17. If M is a G-Rad$_g$-lifting module, then both of $M/Rad_g(M)$ and $M/P_g(M)$ is a G-Rad$_g$-lifting module.

Proof. By Lemmas 1.2.8 and 1.2.36, we deduce that $Rad_g(M)$ (resp. $P_g(M)$) is a fully invariant submodule of M, thus by applying Theorem 5.1.14(3), we get the result. □

Corollary 5.1.18. Let M be a weak duo module and G a direct summand of M. If M is a G-$\mathrm{Rad_g}$-lifting module, then G and M/G are G-$\mathrm{Rad_g}$-lifting modules.

Proof. Let $M = G \oplus C$ for some $C \leq M$. Since M is a weak duo module, then G and C are fully invariant submodules of M. Thus by Theorem 5.1.14(3), $G \cong M/C$ and M/G are G-$\mathrm{Rad_g}$-lifting modules. □

Corollary 5.1.19. Let M be a duo module and G a direct summand of M. Then M is G-$\mathrm{Rad_g}$-lifting if and only if G and M/G are G-$\mathrm{Rad_g}$-lifting modules.

Proof. ⟹) Since any duo module is a weak duo, Corollary 5.1.18 leads to the result.

⟸) Since, $M \cong G \oplus (M/G)$, we deduce the result by Theorem 5.1.10. □

Corollary 5.1.20. Let M be a G-$\mathrm{Rad_g}$-lifting module. If M is distributive (or, duo), then the factor module of M is also G-$\mathrm{Rad_g}$-lifting.

Proof. Clear by Theorem 5.1.14(2, 3). □

Corollary 5.1.21. Let M be a distributive module and $L \leq^{\oplus} M$. Then M is G-$\mathrm{Rad_g}$-lifting if and only if L and M/L are both G-$\mathrm{Rad_g}$-lifting.

Proof. ⟹) If M is a distributive module and L a direct summand of M, then $M = L \oplus N$ for a submodule N of M. From Corollary 5.1.20 $L \cong M/N$ and M/L are G-$\mathrm{Rad_g}$-lifting.

⟸) Since, $M \cong L \oplus (M/L)$, the result is included by Theorem 5.1.11. □

Corollary 5.1.22. A homomorphic image of a distributive (or, duo) G-$\mathrm{Rad_g}$-lifting module is G-$\mathrm{Rad_g}$-lifting.

Proof. Let M be a distributive (duo) and G-$\mathrm{Rad_g}$-lifting module. Assume $f: M \to \grave{M}$ be any homomorphism. By 1st isomorphism theorem, $M/E \cong Imf$

for some submodule E of M. By Corollary 5.1.20, M/E is G-$\mathrm{Rad_g}$-lifting and hence Imf is a G-$\mathrm{Rad_g}$-lifting module. □

Corollary 5.1.23. Let $f: M \to \grave{M}$ be an epimorphism of R-modules M, $\grave{M}$ such that M is a distributive (or, duo) module. If M is a G-$\mathrm{Rad_g}$-lifting module, then $\grave{M}$ is G-$\mathrm{Rad_g}$-lifting.

Proof. Clear by Corollary 5.1.22. □

The following corollary comes consequently from Corollary 5.1.19 and Theorem 5.1.10.

Corollary 5.1.24. Suppose that $M = \bigoplus_{i \in I} M_i$ is a duo module. Then M_i is G-$\mathrm{Rad_g}$-lifting, for $i \in I$, if and only if M is G-$\mathrm{Rad_g}$-lifting.

5.2. Connections with G-$\mathrm{Rad_g}$-lifting modules

This section, will highlight many relations between this concept of G-$\mathrm{Rad_g}$-lifting module and other forms of modules. We have the following:

Proposition 5.2.1. Let M be a G-$\mathrm{Rad_g}$-lifting module. If $Rad_g(M) \ll_g M$ then M is $\mathrm{Rad_g}$-lifting.

Proof. Let $E \leq M$ with $Rad_g(M) \subseteq E$. As M is a G-$\mathrm{Rad_g}$-lifting module, then there is a decomposition $M = T \oplus N$ with $T \leq E$ and $E \cap N \subseteq Rad_g(N)$. From $Rad_g(N) \subseteq Rad_g(M) \ll_g M$, we deduce that $E \cap N \ll_g M$. Thus, by Proposition 4.1.7, M is a $\mathrm{Rad_g}$-lifting module. □

Proposition 5.2.2. The notions "G-$\mathrm{Rad_g}$-lifting" and "$\mathrm{Rad_g}$-lifting" for an R-module M are equivalents when any of the following conditions hold:

(1) If $Rad_g(M) \ll_g M$.

(2) If M is finitely generated (so is cyclic).

Proof. (1) It comes by Remarks and Examples 5.1.3(1) and Proposition 5.2.1.

(2) By Lemma 1.2.12, $Rad_g(M) \ll_g M$. So, the result satisfies by **(1)**. □

Corollary 5.2.3. The notions "G-Rad_g-lifting" and "semisimple" for an R-module M are equivalents when any of the following conditions hold:

(1) $Rad_g(M) = 0$.

(2) M is g-noncosingular.

(3) R is a g-V-ring.

Proof. (1) It comes by Remarks and Examples 5.1.3(4) and Proposition 5.1.6.

(2) By Lemma 1.2.21, $Rad_g(M) = 0$. Hence the result was held by **(1)**.

(3) Since, R is a g-V-ring, i.e, $Rad_g(M) = 0$. So, the result follows by **(1)**. □

Proposition 5.2.4. If M is a (P_g^*)-module, then M is a G-Rad_g-lifting module. If $Rad_g(M) = 0$, then the converse is correct.

Proof. ⟹) By the definitions.

⟸) It comes by Proposition 5.1.6 and Lemma 1.2.52. □

Corollary 5.2.5. Let M be a module. Then the following assertions are equivalent if, $Rad_g(M) = 0$.

(**1**) M is semisimple.

(**2**) M is g-lifting.

(**3**) M is Rad_g-lifting.

(**4**) M is a (P_g^*)-module.

(**5**) M is G-Rad_g-lifting.

(6) M is ⊕-g-Radical supplemented.

(7) M is g-Radical supplemented.

(8) Each summand of M is ⊕-g-Radical supplemented.

Proof. (**1**) ⟺ (**2**) ⟺ (**3**) ⟺ (**4**) from Proposition 4.2.13.

(**4**) ⟺ (**5**) by Proposition 5.2.4.

$(\mathbf{1}) \Leftrightarrow (\mathbf{4}) \Leftrightarrow (\mathbf{6}) \Leftrightarrow (\mathbf{7}) \Leftrightarrow (\mathbf{8})$ from Lemma 1.2.52. □

Proposition 5.2.6. Let M be a uniserial module and $Rad_g(M)$ a minimal submodule of M. If M is a G-$\mathrm{Rad_g}$-lifting module, then M is a $(\mathrm{P_g}^*)$-module.

Proof. Let E be a submodule of M. If $E = 0$, the proof is clear. Now, suppose $E \neq 0$. As $Rad_g(M) \leq M$ and M a uniserial module, either $E \subseteq Rad_g(M)$ or $Rad_g(M) \subseteq E$. Since $Rad_g(M)$ is minimal and $E \neq 0$, we have $Rad_g(M) \subseteq E$. As M is a G-$\mathrm{Rad_g}$-lifting module, then there is a decomposition $M = L \oplus Y$ such that $L \leq E$ and $E \cap Y \subseteq Rad_g(Y)$. Hence M is a $(\mathrm{P_g}^*)$-module. □

From Remarks and Examples 5.1.3(1), Propositions 5.2.1, 5.2.4 and 5.2.6 we deduce the following result.

Corollary 5.2.7. Let M be a uniserial module and $Rad_g(M)$ a minimal and $Rad_g(M) \ll_g M$. Then the following are equivalent.

(1) M is a $\mathrm{Rad_g}$-lifting module.

(2) M is a $(\mathrm{P_g}^*)$-module.

(3) M is a G-$\mathrm{Rad_g}$-lifting module.

Proposition 5.2.8. Let M be a module and $0 \neq E \leq M$ with $E \cap Rad_g(M) = 0$. If E is a G-$\mathrm{Rad_g}$-lifting module, then E is semisimple.

Proof. Since, $Rad_g(E) \subseteq E \cap Rad_g(M)$, then $Rad_g(E) = 0$ and hence E is semisimple by Proposition 5.1.6. □

Proposition 5.2.9. Let M be an indecomposable R-module and $Rad_g(M)$ a proper g-small submodule of M. Then the following cases are equivalent.

(1) M is a local module.

(2) M is a hollow module.

(3) M is a generalized hollow module.

(4) M is a Rad_g-lifting module.

(5) M is a g-local module.

(6) M is a sgrs$^{\oplus}$-module.

(7) M is a G-Rad_g-lifting module.

Proof. (1) ⟹ **(2)** ⟹ **(3)** Clear.

(3) ⟹ **(4)** clear.

$(\mathbf{5}) \Longrightarrow (\mathbf{4})$ By Proposition 4.2.3.

$(\mathbf{5}) \Longrightarrow (\mathbf{1})$ By Lemma 1.2.34.

$(\mathbf{4}) \Leftrightarrow (\mathbf{6})$ By Corollary 4.2.7.

$(\mathbf{4}) \Longrightarrow (\mathbf{7})$ From Remarks and Examples 5.1.3(1).

$(\mathbf{7}) \Longrightarrow (\mathbf{4})$ From Proposition 5.2.1. □

By Lemma 1.2.12 and Proposition 5.2.9, the following result is immediate.

Corollary 5.2.10. Let M be an indecomposable finitely generated R-module with $Rad_g(M) \neq M$. Then the following are equivalent.

(1) M is a local module.

(2) M is a hollow module.

(3) M is a generalized hollow module.

(4) M is a Rad_g-lifting module.

(5) M is a g-local module.

(6) M is a sgrs$^{\oplus}$-module.

(7) M is a G-Rad_g-lifting module.

Proposition 5.2.11. If M is a G-Rad_g-lifting module, then.

(1) The factor $M/Rad_g(M)$ is semisimple.

(2) $M/P_g(M)$ is semisimple whenever, $Rad_g(M)$ is a direct summand of M.

Proof. (1) Suppose that $Rad_g(M) \leq H \leq M$. By Proposition 5.1.2(4), H containing a g-Radical supplement $\grave{C}$ in M, i.e., $M = H + \grave{C}$ and $H \cap \acute{C} \subseteq Rad_g(\acute{C})$, and that implies $(H \cap \acute{C}) + Rad_g(\acute{C}) = Rad_g(\acute{C})$. Since,

$\frac{M}{Rad_g(M)} = \frac{H}{Rad_g(M)} + \frac{\left(\acute{C}+Rad_g(M)\right)}{Rad_g(M)}$ and $\left(\frac{H}{Rad_g(M)}\right) \cap \left(\frac{\acute{C}+Rad_g(M)}{Rad_g(M)}\right) = \frac{H\cap\left(\acute{C}+Rad_g(M)\right)}{Rad_g(M)} = \frac{Rad_g(M)+(H\cap\acute{C})}{Rad_g(M)} = \frac{Rad_g(M)}{Rad_g(M)}$, i.e., $H/Rad_g(M)$ is a direct summand of $M/Rad_g(M)$, and hence $M/Rad_g(M)$ is semisimple.

(2) By Lemma 1.2.37, $P_g(M) = Rad_g(M)$. By **(1)**, $M/P_g(M)$ is semisimple.□

Proposition 5.2.12. Let M be a finitely generated module satisfies DCC on g-small submodules. If M is a G-Rad$_g$-lifting module, then M is Artinian.

Proof. By the same way of proof of Proposition 4.2.21. □

Corollary 5.2.13. Let M be a cyclic module satisfies DCC on g-small submodules. If M is a G-Rad$_g$-lifting module, then M is Artinian.

Proof. Clear by Proposition 5.2.12. □

Proposition 5.2.14. Let M be a G-Rad$_g$-lifting module and $Rad_g(M)$ a proper submodule of M, then there exists a decomposition $M = C_1 \oplus C_2$ such that C_1 is semisimple, $Rad_g(M) \trianglelefteq C_2$ and $C_2/Rad_g(M)$ is semisimple.

Proof. By Proposition 5.2.11(1), we get $M/Rad_g(M)$ is semisimple. Since, $Rad_g(M) \subset M$, we have the result by Lemma 1.2.33. □

Proposition 5.2.15. Assume M is a module over a g-V-ring R. If M is a G-Rad$_g$-lifting R-module, then M is refinable.

Proof. Let E, $C \leq M$ with $M = E + C$. Since R is a g-V-ring, As a result of this, $Rad_g(M) = 0$. Since M is G-Rad$_g$-lifting and $Rad_g(M) \subseteq E$, then there exists a decomposition $M = L \oplus Y$ such that $L \leq E$ and $E \cap Y \subseteq Rad_g(Y)$. By the modular law we deduce that $E = L \oplus (E \cap Y)$. By $E \cap Y \subseteq Rad_g(M) = 0$, $E = L$ that imply $M = L + C$ and $L \leq^{\oplus} M$. Hence M is a refinable module. □

Proposition 5.2.16. Let P be a projective module. Then the following cases are equivalent.

(1) P is G-Rad_g-lifting.

(2) $P/Rad_g(P)$ is semisimple and for every submodule $T \leq P$ containing $Rad_g(P)$, there is a $C \leq^{\oplus} P$ such that $\bar{T} = \bar{C}$, where $\bar{T} = T/Rad_g(P)$.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Let T be a submodule of P such that $Rad_g(P) \subseteq T$. Since, P is a G-Rad_g-lifting module, thus from Proposition 5.2.11, $P/Rad_g(P)$ is a semisimple module. Put $\bar{T} = T/Rad_g(P)$. By Proposition 5.1.2, there is a decomposition $T = C \oplus H$ such that $C \leq^{\oplus} P$ and $H \subseteq Rad_g(P)$. As a result of this, $T \subseteq C + Rad_g(P)$. But $C + Rad_g(P) \subseteq T$, so we deduce that $T = C + Rad_g(P)$. Thus, $T/Rad_g(P) = (C + Rad_g(P))/Rad_g(P)$, imply $\bar{T} = \bar{C}$.

$(\mathbf{2}) \Longrightarrow (\mathbf{1})$ Let $T \leq P$ such that $Rad_g(P) \subseteq T$. As $P/Rad_g(P)$ is semisimple, we have that $P/Rad_g(P) = T/Rad_g(P) \oplus H/Rad_g(P)$ for some submodule H of M. By **(2)**, there exists a direct summand T of P such that $P = T \oplus N$ for some submodule N of P, and $\bar{T} = \bar{C}$, such that $H = N + Rad_g(P)$. As a result of this, $P = T + N + Rad_g(P) = T + N$. As $P = T + N$ is projective, Lemma 1.2.32 implies $P = T' \oplus N$ with $T' \leq T$. Also, $T \cap N \subseteq T \cap H = Rad_g(P)$, hence P is a G-Rad_g-lifting module. □

Proposition 5.2.17. Let M be a G-Rad_g-lifting module with $Rad_g(M) \neq M$. Then $M = W \oplus H$ where W is a g-Radical supplement of $Rad_g(M)$ in M and H a g-Radical.

Proof. Let $Rad_g(M) \neq M$. As $Rad_g(M) \subseteq Rad_g(M)$ and M a G-Rad_g-lifting module, so by Proposition 5.1.2(4), $Rad_g(M)$ containing a g-Radical supplement W in M and $Rad_g(M) \cap W \leq^{\oplus} Rad_g(M)$. Therefore, $M = Rad_g(M) + W$ and $Rad_g(M) \cap W \subseteq Rad_g(W)$. However, $Rad_g(M) =$

$(Rad_g(M) \cap W) \oplus H$ for some $H \leq Rad_g(M)$. Hence, $M = Rad_g(M) + W = W + (Rad_g(M) \cap W) \oplus H = W \oplus H$. By applying Lemma 1.2.10, we get $M = Rad_g(M) + W = Rad_g(W \oplus H) + W = Rad_g(W) \oplus Rad_g(H) + W = Rad_g(H) \oplus W$. Now, by modular law, $H = H \cap (Rad_g(H) \oplus W) = Rad_g(H) \oplus (W \cap H) = Rad_g(H)$, thus H is a g-Radical.□

Remark 5.2.18. The converse of Proposition 5.2.17 is not true, in general, for example, in the $\mathbb{Z}$-module $\mathbb{Z}$ we know that $Rad_g(\mathbb{Z}) = 0 \neq \mathbb{Z}$. Also, we have the only decomposition of $\mathbb{Z}$ as $\mathbb{Z}$-module is $\mathbb{Z} = \mathbb{Z} \oplus (0)$, such that $\mathbb{Z}$ is a g-Radical supplement of $Rad_g(M) = 0$ and its easy (0) is g-Radical, but $M = \mathbb{Z}$ does not be G-Rad_g-lifting $\mathbb{Z}$-module.

Proposition 5.2.19. If M is an indecomposable module, then the following cases are equivalent.

(1) M is G-Rad_g-lifting.

(2) If $E \subset M$ and $Rad_g(M) \subseteq E$, we have $Rad_g(M) = E$.

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$ Suppose $E \leq M$ such that $Rad_g(M) \subseteq E \neq M$. Since, M is a G-Rad_g-lifting module, there is a decomposition $M = T \oplus N$ such that $T \leq E$ and $E \cap N \subseteq Rad_g(N) \subseteq Rad_g(M)$. Thus, $M = E + N$. And since, M is indecomposable, either $N = 0$ or $N = M$. If $N = 0$ then $E = M$ which is a contradiction. Hence, $N = M$ and then $E = E \cap N \subseteq Rad_g(M)$. Therefore, $E = Rad_g(M)$.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$ Suppose $E \leq M$ with $Rad_g(M) \subseteq E$. If $E = M$, then trivially, there is a decomposition $M = M \oplus 0$ such that $M \leq E$ and $E \cap (0) \subseteq Rad_g(M)$. Let $E \neq M$, so by **(2)**, we get $E \subseteq Rad_g(M)$. As a result of this, $M = 0 \oplus M$, $(0) \leq E$ and $E \cap M = E \subseteq Rad_g(M)$. Thus, M is G-Rad_g-lifting. □

Corollary 5.2.20. The following are equivalent for a uniform module M.

(1) M is a G-Rad$_g$-lifting.

(2) If $E \subset M$ and $Rad_g(M) \subseteq E$, we have $Rad_g(M) = E$.

Proof. Since, every uniform module is indecomposable by Lemma 1.2.47, the result comes directly by Proposition 5.2.19. □

As an application of Corollary 5.2.20, we show the following example.

Example 5.2.20. It is well known that $\mathbb{Z}$ as $\mathbb{Z}$-module is uniform. Also, we have that $Rad_g(\mathbb{Z}) \neq n\mathbb{Z}$ for all $0 \neq n\mathbb{Z} \subset \mathbb{Z}$ and $Rad_g(\mathbb{Z}) = 0 \subseteq n\mathbb{Z}$, in fact, this is a direct conclusion from Remarks and Examples 5.1.3(2) and Corollary 5.2.19.

5.3. Rings and localization of G-Rad$_g$-lifting

In this section, we will discuss the localization and rings of G-Rad$_g$-lifting property under some conditions.

Theorem 5.3.1. Let M be a faithful, finitely generated and multiplication module over a commutative ring R with identity. Then M is a G-Rad$_g$-lifting R-module if and only if R is a G-Rad$_g$-lifting ring.

Proof. Suppose M is a G-Rad$_g$-lifting R-module. Let I be any ideal of R such that $Rad_g(R) \subseteq I$. So, $Rad_g(R)M \subseteq IM$, Lemma 2.6.1 imply $Rad_g(M) \subseteq IM$, and, there is a decomposition $M = E \oplus W$ such that $E \leq IM$ and $IM \cap W \subseteq Rad_g(W) \subseteq Rad_g(M)$. Since M is multiplication, we have that $E = JM$ and $W = EM$ for some ideals J and E of R. As a result of this, $M = JM \oplus EM$ such that $JM \leq IM$ and $IM \cap EM \subseteq Rad_g(M)$. From Lemmas 1.2.63, 1.2.64 and Lemma 2.6.1, we conclude that $R = J \oplus E$, $J \leq I$ and $I \cap E \subseteq Rad_g(R)$. Since, R_R is G-Rad$_g$-lifting.

Conversely, assume that R is G-$\mathrm{Rad_g}$-lifting. Let $H = IM$ be a submodule of M with $Rad_g(M) \subseteq IM$, for some ideal I of R. From Lemma 2.6.1, $Rad_g(R)M \subseteq IM$ implies that $Rad_g(R) \subseteq I$ from Lemma 1.2.64. There is a decomposition $R = J \oplus E$ such that $J \leq I$ and $I \cap E \subseteq Rad_g(R)$. So, $M = (J \oplus E)M = JM \oplus EM$ such that $JM \leq IM = H$. By Lemma 1.2.63 and Lemma 2.6.1, we have that $H \cap EM = IM \cap EM = (I \cap E)M \subseteq Rad_g(R)M = Rad_g(M)$. So, M is a G-$\mathrm{Rad_g}$-lifting R-module. □

Theorem 5.3.2. Let M be an R-module and $S \subseteq R$ a multiplicative closed such that $\mathcal{L}(T) \cap S = \emptyset$ for each $T \leq M$. Then M is a G-$\mathrm{Rad_g}$-lifting as R-module if and only if $S^{-1}M$ is a G-$\mathrm{Rad_g}$-lifting as $S^{-1}R$-module.

Proof. Suppose M is a G-$\mathrm{Rad_g}$-lifting as R-module, and let $S^{-1}E \leq S^{-1}M$ as $S^{-1}R$-module such that $Rad_g(S^{-1}M) \subseteq S^{-1}E$. From Lemma 2.3.2, we have that $S^{-1}(Rad_g(M)) \subseteq S^{-1}E$, so $E \leq M$ and $Rad_g(M) \subseteq E$ by Lemma 1.2.65. Then there is a decomposition $M = H \oplus W$ such that $H \leq E$ and $E \cap W \subseteq Rad_g(W)$ for some $W \leq M$. From Lemmas 1.2.71, 1.2.65 and 2.3.2, we deduce that $(S^{-1}H) \oplus (S^{-1}W) = S^{-1}M$ with $S^{-1}H \leq S^{-1}E$ and $(S^{-1}E) \cap (S^{-1}W) \subseteq Rad_g(S^{-1}W)$. Thus, $S^{-1}M$ is a G-$\mathrm{Rad_g}$-lifting as $S^{-1}R$-module.

Conversely, let $S^{-1}M$ be a G-$\mathrm{Rad_g}$-lifting as $S^{-1}R$-module, and let $N \leq M$ such that $Rad_g(M) \subseteq N$. Lemmas 1.2.65 and 2.3.2 implies that $S^{-1}N \leq S^{-1}M$ and $Rad_g(S^{-1}M) = S^{-1}(Rad_g(M)) \subseteq S^{-1}N$. Then there is a decomposition $(S^{-1}C) \oplus (S^{-1}W) = S^{-1}M$ such that $S^{-1}C \leq S^{-1}N$ and $(S^{-1}N) \cap (S^{-1}W) \subseteq Rad_g(S^{-1}W)$ where $S^{-1}W \leq S^{-1}M$. By Lemmas 1.2.71, 1.2.65 and 2.3.2, we have that $C \oplus W = M$ with $C \leq N$ and $N \cap W \subseteq Rad_g(W)$ for some $W \leq M$. This means, M is G-$\mathrm{Rad_g}$-lifting. □

Theorem 5.3.3. Let M be an R-module and $S \subseteq R$ a multiplicative closed such that for any $W \subset M$, $(W:_M s) = W$, for all $s \in S$. Then M is a G-Rad$_g$-lifting as R-module if and only if $S^{-1}M$ is a G-Rad$_g$-lifting as $S^{-1}R$-module.

Proof. Let M be a G-Rad$_g$-lifting as R-module, and $S^{-1}E \leq S^{-1}M$ as $S^{-1}R$-module such that $Rad_g(S^{-1}M) \subseteq S^{-1}E$. From Lemma 2.3.5, As a result of this, $S^{-1}(Rad_g(M)) \subseteq S^{-1}E$ and so $Rad_g(M) \subseteq E$ by Lemma 1.2.67. Then there is a decomposition $M = H \oplus W$ such that $H \leq E$ and $E \cap W \subseteq Rad_g(W)$ for some $W \leq M$. By Lemmas 1.2.71, 1.2.67 and 2.3.5, $S^{-1}M = (S^{-1}H) \oplus (S^{-1}W)$ such that $S^{-1}H \leq S^{-1}E$ and $(S^{-1}E) \cap (S^{-1}W) \subseteq Rad_g(S^{-1}W)$. Hence $S^{-1}M$ is a G-Rad$_g$-lifting as $S^{-1}R$-module.

Conversely, suppose that $S^{-1}M$ be a G-Rad$_g$-lifting as $S^{-1}R$-module, and let $U \leq M$ such that $Rad_g(M) \subseteq U$. From Lemmas 2.3.5 and 1.2.67, $S^{-1}U \leq S^{-1}M$ and $Rad_g(S^{-1}M) = S^{-1}(Rad_g(M)) \subseteq S^{-1}U$. Then there is a decomposition $(S^{-1}D) \oplus (S^{-1}W) = S^{-1}M$ such that $S^{-1}D \leq S^{-1}U$ and $(S^{-1}U) \cap (S^{-1}W) \subseteq Rad_g(S^{-1}W)$ where $S^{-1}W \leq S^{-1}M$. By Lemmas 1.2.71, 1.2.67 and 2.3.5, we have that $M = D \oplus W$ such that $D \leq U$ and $U \cap W \subseteq Rad_g(W)$ where $W \leq M$. So, M is a G-Rad$_g$-lifting as R-module.□

5.4. Principally Generalized Rad$_g$-lifting modules

This section is about given a generalization of G-Rad$_g$-lifting which will be restricted on cyclic submodules that we named PG-Rad$_g$-lifting.

Definition 5.4.1. An R-module M is called principally generalized Rad$_g$-lifting, briefly PG-Rad$_g$-lifting if, any cyclic submodule E of M with $Rad_g(M) \subseteq E$, there is a decomposition $M = T \oplus N$ such that $T \leq E$ and $E \cap N \subseteq Rad_g(N)$. A ring R is called PG-Rad$_g$-lifting if, R_R is PG-Rad$_g$-lifting.

First, we will give many remarks with examples of PG-Rad_g-lifting modules.

Remarks and Examples 5.4.2.

(1) It is easily to ensure that every P-Rad_g-lifting module is PG-Rad_g-lifting, and so every principally g-lifting module is PG-Rad_g-lifting. As application example, $\mathbb{Q}$ and $\mathbb{Z}/p^n\mathbb{Z}$ as $\mathbb{Z}$-modules are P-Rad_g-lifting $\mathbb{Z}$-modules for any prime number p and any $n \in \mathbb{Z}^+$, see Remarks and Examples 4.4.2. Hence, they are PG-Rad_g-lifting modules.

(2) For any R-module M. If $Rad_g(M) = 0$, then M is principally g-lifting if and only if, it is P-Rad_g-lifting if and only if, it is PG-Rad_g-lifting.

(3) Any G-Rad_g-lifting module is PG-Rad_g-lifting. However, if M is a cyclic module over a PID, then we know that every submodule of M is also cyclic. Hence, any cyclic PG-Rad_g-lifting module over a PID is G-Rad_g-lifting.

(4) A PG-Rad_g-lifting module not inherited by its submodules, for instance $\mathbb{Q}$ as $\mathbb{Z}$-module is PG-Rad_g-lifting, while $\mathbb{Z}_\mathbb{Z} \leq \mathbb{Q}_\mathbb{Z}$ is not PG-Rad_g-lifting, to see this: it is well know that $Rad_g(\mathbb{Z}_\mathbb{Z}) = 0$, then for any $n\mathbb{Z} \leq \mathbb{Z}$ we have $Rad_g(n\mathbb{Z}_\mathbb{Z}) \subseteq Rad_g(\mathbb{Z}_\mathbb{Z}) = 0$ for any $n \in \mathbb{Z}^+$. However, if $n\mathbb{Z} \subset \mathbb{Z}$ for any $n \in \mathbb{Z}^+$, then there is only decomposition $\mathbb{Z} = (0) \oplus \mathbb{Z}$ such that $(0) \leq n\mathbb{Z}$ while $n\mathbb{Z} \cap \mathbb{Z} = n\mathbb{Z} \nsubseteq 0 = Rad_g(n\mathbb{Z})$.

(5) Evidently, any principally generalized hollow module is PG-Rad_g-lifting, also every principally semisimple module is PG-Rad_g-lifting, however, the converse is not true in general, as see the $\mathbb{Z}$-module $\mathbb{Z}_{24}$ is PG-Rad_g-lifting but neither principally semisimple nor principally generalized hollow.

Now we will give some appropriate definitions of PG-Rad_g-lifting modules and prove the equivalency among them.

Proposition 5.4.3. The following are equivalent for an R-module M.

(1) M is a PG-$\mathrm{Rad_g}$-lifting R-module.

(2) for any $m \in M$ with $Rad_g(M) \subseteq mR$, can be written as $mR = H \oplus G$, where H is a direct summand of M and $G \subseteq Rad_g(M)$;

(3) for any $m \in M$ with $Rad_g(M) \subseteq mR$, there exists a direct summand C of M such that $C \leq mR$ and $mR/C \subseteq Rad_g(M/C)$;

(4) for any $m \in M$ with $Rad_g(M) \subseteq mR$, mR containing a g-Radical supplement C in M such that $mR \cap C$ is a direct summand of mR;

(5) for any $m \in M$ with $Rad_g(M) \subseteq mR$, there is an $e = e^2 \in End(M)$ with $eM \leq mR$ and $(1-e)mR \subseteq Rad_g(1-e)M$;

(6) for any $m \in M$ with $Rad_g(M) \subseteq mR$, there is a submodule T of M inside mR such that $M = T \oplus C$ and C a g-Radical supplement of mR in M;

(7) for any $m \in M$ with $Rad_g(M) \subseteq mR$, there are principal ideals I and J of R such that $mR = mI \oplus mJ$, where $mI \leq^{\oplus} M$ and $mJ \subseteq Rad_g(M)$.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Suppose $m \in M$ with $Rad_g(M) \subseteq mR$. By **(1)**, there exists a decomposition $M = H \oplus S$ such that $H \leq mR$ and $mR \cap S \subseteq Rad_g(S) \subseteq Rad_g(M)$. By modular law, we have $mR = H \oplus (mR \cap S)$. Put $mR \cap S = G$. Thus $mR = H \oplus G$ such that $H \leq^{\oplus} M$ and $G \subseteq Rad_g(M)$.

$(\mathbf{2}) \Longrightarrow (\mathbf{3})$ Let $m \in M$ such that $Rad_g(M) \subseteq mR$. From **(2)**, $mR = E \oplus W$ such that $E \leq^{\oplus} M$ and $W \subseteq Rad_g(M)$. Suppose, the canonical map $\pi: M \longrightarrow M/E$. As $W \subseteq Rad_g(M)$, we have $(E+W)/E = \pi(W) \subseteq \pi(Rad_g(M)) \subseteq Rad_g(M/E)$, i.e., $mR/E \subseteq Rad_g(M/E)$.

$(\mathbf{3}) \Longrightarrow (\mathbf{4})$ Let $m \in M$ such that $Rad_g(M) \subseteq mR$. By **(3)**, there is a $C \leq^{\oplus} M$ with $C \leq mR$ and $mR/C \subseteq Rad_g(M/C)$, where $M = C \oplus W$ for a $W \leq M$. Since, $M = mR + W$. By modular law, $mR = C \oplus (mR \cap W)$. So, $mR/C \cong mR \cap W$ and $M/C \cong W$. Thus, $mR \cap W \subseteq Rad_g(W)$. Thus, mR containing a g-Radical supplement W in M and $mR \cap W \leq^{\oplus} mR$.

$(\mathbf{4}) \Longrightarrow (\mathbf{5})$ Let $m \in M$ such that $Rad_g(M) \subseteq mR$. By **(5)**, mR containing a g-Radical supplement D in M such that $mR \cap D \leq^{\oplus} mR$. As a result of this, $M = mR + D$ and $mR \cap D \subseteq Rad_g(D)$. So, $mR = (mR \cap D) \oplus H$ for some $H \leq mR$. Thus, $M = (mR \cap D) \oplus H + D = H \oplus D$. Suppose, that $e: M \to H$; $e(h + d) = h$ and $(1 - e): M \longrightarrow D$; $1 - e(h + d) = d$ are projection maps for all $h + d \in M$. Obviously, $e = e^2 \in End(M)$. Since $eM = H \leq mR$ and$(1 - e)mR = mR \cap (1 - e)M = mR \cap D \subseteq Rad_g(D) = Rad_g(1 - e)M$.

$(\mathbf{5}) \Longrightarrow (\mathbf{7})$ Suppose $m \in M$ such that $Rad_g(M) \subseteq mR$. By **(6)**, there exists an $e = e^2 \in End(M)$ such that $eM \leq mR$ and $(1 - e)mR \subseteq Rad_g(1 - e)M$. Consider $M = eM \oplus (1 - e)M$. If $r \in R$ such that $mr = (1 - e)m'$ for some $m' \in M$, so $m' = em' + mr \in mR$, as $eM \leq mR$, and then $mR \cap (1 - e)M \leq (1 - e)mR$. Thus, $mR \cap (1 - e)M = (1 - e)mR$. By modular law, we have $mR = mR \cap (eM \oplus (1 - e)M) = eM \oplus (mR \cap (1 - e)M) = eM \oplus (1 - e)mR$. We put $I = \{a \in R: ma \in eM\}$ and $J = \{b \in R: mb \in (1 - e)mR\}$. It comes $mR = mI \oplus mJ$, where $mI = eM \leq^{\oplus} M$ and $mJ = (1 - e)mR \subseteq Rad_g(1 - e)M$, and thus $mJ \subseteq Rad_g(M)$.

$(\mathbf{7}) \Longrightarrow (\mathbf{1})$ By **(8)**, let $m \in M$ such that $Rad_g(M) \subseteq mR$. Then there exists two ideals I and J of R such that $mR = mI \oplus mJ$, where mI is direct summand of M and $mJ \subseteq Rad_g(M)$. As a result of this, $mR/mI \cong mJ$. Also, $M = mI \oplus H$ for some $H \leq M$. Hence $mR = mI \oplus (mR \cap H)$, by the modular law, and so $mR \cap H \cong mR/mI \cong mJ \subseteq Rad_g(M)$. As $mR \cap H \leq H \leq^{\oplus} M$, then $mR \cap H \subseteq Rad_g(H)$ by Lemma 1.2.7. Hence, **(1)** holds.

$(\mathbf{1}) \Longrightarrow (\mathbf{7})$ Suppose that $m \in M$ such that $Rad_g(M) \subseteq mR$. Since, M is a PG-Rad$_g$-lifting module, so there is a decomposition $M = T \oplus C$ with $T \leq mR$ and $mR \cap C \subseteq Rad_g(C)$. Hence, $M = mR + C$, and then C is a g-Radical supplement of mR in M.

$(\mathbf{7}) \Longrightarrow (\mathbf{1})$ Evident. □

Proposition 5.4.4. If M is a PG-Rad_g-lifting R-module that all its cyclic submodules are indecomposable contains $Rad_g(M)$, then these submodules either direct summand or generalized Radical.

Proof. Let $x \in M$ such that $Rad_g(M) \subseteq xR$. Since M is a PG-Rad_g-lifting module, so by Proposition 5.4.3(2), we get that $xR = T \oplus D$ where $T \leq^{\oplus} M$ and $D \subseteq Rad_g(M)$. According to the hypothesis, $Rad_g(M) \subseteq xR$ and xR is indecomposable. As a result of this, either $xR = T$ or $xR = D \subseteq Rad_g(M)$. Thus, $xR \leq^{\oplus} M$ or $xR = Rad_g(M)$, as required. □

Proposition 5.4.5. Let M be a PG-Rad_g-lifting R-module and suppose E is a direct summand of M contains $Rad_g(M)$, then E is PG-Rad_g-lifting.

Proof. Let $E \leq^{\oplus} M$ such that $Rad_g(M) \subseteq E$. Let $x \in E$ with $Rad_g(E) \subseteq xR$, then by Lemma 1.2.37(1) we have $Rad_g(M) = Rad_g(E)$, thus $Rad_g(M) \subseteq xR$. As M is a PG-Rad_g-lifting module, Proposition 5.4.3(2) implies $xR = T \oplus N$ such that $T \leq^{\oplus} M$ and $N \subseteq Rad_g(M)$. Thus $M = T \oplus D$ for some $D \leq M$. Hence by the modular law, we have $E = T \oplus (E \cap D)$, that means $xR = T \oplus N$ with $T \leq^{\oplus} E$ and $N \subseteq Rad_g(M) = Rad_g(E)$. By Proposition 5.4.3(2), E is PG-Rad_g-lifting. □

Corollary 5.4.6. Let M be a R-module. Then M is PG-Rad_g-lifting if and only if any direct summand of M contains $Rad_g(M)$ is PG-Rad_g-lifting.

Proof. $\Rightarrow$) By Proposition 5.4.5.

$\Leftarrow$) As M is a direct summand of itself with $Rad_g(M) \subseteq M$. By assuming, M is a PG-Rad_g-lifting module. □

Corollary 5.4.7. Let M be a PG-Rad_g-lifting R-module. If $Rad_g(M) \leq^{\oplus} M$, then $Rad_g(M)$ is PG-Rad_g-lifting.

Proof. By Proposition 5.4.5 $Rad_g(M)$ is PG-Rad_g-lifting. □

Now, we will discuss direct sum of PG-Rad_g-lifting modules.

Theorem 5.4.8. Let $M = M_1 \oplus M_2$ be any direct sum of PG-Rad_g-lifting R-modules. If M is a duo (or, distributive) module, then M is PG-Rad_g-lifting.

Proof. Let $x \in M$ such that $Rad_g(M) \subseteq xR$ and $M = M_1 \oplus M_2$ is duo. As, xR is fully invariant in M, Lemma 1.2.14 imply $xR = (M_1 \cap xR) \oplus (M_2 \cap xR)$. We deduce that $M_i \cap xR$ is a cyclic in M_i and $Rad_g(M_i) \subseteq M_i \cap xR$ for each $i = 1,2$. As M_i is PG-Rad_g-lifting, then there is a decomposition $M_i = V_i \oplus W_i$ such that $V_i \leq M_i \cap xR$ and $(M_i \cap xR) \cap W_i = xR \cap W_i \subseteq Rad_g(W_i)$ for $i = 1,2$. Putting $V = V_1 \oplus V_2$ and $W = W_1 \oplus W_2$. Thus, $M = V \oplus W$ such that $V = V_1 \oplus V_2 \leq (M_1 \cap xR) \oplus (M_2 \cap xR) = xR$, and then $xR \cap W = (xR \cap W_1) \oplus (xR \cap W_2) \subseteq Rad_g(W_1) \oplus Rad_g(W_2) = Rad_g(W_1 \oplus W_2) = Rad_g(W)$, as we see in Lemma 1.2.10, so M is PG-Rad_g-lifting. We prove the theorem for distributive by the same way. □

Theorem 5.4.9. Let $\{M_i | \, i \in I\}$ be an infinite family of PG-Rad_g-lifting modules. If $M = \oplus_{i \in I} M_i$ is a duo module, then M is PG-Rad_g-lifting.

Proof. By the same way of proof of Theorem 5.4.15. □

Theorem 5.4.10. If M is a direct sum of PG-Rad_g-lifting R-modules M_1, M_2 such that M_1 is quasi-projective and M_2-projective, then $M = M_1 \oplus M_2$ is PG-Rad_g-lifting.

Proof. Let $x \in R$ such that $Rad_g(M) \subseteq xR$ where $M = M_1 \oplus M_2$. As a result of this, $M_1 \cap (xR + M_2) \leq M_1$ and $Rad_g(M_1) \subseteq M_1 \cap (xR + M_2)$. Since, M_1 is PG-Rad_g-lifting, then there is a decomposition $M_1 = C_1 \oplus C_2$, $C_1 \leq M_1 \cap (xR + M_2)$ and $C_2 \cap (xR + M_2) \subseteq Rad_g(C_2)$. Since $C_1 \oplus C_2 \oplus M_2 \subseteq (xR + M_2) \oplus C_2 \oplus M_2 = xR + (C_2 \oplus M_2)$ then $M = xR + (C_2 \oplus M_2)$. Since, M_1 is a quasi-projective module and M_2-projective, so C_1 is $C_2 \oplus M_2$-projective. By Lemma 1.2.31, there is $E_1 \leq xR$ such that $M = E_1 \oplus (C_2 \oplus M_2)$. It is obvious to confirm that $xR \cap (L + C_2) = L \cap (xR + C_2)$ for any $L \leq M_2$. Moreover, we have that $M_2 \cap (xR + C_2) \leq M_2$ with $Rad_g(M_2) \subseteq M_2 \cap (xR + C_2)$ and M_2 is a PG-Rad_g-lifting module, then there is a decomposition $M_2 = \acute{C}_1 \oplus \acute{C}_2$, $\acute{C}_1 \leq M_2 \cap (xR + C_2) = xR \cap (M_2 + C_2) \leq xR$ and hence $\acute{C}_2 \cap (xR + C_2) \subseteq Rad_g(\acute{C_2})$. We conclude that $M = (E_1 \oplus \acute{C}_1) \oplus (C_2 \oplus \acute{C}_2)$, $E_1 \oplus \acute{C}_1 \leq xR$ and $xR \cap \left(C_2 \oplus \acute{C}_2\right) = \acute{C}_2 \cap (xR + C_2) \subseteq Rad_g(C_2 \oplus \acute{C_2})$. Therefore, we get M is a PG-Rad_g-lifting module. □

Corollary 5.4.11. Let $M = M_1 \oplus M_2$ such that M_1 is a semisimple R-module and M_2 a PG-Rad_g-lifting R-module, and they are relatively projective with M_1, then M is PG-Rad_g-lifting.

Proof. Since M_1 is a semisimple R-module, hence it is principally semisimple. By Remarks and Examples 5.4.2(5), M_1 is PG-Rad_g-lifting and so from Theorem 5.4.10, M is a PG-Rad_g-lifting module. □

Now, we will discuss the factor modules of PG-Rad_g-lifting modules under some terms.

Proposition 5.4.12. Let $M = W \oplus C$ with $Rad_g(M) \subseteq C$, where W, C are submodules of M. If M is PG-Rad_g-lifting, then M/W is PG-Rad_g-lifting.

Proof. Suppose $M = W \oplus C$ such that $Rad_g(M) \subseteq C$. From Proposition 5.4.5, C is a PG-Rad_g-lifting module. Since, $M/W \cong C$, hence M/W is a PG-Rad_g-lifting module. □

Corollary 5.4.13. Assume $M = kerf \oplus N$ is a PG-Rad_g-lifting module and $f: M \to L$ an epimorphism with $Rad_g(M) \subseteq N$. Then L is PG-Rad_g-lifting.

Proof. By Proposition 5.4.12, $M/kerf$ is PG-Rad_g-lifting. By 1st isomorphism theorem $M/kerf \cong L$. Hence, L is PG-Rad_g-lifting. □

Theorem 5.4.14. Let M be a PG-Rad_g-lifting R-module and suppose $E \leq M$. If $(C+E)/E \leq^{\oplus} M/E$ for any $C \leq^{\oplus} M$, then M/E is PG-Rad_g-lifting.

Proof. Let $E \leq xR \leq M$ such that $x \in M$ and $Rad_g(M/E) \subseteq xR/E$. Suppose, that π_E is the natural map form M onto M/E. As, $Rad_g(M) \subseteq M$, we get $\pi_E(Rad_g(M)) \subseteq Rad_g(M/E) \subseteq xR/E$, and then $Rad_g(M) \subseteq xR$. Since, M is PG-Rad_g-lifting, then by Proposition 5.4.3(3), there is a $C \leq^{\oplus} M$ such that $C \leq xR$ and $xR/C \subseteq Rad_g(M/C)$. By the assuming $(C+E)/E \leq^{\oplus} M/E$. Evidently, $(C+E)/E \leq xR/E$. Define, $\psi: M/C \to \frac{M/C}{(E+C)/C}$ as the canonical map. Since, $xR/C \subseteq Rad_g(M/C)$, therefore $xR/(E+C) = \psi(xR/C) \subseteq \psi\big(Rad_g(M/C)\big)$, this imply $xR/(E+C) \subseteq Rad_g(M/(E+C))$. Therefore, M/E is PG-Rad_g-lifting, by Proposition 5.4.3(3). □

Proposition 5.4.15. Let M be a PG-Rad_g-lifting module and E a distributive (or, fully invariant) submodule of M. Then M/E is PG-Rad_g-lifting.

Proof. Assume $T \leq^{\oplus} M$, then $M = T \oplus N$ for some $N \leq M$. It is obvious that $M/E = ((T+E)/E) + ((N+E)/E)$. As E is distributive, so a result of this, $(T+E) \cap (N+E) = (T \cap N) + E = E$. So $((T+E)/E) \cap ((N+E)/E) = 0$, that is $(T+E)/E \leq^{\oplus} M/E$. Thus, by Theorem 5.4.14, M/E is a PG-Rad_g-lifting module. Now we prove the same result for fully invariant submodule. Suppose that $E \leq M$ is fully invariant, thus from Lemma 1.2.19 we get that $M/E = ((T+E)/E) \oplus ((N+E)/E)$, i.e., $(T+E)/E$ is a direct summand of M/E. Hence, M/E is a PG-Rad_g-lifting module by Theorem 5.4.21. □

Corollary 5.4.16. If M is a PG-Rad_g-lifting module, then both $M/Rad_g(M)$ and $M/P_g(M)$ are PG-Rad_g-lifting modules.

Proof. By Lemmas 1.2.8 and 1.2.36, respectively, $Rad_g(M)$ and $P_g(M)$ are fully invariant submodules of M. Hence by Proposition 5.4.8 $M/Rad_g(M)$ and $M/P_g(M)$ are PG-Rad_g-lifting. □

Corollary 5.4.17. Let M be a PG-$\mathrm{Rad_g}$-lifting module. If M is distributive (or, duo), then any factor module of M is also PG-$\mathrm{Rad_g}$-lifting.

Proof. Evident by Proposition 5.4.15. □

Corollary 5.4.18. Let M be a weak duo module and H a direct summand of M. If M is a PG-$\mathrm{Rad_g}$-lifting module, then H and M/H are PG-$\mathrm{Rad_g}$-lifting modules.

Proof. Let $M = H \oplus C$ for some $C \leq M$. Since M is a weak duo module, then H and C are fully invariant submodules of M. Thus, by Proposition 5.4.15, $H \cong M/C$ and M/H are PG-$\mathrm{Rad_g}$-lifting modules. □

Corollary 5.4.19. Suppose that M be a module and $H \leq^{\oplus} M$. Then M is PG-$\mathrm{Rad_g}$-lifting if and only if H and M/H are PG-$\mathrm{Rad_g}$-lifting modules, whenever:

(1) M is a duo module, or

(2) M is a distributive module.

Proof. (1) $\Longrightarrow$) As any duo module is weak duo, Corollary 5.4.18 leads to the result.

$\Longleftarrow$) Since $M \cong H \oplus (M/H)$, we deduce the result by Theorem 5.4.8.

(2) $\Longrightarrow$) Let $H \leq^{\oplus} M$, then $M = H \oplus N$ for some $N \leq M$. From Corollary 5.4.17, $H \cong M/N$ and M/H are PG-$\mathrm{Rad_g}$-lifting.

$\Longleftarrow$) Let $H \leq^{\oplus} M$, then $M = H \oplus N$ for some $N \leq M$. Hence $M/H \cong N$. Thus $M \cong H \oplus (M/H)$, the result comes by Theorem 5.4.8. □

Corollary 5.4.20. Let $f: M \rightarrow \grave{M}$ be any homomorphism. If M is a distributive (or, duo) PG-$\mathrm{Rad_g}$-lifting module, then Imf is also PG-$\mathrm{Rad_g}$-lifting.

Proof. Suppose that M is a distributive and PG-$\mathrm{Rad_g}$-lifting module. Suppose, $f: M \rightarrow \grave{M}$ be any homomorphism. By 1st isomorphism theorem, $M/E \cong Imf$

for some $E \leq M$. By Corollary 5.4.17, M/E is PG-Rad$_g$-lifting and hence Imf is PG-Rad$_g$-lifting. □

From Corollary 5.4.20, we get the following corollary.

Corollary 5.4.21. Let $f: M \to \grave{M}$ be an epimorphism from a distributive (or, duo) module M onto module $\grave{M}$. If M is a PG-Rad$_g$-lifting module, then $\grave{M}$ is also PG-Rad$_g$-lifting.

Corollary 5.4.22. Suppose that $f: M \to \grave{M}$ an homomorphism for a modules M, $\grave{M}$ and $kerf$ is a distributive, or fully invariant submodule of M. If M is a PG-Rad$_g$-lifting module, then Imf is PG-Rad$_g$-lifting also.

Proof. By Proposition 5.4.15 we get $M/kerf$ is a PG-Rad$_g$-lifting module. By 1st isomorphism theorem, $M/kerf \cong Imf$ and so Imf is PG-Rad$_g$-lifting. □

Proposition 5.4.23. If M is a PG-Rad$_g$-lifting R-module, then.

(1) $M/Rad_g(M)$ is principally semisimple.

(2) $M/P_g(M)$ is principally semisimple, whenever $Rad_g(M) \leq^{\oplus} M$

Proof. (1) Suppose $x \in M$ and $Rad_g(M) \leq xR \leq M$. By Proposition 5.4.3(4), xR containing a g-Radical supplement $\acute{C}$ in M, i.e., $M = xR + \acute{C}$ and $xR \cap \acute{C} \subseteq Rad_g(\acute{C})$. Thus $xR \cap \acute{C} \subseteq Rad_g(M)$. As $(xR \cap \acute{C}) + Rad_g(M) = Rad_g(M)$, then we get $\frac{M}{Rad_g(M)} = \frac{xR}{Rad_g(M)} + \frac{(\acute{C}+Rad_g(M))}{Rad_g(M)}$ and $\left(\frac{xR}{Rad_g(M)}\right) \cap \left(\frac{\acute{C}+Rad_g(M)}{Rad_g(M)}\right) = \frac{xR\cap(\acute{C}+Rad_g(M))}{Rad_g(M)} = \frac{Rad_g(M)+(xR\cap\acute{C})}{Rad_g(M)} = \frac{Rad_g(M)}{Rad_g(M)}$, i.e., $\frac{xR}{Rad_g(M)} \leq^{\oplus} \frac{M}{Rad_g(M)}$ and thus $M/Rad_g(M)$ is principally semisimple.

(2) From Lemma 1.2.37(2), we have $P_g(M) = Rad_g(M)$. By **(1)**, $M/P_g(M)$ is principally semisimple. □

Proposition 5.4.24. Let M be a cyclic and PG-Rad$_g$-lifting module over a PID R. Then M is Artinian if, M satisfies DCC on g-small submodules.

Proof. From Remarks and Examples 5.4.2(3) we get M is a G-Rad_g-lifting R-module. On the other hand, M is cyclic, implies finitely generated. Thus, the result comes by Proposition 5.2.13. □

Proposition 5.4.25. Suppose that P be a projective R-module. Then P is PG-Rad_g-lifting if and only if $P/Rad_g(P)$ is principally semisimple and for any cyclic submodule $W \leq P$ contains $Rad_g(P)$, there exists a $C \leq^{\oplus} P$ such that $\overline{W} = \bar{C}$, where $\overline{W} = W/Rad_g(P)$.

Proof. ⟹) Let $x \in P$ such that $Rad_g(P) \subseteq xR$. Since P is a PG-Rad_g-lifting R-module, thus by Proposition 5.4.23 $P/Rad_g(P)$ is principally semisimple. Put $\overline{xR} = xR/Rad_g(P)$. By Proposition 5.4.3(2), there is a decomposition $xR = C \oplus H$ such that $C \leq^{\oplus} P$ and $H \subseteq Rad_g(P)$. As a result of this, $xR \subseteq C + Rad_g(P)$. But $C + Rad_g(P) \subseteq xR$, that mean $xR = C + Rad_g(P)$. Since $xR/Rad_g(P) = (C + Rad_g(P))/Rad_g(P)$, imply $\overline{xR} = \bar{C}$.

⟸) Suppose that $x \in P$ with $Rad_g(P) \subseteq xR$. As $P/Rad_g(P)$ is principally semisimple, we have that $P/Rad_g(P) = xR/Rad_g(P) \oplus H/Rad_g(P)$ for some submodule H of M. Also, by the hypothesis, there is a direct summand T of P such that $\overline{xR} = \bar{T}$, where $P = T \oplus N$ for submodule N of P. Since, $P/Rad_g(P) = \bar{T} \oplus (N + Rad_g(P))/Rad_g(P)$ and hence $H = N + Rad_g(P)$. As a result of this, $P = T + N + Rad_g(P) = xR + N$. As $P = xR + N$ is projective, Lemma 1.2.32 implies $P = T' \oplus N$ with $T' \leq xR$. Also, $xR \cap N \subseteq xR \cap H = Rad_g(P)$, then P is a PG-Rad_g-lifting R-module. □

Corollary 5.4.26. Let R be an arbitrary ring. Then R is PG-Rad_g-lifting if and only if $R/Rad_g(R)$ is principally semisimple and for any cyclic ideal $I \leq R$ contains $Rad_g(R)$, there is a $J \leq^{\oplus} R$ such that $\bar{I} = \bar{J}$, and $\bar{I} = I/Rad_g(R)$.

Proof. Since $R = \langle 1 \rangle$ is a free ring, so it is projective. Hence, the outcome is follows directly by Proposition 5.4.25. □

Proposition 5.4.27. Let M be a PG-Rad$_g$-lifting R-module such that $Rad_g(M)$ is a proper cyclic in M. Then $M = W \oplus H$ where W is a g-Radical supplement of $Rad_g(M)$ in M and H is g-Radical.

Proof. Suppose M is a PG-Rad$_g$-lifting R-module with $Rad_g(M) \neq M$. Since, $Rad_g(M) \subseteq Rad_g(M)$ and cyclic, it comes by Proposition 5.4.3(4), $Rad_g(M)$ containing a g-Radical supplement W in M such that $Rad_g(M) \cap W \leq^{\oplus} Rad_g(M)$. Thus, $M = Rad_g(M) + W$ and $Rad_g(M) \cap W \subseteq Rad_g(W)$. Also, there is a submodule H of $Rad_g(M)$ such that $Rad_g(M) = (Rad_g(M) \cap W) \oplus H$. So, $M = Rad_g(M) + W = W + (Rad_g(M) \cap W) \oplus H = W \oplus H$. By Lemma 1.2.10, $M = Rad_g(M) + W = Rad_g(W) \oplus Rad_g(H) + W = Rad_g(H) \oplus W$. By the modular law, $H = H \cap (Rad_g(H) \oplus W) = Rad_g(H) \oplus (W \cap H) = Rad_g(H)$, which mean H is g-Radical. □

5.5. Connections with PG-Rad$_g$-lifting modules

This section will highlight several connections between the idea of PG-Rad$_g$-lifting module and other forms of modules.

Proposition 5.5.1. Let M be an R-module containing zero generalized Radical. Then M is PG-Rad$_g$-lifting if and only if M is principally semisimple.

Proof. ⟹) Assume that M is a PG-Rad$_g$-lifting R-module and $x \in M$. Since, $Rad_g(M) = 0 \subseteq xR$, then there is a decomposition $M = T \oplus N$ with $T \leq xR$ and $xR \cap N \subseteq Rad_g(N)$. As a result of this, $M = xR + N$ and $xR \cap N \subseteq Rad_g(M) = 0$. Thus, $xR \leq^{\oplus} M$ and then M is principally semisimple.

⟸) Clear by proposition 5.4.2. □

Proposition 5.5.2. Let M be a PG-$\mathrm{Rad_g}$-lifting module containing zero generalized Radical, then all its cyclic submodules are PG-$\mathrm{Rad_g}$-lifting.

Proof. By Proposition 5.5.1, M is a principally semisimple module, thus any cyclic submodule of M is a direct summand that contains $Rad_g(M) = 0$. By Proposition 5.4.8, any cyclic submodule of M is PG-$\mathrm{Rad_g}$-lifting. □

Corollary 5.5.3. Let M be an R-module. If M is an g-noncosingular R-module or, R is a g-V-ring, then any cyclic submodule of M is PG-$\mathrm{Rad_g}$-lifting, whenever M is PG-$\mathrm{Rad_g}$-lifting.

Proof. It follows from Lemma 1.2.21 (resp., definition of g-V-ring) and Proposition 5.5.2. □

Corollary 5.5.4. Let M be a module, $0 \neq E \leq M$ and $E \cap Rad_g(M) = 0$. If E is PG-$\mathrm{Rad_g}$-lifting, then E is principally semisimple.

Proof. Since $Rad_g(E) \subseteq E \cap Rad_g(M)$, then $Rad_g(E) = 0$ and hence E is principally semisimple by Proposition 5.5.1. □

Proposition 5.5.5. let M is PG-$\mathrm{Rad_g}$-lifting R-module and $Rad_g(M) \ll_g M$ then M is P-$\mathrm{Rad_g}$-lifting.

Proof. Suppose that $m \in M$ with $Rad_g(M) \subseteq mR$. As M is PG-$\mathrm{Rad_g}$-lifting, then there is a decomposition $M = T \oplus N$, $T \leq mR$ and $mR \cap N \subseteq Rad_g(N)$. From $Rad_g(N) \subseteq Rad_g(M) \ll_g M$, we deduce that $mR \cap N \ll_g M$. Thus, by Proposition 4.4.3(2), M is P-$\mathrm{Rad_g}$-lifting.□

Corollary 5.5.6. Let M be an R-module M such that $Rad_g(M) \ll_g M$. Then M is PG-$\mathrm{Rad_g}$-lifting if and only if M is P-$\mathrm{Rad_g}$-lifting.

Proof. It comes by Remarks and Examples 5.4.2(1) and Proposition 5.5.5. □

Corollary 5.5.7. If M is a finitely generated R-module, then M is PG-$\mathrm{Rad_g}$-lifting if and only if M is P-$\mathrm{Rad_g}$-lifting.

Proof. By Lemma 1.2.12, we get $Rad_g(M) \ll_g M$. So, the result is followed by Proposition 5.5.2. □

Proposition 5.5.8. If M is an indecomposable R-module, then the following cases are equivalent.

(1) M is PG-Rad_g-lifting.

(2) For all proper cyclic submodule $E \leq M$ containing $Rad_g(M)$, $E = Rad_g(M)$.

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$ Let $x \in M$ such that $Rad_g(M) \subseteq xR \neq M$. By **(1),** then there is a submodule T of M in xR such that $M = T \oplus C$ and C a g-Radical supplement of xR in M, see Proposition 5.4.3(6). Thus, $M = xR + C$ and $xR \cap C \subseteq Rad_g(C)$. As M is indecomposable, either $C = 0$ or $C = M$. If $C = 0$ then $xR = M$, a contradiction. Thus, $C = M$ and then $xR = xR \cap C \subseteq Rad_g(M)$. Hence, $xR = Rad_g(M)$.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$ Let $x \in M$ such that $Rad_g(M) \subseteq xR$. If $xR = M$, then trivially, there is a decomposition $M = M \oplus (0)$ with $M \leq xR$ and $xR \cap (0) \subseteq Rad_g(M)$. Let $xR \neq M$, so by **(2)** we get $xR \subseteq Rad_g(M)$. Thus, $M = (0) \oplus M$, $(0) \leq xR$ and $xR \cap M = xR \subseteq Rad_g(M)$. □

Corollary 5.5.9. The following are equivalent for a uniform R-module M.

(1) M is a PG-Rad_g-lifting.

(2) For all proper cyclic $E \leq M$ containing $Rad_g(M)$, $E = Rad_g(M)$.

Proof. As every uniform module is indecomposable, Proposition 5.5.8 directly yields the result. □

Proposition 5.5.10. Let M be an R-module with $Rad_g(M) = 0$. Then the following are equivalent.

(1) M is principally semisimple.

(2) M is a P-(P_g^*)-module.

(3) M is PG-Rad_g-lifting

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Clear by Proposition 2.5.14.

$(\mathbf{2}) \Longrightarrow (\mathbf{3})$ Clear by their definitions. $(\mathbf{3}) \Longrightarrow (\mathbf{1})$ by Proposition 5.5.1. □

Corollary 5.5.11. The terms "PG-Rad_g-lifting" and "principally semisimple" for an R-module M are equivalent for any of the following conditions hold:

(1) M is g-noncosingular.

(2) R is a g-V-ring.

Proof. (1) By Lemma 1.2.21, we have $Rad_g(M) = 0$. Hence, the result was held by Proposition 5.5.10.

(2) Since R is a g-V-ring, hence $Rad_g(M) = 0$. Hence the result follows by Proposition 5.5.10. □

Proposition 5.5.12. Let M be a uniserial R-module. If M is PG-Rad_g-lifting and $Rad_g(M)$ is minimal in M. Then,

(1) Every nonzero cyclic submodule containing $Rad_g(M)$.

(2) Every cyclic submodule of M is a direct summand.

(3) M is principally semisimple.

Proof. (1) Let $0 \neq x \in M$. Since $Rad_g(M) \leq M$ and M a uniserial, then either $xR \subseteq Rad_g(M)$ or $Rad_g(M) \subseteq xR$. Since, $Rad_g(M)$ is minimal and $xR \neq 0$, we have $Rad_g(M) \subseteq xR$.

(2) Let E be any cyclic submodule of M. If $E = 0$, then E trivially a direct summand. Suppose $E \neq 0$, by **(1)** $Rad_g(M) \subseteq E$. As M is a PG-Rad_g-lifting module, then by Proposition 5.4.3(6), there is a decomposition $M = T \oplus N$ such that $T \leq E$ and $E \cap N \subseteq Rad_g(N)$. Therefore, $M = E + N$ and $E \cap N \subseteq Rad_g(M)$. As $Rad_g(M)$ is minimal in M, we have that $E \cap N = 0$. Thus $E \leq^{\oplus} M$.

(3) From **(2)** we get that every cyclic submodule is a direct summand of M. This imply M is a principally semisimple module. □

Proposition 5.5.13. Let M be a uniserial R-module and $Rad_g(M)$ a minimal in M. If M is PG-Rad$_g$-lifting and $0 \neq m \in M$, then mR is PG-Rad$_g$-lifting.

Proof. Suppose that M is a PG-Rad$_g$-lifting module and $0 \neq m \in M$. From Proposition 5.5.12(1,2), mR containing $Rad_g(M)$, respectively, $mR \leq^{\oplus} M$. Hence by Proposition 5.4.8 mR is PG-Rad$_g$-lifting.

Proposition 5.5.14. Let M be a uniserial module and $Rad_g(M)$ a minimal submodule of M. Then the following are equivalent.

(1) M is a principally semisimple module.

(2) M is a P-(P_g^*)-module.

(3) M is a PG-Rad$_g$-lifting module.

Proof. $(\mathbf{1}) \Leftrightarrow (\mathbf{2})$ By Corollary 2.5.15.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$ Clear by definitions.

$(\mathbf{3}) \Rightarrow (\mathbf{1})$ By Proposition 5.5.12. □

Now, we will give some cases to make PG-Rad$_g$-lifting modules inherit their direct summands as well.

Proposition 5.5.15. Consider M be an R-module. If $Rad_g(M) = 0$ or, M is g-noncosingular. Then the following are equivalent.

(1) M is principally semisimple.

(2) M is principally g-lifting.

(3) M is P-Rad$_g$-lifting.

(4) M is P-(P_g^*)-module.

(5) M is PG-Rad$_g$-lifting.

(6) M is principally ⊕-g-supplemented.

(7) M is a principally sgrs$^{\oplus}$-module.

(8) M is ⊕-PG-Rad$_g$-supplemented.

(9) M is ⊕-PG-Radical supplemented.

(10) M is PG-Radical supplemented.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2}) \Longrightarrow (\mathbf{3})$ By Proposition 4.5.5.

$(\mathbf{3}) \Longrightarrow (\mathbf{4})$ Let $m \in M$. As $Rad_g(M) = 0 \subseteq mR$, so there is a decomposition $M = T \oplus N$ with $T \leq mR$ and $mR \cap N \ll_g N$. Thus $mR \cap N \ll_g Rad_g(N)$ and hence, **(4)** hold.

$(\mathbf{4}) \Longrightarrow (\mathbf{5})$ By Proposition 5.5.10.

$(\mathbf{5}) \Longrightarrow (\mathbf{6})$ Assume $x \in M$. As $Rad_g(M) = 0 \subseteq xR$ and M a PG-Rad_g-lifting module, then by Proposition 5.4.3(4) there exists $M = T \oplus N$ with $T \leq xR$ and $xR \cap N \subseteq Rad_g(N) \subseteq Rad_g(M) = 0$. So that, $M = xR + N$ and $xR \cap N = 0 \ll_g N$. Thus M is principally ⊕-g-supplemented.

$(\mathbf{6}) \Longrightarrow (\mathbf{7}) \Longrightarrow (\mathbf{8}) \Longrightarrow (\mathbf{1})$ By Lemma 1.2.61.

$(\mathbf{1}) \Leftrightarrow (\mathbf{9}) \Leftrightarrow (\mathbf{10})$ By Proposition 3.2.28. □

Corollary 5.5.16. Let M be a cyclic module over a PID R with $Rad_g(M) = 0$. Then the following are equivalent.

(1) M is semisimple.

(2) M is principally semisimple.

(3) M is g-lifting.

(4) M is principally g-lifting.

(5) M is ⊕-g-supplemented.

(6) M is principally ⊕-g-supplemented.

(7) M is a $\text{sgrs}^{\oplus}$-module.

(8) M is a principally $\text{sgrs}^{\oplus}$-module.

(9) M is ⊕-G-Rad_g-supplemented.

(10) M is ⊕-PG-Rad_g-supplemented.

(11) M is Rad_g-lifting.

(12) M is P-Rad_g-lifting.

(13) M is G-Rad_g-lifting.

(14) M is PG-Rad_g-lifting.

(15) M is a (P_g^*)-module.

(16) M is a P-(P_g^*)-module.

(17) M is ⊕-g-Radical supplemented.

(18) M is principally ⊕-g-Radical supplemented.

(19) M is g-Radical supplemented.

(20) M is principally g-Radical supplemented.

Proof.$(\mathbf{1}) \Leftrightarrow (\mathbf{11}) \Leftrightarrow (\mathbf{13}) \Leftrightarrow (\mathbf{15}) \Leftrightarrow (\mathbf{17}) \Leftrightarrow (\mathbf{19})$ By Proposition 5.2.19.

$(\mathbf{2}) \Leftrightarrow (\mathbf{4}) \Leftrightarrow (\mathbf{6}) \Leftrightarrow (\mathbf{8}) \Leftrightarrow (\mathbf{10}) \Leftrightarrow (\mathbf{12}) \Leftrightarrow (\mathbf{14}) \Leftrightarrow (\mathbf{16}) \Leftrightarrow (\mathbf{18}) \Leftrightarrow (\mathbf{20})$ By Proposition 5.5.15.

$(\mathbf{1}) \Leftrightarrow (\mathbf{2}) \Leftrightarrow (\mathbf{3}) \Leftrightarrow (\mathbf{4}) \Leftrightarrow (\mathbf{5}) \Leftrightarrow (\mathbf{6}) \Leftrightarrow (\mathbf{7}) \Leftrightarrow (\mathbf{8}) \Leftrightarrow (\mathbf{9}) \Leftrightarrow (\mathbf{10})$ From Lemma 1.2.61. □

Proposition 5.5.17. Suppose the following cases for an R-module M:

(1) M is PG-Rad_g-lifting.

(2) $M/Rad_g(M)$ is principally semisimple.

Then (1) ⟹ (2). If M is a refinable R-module, then (2) ⟹ (1).

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ By Proposition 5.4.30.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$ Suppose that $m \in M$ such that $Rad_g(M) \subseteq mR$. As $M/Rad_g(M)$ is principally semisimple and $mR/Rad_g(M)$ is cyclic in $M/Rad_g(M)$, hence there is $D \leq M$ such that $M/Rad_g(M) = (mR/Rad_g(M)) \oplus (D/Rad_g(M))$, where $D/Rad_g(M) \leq M/Rad_g(M)$. Thus, $M = mR + D$ and $mR \cap D = Rad_g(M)$. As $M = mR + D$ is refinable, then there exists a direct summand T of M such that $T \leq mR$ and $M = T + D$. Suppose, $\pi: M \to M/T$ is a canonical map. As $mR \cap D \subseteq Rad_g(M)$, then $\pi(mR \cap D) \subseteq \pi(Rad_g(M))$ and thus $\frac{(mR\cap D)+T}{T} \subseteq Rad_g\left(\frac{M}{T}\right)$. By modular law, we have that $\frac{(T+D)\cap mR}{T} = \frac{M\cap mR}{T} = \frac{mR}{T} \subseteq Rad_g\left(\frac{M}{T}\right)$. Thus M is PG-Rad_g-lifting, by Proposition 5.4.3(3).□

Corollary 5.5.18. Let M be a refinable and cyclic module over a PID R. Then the following are equivalent.

(1) M is a principally sgrs$^{\oplus}$-module.

(2) M is a sgrs$^{\oplus}$-module.

(3) M is PG-$\mathrm{Rad_g}$-lifting.

(4) M is G-$\mathrm{Rad_g}$-lifting.

(5) $M/Rad_g(M)$ is principally semisimple.

Proof. $(\mathbf{1}) \Leftrightarrow (\mathbf{2})$ By Lemma 1.2.54.

$(\mathbf{1}) \Leftrightarrow (\mathbf{5})$ By Lemma 1.2.55.

$(\mathbf{3}) \Leftrightarrow (\mathbf{4})$ By Remarks and Examples 5.4.2(3).

$(\mathbf{3}) \Leftrightarrow (\mathbf{5})$ By Proposition 5.5.17.□

Corollary 5.5.19. If M is a nonzero indecomposable R-module with $Rad_g(M) = 0$, then the following are equivalent.

(1) M is principally semisimple.

(2) M is principally g-lifting.

(3) M is P-$\mathrm{Rad_g}$-lifting.

(4) M is principally ($\mathrm{P_g}^*$)-module.

(5) M is PG-$\mathrm{Rad_g}$-lifting.

(6) M is principally ⊕-g-supplemented.

(7) M is a principally sgrs$^{\oplus}$-module.

(8) M is ⊕-PG-$\mathrm{Rad_g}$-supplemented.

(9) M is principally ⊕-g-Radical supplemented.

(10) M is principally g-Radical supplemented.

(11) M is local.

(12) M is hollow.

(13) M is principally generalized hollow.

Proof. $(\mathbf{1}) \Leftrightarrow \cdots \Leftrightarrow (\mathbf{10})$ By Proposition 5.5.15.

$(\mathbf{1}) \Leftrightarrow (\mathbf{11}) \Leftrightarrow (\mathbf{12}) \Leftrightarrow (\mathbf{13})$ By Corollary 5.2.9.

5.6. Rings and localization of PG-$\mathrm{Rad_g}$-lifting

We are going to study the behavior of PG-$\mathrm{Rad_g}$-lifting modules as rings and we discussed under localization.

Theorem 5.6.1. Let M be a faithful, finitely generated and multiplication module over a commutative ring R with unity. Then M is a PG-$\mathrm{Rad_g}$-lifting R-module if and only if R is a PG-$\mathrm{Rad_g}$-lifting ring.

Proof. Suppose that M is a PG-$\mathrm{Rad_g}$-lifting R-module. Let I be any principal ideal of R such that $Rad_g(R) \subseteq I$. As M is a finitely generated multiplication R-module, then IM is a cyclic submodule of M. Thus, $Rad_g(R)M \subseteq IM$. So, by Lemma 2.6.1 $Rad_g(M) \subseteq IM$, and there exists a decomposition $M = E\oplus W$ such that $E \leq IM$ and $IM \cap W \subseteq Rad_g(W) \subseteq Rad_g(M)$. As M is multiplication, we have that $E = JM$ and $W = EM$ for some ideals J and E of R. As a result of this, $M = JM\oplus EM$ such that $JM \leq IM$ and $IM \cap EM \subseteq Rad_g(M)$. From Lemmas 1.2.63, 1.2.64 and 2.6.1, we conclude that $R = J\oplus E, J \leq I$ and $I \cap E \subseteq Rad_g(R)$. Since, R_R is PG-$\mathrm{Rad_g}$-lifting.

Conversely, suppose that R is PG-$\mathrm{Rad_g}$-lifting. Let $H = IM$ be a cyclic submodule of M with $Rad_g(M) \subseteq IM$, for some ideal I of R. Because M is a finitely generated multiplication R-module, then I is cyclic in R. By Lemma 2.6.1, $Rad_g(R)M \subseteq IM$ implies $Rad_g(R) \subseteq I$ by Lemma 1.2.64. There is a decomposition $R = J\oplus E$ such that $J \leq I$ and $I \cap E \subseteq Rad_g(R)$. So, $M = (J\oplus E)M = JM\oplus EM$ such that $JM \leq IM = H$. By Lemma 1.2.63 and 2.6.1, we have that $H \cap EM = IM \cap EM = (I \cap E)M \subseteq Rad_g(R)M = Rad_g(M)$. Hence, M is PG-$\mathrm{Rad_g}$-lifting. □

Theorem 5.6.2. Let M be an R-module and $S \subseteq R$ a multiplicative closed such that $\mathcal{L}(T) \cap S = \emptyset$ for each $T \leq M$. Then M is a PG-$\mathrm{Rad_g}$-lifting as R-module if and only if $S^{-1}M$ is a PG-$\mathrm{Rad_g}$-lifting as $S^{-1}R$-module.

Proof. Suppose M is a PG-$\mathrm{Rad_g}$-lifting as R-module and let $S^{-1}E$ be a cyclic submodule of $S^{-1}M$ as $S^{-1}R$-module such that $Rad_g(S^{-1}M) \subseteq S^{-1}E$. From Lemma 2.3.2, we have that $S^{-1}(Rad_g(M)) \subseteq S^{-1}E$, so E is cyclic in M such that $Rad_g(M) \subseteq E$, see Lemma 1.2.66. Then there exists a decomposition $M = H \oplus W$ such that $H \leq E$ and $E \cap W \subseteq Rad_g(W)$ for some $W \leq M$. From Lemmas 1.2.71, 1.2.65 and 2.3.2, we get $(S^{-1}H) \oplus (S^{-1}W) = S^{-1}M$ with $S^{-1}H \leq S^{-1}E$ and $(S^{-1}E) \cap (S^{-1}W) \subseteq Rad_g(S^{-1}W)$. Hence, $S^{-1}M$ is a PG-$\mathrm{Rad_g}$-lifting as $S^{-1}R$-module.

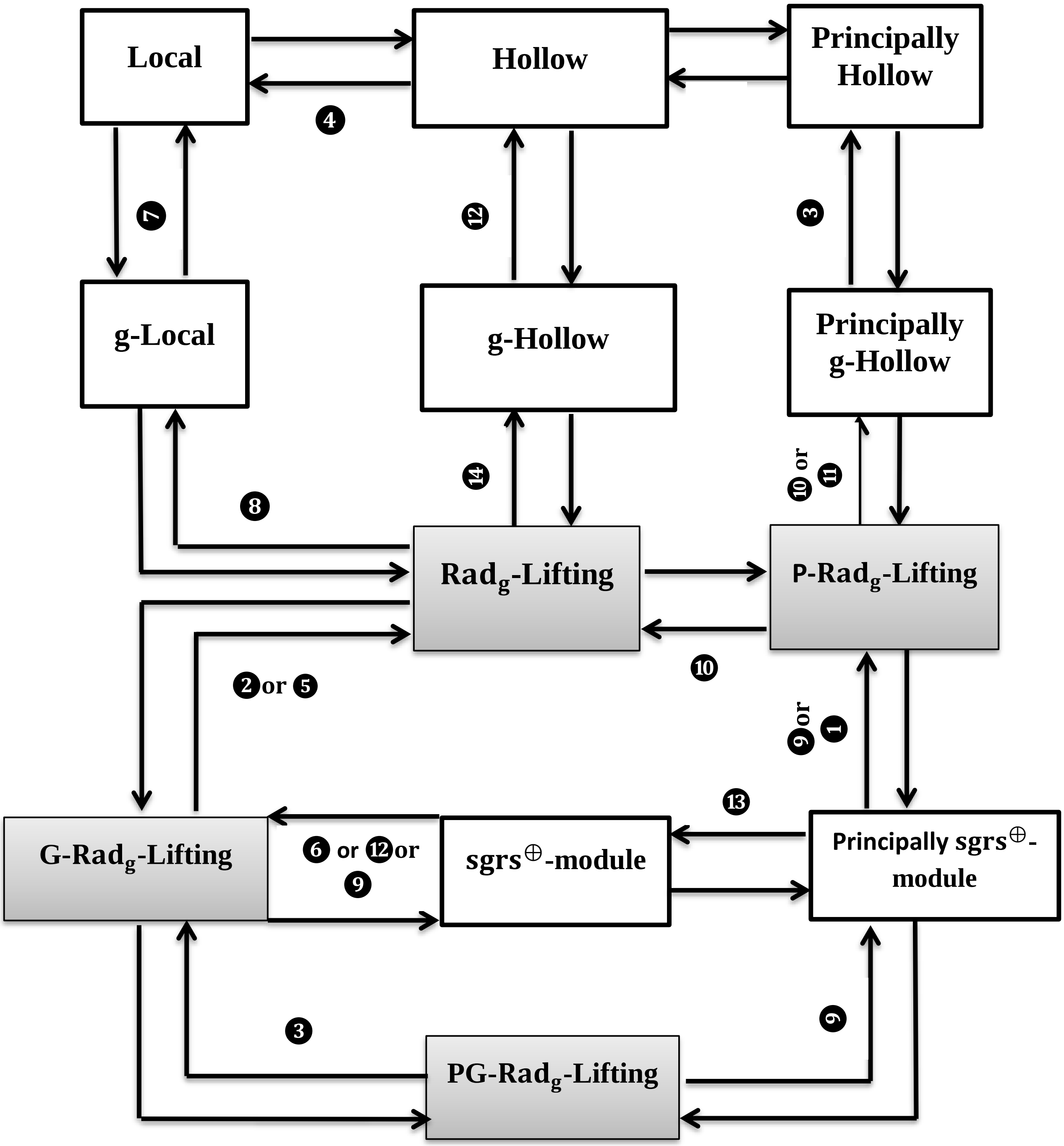
Conversely, suppose $S^{-1}M$ is a PG-$\mathrm{Rad_g}$-lifting as $S^{-1}R$-module, and let E be a cyclic submodule of M such that $Rad_g(M) \subseteq E$. Lemma 1.2.66 and 2.3.2 implies $S^{-1}E$ is a cyclic submodule of $S^{-1}M$ and $Rad_g(S^{-1}M) = S^{-1}(Rad_g(M)) \subseteq S^{-1}E$. Then there is a decomposition $(S^{-1}C) \oplus (S^{-1}W) = S^{-1}M$ such that $S^{-1}C \leq S^{-1}E$ and $(S^{-1}E) \cap (S^{-1}W) \subseteq Rad_g(S^{-1}W)$ where $S^{-1}W \leq S^{-1}M$. By Lemmas 1.2.71, 1.2.65 and 2.3.2, we have that $C \oplus W = M$ such that $C \leq E$ and $E \cap W \subseteq Rad_g(W)$ for a $W \leq M$, that means M is a PG-$\mathrm{Rad_g}$-lifting as R-module. □

Theorem 5.6.3. Suppose that M be an R-module and $S \subseteq R$ a multiplicative closed set such that for any $W \subset M$, $(W :_M s) = W$, for all $s \in S$. Then M is a PG-$\mathrm{Rad_g}$-lifting as R-module if and only if $S^{-1}M$ is a PG-$\mathrm{Rad_g}$-lifting as $S^{-1}R$-module.

Proof. Let M be a PG-$\mathrm{Rad_g}$-lifting as R-module and let $S^{-1}E$ be a cyclic submodule of $S^{-1}M$ as $S^{-1}R$-module such that $Rad_g(S^{-1}M) \subseteq S^{-1}E$. From Lemma 2.3.5, As a result of this, $S^{-1}(Rad_g(M)) \subseteq S^{-1}E$ and hence E is a cyclic submodule of M with $Rad_g(M) \subseteq E$ by Lemmas 1.2.67 and 1.2.68. Then there is a decomposition $M = H \oplus W$ such that $H \leq E$ and $E \cap W \subseteq Rad_g(W)$ for some $W \leq M$. By Lemmas 1.2.71, 1.2.67 and 2.3.5, we deduce $S^{-1}M = (S^{-1}H) \oplus (S^{-1}W)$ such that $S^{-1}H \leq S^{-1}E$ and $(S^{-1}E) \cap (S^{-1}W) \subseteq Rad_g(S^{-1}W)$. Hence, $S^{-1}M$ is a PG-Rad$_g$-lifting as $S^{-1}R$-module.

Conversely, suppose that $S^{-1}M$ is a PG-Rad$_g$-lifting as $S^{-1}R$-module, and let U be a cyclic submodule of M such that $Rad_g(M) \subseteq U$. From Lemma 2.3.5, 1.2.67 and 1.2.68, $S^{-1}U$ is a cyclic submodule of $S^{-1}M$ and $Rad_g(S^{-1}M) = S^{-1}(Rad_g(M)) \subseteq S^{-1}U$. Then there exists a decomposition $(S^{-1}D) \oplus (S^{-1}W) = S^{-1}M$ such that $S^{-1}D \leq S^{-1}U$ and $(S^{-1}U) \cap (S^{-1}W) \subseteq Rad_g(S^{-1}W)$ where $S^{-1}W \leq S^{-1}M$. By Lemmas 1.2.71, 1.2.67 and 2.3.5, we deduce that $M = D \oplus W$ such that $D \leq U$ and $U \cap W \subseteq Rad_g(W)$ where $W \leq M$. Therefore, M is a PG-Rad$_g$-lifting as R-module. □

Diagram of Some Important Implications and Their Reverses for Chapters Four & Five

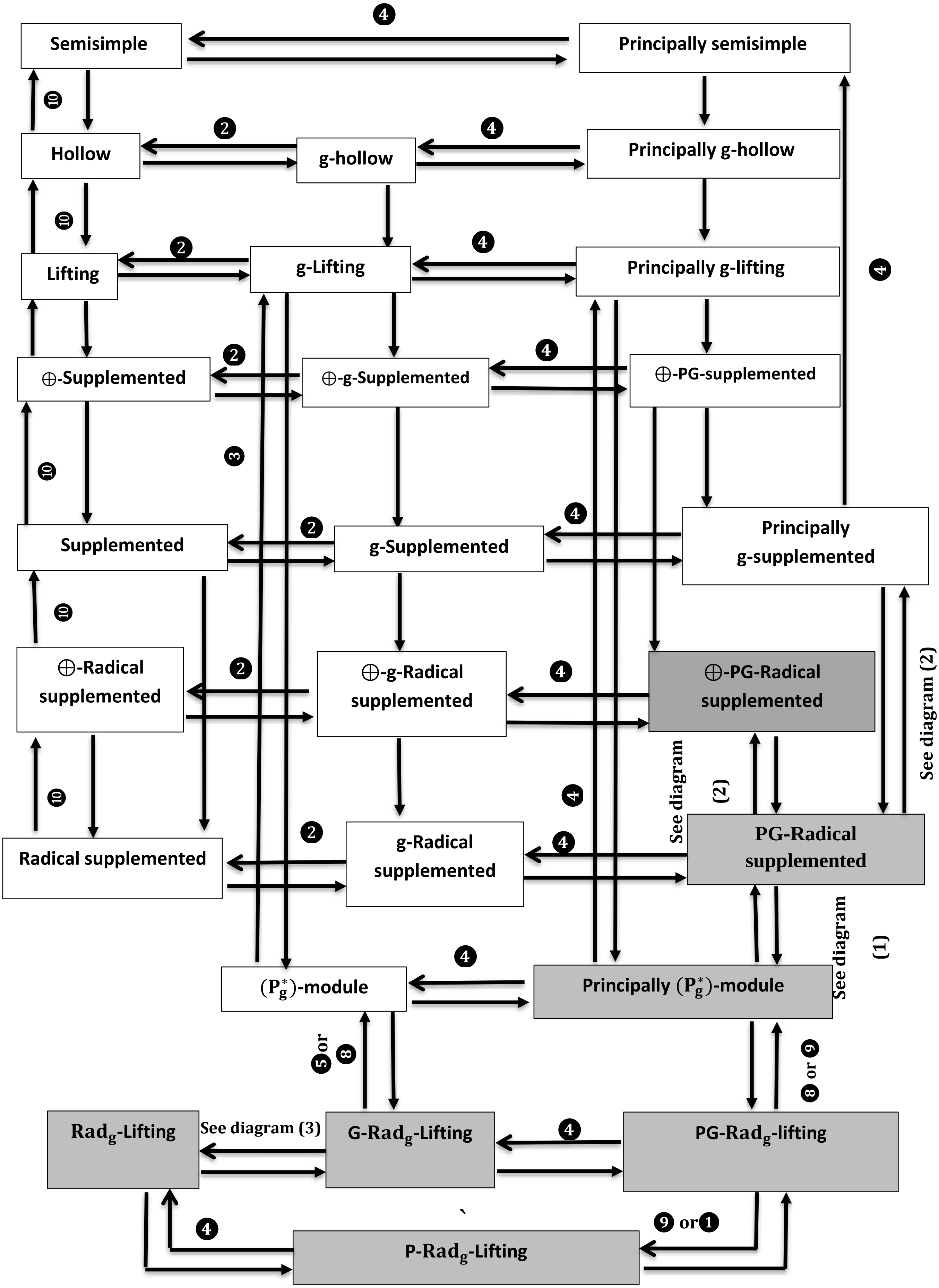


Local
Hollow
Principally Hollow
g-Local
g-Hollow
Principally g-Hollow
Rad$_g$-Lifting
P-Rad$_g$-Lifting
G-Rad$_g$-Lifting
sgrs$^\oplus$-module
Principally sgrs$^\oplus$-module
PG-Rad$_g$-Lifting
4
7
12
3
8
14
10 or 11
10
2 or 5
9 or 1
13
6 or 12 or 9
3
9

Diagram (3)

Symbol	Condition	Related results
❶	If M is indecomposable	Proposition 4.2.6
❷	If $Rad_g(M) \ll_g M$ or finitely generated	Proposition 5.2.2
❸	If M be a cyclic module over a PID R with $Rad_g(M) = 0$	Corollary 5.5.16
❹	If M is indecomposable (or finitely generated) & $Rad_g(M)$ is proper g-small in M	Proposition 5.2.9 & 5.2.10
❺	If M is uniserial module and $Rad_g(M)$ a minimal and g-small submodule of M	Corollary 5.2.7
❻	If M be a refinable and cyclic module over a PID R.	Corollary 5.5.18
❼	If M is indecomposable	Lemma 1.2.34
❽	If M is indecomposable with $Rad_g(M) \neq M$	Corollary 4.2.32
❾	If $Rad_g(M) = 0$ or, M is e-noncosingular.	Corollary 5.5.15
❿	If M is an indecomposable cyclic over a PID ring	Corollary 4.5.21
⓫	If M is a nonzero indecomposable R-module with $Rad_g(M) = 0$	Corollary 4.5.22
⓬	If M is an indecomposable & $Rad_g(M)$ a proper g-small submodule of M	Proposition 5.2.9
⓭	If M be a cyclic module over a PID R	Lemma 1.2.54
⓮	If M is indecomposable with $Rad_g(M) \neq M$	Corollary 4.2.33

General Diagram



Semisimple
Principally semisimple
Hollow
g-hollow
Principally g-hollow
Lifting
g-Lifting
Principally g-lifting
⊕-Supplemented
⊕-g-Supplemented
⊕-PG-supplemented
Supplemented
g-Supplemented
Principally g-supplemented
⊕-Radical supplemented
⊕-g-Radical supplemented
⊕-PG-Radical supplemented
Radical supplemented
g-Radical supplemented
PG-Radical supplemented
See diagram (2)
See diagram (2)
See diagram (1)
(P_g^*)-module
Principally (P_g^*)-module
5 or 8
8 or 9
Rad_g-Lifting
See diagram (3)
G-Rad_g-Lifting
PG-Rad_g-lifting
9 or 1
P-Rad_g-Lifting

Symbols	The conditions	Related results
❶	If M is R-module with $Rad_g(M) = 0$	Proposition 5.5.15
❷	M is indecomposable	Lemma 1.2.11
❸	If $Rad_g(M) \ll_g M$	[15, Proposition 3.6]
❹	If M is cyclic over a PID ring & $Rad_g(M) = 0$.	Corollary 5.5.15
❺	If $Rad_g(M) = 0$	Corollary 5.2.5
❽	If M is uniserial & $Rad_g(M)$ is minimal	Proposition 5.5.6
❾	$Rad_g(M) = 0$ or, R is g-V-ring or, M is g-noncosingular	Proposition 5.5.15
❿	If $Rad_g(M) = 0$.	Lemma 1.2.61 & 1.2.52 &1.2.58

Conclusion

In this work, we presented seven different concepts, they are: principally (P_g^*)-modules, PG-radical supplemented modules, $\oplus$-PG-radical supplemented modules, Rad_g-lifting modules, P-Rad_g-lifting modules, G-Rad_g-lifting modules and PG-Rad_g-lifting modules. As seen, the first three ideas was a generalizations of (P_g^*)-module, g-radical supplemented and$\oplus$-g-radical supplemented respectively. And the second four concepts was a generalizations of g-lifting and principally g-lifting modules.

It was obvious that every (P_g^*)-module is principally (P_g^*)-module and every g-radical supplemented module is PG-radical supplemented, every $\oplus$-g-radical supplemented module is $\oplus$-PG-radical supplemented, every g-lifting module is Rad_g-lifting, every principally g-lifting module is P-Rad_g-lifting and every PG-Rad_g-lifting module is G-Rad_g-lifting but the converse is not true in general. We found some counter examples to show that the converse is not true and we faced some difficulties to find other couture examples.

So that we found that these notions, (principally (P_g^*)-module with (P_g^*)-module), (g-radical supplemented module with PG-radical supplemented module), ($\oplus$-g-radical supplemented module with $\oplus$-PG-radical supplemented module), (Rad_g-lifting module with P-Rad_g-lifting module) and (G-Rad_g-lifting module with PG-Rad_g-lifting module) were all equivalents if, the module is cyclic over PID rings.

Finally, we found that, there is an equivalency among our seven concepts with each other under special conditions as you can see that in sections of connections that we added in all chapters of this dissertation.

Future Works

As a future plane we may give the idea of strongly G-Rad_g-supplemented and principally strongly G-Rad_g-supplemented modules and find the relations between these two definitions and our seven definitions, so that with many different other concepts.

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الخلاصة

ان الهدف الأساسي والأول لهذه الأطروحة هو فحص الخصائص والميزات والعلاقات المرتبطة بالأفكار السبعة حيث يتم تغطية هذا الهدف من خلال مجموعة متنوعة من الأمثلة والنتائج الهامة أيضا كعلاقة بين المفاهيم السبعة. الهدف الثاني من هذه الاطروحة هو النظر في الروابط بين تلك الافكار وعدد من الأنواع الأخرى من المقاسات، مثل: المقاسات الغير القابلة للتحليل، الشبه البسيطة، المجوفة المعممة، مقاسات الرفع من النمط-g، المكملة الجذرية من النمط-g، التوزيع، المنتظمة، ديو الضعيفة، الاسقاطية، ... الخ. حيث تم استكشاف جميع سمات هذه المقاسات والتوصيفات والأمثلة .هناك الكثير من الصلات المباشرة لأنواع أخرى من المفاهيم تم توفيرها .أيضًا ، نقدم بعض الشروط مثل الحالة التي تجعل فئات المقاسات السبع مغلقة تحت الجمع المباشر.

هذه الدراسة قدّمت العديد من المفاهيم المختلفة في نظرية المقاس. أولاً ، قدمنا مفهوم المقاسات التكميلية الجذرية المعممة الدائرية، حيث يُقال إن المقاسM على الحلقة R هو مقاس مكمل جذري معمم دائري، مكمل جذري من النمط PG، إذا كان لكل مقاس جزئي دائري N لـM فانه يوجد مقاس جزئي مكمل X في M .بعد ذلك نقدم مفهوم المقاسات الأساسية من النمط -($\mathrm{P_g}^*$)، حيث يُقال أن المقاس M على الحلقة R أساسيًا من النمط -($\mathrm{P_g}^*$) إذا كان لكل عنصر$m \in M$ ، فانه يوجد $D \leq^{\oplus} M$ بحيث ان $D \leq mR$ و $mR/D \subseteq Rad_g(M/D)$. لاحقا نقدم مفهوم المقاسات الجذرية التكميلية المعممة الدائرية، حيث يُقال إن المقاسM على الحلقة R هو مقاس جذري تكميلي معمم دائري إذا كان لأي مقاس جزئي دائريN فيM ، فانه يوجد $X \leq^{\oplus} M$ بحيث ان $M = N + X$ و $N \cap X \subseteq Rad_g(X)$. علاوة على ذلك ، قدمنا فكرة مقاسات الرفع من النمط $\mathrm{Rad_g}$-، حيث يُقال إن المقاسM على الحلقة R هو مقاس رفع من النمط $\mathrm{Rad_g}$- إذا كان لأي مقاس جزئي دائري N فيM بحيث ان $Rad_g(M) \subseteq N$ فانه يوجد $M = A \oplus B$ بحيث ان $A \leq N$ و $N \cap B \ll_g B$. ثم قدمنا مفهوم نفس مفهوم مقاسات الرفع من النمط $\mathrm{Rad_g}$- الذي يقتصر على المقاسات الجزئية الدائرية.

فيما بعد قدمنا تعريف مقاسات الرفع من النمط -$\mathrm{G\text{-}Rad_g}$، إذا كان لأي مقاس جزئي دائري N فيM، بحيث ان $Rad_g(M) \subseteq N$ فانه يوجد $M = A \oplus B$ بحيث ان $A \leq N$ و $N \cap B \leq Rad_g(B)$. كذلك قدمنا مفهوم نفس مفهوم مقاسات الرفع من النمط -$\mathrm{G\text{-}Rad_g}$ الذي يقتصر على المقاسات الجزئية الدائرية.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل – كلية التربية للعلوم الصرفة

قسم الرياضيات

بعض أنواع مقاسات الرفع من النمط -g و تعميماتها

أطروحة

مقدمة الى مجلس كلية التربية للعلوم الصرفة في جامعة بابل كجزء
من متطلبات نيل درجة دكتوراه في فلسفة التربية / الرياضيات

من قبل

رشا نجاح مرزه حمزه

بأشراف

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