Republic of Iraq
Ministry of Higher Education
and Scientific Research
University of Babylon
College of Education for pure Science
Department of Mathematics

New Nano Topological Structures

A Thesis
Submitted to the Council of the College of Education, for Pure Sciences in the University of Babylon
In Partial Fulfillment of the Requirements for the Degree of Master in Education /Mathematics

By

Ghufran Ali Hussien Abd-Al Razaq

Supervised By

Prof. Zahir Dobeas Azawe AL-Nafie (Ph. D.)

2023 A. D. 1445 A. H.

بسم الله الرحمن الرحيم

﴿إِنَّ اللَّهَ وَمَلَائِكَتَهُ يُصَلُّونَ عَلَى النَّبِيِّ يَا أَيُّهَا الَّذِينَ آمَنُوا صَلُّوا عَلَيْهِ وَسَلِّمُوا تَسْلِيمًا﴾

صدق الله العلي العظيم

سورة الأحزاب آية (٥٦)

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Dedication

To my parents…

May God protect you for me and keep you alive and proud. I cherish and am proud of you in every place and time.

To whose presence is the great support, to those with whom I participate my happy, sad and all my moments, my brothers and sisters.

To my distinguished teachers…

From whom I derived the letters, and I learned how to pronounce word and formulate phrases.

To colleagues.

who have spared no effort to provide me with information and data.

To everyone who supported me to reach this success.

Ghufran Ali Hussein

Acknowledgem

First and foremost, I thank the Almighty Allah for helping me to complete this thesis. Our thanks go to Prophet Mohammed and Ahlul Bayt (blessings of Allah be upon them all).

I would like to introduce my deep gratitude to my supervisor Prof. Dr. Zahir Dobeas AL-Nafie for his continuous support to my research, for patience, encouragement and invaluable suggestions throughout the entire period of my work.

Also, my thanks to all members of Mathematics Department, College of Education for Pure Sciences, University of Kerbala, who are credited for my arrival at this stage.

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Symbol	Description
$\mathcal{P}O$	**Pre-open Set**
$\mathcal{P}C$	**Pre-closed Set**
$\mathcal{P}Int(A)$	**Pre-interior of Set**
$\mathcal{P}Cl(A)$	**Pre-closure of Set**
$\mathbb{N}_X$	**neighborhood (nbd)**
$S\mathcal{P}O$	**Semi Pre-open Set**
$S\mathcal{P}C$	**Semi Pre-closed Set**
$S\mathcal{P}\ Cl(A)$	**Semi Pre-closure Set**
NTS	**Nano Topological Space**
$N\ Int(A)$	**Nano interior Set**
$N\ Cl(A)$	**Nano closure Set**
$N_{\mathcal{P}O}$	**Nano Pre-open Set**
$N_{\mathcal{P}C}$	**Nano Pre-closed Set**
N_{RO}	**Nano regular-open**
N_{RC}	**Nano regular-closed**
$N_{\mathcal{P}O}Int(A)$	**Nano Pre-interior Set**
$N_{\mathcal{P}C}Cl(A)$	**Nano Pre-closure Set**

List of Symbols

Symbols	Description
$N_{\mathcal{PO}} - T_0$	Nano Pre-T_0
$N_{\mathcal{PO}} - T_1$	Nano Pre-T_1
$N_{\mathcal{PO}} - T_2$	Nano Pre-T_2
NSS	Nano Sub-Space
$N_{\mathcal{PO}}$ −Paracompact	Nano Pre-open Paracompact
$N_{\mathcal{PO}}$ −regular space	Nano Pre-open-regular space
$N_{\mathcal{PO}}$ −normal space	Nano Pre-open-normal space
$C\ N_{\mathcal{PO}}$ −Paracompact	Countable Nano Pre-open Paracompact
$W\ N_{\mathcal{PO}}$ −Paracompact	Weakly Nano Pre-open Paracompact
$S\ N_{\mathcal{PO}}$ −Paracompact	Strongly Nano Pre-open Paracompact
$PN_{\mathcal{PO}}$ −regular	Point Nano Pre-open-regular

Abstract

In this work, a new class of nano topological spaces is defined. This class possesses the paracompactness property. This lead to study new classes that are more general than the nano topological spaces. A new class of paracompactness is discassed in the nano topological spaces, termed pre-open sets, that presented and explored in this work. Since nano topology contains at most five sets this leads us to establish new structures of nano open sets, nano open covers, and nano locally refinement coves new requirement concepts are discussed like nano pre-open covers, nano pre-T_0,T_1,T_2,T_3 ,T_4 and countable nano pre-open paracompact. Many new theorems are stated and proved. So the idea of thesis crystallized in two paragraphs: In the first paragraph define a new types of paracompactness of nano topological space modeled by pre-open sets structures are constructed, investigated and studied, $N_{\mathcal{PO}}$ −Paracompactness of these nano spaces are defined and studied. In second paragraph, new types of the nano topological space by using the nano pre-open subsets are constructed, investigated and studied, such as the countable nano pre-open paracompact and weakly ,strongly $\mathrm{N}_{\mathcal{PO}}$ −Paracompact, and the relation between them.

Introduction

Nano topological spaces, also known as nano-spaces, are concept that combines the principles of topology and nano science. In traditional topology, the focus is on the properties of sets and their relationships, while in nano science, the focus is on the behavior and properties of materials at the nanoscale [31].

In nano topological spaces, the idea is to study the topological properties of sets and structures at the nano concept. This involves analyzing the behavior of objects that exhibit unique properties and phenomena at the nano level, topological properties are known in nano-topological spaces that involve the study of topological properties and relationships at the nano concept, this can include properties such as connectivity, compactness, continuity, and convergence, but with a focus on nano objects. Nano topological spaces are also considered the unique phenomena and behaviors that arise at the nanoscale. This can include quantum effects, surface effects, size-dependent properties, and other nanoscale-specific phenomena, which may influence the topological properties of nanostructures [32].

The study of nano topological spaces has potential applications in various fields, including nanoelectronics, nanophotonics, nanomaterials, and nanomedicine, [33] understanding the topological properties of nanostructures can help in designing and optimizing nanoscale devices, materials, and systems. It is worth noting that the concept of nano-topological spaces is still relatively new and an ongoing area of research. Further exploration and development are needed to fully understand and characterize the topological properties and behaviors of nanostructures.

Introduction

Nano topologies proposed by Lellis Thivagar et al. [26],[27] and they are introduced in terms of a subset X of the universal set U defined by the lower and upper approximations of this subset.

The pair (U, R) containing a set U and an equivalence relation R on it is called the space of approximation. In the same equivalence class, elements are called indistinguishable from each other.

This setting generates a nano topology $\tau_R(X)$ on U containing nano open sets and their complements are called closed nano sets. $(U, \tau_R(X))$ called a nano topological space. Nano pre-neighborhoods defined in [24] nano pre interior, nano pre closure in nano topological spaces and investigated many properties related among them.

If $A \subseteq U$, a nano interior of A is the largest nano open set contained in A and is denoted by $N\,Int\,(A)$ while its nano closure is the smallest nano closed set containing A and denoted by $N\,Cl\,(A)$. Since the nano topology has applications in some medical and physics models used to solve real-world problems [2] and [5] we tried to modify these topologies by adding a new class of topologies to these topologies hopping to get new results capplicable in medicine and life in the future.

Tietze [29] and [1] is credited with the original definition of the term "normal property" in topological space. Aarts [7], then focused their attention to this matter. The expansions of the concepts of pairwise regularity, pairwise normality, and total normality were recently introduced by Dorsett [9], and Jasiml [28].

Introduction

One of the basic concepts of topology is the paracompact spaces. This topic generalizes both compact and metrizable spaces [3] , [4] , [5], [8] and [12], they derived their characterization in terms of open cover that is refinement and locally finite, it is another open cover of the space that contains the open sets that intersect the neighbourhood of any element in the space finitely and these sets are contained at the same time any open cover of the space [11]. Since the nano topology has at most five elements, this makes the study of paracompactness most difficult in this topology and this reason leads us to study these spaces via the pre-open sets structures derived in nano spaces.

Dieudonne, J. [8] introduced the paracompact spaces, that contain the compact spaces and metric spaces as special cases. Dowker, C. H. [10] and Katetov, M. [13] generalized the notion of paracompactness to introduce the countably paracompact spaces.

This work is divided into three main courses : first gives a concise exposition fundamentals of the nano topological spaces theory, second complemented by a survey of the concept of nano pre-open sets in nano topological spaces. It has a property paracompact with the addition of some new concepts, and the third completes it to prove that these nano topological spaces are also countable, while mentioning some new concepts.

Chapter one deals with the fundamental and basic definitions and examples in topological spaces and nano topological spaces. In chapter two, we defined the nano locally finite and nano pre-open refinement and studied many nano topological structures on it, like nano pre-Paracompactness. Also many relations between these spaces and many important examples are considered. In third chapter we defined the countable nano pre-Paracompactness, weak and strong nano pre-Paracompactness. Also many relations between these spaces and many important examples are considered. As for chapter four, it includes conclusions and future works.

Introduction

In this chapter, fundamental definitions in topological spaces and nano topological spaces are given and investigated. Some facts and types of these spaces are also considered and the relation among them are explained.

1.1 Basic Definitions

In the section, the fundamentals of the topological spaces are defined and given some examples of them, and we mention some important theorems.

Definition 1.1.1[11]

let T be a collection of subsets of a set X, satisfying the following three conditions:

(i) $\emptyset \in T$ and $X \in T$.

(ii) If $A, B \in T$, then $A \cap B \in T$.

(iii) $\forall A_\lambda \in T$, then $\forall \cup_{\lambda \in \Lambda} A_\lambda \in T, \lambda \in \wedge$.

Then T is a topology on X, then the pair (X, T) called a **topological space.**

Example 1.1.2

Let $X = \{\alpha, \beta, \delta, \Omega\}$ and $T = \{\emptyset, X, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, \{\alpha, \Omega\}, \{\beta, \delta\}, \{\alpha, \beta, \delta\}, \{\alpha, \beta, \Omega\}\}$ is a topology on X,since the conditions are satisfied.

Definition1.1.3[11]

Let (X, T) be any topological space. Then the members of T are said to be an **open sets**.

Examples1.1.4[11]:

1) In all topologies, $\emptyset, X$ are open.

2) In the discrete topology, all subsets are open.

Definition1.1.5[11]

A subset S of a topological space X is said to be a **closed set** in (X, T) if its complement in X, namely $X - S$, is open in (X, T).

Examples 1.2.6:

1) In all topologies, $\emptyset, X$ are closed:

since $\emptyset^c = X$ is open, then $\emptyset$ is closed, $X^c = \emptyset$ is open, then X is closed.

2) In the discrete topology, all subsets are closed.

3) In $X = \{\alpha, \beta, \delta, \Omega\}, T = \{\emptyset, X, \{\alpha, \delta\}, \{\alpha\}, \{\alpha, \Omega\}, \{\Omega\}, \{\alpha, \delta, \Omega\}\}$, we have

$X, \emptyset, \{\beta, \Omega\}, \{\beta, \delta, \Omega\}, \{\beta, \delta\}, \{\alpha, \beta, \delta\}, \{\beta\}$ are closed.

Definition1.1.7[11][15]

Let $\mathrm{N} \subseteq X$, N is called **neighborhood** (nbd) of x if there exist

$u \in T$, s.t $x \in u \subseteq \mathrm{N}$.

The family of all nbd of x denoted by N_X.

$$\mathrm{N}_X = \{\mathrm{N} \subseteq X \, ; \, \mathrm{N} \text{ is } nbd \text{ of } x\}$$

Example1.1.8:

Let $X = \{\alpha, \beta, \delta\}$ and let $T = \{\emptyset, X, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}\}$, then

$\mathrm{N}(\alpha) = \{\{\alpha\}, \{\alpha, \beta\}, \{\alpha, \delta\}, X\}$.

$\mathrm{N}(\beta) = \{\{\beta\}, \{\beta, \alpha\}, \{\beta, \delta\}, X\}$.

$\mathrm{N}(\delta) = \{X\}$.

Definition 1.1.9[11]

A family $\{A_s\}_{s\in S}$ of subsets of a set X is called point finite (point countable) if for every $x \in X$ the set $\{s \in S, \in x \in A_s\}$ is finite.

Definition1.1.10[11]

Let $A \subseteq X$. We define a closure of A in (X, T) by the set

$Cl(A) = \bigcap\{F \subseteq X, F^c \in T \; and \; A \subseteq F\}$.

Theorem1.1.11[11]:

Let A, B are a subsets of X, then:

1. $A \subseteq Cl(A)$.
2. $Cl(Cl(A)) = Cl(A)$.
3. $Cl(A \cup B) = Cl(A) \cup Cl(B)$.
4. If $X \setminus A$ is open, then $Cl(A) = A$.
5. $Cl(\emptyset) = \emptyset$ and $Cl(X) = X$.

Example1.1.12

(1) Since $\emptyset$ and X are closed sets, then, $Cl(\emptyset) = \emptyset$, and $Cl(X) = X$.

(2) Let $X = \{\alpha, \beta, \delta\}$ and $A = \{\alpha, \beta\}$ with $A \subseteq X$ and $T = \{\emptyset, X, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}\}$,then

$F = \{X, \emptyset, \{\beta, \delta\}, \{\alpha, \delta\}, \{\delta\}\}$, and $Cl(A) = X$.

(3) Let $X = \{a, b, c, d, e\}$ and $T = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$.

The closed sets are $\emptyset, X, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}$ and $\{a\}$.

So the smallest closed set containing $\{b\}$ is $\{b, e\}$; that is, $Cl(\{b\}) = \{b, e\}$. Similarly $\{a, c\} = X$,and $Cl(\{b, d\}) = \{b, c, d, e\}$.

Definition 1.1.13[11]

Let A be a subset of a topological space X, then the **interior of $\boldsymbol{A}$** denoted by $Int(A)$ is defined as a set;

$$Int(A) = \cup \{G \subseteq X \,, \ \ G \in T \ \text{ and } G \subseteq A\} .$$

Theorem1.1.14[11][20]

Let A, B be two subsets of a topological space X, then,

(1) $Int(X) = X$.

(2) $Int(A) \subseteq A$.

(3) $Int(A \cap B) = Int(A) \cap Int(B)$.

(4) $Int(Int(A)) = Int(A)$.

Examples1.1.15.

(1) $X = \{1,2,3\}, T = \{X, \emptyset, \{1,2\}, \{1\}, \{2\}\}$.

If $A = \{1,3\}$, then $Int(A) = \{1\}$.

(2) $X = \{1,2,3,4,5,6\}$, $T = \{X, \emptyset, \{3\}, \{4\}, \{3,4\}\}$.

If $A = \{2,3,4\}$, then $Int(A) = \{3,4\}$.

1.2 Pre-open and Pre--closed sets in a topological space.

Before any thing, by X we mean a topological space X with topology T

Definition 1.2.1[17][18]

Let A a subset of X. Then A is called a **Pre-open set** if,

$A \subseteq Int\left(Cl\,(A)\right)$ .The set of all Pre-open sets in X is denoted by $\mathcal{PO}(X)$.

Properties of Pre-open sets [17]:

(1) $\emptyset, X$ are Pre-open sets.

(2) Every open set in a topological space is a Pre-open set but a Pre-open set may not be an open set.

(3) An union of any number of Pre-open sets is Pre-open set.

Definition 1.2.2[17][18]

Let $A \subseteq X$. Then A is called a **Pre-closed set** if $Cl\,(Int\,(A)) \subseteq A$.

The set of all Pre-closed sets in a topological space is denoted by $\mathcal{PC}(X)$.

Properties of Pre-closed sets [17]

(1) $\emptyset, X$ are Pre-closed sets.

(2) Every closed set of a topological space is a Pre-closed set.

(3) The intersection of any number of Pre-closed sets is a Pre-closed set.

Remarks1.2.4[29]

(1) The union of any family of Pre-open sets is a Pre-open.

(2) The intersection of two Pre-open sets may not be Pre-open set.

(3) The intersection of any family of Pre-closed sets is Pre-closed .

(4) The union of two Pre-closed sets may not be Pre-open.

(5) Every open set is Pre-open but the converse in general not true.

(6) Every closed set is Pre-closed but the converse in general not true.

Definition1.2.5 [12]

The union of all Pre-open sets, contained in A called **Pre-interior of A** and it is denoted by $\mathcal{P}Int(A)$.

Definition 1.2.6 [12]:

The intersection of all Pre-closed sets containing a set A is called **Pre-closure of A** , and is denoted by $\mathcal{P}Cl(A)$.

Definition 1.2.7 [6]

(1) A subset A of a topological space X is called a **Semi Pre-open** (denoted by $S\mathcal{P}O$) open set if there exists a Pre-open set U in X such that $U \subseteq A \subseteq \mathcal{P}Cl(U)$.

(2) Complement of a semi Pre-open set called a semi Pre-closed set.

(3) Family of all semi Pre-open sets of X is denoted by $S\mathcal{P}O(X)$.

(4) Family of all semi Pre-closed sets of X is denoted by $S\mathcal{P}C(X)$.

Definition 1.2.8 [6]:

The intersection of all semi Pre-closed sets containing a set A is called the **Semi Pre-closure of A**, and is denoted by $S\mathcal{P}Cl(A)$.

Definition 1.2.11

A subset A of a space (X, T) is called:

(1) Semi-open [16] if $A \subseteq Cl(Int(A))$.

(2) Pre- open [17] if $A \subseteq Int(Cl(A))$.

(3) α $-$open [19] if $A \subseteq Int(Cl(Int(A)))$.

(4) Regular open [19] if $A = Int\ (Cl(A))$.

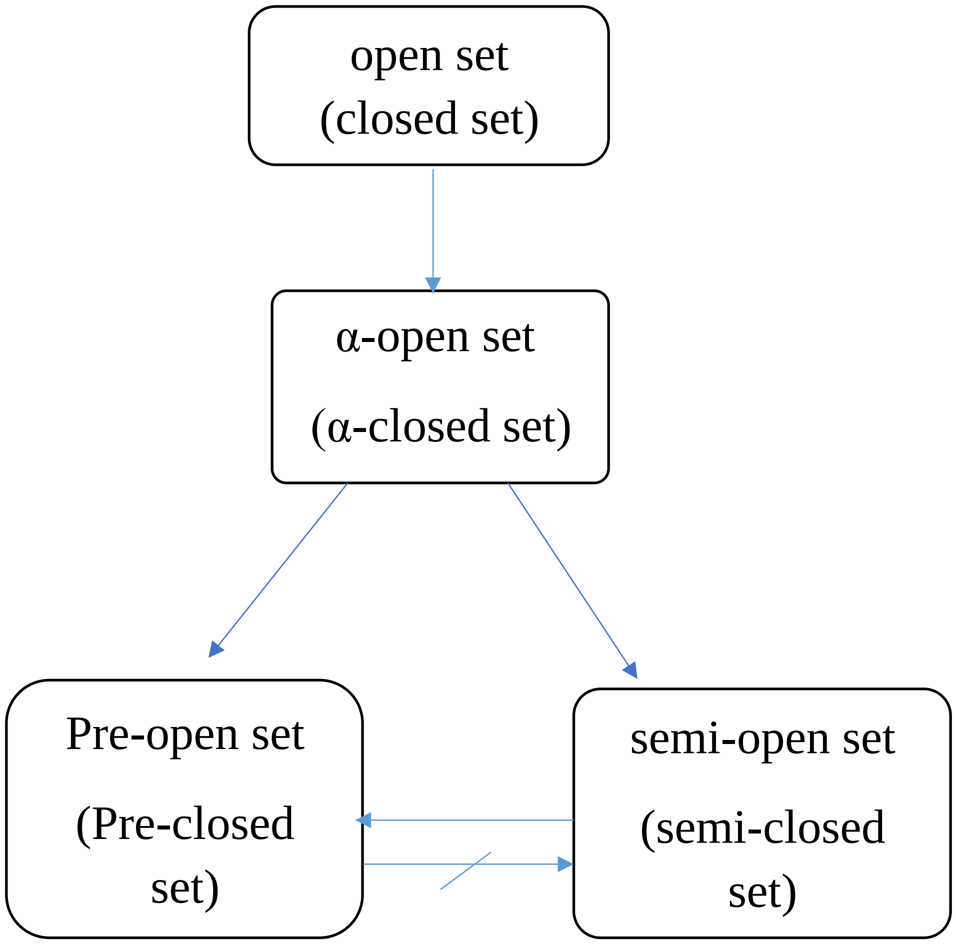


Figure 1.1 : The following diagram show many relations

1.3 Paracompact Topological Spaces

Definition 1.3.1[22]

A collection $\{k_i\}_{i\in\lambda}$ of subsets of X is called a **cover of X** iff $X = \bigcup_{i\in\lambda} K_i$.

Remarks 1.3.2[22]

(1) If $\{k_i\}_{i\in\lambda}$ is cover of X and λ is finite set ,then $\{k_i\}_{i\in\lambda}$ is called a finite cover of X.

(2) If $\forall\ k_i$ is open or closed in X and $\{k_i\}_{i\in\lambda}$ is called a open or closed cover of X.

(3) If $M = \{k_i\}_{i\in\lambda}$ is a cover of X and $\{H_j\}_{j\in\lambda}$ be a sub family of M, then $\{H_j\}_{j\in\lambda}$ is called a sub cover of X.

Definition 1.3.3[14]

An open cover $\{k_i \in T\}_{i\in\lambda}$ a family of subsets of a topological space X is called a **locally finite** or (neighbourhood finite) if and only if for each $x \in X$, there exist a neighbourhood U such that the set $\{ i \in \lambda : U \cap k_i \neq \emptyset \}$ is finite.

Remark 1.3.4[14]: Every locally finite collection is point finite.

Definition 1.3.5[14]

If $\mathcal{A}$ and β are covers of the topological space (X , T) we say that **β refines of $\mathcal{A}$** and write $\beta \subset \mathcal{A}$ if and only if for each $U \in \mathcal{A}$ is contained in some $B \in \beta$. Then we say that β is a refinement of $\mathcal{A}$.

Definition 1.3.6[14]

Any sub covering of a given covering is a refinement of that covering.

Definition 1.3.7[4]

A topological space (X, T) is called a **paracompact** if every open cover of X has a refinement by a locally finite open cover.

Example1.3.8

Let $U = \{a, b, c, d\},\ T = \{\emptyset, U, \{a\}, \{a, b, d\}, \{b, d\}\}$.

The closed set $= \{U, \emptyset, \{b, c, d\}, \{c\}, \{a, c\}\}$.

$\mathcal{A}_1 = \{U, \{a\}\}$. $\mathcal{A}_4 = \{U, \{a\}, \{a, b, d\}\}$.

$\mathcal{A}_2 = \{U, \{a, b, d\}\}$. $\mathcal{A}_5 = \{U, \{a\}, \{b, d\}\}$.

$\mathcal{A}_3 = \{U, \{b, d\}\}$. $\mathcal{A}_6 = \{U, \{a, b, d\}, \{b, d\}\}$.

$\mathcal{A}_7 = \{U, \emptyset\}$.

If $\forall\, B \in \beta\ \exists\, A \in \mathcal{A}\ \ni\ B \subseteq A$.

Let $\beta_1 = \{\{b\}, \{a, c, d\}\}$. $\beta_6 = \{\{a, b, d\}, \{a, c, d\}\}$.

$\beta_2 = \{\{b, d\}, \{a, c, d\}\}$. $\beta_7 = \{\{b, d\}, \{a, b, c\}\}$.

$\beta_3 = \{\{d\}, \{a, b, c\}\}$. $\beta_8 = \{\{a, b, d\}, \{a, b, c\}\}$.

$\beta_4 = \{\{a, b\}, \{a, c, d\}\}$. $\beta_9 = \{U, \{a\}\}$.

$\beta_5 = \{\{a, d\}, \{a, b, c\}\}$. $\beta_{10} = \{U, \emptyset\}$.

β_2 is a refinement of $\mathcal{A}_3$, $\mathcal{A}_5$ and $\mathcal{A}_6$. β_6 is a refinement of $\mathcal{A}_2$, $\mathcal{A}_4$.

β_7 is a refinement of $\mathcal{A}_3$, $\mathcal{A}_6$. β_8 is a refinement of $\mathcal{A}_2$. β_9 is a refinement of $\mathcal{A}_1$, β_{10} is a refinement of $\mathcal{A}_7$.

$\therefore$ (X, T) is a Paracompact space.

1.4 Nano Topological Spaces

In this section we will give and investigate some nano topological structures like nano open set, nano closed set, nano interior and nano closure.

Definition 1.4.1[27][21]

Let U be a nonempty finite set of objects. The U called the universe and R be an equivalence relation on U. Then U is divided in to disjoint equivalence classes. Elements in the same equivalence class are said to be indiscernible with one an-other. The pair (U, R) is said to be the **approximation space**. If $X \subseteq U$, then:

(1) The lower approximation of X with respect to R is the setof all objects, which can be for certain classified as X with respect to R and is denoted by $L_R(X)$, $L_R(X) = \cup_{x \in U}\{R(x): R(x) \subseteq X\}$.

where $R(x)$ denotes the equivalence class determined by $x \in U$.

(2) The upper approximation of X with respect to R is the set of all objects which can be classified as X with respect to R and is denoted by $U_R(X)$, $U_R(X) = \cup_{x \in U} \{R(x) : R(x) \cap X \neq \emptyset\}$.

(3) The boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not-X with respect to R and it's denoted by $B_R(X)$, $B_R(X) = U_R(X) - L_R(X)$.

Definition 1.4.2[30]

An equivalence relation on a set X is a binary relation R on X satisfying the three properties:

(1) $a\ R\ a$ for all $a \in X$ (reflexivity).

(2) $a\ R\ b$ implies $b\ R\ a$ for all $a\ ,b\ \in\ X$ (symmetry).

(3) if $a\ R\ b$ and $b\ R\ c$ then $a\ R\ c$ for all $a\ ,b\ ,c\ \in\ X$ (transitivity).

The equivalence class of an element a is often denoted $[a]$ or $[\ a\ ]\ R$, and is defined as the set $\{\ x\ \in\ X\ :\ a\ R\ x\ \}$ of elements that are related to a by R .

Example1.4.3

Let $U\ =\ \{a\ ,b\ ,c\ ,d\}\ ,X\ =\ \{a\ ,b\}, U/R\ =\ \{\{a\}\ ,\{b\}\ ,\{c\ ,d\}\}$.

$$L_R(X) = \{a,b\}\ , U_R(X)\ =\ \{a\}\ \cup\ \{b\}\ =\ \{a,b\}\ ,$$

$$B_R(X)\ =\ U_R(X)\ -\ L_R(X)\ =\ \{a,b\}\ -\ \{a,b\} =\ \emptyset.$$

Proposition 1.4.4[26]

If (U,R) is an approximation space and $X,Y \subseteq U$, then

1. $L_R(X) \subseteq\ X\ \subseteq U_R(X)$.

2.$L_R(\emptyset)\ =\ U_R(\emptyset)\ =\ \emptyset$ and $L_R(U) =\ U_R(U) =\ U$.

3. $U_R(X\ \cup\ Y) =\ U_R(X) \cup\ U_R(Y)$.

4. $U_R(X\ \cap\ Y) \subseteq\ U_R(X) \cap\ U_R(Y)$.

5.$L_R(X\ \cup\ Y) \supseteq\ L_R(X) \cup\ L_R(Y)$.

6. $L_R(X\ \cap\ Y) = L_R(X) \cap\ L_R(Y)$.

7. $L_R(X)\ \subseteq\ L_R(Y)$ and $U_R\ (X)\ \subseteq\ U_R(Y)$ whenever $X\ \subseteq\ Y$.

8. $U_R(X^c)\ =\ [L_R(X)]^c$ and $L_R(X^c)\ =\ [U_R\ (X)]^c$.

9. $U_R U_R(X) =\ L_R U_R(X) =\ U_R(X)$.

10. $L_R L_R(X) = U_R L_R(X) =\ L_R(X)$.

11. $L_R(X) \cup B_R(X) =\ U_R(X)$.

Definition 1.4.5[27][21]

Let U be the universe and $X \subseteq U, R$ be an equivalence relation on U. Then $\tau_R(X) = \{U, \emptyset, L_R(X), U_R(X), B_R(X)\}$.

Then $\tau_R(X)$ satisfies the following axioms:

(1) U and $\emptyset \in \tau_R(X)$.

(2) Union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.

(3) Intersection of the elements of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$.

There for , $\tau_R(X)$ represented a topology named the Nano topology on U with respect to X and $(U, \tau_R(X))$ is a **Nano topological space** and it's denoted by NTS.The elements of $\tau_R(X)$ are called as Nano-open sets.

Remark 1.4.6[21]

If $\tau_R(X)$ is the NTS on U with respect to X, then the set $B = \{U, \emptyset, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.

Example1.4.7

Let $U = \{a, b, c, d\}, X = \{a, b\}, U/R = \{\{a\}, \{c\}, \{b, d\}\}$.

$$L_R(X) = \{a\}, U_R(X) = \{a\} \cup \{b, d\} = \{a, b, d\},$$

$$B_R(X) = U_R(X) - L_R(X) = \{a, b, d\} - \{a\} = \{b, d\}.$$

$$\tau_R(X) = \{U, \emptyset, L_R(X), U_R(X), B_R(X)\} = \{U, \emptyset, \{a\}, \{a, b, d\}, \{b, d\}\}.$$

Note: From now up all $(U, \tau_R(X))$ is a NTS with respect to X.

Definition 1.4.8[27]

Let $(U, \tau_R(X))$ is a NTS and if $A \subseteq U$, then the **Nano-interior of A** will represented the union of all Nano-open subsets of A and it is denoted by $N\ Int(A)$.

The **Nano-closure of A** will represented the intersection of all Nano-closed sets containing A and it is denoted by $N\ Cl(A)$.

Example 1.4.9

Let $U = \{1, 2, 3, 4\}, X = \{1, 2\}, U/R = \{\{1\}, \{3\}, \{2, 4\}\}$.

$$L_R(X) = \{1\}, U_R(X) = \{1\} \cup \{2, 4\} = \{1, 2, 4\},$$

$$B_R(X) = U_R(X) - L_R(X) = \{1, 2, 4\} - \{1\} = \{2, 4\}.$$

$$\tau_R(X) = \{U, \emptyset, L_R(X), U_R(X), B_R(X)\} = \{U, \emptyset, \{1\}, \{1, 2, 4\}, \{2,4\}\}.$$

Nano open sets = $\{U, \emptyset, \{1\}, \{1, 2, 4\}, \{2, 4\}\}$,

Nano closed sets= $\{U, \emptyset, \{2, 3, 4\}, \{3\}, \{1, 3\}\}$.

Let $A = \{1, 2\}, N\ Int(A) = \{1\} \cup \emptyset = \{1\}, N\ Cl(A) = U$.

Definition 1.4.10

A subset A of a $NTS(U, \tau_R(X))$ is called:

(1) Nano-regular open if $N\ Int\big(N\ Cl(A)\big) = A$ [24].

(2) Nano α −open if $A \subseteq N\ Int\ (N\ Cl\ (N\ Int(A)))$ [24] and [28].

(3) Nano Pre-open if $A \subseteq N\ Int(N\ Cl(A))$ [24] and [28].

(4) Nano Semi-open if $A \subseteq N\ Cl(N\ Int(A))$ [24] and [28].

We denote by $N_{RO}(U, X)$ (resp. $\tau_R\ (X), N_{\mathcal{PO}}(U, X)$ and $N_{SO}(U, X)$ the family of Nano-regular open (resp. Nano α −open, Nano Pre-open and Nano Semi-open) sets in U .

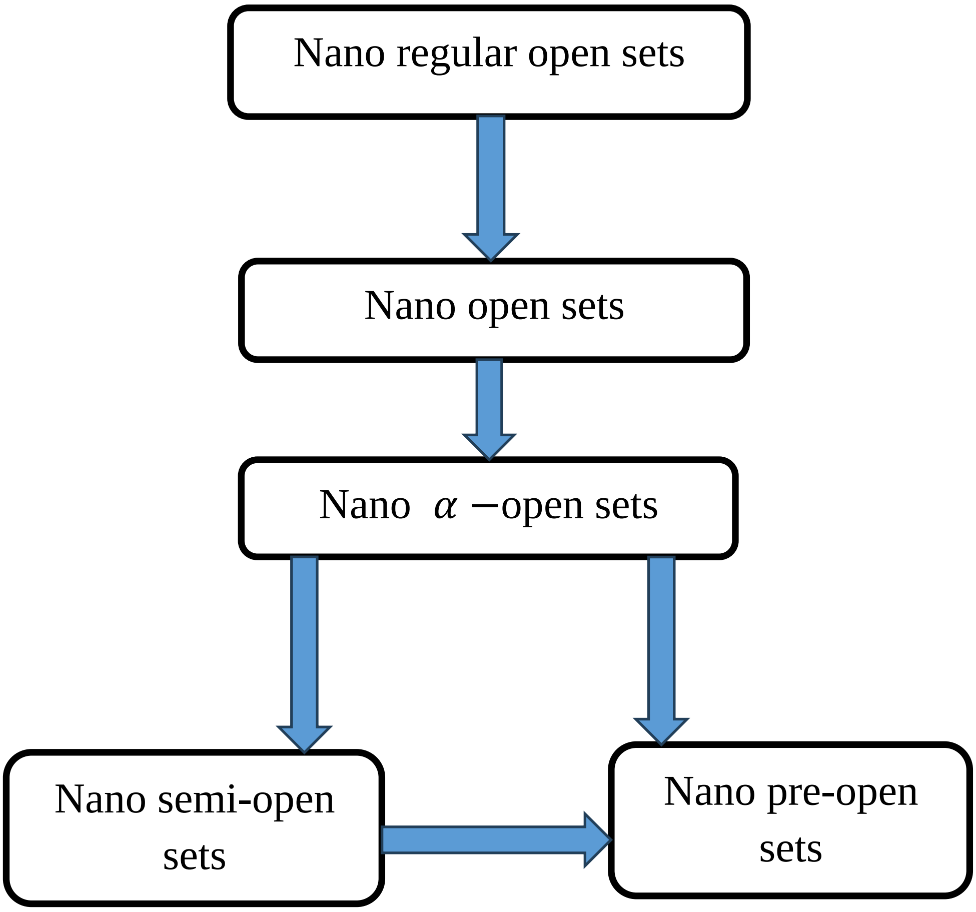


Figure 1.2 :The relationships between the other nano near sets.

Definition 1.4.11[27]

The complement of Nano Pre-open is a **Nano Pre-closed**, and it is denoted by $N_{\mathcal{P}C}(U)$.

Example 1.4.12

Let $U = \{\alpha, \beta, \delta\}, X = \{\alpha\}, U/R = \{\{\alpha\}, \{\beta\}, \{\beta, \delta\}\}$.

$\tau_R(X) = \{U, \emptyset, \{\alpha\}\}$.

The Nano closed sets $F = \{\emptyset, U, \{\beta, \delta\}\}$.

$N_{\mathcal{P}O}(U, X) = \{U, \emptyset, \{\alpha\}, \{\alpha, \beta\}, \{\alpha, \delta\}\}$.

$N_{\mathcal{P}C}(U, X) = \{\emptyset, U. \{\beta, \delta\}, \{\delta\}, \{\beta\}\}$.

Definition 1.4.13[27]

The complement of Nano is a **Nano regular-closed**, and it is denoted by $N_{RF}(U)$ or $N_{RC}(U)$.

Definition 1.4.14[24]

The union of all $N_{\mathcal{P}O}$ sets that contained in A is called the **Nano Pre-interior of A** and it is denoted by $N_{\mathcal{P}O}\ Int(A)$.

Definition 1.4.15[24]

The intersection of $N_{\mathcal{P}C}$ sets containing a set A is called the **Nano Pre-closure of A** and is denoted by $N_{\mathcal{P}C}Cl(A)$.

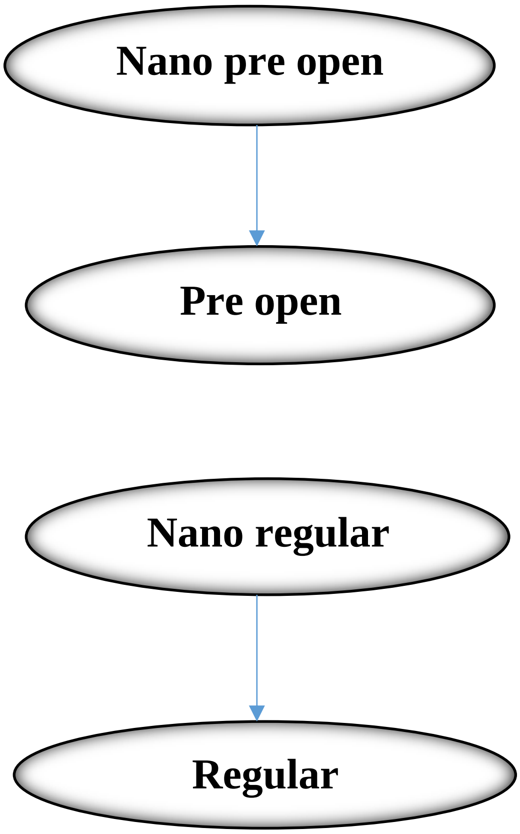


Figure 1.3 :The relationships between the nano and general sets.

Definition 1.4.16[24]

The family of all nano pre-neighbourhoods of the point $x \in U$ is called **nano pre-neighbourhood** of U and denoted by $N_{\mathcal{P}O} - nbd$.

Example 1.4.17.

Let $U = \{e_1, e_2, e_3, e_4\}, U/R = \{\{e_1, e_3\}, \{e_2\}, \{e_4\}\}, X = \{e_2, e_3\}$ and

$\tau_R(X) = \{U, \emptyset, \{e_2\}, \{e_1, e_2, e_3\}, \{e_1, e_3\}\}$.

$N_{\mathcal{P}O}(U, X) = \{U, \emptyset, \{e_1\}, \{e_2\}, \{e_4\}, \{e_1, e_2\}, \{e_1, e_4\},$

$\{e_2, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}\}$.

Then $N_{\mathcal{P}O} - nbd(e_1) = \{U, \emptyset, \{e_1\}, \{e_1, e_2\}, \{e_1, e_4\}, \{e_1, e_2, e_3\},$

$\{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}\}$.

$N_{\mathcal{P}O} - nbd(e_2) = \{U, \emptyset, \{e_2\}, \{e_1, e_2\}, \{e_2, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}\}$.

$N_{\mathcal{P}O} - nbd(e_3) = \{U, \emptyset, \{e_1, e_2, e_3\}, \{e_1, e_3, e_4\}\}$.

$N_{\mathcal{P}O} - nbd(e_4) = \{U, \emptyset, \{e_4\}, \{e_1, e_4\}, \{e_2, e_4\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}\}$.

Definition 1.4.18[25]

A $NTS\ U$ is called **Nano-$\boldsymbol{T_0}$** (or $N - T_0$) space for $a, b \in U$and $a \neq b$, there exists a Nano-open set $\mathcal{G}$ such that , $a \in \mathcal{G}$ and $b \notin \mathcal{G}$.

Definition 1.4.19[25]

A $NTS\ U$ is called a **Nano Semi-$\boldsymbol{T_0}$** (or $N_{SO} - T_0$) space for $a, b \in U$ and $a \neq b$, there exists a Nano Semi-open set $\mathcal{G}$ such that $a \in \mathcal{G}$ and $b \notin \mathcal{G}$.

Definition 1.4.20[25]

A $NTS\ U$ is called **Nano Pre-$\boldsymbol{T_0}$** (or $N_{\mathcal{P}O} - T_0$) space for $a, b \in U$ and $a \neq b$, there exists a $N_{\mathcal{P}O}$ set $\mathcal{G}$ such that $a \in \mathcal{G}$ and $b \notin \mathcal{G}$.

Example 1.4.21

Let $U = \{1,2,3\}$, $\tau_R(X) = \{\emptyset, U, \{1\}, \{1,2\}, \{2\}\}$

$N_{\mathcal{PO}}(U,X) = \{\emptyset, U, \{1\}, \{2\}, \{1,2\}\}$.

$1 \neq 2 \Longmapsto 1 \in \{1\} \wedge 2 \notin \{1\}$.

$1 \neq 3 \Longmapsto 1 \in \{1\} \wedge 3 \notin \{1\}$.

$2 \neq 3 \Longmapsto 2 \in \{2\} \wedge 3 \notin \{2\}$.

$\therefore \left(U, \tau_R(X)\right)$ is a $N_{\mathcal{PO}} - T_0$ space.

Theorem 1.4.22[25]

If $(U, \tau_R(X))$ be $N - T_0$ space, then it is $N_{\mathcal{PO}} - T_0$.

The following example shows that a converse of theorem (1.5.22) is not true.

Example 1.4.23

Let $U = \{\alpha, \beta, \delta, \Omega\}$, $U/R = \{\{\alpha\}, \{\beta, \Omega\}, \{\delta\}\}, X = \{\alpha, \beta\}$ and $\tau_R(X) = \{U, \emptyset, \{\alpha\}, \{\alpha, \beta, \Omega\}, \{\beta, \Omega\}\}$ be a Nano topology on U. We have

$N_{SO}(U,X) = \{U, \emptyset, \{\alpha\}, \{\alpha, \delta\}, \{\beta, \Omega\}, \{\alpha, \beta, \Omega\}, \{\beta, \delta, \Omega\}\}$,

$N_{\mathcal{PO}}(U,X)$
$= \{U, \emptyset, \{\alpha\}, \{\beta\}, \{\Omega\}, \{\alpha, \beta\}, \{\alpha, \Omega\}, \{\beta, \Omega\}, \{\alpha, \beta, \delta\}, \{\alpha, \beta, \Omega\}, \{\alpha, \delta, \Omega\}\}$.

(1) Let $a = \{\alpha, \delta\}$ and $b = \{\Omega\}$ then it is a $N_{SO} - T_0$ space but not $N - T_0$ space .

(2) Let $a = \{\beta\}$ and $b = \{\delta\}$ then it is a $N_{\mathcal{PO}} - T_0$ space but not $N - T_0$ space.

Theorem 1.4.24[25]

Every $N_{SO} - T_0$ space is a $N_{\mathcal{P}O} - T_0$ space.

Example 1.4.25

From the example 1.4.23, Let $a = \{\beta\}$ and $b = \{\delta\}$
then it is $N_{\mathcal{P}O} - T_0$ space but not $N_{SO} - T_0$ space.

Definition 1.4.26[25]

A $NTS\ U$ is called **Nano-T_1** (or $N - T_1$), if for $a, b \in U$ and $a \neq b$, there exists a Nano-open sets $\mathcal{G}$ and $\mathcal{H}$ such that $a \in \mathcal{G}, b \notin \mathcal{G}$ and $b \in \mathcal{H}, a \notin \mathcal{H}$.

Definition 1.4.27[25]

A $NTS\ U$ is called **Nano Semi-T_1** (or$N_{SO} - T_1$), if for $a, b \in U$ and $a \neq b$, there exists a N_{SO} sets $\mathcal{G}$ and $\mathcal{H}$ such that $a \in \mathcal{G}, b \notin \mathcal{G}$ and $b \in \mathcal{H}, a \notin \mathcal{H}$.

Definition 1.4.28[25]

A $NTS\ U$ is called **Nano Pre-T_1** (or $N_{\mathcal{P}O} - T_1$) space for $a, b \in U$ and $a \neq b$, there exists a $N_{\mathcal{P}O}$ sets $\mathcal{G}$ and $\mathcal{H}$such that $a \in \mathcal{G}, b \notin \mathcal{G}$ and $b \in \mathcal{H}, a \notin \mathcal{H}$.

Theorem 1.4.29[25]

Every $N - T_1$ space is $N_{\mathcal{P}O} - T_1$ (resp.$N_{SO} - T_1$) space.
Converse of theorem above is not true as the following.

Example 1.4.30

Let $U = \{\alpha, \beta, \delta, \Omega\}$, $U/R_1 = \{\{\alpha\}, \{\beta, \Omega\}, \{\delta\}\}, X_{\mathcal{G}} = \{\alpha, \beta\}$ and $\tau_R(X_{\mathcal{G}}) = \{U, \emptyset, \{\alpha\}, \{\alpha, \beta, \Omega\}, \{\beta, \Omega\}\}$ be a nano topology on U. We have

$N_{SO}(U, X_{\mathcal{G}}) = \{U, \emptyset, \{\alpha\}, \{\alpha, \delta\}, \{\beta, \Omega\}, \{\alpha, \beta, \Omega\}, \{\beta, \delta, \Omega\}\}$,

$N_{\mathcal{P}O}(U, X_{\mathcal{G}})$

$= \{U, \emptyset, \{\alpha\}, \{\beta\}, \{\Omega\}, \{\alpha, \beta\}, \{\alpha, \Omega\}, \{\beta, \Omega\}, \{\alpha, \beta, \delta\}, \{\alpha, \beta, \Omega\}, \{\alpha, \delta, \Omega\}\}$.

$U/R_2 = \{\{\alpha, \beta\}, \{\delta\}, \{\Omega\}\}$, $X_{\mathcal{H}} = \{\beta, \delta\}$ and

$\tau_R(X_{\mathcal{H}}) = \{U, \emptyset, \{\delta\}, \{\alpha, \delta, \Omega\}, \{\alpha, \Omega\}\}$. we have

$N_{SO}(U, X_{\mathcal{H}}) = \{U, \emptyset, \{\delta\}, \{\alpha, \Omega\}, \{\beta, \delta\}, \{\alpha, \beta, \Omega\}, \{\alpha, \delta, \Omega\}\}$,

$N_{\mathcal{P}O}(U, X_{\mathcal{H}}) = \{U, \emptyset, \{\alpha\}, \{\delta\}, \{\Omega\}, \{\alpha, \delta\}, \{\alpha, \Omega\}, \{\beta, \Omega\},$
$\{\alpha, \beta, \delta\}, \{\alpha, \delta, \Omega\}\}$,

(1) Let $a = \{\alpha, \delta\}$ and $b = \{\alpha, \beta\}$ then it is $N_{SO} - T_1$ space but not $N - T_1$ space.

(2) Let $a = \{\beta\}$ and $b = \{\delta\}$ then it is $N_{\mathcal{P}O} - T_1$ spacebut not $N - T_1$ space.

Definition 1.4.31[25]

A space U is called a **Nano-T_2** (or $N - T_2$) space if for $a, b \in U$ and $a \neq b$, there exists disjoint Nano open sets $\mathcal{G}$ and $\mathcal{H}$ such that $a \in \mathcal{G}$ and $b \in \mathcal{H}$.

Definition 1.4.32[25]

A space U is called a **Nano Semi-T_2** (or$N_{SO} - T_2$) space for $a, b \in U$ and $a \neq b$, there exists disjoint N_{SO} sets $\mathcal{G}$ and $\mathcal{H}$ such that $a \in \mathcal{G}$ and $b \in \mathcal{H}$.

Definition 1.4.33[25]

A space U is called a **Nano Pre- T_2** (or $N_{\mathcal{P}O} - T_2$) space if for $a, b \in U$ and $a \neq b$, there exists disjoint $N_{\mathcal{P}O}$ sets $\mathcal{G}$ and $\mathcal{H}$ such that $a \in \mathcal{G}$ and $b \in \mathcal{H}$.

Example 1.4.34

Let $U = \{a, b, c\}$, $U/R = \{\{a\}, \{b\}, \{b, c\}\}$, $X = \{a, c\}$ and

$\tau_R(X) = \{\emptyset, U, \{a\}, \{b, c\}\}$.

The nano closed set $F = \{U, \emptyset, \{b, c\}, \{a\}\}$.

$N_{\mathcal{P}O}(U, X) = \{\emptyset, U, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

$a \neq b \Rightarrow a \in \{a\} \wedge b \in \{b\} \Rightarrow \{a\} \cap \{b\} = \emptyset$.

And by the same way the other choices are hold and then,

$(U, \tau_R(X))$ be a $N_{\mathcal{P}O} - T_2$ space.

Next, we shall characterize $N_{\mathcal{P}O} - T_2$ spaces as in the following.

Theorem 1.4.35[25]

Every $N - T_2$ space is a $N_{\mathcal{P}O} - T_2$ (resp. $N_{SO} - T_2$) space.

Example 1.4.36

From the example 1.4.30.

(1) Let $a = \{\alpha, \delta\}$ and $b = \{\alpha, \Omega\}$ then it is $N_{SO} - T_2$ space but not $N - T_2$ space.

(2) Let $a = \{\beta\}$ and $b = \{\delta\}$ then it is $N_{\mathcal{P}O} - T_2$ spacebut not $N - T_2$ space.

Theorem 1.4.37[25]

Every $N_{SO} - T_2$ space is $N_{\mathcal{P}O} - T_2$ space.

Converse of above theorem is need not be true in general.

Example 1.4.38

From the example 1.4.34

$N_{SO}(U,X) = \{U, \emptyset, \{a\}, \{b,c\}\}$,

$a \neq b \Rightarrow a \in \{a\} \wedge b \in \{b,c\}, \{a\} \cap \{b,c\} = \emptyset$.

$a \neq c \Rightarrow a \in \{a\} \wedge c \in \{b,c\}, \{a\} \cap \{b,c\} = \emptyset$.

$b \neq c \Rightarrow b \in \{b,c\} \wedge c \in \{b,c\}, \{b,c\} \cap \{b,c\} \neq \emptyset$.

$\therefore (U, \tau_R(X))$ is not $N_{SO} - T_2$.

1.5 Nano Sub-Space (NSS)

Let $(U, \tau_R(X))$ be a NTS and let $Z \subset U$. We may construct a nano topology $\tau_{RZ}(X)$ for Z which is called the relative nano topology.The nano topology is defined as follows :

Definition 1.5.1

Let $(U, \tau_R(X))$ be a NTS and let $Z \subset U$ .The $\tau_R(X)$-relative nano topology for Z is the collection $\tau_{RZ}(X)$ given by

$$\tau_{RZ}(X) = \{E \cap Z : E \in \tau_R(X)\}.$$

The Nano topological space (Z , $\tau_{RZ}(X)$) is called is a **Nano Sub-Space** of $(U, \tau_R(X))$ and denoted by (NSS). The nano topology $\tau_{RZ}(X)$ on Z is said to be induced by $\tau_R(X)$.

In order that the above definition may be consistent ,we must show that $\tau_{RZ}(X)$ is actually a nano topology for Z .This we prove in the following theorem .

Theorem 1.5.2

Let $(U, \tau_R(X))$ be a NTS and let $Z \subset U$,then the collection $\tau_{RZ}(X) = \{ E \cap Z : E \in \tau_R(X)\}$ is a Nano topology on Z .

Proof:

We have by definition of relative Nano topology

$$\tau_{RZ}(X) = \{E \cap Z : E \in \tau_R(X)\}$$

Now, we shall find the intersection of Z with each member of $\tau_R(X)$

$$\tau_R(X) = \{U, \emptyset , L_R(X), U_R(X), B_R(X)\}$$

and we shall get the member of $\tau_{RZ}(X)$.

$\emptyset \cap Z = \emptyset$, $U \cap Z = Z$, $L_R(X) \cap Z = L_R(X)$,

$U_R(X) \cap Z = U_R(X)$, $B_R(X) \cap Z = B_R(X)$

$\therefore \tau_{RZ}(X) = \{\emptyset, Z, L_R(X), U_R(X), B_R(X)\}$

The $\tau_{RZ}(X)$ is a NTS for $Z \subset U$ and is called $\tau_R(X)$-relative nano topology.

Example 1.5.3

Let $U = \{1,2,3,4\}$ and $\tau_R(X) = \{U, \emptyset, \{1\}, \{1,2,4\}, \{2,4\}\}$ and $Z \subset U$ such that $Z = \{1,2,3\}$.

We have by definition of relative nano topology

$$\tau_{RZ}(X) = \{E \cap Z : E \in \tau_R(X)\}$$

Now ,we shall find the intersection of Z with each member of

$\tau_R(X)$ and we shall get the member of $\tau_{RZ}(X)$.

$$U \cap Z = Z, \quad \emptyset \cap Z = \emptyset, \quad \{1\} \cap Z = \{1\}, \quad \{1,2,4\} \cap Z = \{1,2\}, \quad \{2,4\} \cap Z = \{2\}$$

$\tau_{RZ}(X) = \{Z, \emptyset, \{1\}, \{1,2\}, \{2\}\}$.

The $\tau_{RZ}(X)$ is a nano topology for $Z \subset U$ and is called $\tau_R(X)$-relative nano topology.

Theorem 1.5.4

Let (Z, E)be a subspace of a NTS $(U, \tau_R(X))$ and let (W, V)be a NSS of (Z, E) .Then (W, V) is a NSS of $(U, \tau_R(X))$.

Also if U be a NTS and Z and W be a NSS of $(U, \tau_R(X))$. Then the Nano topology with Z has a NSS of $(U, \tau_R(X))$ is the same as that which it has as a NSS of W .

Proof:

Since $W \subset Z \subset V$,we have $W \subset V$ .We have to show that $\tau_{RW}(X) = V$. Let $V' \in V$. Since (W, V) is a NSS of (Z, E), there exists $E' \in E$ such that $V' = E' \cap W$. Again since (Z, K) is a NSS of $\left(U, \tau_R(X)\right)$ there exists $G \in \tau_R(X)$ such that $E' = G \cap Z$. Thus

$$V' = E' \cap W = (G \cap Z) \cap W$$

$$= G \cap (Z \cap W)$$

Since $W \subset Z$, there fore $V' = G \cap W$.

Hence by definition of $\tau_{RZ}(X)$, $V' \in \tau_{RZ}(X)$.

$$\therefore V \subset \tau_{RZ}(X) . \quad \ldots\ldots.(1)$$

Conversely, let $D \in \tau_{RZ}(X)$ .Then by definition of $\tau_{RZ}(X)$,there exists $G \in \tau_{RZ}(X)$, such that $D = G \cap W$.

Since (Z, E) is a NSS of $\left(U, \tau_R(X)\right)$, we have $G \cap Z \in E$ .Again since (W, V) is a NSS of (Z, E) ,we have $(G \cap Z) \cap W \in V$.That is ,

$$G \cap (Z \cap W) \in V$$

Since $W \subset Z$, there fore $G \cap W \in V$

But $D = G \cap W$ as proved above. Hence $D \in V$.If the following that

$$\tau_{RZ}(X) \subset V \quad \ldots\ldots.(2)$$

From (1) and (2). We have $\tau_{RZ}(X) = V$.

Theorem 1.5.5

Let $(Z, \tau_{RZ}(X))$ be a NSS of $(U, \tau_R(X))$, then :

(1) A subset A of Z is N-closed in Z if and only if there exists a set N-closed (F) subset U such that $A = F \cap Z$.

(2) For every $A \subset Z$, then $N\ Cl_Z(A) = N\ Cl_U(A) \cap Z$.

Proof:

(1) A is N-closed in Z.

$\leftrightarrow Z - A$ is nano open in Z.

$\leftrightarrow Z - A = \mathcal{G} \cap Z$ for some nano open subset $\mathcal{G}$ of U [by theorem 1.5.2]

$\leftrightarrow A = Z - (\mathcal{G} \cap Z) = (Z - \mathcal{G}) \cup (Z - Z)$ [De-Morgan Law]

$\leftrightarrow A = Z - \mathcal{G}$ [$\therefore Z - Z = \emptyset$]

$\leftrightarrow A = Z \cap \mathcal{G}^c$ where $\mathcal{G}^c$denotes the complement of $\mathcal{G}$ in U.

$\leftrightarrow A = Z \cap F$ where $F = \mathcal{G}^c$is N-closed in U since $\mathcal{G}$ is nano open in Z.

(2) By definition.

$Cl_Z(A) = \cap\{B: B$ is N −closed in Z and $A \subset B\}$

$= \cap\{F \cap Z: Fis\ N - closed\ in\ U\ and\ A \subset F \cap Z\}$ [by (1)]

$= \cap\{F \cap Z: FisN - closed\ in\ U\ and\ A \subset F\ \}$

$= [\cap\{F: F\ is\ N - closed\ in\ U\ and\ A \subset F\}]\ \cap Z$

$= Cl_U(A) \cap Z.$

Theorem 1.5.6

Let Z be a NSS of a NTS of U . If $A \subset Z$ is a N-open (a N- closed) in U, such that $A \subset Z$,then A is also a N-open (a N- closed) in Z .

Proof:

Since $A \subset Z$, we have $A = A \cap Z$ so that A is the intersection of Z with a set N-open (N-closed) in U , namely A .Hence by theorem 1.5.2 and 1.5.5 , A is a N-open (N- closed) in Z .

1.6 $N_{\mathcal{PO}}$ −regular and $N_{\mathcal{PO}}$ −normal spaces

In this section we introduce a space called $N_{\mathcal{PO}}$ −regular space and we introduce a space called $N_{\mathcal{PO}}$ −normal space.

Definition 1.6.1[23]

A $NTS\left(\mathrm{U}, \tau_R(\mathrm{X})\right)$ is said to be $\boldsymbol{N_{\mathcal{PO}}}$ **−regular space** if and only if for each $N_{\mathcal{PC}}$, for each $p \notin N_{\mathcal{PC}}$ there exists $\mathcal{G}, \mathcal{H} \in N_{\mathcal{PO}}(U, X)$, $p \in \mathcal{G}$ and $N_{\mathcal{PC}} \subseteq \mathcal{H}$, $\mathcal{G} \cap \mathcal{H} = \emptyset$.

Example 1.6.2

Let $U = \{\alpha, \beta, \delta\}$ and $U/R = \{\{\alpha\}, \{\beta\}, \{\beta, \delta\}\}$, $X = \{\alpha, \delta\}$

$$\tau_R(X) = \{\emptyset, U, \{\alpha\}, \{\beta, \delta\}\}.$$

The nano closed set $F = \{U, \emptyset, \{\beta, \delta\}, \{\alpha\}\}$.

$N_{\mathcal{PO}}(U, X) = \{\emptyset, U, \{\alpha\}, \{\beta\}, \{\delta\}, \{\alpha, \beta\}, \{\alpha, \delta\}, \{\beta, \delta\}\}$.

$N_{\mathcal{PC}}(U, X) = \{U, \emptyset, \{\beta, \delta\}, \{\alpha, \delta\}, \{\alpha, \beta\}, \{\delta\}, \{\beta\}, \{\alpha\}\}$.

$\alpha \notin \{\beta, \delta\} \Rightarrow \alpha \in \{\alpha\} \wedge \{\beta, \delta\} \subseteq \{\beta, \delta\} \ni \{\alpha\} \cap \{\beta, \delta\} = \emptyset$.

$\beta \notin \{\alpha\} \Rightarrow \beta \in \{\beta, \delta\} \wedge \{\alpha\} \subseteq \{\alpha\} \ni \{\beta, \delta\} \cap \{\alpha\} = \emptyset$.

$\delta \notin \{\alpha\} \Rightarrow \delta \in \{\beta, \delta\} \wedge \{\alpha\} \subseteq \{\alpha\} \ni \{\beta, \delta\} \cap \{\alpha\} = \emptyset$.

$\therefore (U, \tau_R(X))$ is a $N_{\mathcal{PO}}$ −regular space.

Theorem 1.6.3[23]

Every Nano regular space is $N_{\mathcal{PO}}$ − regular.

Proof:

Let $\mathcal{F}$ be a Nano closed set and $x \notin \mathcal{F}$ be a point of a Nano regular space $\left(U, \tau_R(X)\right)$ Since U is a Nano regular space there exist two disjoint nano open sets $\mathcal{G}$ and $\mathcal{H}$ such that $x \in \mathcal{G}$ and $\mathcal{F} \subset \mathcal{H}$. since every Nano open set is $N_{\mathcal{PO}}$ set, $\mathcal{G}$ and $\mathcal{H}$ are $N_{\mathcal{PO}}$ sets, $x \notin N_{\mathcal{PC}}$ such that $x \in \mathcal{G}$ and $N_{\mathcal{PC}} \subset \mathcal{H}$. Hence $\left(U, \tau_R(X)\right)$ is a $N_{\mathcal{PO}}$ − regular space.

Remark 1.6.4[23]

Every $N_{\mathcal{PO}}$ − regular space need not be Nano regular as given in the following example.

Example 1.6.5

Let $U = \{\alpha,\beta,\delta,\Omega\}$ with $U/R = \{\{\alpha,\delta\},\{\beta\},\{\Omega\}\}$ and $X = \{\alpha,\delta\}$. Then the Nano topology is $\tau_R(X) = \{U,\emptyset,\{\alpha,\delta\}\}$. Also $(U,\tau_R(X))$ is $N_{\mathcal{PO}}$-regular space but not Nano regular space.

$\tau_R(X) = \{U,\emptyset,\{\alpha,\delta\}\}$.

The nano closed set $F = \{\emptyset, U, \{\beta,\Omega\}\}$.

$\alpha \notin \{\beta,\Omega\} \Rightarrow \alpha \in \{\alpha,\delta\} \wedge \{\beta,\Omega\} \subseteq U \ni \{\alpha,\delta\} \cap U \neq \emptyset$.

$\therefore (U,\tau_R(X))$ is not nano regular space.

$N_{\mathcal{PO}}(U,X) = \{\emptyset, U, \{\alpha\},\{\delta\},\{\Omega\},\{\alpha,\beta\},\{\alpha,\delta\},\{\alpha,\Omega\},\{\beta,\delta\},\{\beta,\Omega\},$
$\{\delta,\Omega\},\{\alpha,\beta,\delta\},\{\alpha,\beta,\Omega\},\{\alpha,\delta,\Omega\},\{\beta,\delta,\Omega\}\}$.

$N_{\mathcal{PC}}(U,X) = \{U,\emptyset,\{\beta,\delta,\Omega\},\{\alpha,\beta,\Omega\},\{\alpha,\beta,\delta\},\{\delta,\Omega\},\{\beta,\Omega\},\{\beta,\delta\},$
$\{\alpha,\Omega\},\{\alpha,\delta\},\{\alpha,\beta\},\{\Omega\},\{\delta\},\{\beta\},\{\alpha\}\}$.

$\alpha \notin \{\beta,\Omega\} \Rightarrow \alpha \in \{\alpha\} \wedge \{\beta,\Omega\} \subseteq \{\beta,\Omega\} \ni \{\alpha\} \cap \{\beta,\Omega\} = \emptyset$.

$\beta \notin \{\alpha,\Omega\} \Rightarrow \beta \in \{\beta,\delta\} \wedge \{\alpha,\Omega\} \subseteq \{\alpha,\Omega\} \ni \{\beta,\delta\} \cap \{\alpha,\Omega\} = \emptyset$.

$\delta \notin \{\alpha\} \Rightarrow \delta \in \{\beta,\delta\} \wedge \{\alpha\} \subseteq \{\alpha\} \ni \{\beta,\delta\} \cap \{\alpha\} = \emptyset$.

$\Omega \notin \{\alpha,\beta\} \Rightarrow \Omega \in \{\Omega\} \wedge \{\alpha,\beta\} \subseteq \{\alpha,\beta\} \ni \{\Omega\} \cap \{\alpha,\beta\} = \emptyset$.

$\therefore (U,\tau_R(X))$ is $N_{\mathcal{PO}}$ −regular space.

Definition1.6.6[7]

A NTS $(U,\tau_R(X))$ is said to be a $\boldsymbol{N_{\mathcal{PO}}}$ **−normal space** if and only if for each $F_1, F_2 \in N_{\mathcal{PC}}$ sets of U such that $F_1 \cap F_2 = \emptyset$,there exists $\mathcal{G},\mathcal{H} \in N_{\mathcal{PO}}$.Then $F_1 \subseteq \mathcal{G}$ and $F_2 \subseteq \mathcal{H}$, $\mathcal{G} \cap \mathcal{H} = \emptyset$.

Example1.6.7

Let $U = \{\alpha,\beta,\delta\}$ and $U \backslash R = \{\{\alpha\},\{\beta\},\{\beta,\delta\}\}$, $X = \{\alpha,\delta\}$.

$\tau_R(X) = \{\emptyset, U, \{\alpha\},\{\beta,\delta\}\}$.

Is $(\mathrm{U},\tau_R(\mathrm{X}))$ is a $N_{\mathcal{PO}}$ −normal space.

The nano closed sets $F = \{U,\emptyset,\{\beta,\delta\},\{\alpha\}\}$.

$N_{\mathcal{PO}}(U,X) = \{\emptyset, U, \{\alpha\},\{\beta\},\{\delta\},\{\alpha,\beta\},\{\alpha,\delta\},\{\beta,\delta\}\}$.

$N_{\mathcal{PC}}(U,X) = \{U,\emptyset,\{\beta,\delta\},\{\alpha,\delta\},\{\alpha,\beta\},\{\delta\},\{\beta\},\{\alpha\}\}$.

$\{\alpha\} \subseteq \{\alpha\}$ and $\{\beta,\delta\} \subseteq \{\beta,\delta\}$, $\{\alpha\} \cap \{\beta,\delta\} = \emptyset$.

$\{\beta\} \subseteq \{\beta\}$ and $\{\alpha\} \subseteq \{\alpha\}$, $\{\beta\} \cap \{\alpha\} = \emptyset$.

$\{\delta\} \subseteq \{\delta\}$ and $\{\beta\} \subseteq \{\beta\}$, $\{\delta\} \cap \{\beta\} = \emptyset$.

$\therefore (U, \tau_R(X))$ is a $N_{\mathcal{P}O}$ −normal space.

Example1.6.8

Let $U = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\frac{U}{R} = \{\{\lambda_1\}, \{\lambda_2\}, \{\lambda_2, \lambda_3\}\}$, $X = \{\lambda_1\}$, $\tau_R(X) = \{\emptyset, U, \{\lambda_1\}\}$.

The nano closed set $F = \{U, \emptyset, \{\lambda_2, \lambda_3\}\}$.

$N_{\mathcal{P}O}(U, X) = \{\emptyset, U, \{\lambda_1\}, \{\lambda_1, \lambda_2\}, \{\lambda_1, \lambda_3\}\}$.

$N_{\mathcal{P}C}(U, X) = \{U, \emptyset, \{\lambda_2, \lambda_3\}, \{\lambda_3\}, \{\lambda_2\}\}$.

$\{\lambda_2\} \subseteq \{\lambda_1, \lambda_2\}$ and $\{\lambda_3\} \subseteq \{\lambda_1, \lambda_3\}$, $\{\lambda_1, \lambda_2\} \cap \{\lambda_1, \lambda_3\} \neq \emptyset$.

$\therefore (U, \tau_R(X))$ is not a $N_{\mathcal{P}O}$ −normal space.

Theorem1.6.9[7]

Every Nano normal space is a $N_{\mathcal{P}O}$ −normal space**.**

Proof:

Let $(U, \tau_R(X))$ is a Nano normal space and F_1, F_2 are two disjoint pair of nano closed sets. Since $(U, \tau_R(X))$ is a Nano normal there exists disjoint $N_{\mathcal{P}O}$ sets $\mathcal{G}$ and $\mathcal{H}$ such that $F_1 \subset \mathcal{G}$ and $F_2 \subset \mathcal{H}$ since every nano open set is $N_{\mathcal{P}O}$, $\mathcal{G}$ and $\mathcal{H}$ are $N_{\mathcal{P}O}$ sets. Hence $(U, \tau_R(X))$ is a $N_{\mathcal{P}O}$ −normal space**.**

The following example shows that in general the converse theorem 1.6.9 is not true.

Example1.6.10

Let $U = \{\alpha, \beta, \delta, \Omega\}$ with $U/R = \{\{\alpha, \beta\}, \{\delta\}. \{\Omega\}\}$ and $X = \{\alpha, \beta\}$. Then the Nano topology is $\tau_R(X) = \{U, \emptyset, \{\alpha, \beta\}\}$.

Then $(U, \tau_R(X))$ is $N_{\mathcal{P}O}$ −normal space but not Nano normal space.

Theorem1.6.11[7]

If a $NTS\ U$ is a $N_{\mathcal{PO}}$ $-$normal space then for every pair of $N_{\mathcal{PO}}\ \mathcal{G}$ and $\mathcal{H}$ whose union is U, there exists $N_{\mathcal{PC}}$ sets $\mathcal{A}$ and β such that and $\mathcal{A} \subset \mathcal{G}$, $\beta \subset \mathcal{H}$ and $\mathcal{A} \cup \beta = U$.

Proof:

Let $\mathcal{G}$ and $\mathcal{H}$ be a pair of $N_{\mathcal{PO}}$ sets in a $N_{\mathcal{PO}}$ $-$normal space U such that $\mathcal{G} \cup \mathcal{H} = U$.Then $U - \mathcal{G}$, $U - \mathcal{H}$ are disjoint Nano closed sets. Since U is $N_{\mathcal{PO}}$ $-$normal space, there exists two disjoint $N_{\mathcal{PO}}$ sets $\mathcal{G}_1$ and $\mathcal{H}_1$ such that $U - \mathcal{G} \subset \mathcal{G}_1$ and $U - \mathcal{H} \subset \mathcal{H}_1$.

Let $\mathcal{A} = U - \mathcal{G}_1$, and $\beta = U - \mathcal{H}_1$. Then $\mathcal{A}$, β are $N_{\mathcal{PC}}$ sets such that $\mathcal{A} \subset \mathcal{G}$, $\beta \subset \mathcal{H}$ and $\mathcal{A} \cup \beta = U$.

Introduction

In this chapter we define a new types of Paracompactness of Nano Topological Space Modeled by Pre-open Sets Structures are constructed, investigated and studied, $N_{\mathcal{PO}}$ −Paracompactness of these Nano Spaces are defined and studied. Also a new type of Nano Topological Pre-Open Paracompactness is define.

2.1 $N_{\mathcal{PO}}$ −Paracompact spaces.

Definition 2.1.1

Let $(U, \tau_R(X))$ be a NTS. A nano open cover a family $\{k_i \in T\}_{i\in\lambda}$ of $N_{\mathcal{PO}}$ subsets of a $NTS\ U$ is called a **Nano locally finite** or (a Nano neighbourhood finite) if and only if for each point $x \in X$ there exists a Nano neighbourhood U such that the set $\{ k_i , U \cap k_i \neq \emptyset ;\ i \in \lambda\}$ is a nano finite sets.

Theorem 2.1.2

For every nano locally finite family $\{k_i\}_{i\in\lambda}$ we have the equality
$N\ Cl\ (\bigcup_{i\in\lambda} k_i) = \bigcup_{i\in\lambda} N\ Cl\ (k_i)$.

Proof:

If $k_i \subset \bigcup_{i\in\lambda} k_i$ then $N\ Cl\ (k_i) \subset N\ Cl\ (\bigcup_{i\in\lambda} k_i)$, for every $i \in \lambda$; therefore we have $N\ Cl\ (k_i) \subset N\ Cl\ (\bigcup_{i\in\lambda} k_i)$.To prove the reverse inclusion, let us note that, by nano local finiteness of $\{k_i\}_{i\in\lambda}$, for every $x \in N\ Cl\ (\bigcup_{i\in\lambda} k_i)$, there exists a nano neighbourhood U such that the set $\mathcal{S}_0 = \{ k_i , U \cap k_i \neq \emptyset ;\ i \in \lambda\}$ is a nano finite. From it followes that $x \notin N\ Cl\ (\bigcup_{i\in\lambda\setminus\mathcal{S}_0} k_i)$, since

$$x \in N\ Cl\ (\bigcup_{i\in\lambda} k_i) = N\ Cl\ (\bigcup_{i\in\mathcal{S}_0} k_i) \cup N\ Cl\ (\bigcup_{i\in\lambda\setminus\mathcal{S}_0} k_i) ,$$

We have $x \in N\,Cl\,(\bigcup_{i\in\mathcal{S}_0} k_i) = \bigcup_{i\in\mathcal{S}_0} N\,Cl\,(k_i) \subset \bigcup_{i\in\lambda} N\,Cl\,(k_i)$.

Corollary 2.1.3

Let V be a nano locally finite family and $V = \cup\,\mathcal{V}$. If all members $\mathcal{V}$ are $N_{\mathcal{P}C}$, then V is a $N_{\mathcal{P}C}$ set and if all members $\mathcal{V}$ are $N_{\mathcal{P}C}$ and $N_{\mathcal{P}O}$, then V is a $N_{\mathcal{P}C}$-and-$N_{\mathcal{P}O}$ set.

Being clear, it does not need proof.

Theorem 2.1.4

If $\{k_i\}_{i\in\lambda}$ is a nano locally finite system of sets in X, then $\{N\,Cl\,(k_i)\}_{i\in\lambda}$

also is a nano locally finite.

Proof:

Given $h \in X$ and find an open nano neighbourhood U of h such that $U \cap k_i = \emptyset$, except for finitely is a many nano locally finite system of sets many i.

since $\{k_i\}_{i\in\lambda}$ is a nano locally finite system of sets.

$\Rightarrow U \cap N\,Cl\,(k_i) = \emptyset$ for every i.

$\Rightarrow U \cap N\,Cl\,(k_i) \neq \emptyset$ for every i.

$\{N\,Cl\,(k_i)\}_{i\in\lambda}$is a nano locally finite.

Definition 2.1.5

If $\mathcal{A}$ is a nano open cover of the $NTS\ (U, \tau_R(X))$and β is a $N_{\mathcal{P}O}$cover of the $NTS\ (U, \tau_R(X))$we say that β refines $\mathcal{A}$ and write $\beta \subset \mathcal{A}$ if and only if for each $U \in \mathcal{A}$ is contained in some $B \in \beta$. Then we say that β is a nano pre-open refinement and denoted by $\boldsymbol{N_{\mathcal{P}O}}$ **refinement of** $\boldsymbol{\mathcal{A}}$.

Definition 2.1.6

Let $(U,\tau_R(X))$ be a NTS is called a **Nano Pre-open Paracompact** if X is a nano Pre-open Hausdorff space ($N_{\mathcal{PO}}-T_2$space) and every open cover $\mathcal{V}$ of X has a nano locally finite $N_{\mathcal{PO}}$ refinement cover $\mathcal{M}$ and it is denoted by $N_{\mathcal{PO}}$ −Paracompact space.

Example 2.1.7

Let $U=\{a,b,c,d\}$ with $U/R=\{\{a\},\{c\},\{b,d\}\}$ and $X=\{a,b\}$.

$\tau_R(X)=\{\emptyset,U,\{a\},\{a,b,d\},\{b,d\}\}$.

The nano closed set $F=\{U,\emptyset,\{b,c,d\},\{c\},\{a,c\}\}$.

$N_{\mathcal{PO}}(U,X)=\{\emptyset,U,\{a\},\{b\},\{d\},\{a,b\},\{a,d\},\{b,d\},\{a,b,c\}\{a,b,d\},$

$\{a,c,d\}\}$.

Let $\mathcal{G}=\{b\}$ and $\mathcal{H}=\{c\}$ then it is $N_{\mathcal{PO}}-T_2$ space but not $N-T_2$ space.

Let $\mathcal{A}$ is a nano open cover and β is a $N_{\mathcal{PO}}$ of U,we say that β is a $N_{\mathcal{PO}}$ refinement of $\mathcal{A}$.

$\mathcal{A}_1=\{U,\{a\}\}$. $\mathcal{A}_4=\{U,\{a\},\{a,b,d\}\}$.

$\mathcal{A}_2=\{U,\{a,b,d\}\}$. $\mathcal{A}_5=\{U,\{a\},\{b,d\}\}$.

$\mathcal{A}_3=\{U,\{b,d\}\}$. $\mathcal{A}_6=\{U,\{a,b,d\},\{b,d\}\}$.

$\mathcal{A}_7=\{U,\emptyset\}$.

If $\forall B\in\beta\ \exists A\in\mathcal{A}\ \ni\ B\subseteq A$.

Let $\beta_1=\{\{b\},\{a,c,d\}\}$. $\beta_6=\{\{a,b,d\},\{a,c,d\}\}$.

$\beta_2=\{\{b,d\},\{a,c,d\}\}$. $\beta_7=\{\{b,d\},\{a,b,c\}\}$.

$\beta_3=\{\{d\},\{a,b,c\}\}$. $\beta_8=\{\{a,b,d\},\{a,b,c\}\}$.

$\beta_4 = \{\{a,b\},\{a,c,d\}\}$. $\beta_9 = \{U,\{a\}\}$.

$\beta_5 = \{\{a,d\},\{a,b,c\}\}$. $\beta_{10} = \{U,\emptyset\}$.

β_2 is a $N_{\mathcal{PO}}$ refinement of $\mathcal{A}_3$, $\mathcal{A}_5$ and $\mathcal{A}_6$.

β_6 is a $N_{\mathcal{PO}}$ refinement of $\mathcal{A}_2$, $\mathcal{A}_4$.

β_7 is a $N_{\mathcal{PO}}$ refinement of $\mathcal{A}_3$, $\mathcal{A}_6$.

β_8 is a $N_{\mathcal{PO}}$ refinement of $\mathcal{A}_2$.

β_9 is a $N_{\mathcal{PO}}$ refinement of $\mathcal{A}_1$.

β_{10} is a $N_{\mathcal{PO}}$ refinement of $\mathcal{A}_7$.

$\therefore (U,\tau_R(X))$ *is a* $N_{\mathcal{PO}}$ −Paracompact .

Theorem 2.1.8

Every $N_{\mathcal{PO}}$ −Paracompact space is Paracompact.

Proof:

Let U be a $N_{\mathcal{PO}}$ −Paracompact space, then U is a $N_{\mathcal{PO}}$ Hausdorff ($N_{\mathcal{PO}} - T_2$) and any open cover of $\mathcal{V}$ has a nano locally finite $N_{\mathcal{PO}}$refinement, since any open set is Pre-open then any nano locally finite open refinement is locally finite Pre-open refinement which means that it is Paracompact U.

Remark 2.1.9

Converse of theorem 2.1.8 is not necessarily true in general.

Example 2.1.10

Let $U = \{\alpha,\beta,\delta\}, T = \{\emptyset, U,\{\alpha\},\{\beta\},\{\alpha,\beta\}\}$.

Let $\mathcal{A}$, β are two open cover of U,we say that β is refinement of $\mathcal{A}$.

$\mathcal{A}_1 = \{U,\{\alpha\}\}$, $\mathcal{A}_2 = \{U,\{\beta\}\}$, $\mathcal{A}_3 = \{U,\{\alpha,\beta\}\}$ and

$\beta_1 = \{\{\alpha\},\{\beta\}\} \subset \mathcal{A}_1, \mathcal{A}_2,\ \ \beta_2 = \{\{\alpha,\beta\}\} \subset \mathcal{A}_3$

$\therefore$ (X,T) is Paracompact space but not $N_{\mathcal{P}O}$ $-$Paracompact space.

Let $U = \{\alpha,\beta,\delta\}$,$U/R = \{\{\alpha\},\{\beta\},\{\beta,\delta\}\}$ and $X = \{\beta\}$,

$\tau_R(X) = \{\emptyset, U, \{\beta\},\{\beta,\delta\},\{\delta\}\}$.

The nano closed set $F = \{U, \emptyset, \{\alpha,\delta\},\{\alpha\},\{\alpha,\beta\}\}$

$N_{\mathcal{P}O}(U,X) = \{\emptyset, U, \{\beta\},\{\delta\}\}$

$\therefore$ $N_{\mathcal{P}O}(U,X)$ is not NTS .

$\therefore$ Paracompact is not $N_{\mathcal{P}O}$ $-$Paracompact space.

Lemma2.1.11

Let X be a $N_{\mathcal{P}O}$ $-$Paracompact space and $\mathcal{A}$ is an open cover and β is $N_{\mathcal{P}C}$ subsets of X . If for every $a \in \mathcal{A}$ there exist $N_{\mathcal{P}O}$ sets U_a, V_a such that $\mathcal{A} \subset U_a$, $a \in V_a$ and $U_a \cap V_a = \emptyset$, then there also exist $N_{\mathcal{P}O}$ sets U, V such that $\mathcal{A} \subset U$, $\beta \subset V$ and $U \cap V = \emptyset$.

Proof:

The family $\{\{V_a\}_{a\in\beta} \cup \{X\setminus \beta\}\}$ is a cover of X , so that it has a nano locally finite $N_{\mathcal{P}O}$ refinement $\{\mathcal{W}_s\}_{s\in\mathcal{S}}$. Letting $\mathcal{S}_0 = \{s \in \mathcal{S} : \mathcal{W}_s \cap \beta \neq \emptyset\}$ we have $\mathcal{A} \cap N\ Cl(\mathcal{W}_s) = \emptyset$ for every $s \in \mathcal{S}_0$ and $\beta \subset \bigcup_{s\in\mathcal{S}_0} \mathcal{W}_s$.

By virtue of Theorem 2.1.2. (For every a nano locally finite family $\{k_i\}_{i\in\lambda}$ we have the equality $N\ Cl\ (\bigcup_{i\in\lambda} k_i) = \bigcup_{i\in\lambda} N\ Cl\ (k_i)$),

the set $U = X \setminus \bigcup_{s\in\mathcal{S}_0} N\ \ Cl\ (\mathcal{W}_s)$ is $N_{\mathcal{P}O}$; one readily sees that U and $V = \bigcup_{s\in\mathcal{S}_0} N\ \mathcal{W}_s$, have all the required properties.

Theorem 2.1.12

Every closed a nano subspace of a $N_{\mathcal{P}O}$ −Paracompact space is Paracompact.

Proof:

Let Z be a closed a nano subspace of the $N_{\mathcal{P}O}$ −Paracompact space of X, let $\mathcal{A}$ be a covering of Z by sets $N_{\mathcal{P}O}$ in Z. For every $a \in \mathcal{A}$, choose a $N_{\mathcal{P}O}$ set $\mathcal{A}'$ of X such that $\mathcal{A}' \cap Z = \mathcal{A}$. Cover X by the $N_{\mathcal{P}O}$ sets $\mathcal{A}'$, along with the $N_{\mathcal{P}O}$ set Z^c. Let β be a nano locally finite $N_{\mathcal{P}O}$refinement of this covering that covers X. The collection $C = \{\beta \cap Z|\ b \in \beta\}$ is the required a nano locally finite $N_{\mathcal{P}O}$refinement of $\mathcal{A}$.

Lemma 2.1.13

If for any open cover of a $N_{\mathcal{P}O}$ −regular space X there is a nano locally finite $N_{\mathcal{P}O}$ refinement, then for every open cover $\{U_a\}_{a\in\beta}$ of the nano space X there exists a nano closed locally finite cover $\{V_a\}_{a\in\beta}$ of X such that $V_a \subset U_a$, for every $a \in \beta$.

Proof:

By a $N_{\mathcal{P}O}$ −regular space of X, there exists an open cover $\mathcal{W}$ of the nano space X such that $\{N\ Cl\ (W): W \in \mathcal{W}\}$ is a $N_{\mathcal{P}O}$ refinement of $\{U_a\}_{a\in\beta}$. Take a nano locally finite $N_{\mathcal{P}O}$ refinement $\{\mathcal{A}_t\}_{t\in T}$ of the cover $\mathcal{W}$, for every $t \in T$ choose an $a(t) \in \beta$ such that $N\ Cl\ (\mathcal{A}_t) \subset U_{a(t)}$, and let $V_a = \bigcup_{a(t)=a} N\ Cl\ (\mathcal{A}_t)$. From Theorems 2.1.2 and Corollary 2.1.3 it follows readily that $\{V_a\}_{a\in\beta}$ is a nano closed locally finite cover of X and the definition of the V_a's implies that $V_a \subset U_a$, for every $a \in \beta$.

Remark 2.1.14

Let us note that if the cover $\{\mathcal{A}_t\}_{t\in T}$in the last proof is a$N_{\mathcal{P}O}$, then the sets $K_a = \bigcup_{a(t)=a} \mathcal{A}_t$ are $N_{\mathcal{P}O}$ set and $N\ Cl\ (K_a) = V_a$. Hence, for every open cover $\{U_a\}_{a\in\beta}$ of a $N_{\mathcal{P}O}$ $-$Paracompact space there exists a nano locally finite $N_{\mathcal{P}O}$ cover $\{K_a\}_{a\in\beta}$ such that $N\ Cl\ (K_a) \subset U_a$, for every $a \in \beta$.

Theorem2.1.15

Every $N_{\mathcal{P}O}$ $-$Paracompact space is $N_{\mathcal{P}O}$ $-$normal.

Proof:

Let X be $N_{\mathcal{P}O}$ $-$Paracompact space is a $N_{\mathcal{P}O}$ $-$normal.We shall show first that a $N_{\mathcal{P}O}$ $-$Paracompact space is $N_{\mathcal{P}O}$ $-$regular space. Suppose $\mathcal{F}$ is a $N_{\mathcal{P}C}$ set in a $N_{\mathcal{P}O}$ $-$Paracompact space X and for each $x \notin \mathcal{F}, y \in \mathcal{F}$ there exists $N_{\mathcal{P}O}$ set G containing y such that $x \notin N\ Cl\ (G)$. Then the $N_{\mathcal{P}O}$ sets G: $y \in F$ to gether with the set $\mathcal{F}^C$, form an open cover of X. Let $\mathcal{w}$ be a nano locally finite $N_{\mathcal{P}O}$refinement and

$$\mathcal{V} = \cup \{\mathcal{W} \in \mathcal{w};\ \mathcal{W} \cap \mathcal{F} \neq \emptyset\}$$

Then $\mathcal{V}$ is a $N_{\mathcal{P}O}$ and contains $\mathcal{F}$, and

$Cl(\mathcal{V}) = \cup \{\ Cl(\mathcal{W}); \mathcal{W} \cap \mathcal{F} \neq \emptyset\}$ [By a result proved earlier]

But each set $\mathcal{W}$ is contained in some G since $\mathcal{w}$ is a $N_{\mathcal{P}O}$refinement and hence, $Cl(\mathcal{W})$is contained in G. hence $x \notin Cl(\mathcal{W})$. (since $x \notin Cl(G)$). Thus $x \notin Cl(\mathcal{V})$. But $\mathcal{V} \supseteq \mathcal{F}$.Thus x and $\mathcal{F}$ are seperated by $N_{\mathcal{P}O}$ sets in X. Hence the nano space is a $N_{\mathcal{P}O}$ $-$regular space.

Now we will prove that the nano space is $N_{\mathcal{P}O}$ $-$normal.

Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint, $N_{\mathcal{P}C}$ sets of X. Since the nano space is $N_{\mathcal{PO}}$ −regular space , then to each $y \in \mathcal{F}$, there exists a $N_{\mathcal{PO}}$ set G containing y and $Cl(G) \cap \mathcal{F}_2 = \emptyset$. Then the $N_{\mathcal{PO}}$ sets G together with $\mathcal{F}^C$ form an open cover of X. Let $\mathcal{w}$ be a nano locally finite $N_{\mathcal{PO}}$refinement and

$$\mathcal{V} = \{\mathcal{W} \in \mathcal{w}; \mathcal{W} \cap \mathcal{F} \neq \emptyset\}$$

Then $\mathcal{V}$ is a $N_{\mathcal{PO}}$ and contains $\mathcal{F}$ and

$$Cl(\mathcal{V}) = \cup \{Cl(\mathcal{W}); \mathcal{W} \cap \mathcal{F} \neq \emptyset\}$$

But each that $\mathcal{W}$ is contained in some G (since $\mathcal{w}$ is a $N_{\mathcal{PO}}$ refinement) and hence each $Cl(\mathcal{W})$ is contained in$Cl(G)$.Thus there is a$N_{\mathcal{PO}}$ set $\mathcal{V}$ such that

$\mathcal{F}_1 \subset \mathcal{V}$ and $Cl(\mathcal{V}) \cap \mathcal{F}_2 = \emptyset$

Thus X is a $N_{\mathcal{PO}}$ −normal.

<u>Lemma2.1.16</u>

If X is a $N_{\mathcal{PO}} - T_1$ space and for every $N_{\mathcal{PC}}$ set $N_{\mathcal{PC}} \subset X$ and every $N_{\mathcal{PO}}$ $\mathcal{W} \subset X$ that contains $N_{\mathcal{PC}}$there exists a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$of $N_{\mathcal{PO}}$ subsets of X Such that $N_{\mathcal{PC}} \subset \bigcup_{i=1}^{\infty} \mathcal{W}_i$ and $N\ Cl\ (\mathcal{W}_i) \subset \mathcal{W}$ for

$i = 1,2,\ldots$,then the nano space X is $N_{\mathcal{PO}}$ −normal.

Proof:

Let $\mathcal{A}$ and β be disjoint $N_{\mathcal{PC}}$ subsets of X. Letting $N_{\mathcal{PC}} = \mathcal{A}$ and $\mathcal{W} = X \setminus \beta$we obtain a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of $N_{\mathcal{PO}}$subsets of X such that

$\mathcal{A} \subset \bigcup_{i=1}^{\infty} \mathcal{W}_i$ and $\beta \cap N\ Cl\ (\mathcal{W}_i) = \emptyset$ for $i = 1,2,\ldots$ **…….(1)**

Letting $N_{\mathcal{PC}} = \beta$ and $\mathcal{W} = X \setminus \mathcal{A}$we obtain a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$of $N_{\mathcal{PO}}$subsets of X such that

$\beta \subset \bigcup_{i=1}^{\infty} \mathcal{V}_i$ and $\mathcal{A} \cap N\ Cl(\mathcal{V}_i) = \emptyset$ for $i = 1,2,\ldots$**…….(2)**

Let

$\mathcal{G}_i = \mathcal{W}_i \setminus \bigcup_{j \le i} N\ Cl\ (\mathcal{V}_j)$ and $\mathcal{H}_i = \mathcal{V}_i \setminus \bigcup_{j \le i} N\ Cl\ (\mathcal{W}_j)$. **.(3)**

The sets $\mathcal{G}_i$and $\mathcal{H}_i$ are $N_{\mathcal{P}O}$; moreover, (1) and (2) imply that

$\mathcal{A} \subset U = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ and $\beta \subset \mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{H}_i$.

To complete the proof we have to show that the $N_{\mathcal{P}O}$sets $\mathcal{K}$ and $\mathcal{V}$ are disjoint.

Since (**3**) implies that $\mathcal{G}_i \cap \mathcal{V}_j = \emptyset$ for $j \le i$, we have $\mathcal{G}_i \cap \mathcal{H}_j = \emptyset$ for

$j \le i$. Similarly, $\mathcal{H}_j \cap \mathcal{W}_i = \emptyset$ for $j \le i$ and $\mathcal{G}_i \cap \mathcal{H}_j = \emptyset$ for . Thus $\mathcal{G}_i \cap \mathcal{H}_j = \emptyset$ for $i, j = 1, 2, \ldots$ and therefore $\mathcal{K} \cap \mathcal{V} = \emptyset$.

Theorem2.1.17

Every countable $N_{\mathcal{P}O}$ −regular space is $N_{\mathcal{P}O}$ −normal.

Proof:

Every countable $N_{\mathcal{P}O}$ −regular space X satisfies the condition in Lemma 2.1.16 because for any $x \in N_{\mathcal{P}C}$ there is a $N_{\mathcal{P}O}\mathcal{K}_x$,such that $x \in \mathcal{K}_x \subset N\ Cl\ (\mathcal{K}_x) \subset \mathcal{W}$, the family of $\mathcal{K}_x$'s is countable and $N_{\mathcal{P}C} \subset \bigcup_{x \in N_{\mathcal{P}C}} \mathcal{K}_x$.

Lemma 2.1.18.

If any open cover $\mathcal{V}$ of a NTS of X has a nano locally finite a $N_{\mathcal{P}C}$ refinement, then it is also a nano locally finite $N_{\mathcal{P}O}$ refinement cover $\mathcal{M}$.

Proof:

Put U an open cover of the nano space X; take a nano locally finite $N_{\mathcal{P}O}$ refinement $\mathcal{M} = \{M_a\}_{a \in \beta}$of U and for every $x \in X$ choose a nano neighbourhood V_x of the point x which meets only finitely many members of $\mathcal{M}$. Let $\mathcal{F}$ be a nano locally finite $N_{\mathcal{P}C}$ refinement of the open cover $\{ V_x \}_{x \in X}$ and for every $a \in \beta$ let

$$\mathcal{W}_a = X \setminus \bigcup\{F \in \mathcal{F} : F \cap M_a = \emptyset\}.$$

Clearly, the set $\mathcal{W}_a$ is a $N_{\mathcal{P}O}$ and contains M , furthermore for every $a \in \beta$ and any $F \in \mathcal{F}$ we have

(1) $\mathcal{W}_a \cap F \neq \emptyset$ if and only if $M_a \cap F \neq \emptyset$.

For every $a \in \beta$ take a $U(a) \in \mathcal{V}$ such that $M_a \subset U(a) \neq \emptyset$ and let $V_a = \mathcal{W}_a \cap U(a)$. The family = $\{V_a\}_{a\in\beta}$ set is a$N_{\mathcal{P}O}$ refinement of the cover U. Since every point $x \in X$ has a nano neighbourhood intersect finitely many members of $\mathcal{F}$ and every member of $\mathcal{F}$ intersect finitely many members of $\mathcal{M}$, it follows from (1) that the cover $\{V_a\}_{a\in\beta}$ is a nano locally finite.

Theorem 2.1.19.

Let $(\mathrm{U}, \tau_R(\mathrm{X}))$ be a NTS.If $N_{\mathcal{P}O}$ is $\mathcal{P}O$ −Paracompact space, then $(U, \tau_R(X))$ is a $N_{\mathcal{P}O}$ −Paracompact space.

Proof:

Let U be an open cover of $(U, \tau_R(X))$. Then U is an open cover of the $\mathcal{P}O$ −Paracompact space $N_{\mathcal{P}O}$ (U, X) and so it has a nano locally finite $N_{\mathcal{P}O}$ refinement $\mathcal{M}$ in $N_{\mathcal{P}O}$ (U, X) . Now for each$M \in \mathcal{M}$, choose $U_M \in U$ such that $M \subseteq U_M$. One can easily show that the collection $\{U_M \cap N\ Int(N\ Cl\ (M))\}$ is a nano locally finite $N_{\mathcal{P}O}$refinement of U in $(U, \tau_R(X))$.

2.2 $N_{\mathcal{PO}}$ point-finite

Definition 2.2.1

A family $\{\mathcal{A}_s\}_{s\in S}$ of $N_{\mathcal{PO}}$ subsets of a set X is called $N_{\mathcal{PO}}$ point-finite ($N_{\mathcal{PO}}$ point- countable) if for every $z \in X$ the $N_{\mathcal{PO}}$ set $\{s \in S: z \in \mathcal{A}_s\}$ is finite.

Remark 2.2.2

Clearly every nano locally finite cover is $N_{\mathcal{PO}}$ point-finite.

Theorem2.2.3

For every $N_{\mathcal{PO}}$ point-finite open cover $\{\mathfrak{U}_s\}_{s\in S}$ of a $N_{\mathcal{PO}}$ $-$normal space X ,there exists an open cover $\{\mathcal{V}_s\}_{s\in S}$ of X such that $N\ Cl(\mathcal{V}_s) \subset \mathfrak{U}_s$, for every $s \in S$.

Proof:

Let $\mathcal{H}$ be the family of all functions H from $N_{\mathcal{PO}}$ set S to the NTS 0 of the nano space X subject to the conditions:

$H(s) = \mathfrak{U}_s$ or $N\ Cl(H(s)) \subset \mathfrak{U}_s$, ... (1)

And

$\bigcup_{s\in S} H(s) = X$. ……..(2)

Let us order the family $\mathcal{H}$ by defining that $H_1 \leq H_2$ whenever $H_2(s) = H_1(s)$ for every $s \in S$ $\ni$ $H_1(s) \neq \mathfrak{U}_s$. We shall show that for each linearly ordered subfamily $\mathcal{H}_0 \subset \mathcal{H}$ the formula

$\mathcal{H}_0(s) = \bigcap_{H\in\mathcal{H}_0} H(s)$ for $s \in S$ defines a member of H. Condition (1) is clearly satisfied for $H = H_0$; we shall verify condition (2). Take a point $z \in X$; as $\{\mathfrak{U}_s\}_{s\in S}$ is $N_{\mathcal{PO}}$ point- finite, there exists a finite $N_{\mathcal{PO}}$ set $S_0 = \{s_1, s_2, \dots, s_k\} \subset S$ such that $z \in \mathfrak{U}_{s_i}$ for $i = 1,2,\dots,k$ and $z \notin \mathfrak{U}_s$ for $s \in$

$S \setminus S_0$. If $H_0(s_i) = \mathfrak{U}_{s_i}$ for some $s_i \in S_0$, then $z \in H_0(s_i) \subset \bigcup_{s\in S} H_0(s)$. Assume now that for $i = 1, 2, \ldots, k$ there exists a $H_i \in \mathcal{H}_0$such that $H_i(s_i) \neq \mathfrak{U}_{s_i}$. The family $\mathcal{H}_0$ is ordered linearly, there exists $j \leq k \ni H_i \leq H_j$ for $i = 1,2, \ldots, k$. Applying (2) to H_j we find an $i_0 \leq k \ni z \in H_j\left(s_{i_0}\right) = H_0(s_{i_0})$, so that also in this case $z \in \bigcup_{s\in S} H_0(s)$. One easily sees that $H \leq H_0$ for every $H \in \mathcal{H}_0$.

From the Kuratowski-Zorn lemma, there exists a maximal element H in $\mathcal{H}_i$ to finish the proof it enought to prove that $N\ Cl(H(s)) \subset \mathfrak{U}_s$, for every $s \in S$. Assume that $N\ Cl(H(s_0)) \cap (X \setminus \mathfrak{U}_{s_0}) \neq \emptyset$. The set $\mathcal{A} = X \setminus \cup \{H(s): s \in S \setminus \{s_0\}\} \subset H(s_0)$ is $N_{\mathcal{PC}}$. By the $N_{\mathcal{PO}}$ $-$normality of X there exists a $N_{\mathcal{PO}}$ set $\mathfrak{U}$ such that $\mathcal{A} \subset \mathfrak{U} \subset N\ Cl(\mathfrak{U}) \subset H(s_0)$. Since from (1) it follows that $H(s_0) = \mathfrak{U}_{s_0}$, the formula

$$H_0(s) = \begin{cases} \mathfrak{U} & for\ \ s = s_0 \ , \\ H(s) & for\ \ s \neq s_o, \end{cases}$$

defines a function $H_0 \in \mathcal{H}$ such that $H \leq H_0$ and $H \neq H_0$. This contradiction to maximality of Hshows that $N\ Cl(H(s)) \subset \mathfrak{U}_s$for every $s \in S$.

2.3 σ −nano locally finite

Definition 2.3.1

A collection A of subsets of NTS is called **σ −nano locally finite** if and only if $A = \bigcup_{n=1}^{\infty} \mathcal{A}_n$, where each $\mathcal{A}_n$ is a nano locally finite collection in X, hence if and only if A is the countable union of nano locally finite collections family in X.

Definition2.3.2

A collection A of subsets of NTS is called **σ −nano discrete** if and only if A is the countable union of nano discrete collections in X.

If A is a σ −nano locally finite cover of X, the subcollections $\mathcal{A}_n$,which are nano locally finite and make up A will not usually be covers.

Theorem 2.3.3

Every open σ −nano locally finite cover $\mathcal{V}$ of a NTS X has a nano locally finite $N_{\mathcal{PO}}$refinement cover $\mathcal{M}$.

Proof:

let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$, where $\mathcal{V}_i = \{V_a\}_{a\in\beta_i}$ is a nano locally finite family of $N_{\mathcal{PO}}$sets and $\beta_i \cap \beta_j = \emptyset$ whenever $i \neq j$ .For each $\beta_0 \in \beta i$ let

$$\boldsymbol{A_{a_0} = V_{a_0} \setminus \bigcup_{w<i} \bigcup_{a \in \beta_w} V_a}$$

the family $S = \{S_a\}_{a\in\beta}$, where $\beta = \bigcup_{i=1}^{\infty} \beta_i$, covers X and is a $N_{\mathcal{PO}}$refinement of $\mathcal{V}$. We shall show that S is a nano locally finite. Consider a point $x \in X$, denote by w the smallest natural number such that $x \in \bigcup_{a \in \beta_w} V_a$, and take an $a_0 \in \beta_w$ satisfying $x \in V_{a0}$; clearly V_{a0} , is a nano neighbourhood of x disjoint from all sets S_a, with $a \in \bigcup_{w<i} \beta_i$. Since the families $\mathcal{V}_i$are nano locally finite , for every $i \leq w$, there exists a nano pre-

neighbourhood U_i of x intersect finitely many member of $\mathcal{V}_i$. The nano pre-neighbourhood $U_1 \cap U_2 \cap \dots \dots \cap U_w \cap V_{a_0}$ of the point x intersec finitely many members of $\mathcal{V}$ **.**

Theorem2.3.4

For every $N_{\mathcal{P}O}$ −regular space X the following conditions are equivalent:

(1) The nano space X is $N_{\mathcal{P}O}$ −Paracompact space.

(2) Every open cover of the nano space X has an open σ −nano locally finite $N_{\mathcal{P}O}$refinement.

(3) Every open cover of the nano topological space X has a nano locally finite $N_{\mathcal{P}O}$refinement.

(4) Every open cover of the nano topological space X has a closed nano locally finite $N_{\mathcal{P}O}$refinement.

Proof:

Suppose X is $N_{\mathcal{P}O}$ −Paracompact and $\mathcal{V}$ an open cover of X. Let $\mathcal{M} = \{M_a\}_{a\in\beta}$ be a nano locally finite $N_{\mathcal{P}O}$refinement of $\mathcal{V}$. By virtue of Lemma 2.1.13 there exists a $N_{\mathcal{P}C}$ cover $\{V_a\}_{a\in\beta}$ of the nano topological space X such that $V_a \subset M_a$ for every $a \in \beta$, (2), (3), (4) proof of The theorem 2.3.3 and the lemma follows from 2.1.13 and 2.1.18.

2.4 Theorem on Characterization of $N_{\mathcal{P}O}$ −Paracompactness

In this section, we define and study the notions of nano$-T_3$ spaces , nano Pre$-T_3$ space in nano topological spaces ,Theorem on Characterization of $N_{\mathcal{P}O}$ −Paracompactness.

Definition2.4.1

A space U is called a **nano$-T_3$** (or $N - T_3$) space if and only if nano regular space and nano$-T_1$ space.

Definition2.4.2

A space U is called a **nano Pre$-T_3$** (or $N_{\mathcal{P}O} - T_3$) space if and only if $N_{\mathcal{P}O}$ −regular space and $N_{\mathcal{P}O} - T_1$.

Example2.4.3

Let $U = \{\alpha, \beta, \delta\}$ and $U/R = \{\{\alpha\}, \{\delta\}, \{\beta, \delta\}\}$, $X = \{\alpha, \beta\}$

$\tau_R(X) = \{\emptyset, U, \{\alpha\}, \{\beta, \delta\}\}$.

The nano closed set $F = \{U, \emptyset , \{\beta, \delta\}, \{\alpha\}\}$.

$N_{\mathcal{P}O}(U, X) = \{\emptyset, U, \{\alpha\}, \{\beta\}, \{\delta\}, \{\alpha, \beta\}, \{\alpha, \delta\}, \{\beta, \delta\}\}$.

$N_{\mathcal{P}C} = \{U, \emptyset, \{\beta, \delta\}, \{\alpha, \delta\}, \{\alpha, \beta\}, \{\delta\}, \{\beta\}, \{\alpha\}\}$.

$\alpha \notin \{\beta, \delta\} \Rightarrow \alpha \in \{\alpha\} \wedge \{\beta, \delta\} \subseteq \{\beta, \delta\} \ni \{\alpha\} \cap \{\beta, \delta\} = \emptyset$.

$\beta \notin \{\alpha\} \Rightarrow \beta \in \{\beta, \delta\} \wedge \{\alpha\} \subseteq \{\alpha\} \ni \{\beta, \delta\} \cap \{\alpha\} = \emptyset$.

$\delta \notin \{\alpha\} \Rightarrow \delta \in \{\beta, \delta\} \wedge \{\alpha\} \subseteq \{\alpha\} \ni \{\beta, \delta\} \cap \{\alpha\} = \emptyset$.

$\therefore (U, \tau_R(X))$ is $N_{\mathcal{P}O}$ −regular space.

$\alpha \neq \beta \Rightarrow \alpha \in \{\alpha\} \wedge \beta \notin \{\alpha\}$ and $\beta \in \{\beta\} \wedge \alpha \notin \{\beta\}$

$\alpha \neq \delta \Rightarrow \alpha \in \{\alpha\} \wedge \delta \notin \{\alpha\}$ and $\delta \in \{\delta\} \wedge \alpha \notin \{\delta\}$

$\beta \neq \delta \Rightarrow \beta \in \{\beta\} \wedge \ \delta \notin \{\beta\}$ and $\delta \in \{\delta\} \ \wedge \ \beta \notin \{\delta\}$

$\therefore \left(U, \tau_R(X)\right)$ is $N_{\mathcal{PO}} - T_1$.

$\therefore \left(U, \tau_R(X)\right)$ is $N_{\mathcal{PO}} - T_3$.

Theorem 2.4.4

If X is a $N_{\mathcal{PO}} - T_3$, the following are equivalent :

(1) X is a $N_{\mathcal{PO}}$ −Paracompact.

(2) Each open cover of X has an open σ −nano local finite $N_{\mathcal{PO}}$ refinement.

(3) Each open cover has a nano locally finite $N_{\mathcal{PO}}$ refinement (not necessarily open).

(4) Each open cover of X has a closed nano locally finite $N_{\mathcal{PO}}$ refinement.

Proof:

(1) $\Rightarrow$ (2)

Since X is a $N_{\mathcal{PO}}$ −Paracompact, each open cover of X has an open a nano locally finite$N_{\mathcal{PO}}$ refinement. Also, a nano locally finite cover is σ −nano locally finite. It follows therefore that each open cover of X has an open σ −nano locally finite $N_{\mathcal{PO}}$ refinement.

(2) $\Rightarrow$ (3)

Let $\mathcal{V}$ be an open cover of X. By (2), there is a $N_{\mathcal{PO}}$ refinement $\mathcal{M}$ of $\mathcal{V}$ such that $\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$,where each $\mathcal{M}_n$ is a nano locally finite collection of $N_{\mathcal{PO}}$ sets, say

$$\mathcal{M}_n = \left\{M_{n_\beta}; \beta \in B\right\}. \text{ For each } n,$$

Let $\mathcal{W}_n = \bigcup_\beta M_{n_\beta}$ .Then $\{\mathcal{W}_1, \mathcal{W}_2, \dots \dots\}$

Covers X .Define $\mathcal{A}_1 = \mathcal{W}_n - \bigcup_{i<n} \mathcal{W}_i$

Then $\{\mathcal{A}_n ; n \in N\}$ is a nano locally finite $N_{\mathcal{P}O}$ refinement of $\{\mathcal{W}_n ; n \in N\}$. Now consider $\{\mathcal{A}_n \cap M_{n_\beta}\}$, $\beta \in B$.

This is a nano locally finite $N_{\mathcal{P}O}$ refinement of $\mathcal{M}$ and since $\mathcal{M}$ is a $N_{\mathcal{P}O}$ refinement of $\mathcal{V}$.Thus each open cover of X has a nano locally finite $N_{\mathcal{P}O}$ refinement.

(3) $\Rightarrow$ (4).

Let $\mathcal{V}$ be an open cover of X. For each $x \in X$, pick some U_X, in $\mathcal{V}$ such that $x \in U_X$, and [as X is a $N_{\mathcal{P}O} - T_3$ space] by $N_{\mathcal{P}O}$ −regularity, find an open nano pre-neighbourhood M_X, of X such that $N\ Cl\ (M_X) \subset U_X$, Now $\{M_X; x \in X\}$ is an open cover of X and so by part (3) has a nano locally finite $N_{\mathcal{P}O}$ refinement $\{\mathcal{A}_\beta, \beta \in B\}$. Then $\{N\ Cl\ (\mathcal{A}_\beta); \beta \in B\}$ is still a nano locally finite by the (proved earlier) and for each β, if $\mathcal{A}_\beta \subset M_X$, then $N\ Cl\left(\mathcal{A}_\beta\right) \subset N\ Cl\ (M_X) \subset U$, CU for some $U \in \mathcal{V}$. It follows that $N\ Cl\ (\mathcal{A}_\beta)\ ; \beta \in B\}$ is a closed nano locally finite $N_{\mathcal{P}O}$ refinement of $\mathcal{V}$.

(4) $\Rightarrow$(1)

Let $\mathcal{V}$ be an open cover of X, $\mathcal{M}$ a closed nano locally finite $N_{\mathcal{P}O}$ refinement. For each $x \in X$, let $\mathcal{W}_X$, be a nano pre- neighbourhood of x meeting only finitely many$M \in \mathcal{M}$. Now let $\mathcal{A}$ be a closed nano locally finite $N_{\mathcal{P}O}$ refinement of $\{w_X; x \in X\}$. For each $M \in \mathcal{M}$, let

$$M^\star = X - U\ \{A \in \mathcal{A}\ ;\ A \cap M = \emptyset\}.$$

Then $\{M^\star, M \in \mathcal{M}\}$ is an open cover (The sets $M^\star$are open by the result proved above) and is also nano locally finite. For consider $x \in X$, there is a nano pre-neighbourhood U of x meeting only $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$,say from $\mathcal{A}$. But whenever $U \cap\ M^\star \neq \emptyset$,we have $\mathcal{A}_k \cap M^\star \neq \emptyset$for some $K\ =\ 1, 2, \dots, n$ which implies $\mathcal{A}_k \cap M \neq \emptyset$. Since each $\mathcal{A}_k$ meets only finitely many M, we must then have$U \cap M^{\star} \neq \emptyset$ for all but finitely many of the $M^{\star}$. Hence $\{M^{\star} ; M \in \mathcal{M}\}$ is a nano locally finitely.

Now for each $M \in \mathcal{M}$, pick $U \in \mathcal{V}$ such that $M \subset U$ and form the set $U \cap M^{\star}$The collection of nano sets which results, as M ranges through $\mathcal{M}$, serves as an open nano locally finite $N_{\mathcal{P}O}$ refinement of $\mathcal{V}$.

<u>Corollary2.4.5</u>

Every Lindelof $N_{\mathcal{P}O} - T_3$ space X is $N_{\mathcal{P}O}$ −Paracompact.

Proof:

Since X is Lindelof, every open covering of X has a countable subcovery and A countable mas a subcover is a σ −nano locally finite $N_{\mathcal{P}O}$ refinement.Thus each open cover of X has an open σ −nano locally finite $N_{\mathcal{P}O}$ refinement. Then by the above theorem, X is $N_{\mathcal{P}O}$ −Paracompact.

2.5 Nano Pre-open Star

Definition2.5.1

Let $(U,\tau_R(X))$ be a NTS and $\mathcal{V}=\{V_a\}_{a\in\beta}$ be a cover of a $(U,\tau_R(X))$,such that $U=\cup\,\mathcal{V}$,we say that is $\boldsymbol{N_{\mathcal{PO}}}$**star** of a set $\mathcal{M}\subset U$ with respect to $\mathcal{V}$ is the union of all the set $V_a\in\mathcal{V}$ that intersect $\mathcal{M}$, that is

$$N_{\mathcal{PO}}St\ (\mathcal{M},\mathcal{V}) = \bigcup\ \{V_a:\ \mathcal{M}\cap V_a\neq\emptyset\}\ .$$

Given a one-point $z\in X$,we write $N_{\mathcal{PO}}\ St\ (z,\mathcal{V})$ instead of $N_{\mathcal{PO}}\ St\ (\{z\},\mathcal{V})$.

Definition2.5.2

Let we say that a cover $\mathcal{W}=\{W_t\}_{t\in N_{\mathcal{PO}}}$of a set X is a $\boldsymbol{N_{\mathcal{PO}}}$**star** $\boldsymbol{N_{\mathcal{PO}}}$ **refinement** of another cover $\mathcal{V}=\{V_a\}_{a\in\beta}$ of the same set X if for every $t\in\ N_{\mathcal{PO}}$ there exists an $a\ \in\beta$ such that $N_{\mathcal{PO}}St\ (W_t,\mathcal{W})\ \subset V_a$.

Definition2.5.3

If for each $x\in X$ there exists an $a\ \in\beta$such that $N_{\mathcal{PO}}\ St\ (x,\mathcal{W})\subset V_a$, then we say that W is a **barycentric** $\boldsymbol{N_{\mathcal{PO}}}$**refinement** of $\mathcal{V}$. Clearly, every $N_{\mathcal{PO}}$star $N_{\mathcal{PO}}$refinement is a barycentric $N_{\mathcal{PO}}$refinement and every barycentric $N_{\mathcal{PO}}$refinement is a $N_{\mathcal{PO}}$refinement.

Properties of $\boldsymbol{N_{\mathcal{PO}}}$ star

(1) $\mathcal{M}\subseteq N_{\mathcal{PO}}St(\mathcal{M},\mathcal{V})$.

(2) If $\mathcal{M}\subseteq\mathcal{H}$, then $N_{\mathcal{PO}}St(\mathcal{M},\mathcal{V})\subseteq N_{\mathcal{PO}}St(\mathcal{H},\mathcal{V})$.

(3) For any cover $\mathcal{V}$of X, the sets $\mathcal{V}^*=\{St(V_a,\mathcal{V})|V_a\in\mathcal{V}\}$ and $\mathcal{V}^b=\{St(z,\mathcal{V})\mid z\in X\}$ are both covers of X .

(4)$\mathcal{V}$ is $N_{\mathcal{PO}}$refinement of $\mathcal{V}^b$and $\mathcal{V}^b$is $N_{\mathcal{PO}}$refinement of $\mathcal{V}^*$.

Example2.5.4

Let $U = \{\alpha, \beta, \delta\}$ and $U/R = \{\{\alpha\}, \{\beta\}, \{\beta, \delta\}\}$, $X = \{\alpha, \delta\}$

$$\tau_R(X) = \{\emptyset, U, \{\alpha\}, \{\beta, \delta\}\} .$$

The nano closed set $F = \{U, \emptyset , \{\beta, \delta\}, \{\alpha\}\}$.

$N_{\mathcal{PO}}(U, X) = \{\emptyset, U, \{\alpha\}, \{\beta\}, \{\delta\}, \{\alpha, \beta\}, \{\alpha, \delta\}, \{\beta, \delta\}\}$.

$\{\alpha\} \subset U \Rightarrow \{\alpha\} \cap \{\alpha\} = \{\alpha\}$.

$\{\beta\} \subset U \Rightarrow \{\beta\} \cap \{\beta, \delta\} = \{\beta\}$.

$\{\delta\} \subset U \Rightarrow \{\delta\} \cap \{\beta, \delta\} = \{\delta\}$.

$\{\alpha, \beta\} \subset U \Rightarrow \{\alpha, \beta\} \cap \{\beta, \delta\} = \{\beta\}$.

$\{\alpha, \delta\} \subset U \Rightarrow \{\alpha, \delta\} \cap \{\alpha\} = \{\alpha\}$.

$\{\beta, \delta\} \subset U \Rightarrow \{\beta, \delta\} \cap \{\beta, \delta\} = \{\beta, \delta\}$.

$N_{\mathcal{PO}}St(\mathcal{M}, \mathcal{V}) = U$,

$\forall \{\alpha\} \in \{\alpha\} \ \exists \ \alpha \in \{\alpha\} \ni N_{\mathcal{PO}} \, St(\alpha, \mathcal{W}) \subset V_a$.

$\{\alpha\} \subset U \Rightarrow \{\alpha\} \cap \{\alpha\} = \{\alpha\} \subset \{\alpha\}$.

$\therefore N_{\mathcal{PO}} \, St(\alpha, \mathcal{W}) \subset V_a$.

$\therefore (U, \tau_R(X))$ is $N_{\mathcal{PO}}$ Star $N_{\mathcal{PO}}$ refinement.

$\forall \alpha \in X \ \exists \{\alpha\} \in \{\alpha\} \ni N_{\mathcal{PO}} \, St(\alpha, \mathcal{W}) \subset V_a$.

$\{\alpha\} \subset U \Rightarrow \{\alpha\} \cap \{\alpha\} = \{\alpha\} \subset \{\alpha\}$.

$\therefore (U, \tau_R(X))$ is barycentric $N_{\mathcal{PO}}$ refinement.

Theorem 2.5.5

A barycentric $N_{\mathcal{PO}}$ refinement z of a barycentric $N_{\mathcal{PO}}$ refinement v of μ.

Proof:

Suppose $\mathcal{Z}_0 \in z$. Choose a fixed $y_0 \in \mathcal{Z}_0$. For each $\mathcal{Z} \in z$ such that $\mathcal{Z} \cap \mathcal{Z}_0 \neq \emptyset$ choose a $w \in \mathcal{Z} \cap \mathcal{Z}_0$.

Then $$\mathcal{Z} \cup \mathcal{Z}_0 \subseteq \bigcup \{\mathcal{Z}_1 ; w \in \mathcal{Z}_1, \mathcal{Z}_1 \in w\}$$

$$= St\,(w, z) \subset \text{some } \mathcal{V} \in v$$

because z is a barycentric $N_{\mathcal{PO}}$refinement of$v$. Now since each such $\mathcal{V}$ containsy_0, we conclude that

$$N_{\mathcal{PO}}St\,(\mathcal{Z}_0 , z) = \bigcup \{\mathcal{Z} \in z\,;\, \mathcal{Z}_0 \cap \mathcal{Z} \neq \emptyset\}$$

$$N_{\mathcal{PO}}St(\mathcal{Z}_0 , v) \text{ some } U \in \mu,$$

since v is a barycentric $N_{\mathcal{PO}}$ refinement of μ .Thus for each $\mathcal{Z}_0 \in z$, there is some $U \in \mu$ such that $N_{\mathcal{PO}}\ St\ (\mathcal{Z}_0 , z)$. Hence z is a star $N_{\mathcal{PO}}$ refinement of μ. This completes the proof.

Theorem 2.5.6

A barycentric $N_{\mathcal{PO}}$ refinement of any $N_{\mathcal{PO}}$ refinement of $\mathcal{M}$ is a berycentric $N_{\mathcal{PO}}$refinement of $\mathcal{M}$.

Infact, let

$$z = \{\mathcal{Z}_\alpha\,; \alpha \epsilon A\}$$

be barycentric $N_{\mathcal{PO}}$ refinement of a family

$$v = \{\mathcal{V}_\alpha\,; \alpha \in A\}$$

where v is a $N_{\mathcal{PO}}$refinement of $\mathcal{M}$. Let $x \in X$, then

$$N_{\mathcal{PO}}St(x,z) = \cup \{\mathcal{Z}_\alpha \,; x\epsilon\mathcal{Z}_\alpha : \mathcal{Z}_\alpha \epsilon z\}$$

$$\subseteq \mathcal{V}_\alpha \, \{\mathcal{V}_\alpha \in v\}$$

since z is a barycentric $N_{\mathcal{PO}}$refinement of v, But since v is a $N_{\mathcal{PO}}$ refinement of μ , to each $\mathcal{V}_\alpha \in v$, there is some $U \in \mu$such that $\mathcal{V}_\alpha \subseteq U$. Thus

$$St(x,z) \subseteq U \text{ for some } U \in \mu \,.$$

z is a barycentric $N_{\mathcal{PO}}$ refinement of μ .

Lemma2.5.7

If an open cover U of a $NTS\ X$ has a closed nano locally finite $N_{\mathcal{PO}}$refinement, then U has also an open barycentric $N_{\mathcal{PO}}$ refinement.

Proof:

Let $\digamma = \{\mathcal{F}_t\}_{t\in \tau_R}$ be a closed nano locally finite $N_{\mathcal{PO}}$ refinement of $\mathfrak{U} = \{U_a\}_{a\in\beta}$.For every $a \in \beta$ choose an $a(t) \in \beta$ such that $\mathcal{F}_t \subset U_{a(t)}$.

It follows from nano local finiteness of $\digamma$ that the $N_{\mathcal{PO}}$set

$\tau_R(x) = \{t \in \tau_R : x \in \mathcal{F}_t\}$ is nano finite for every $x \in X$, and this implies that the set

$\mathcal{V}_x = \bigcup_{t\in T(x)} U_{a(t)} \cap (X \setminus \bigcup_{t\notin \tau_R(x)} \mathcal{F}_t)$.......(1)

is a $N_{\mathcal{PO}}$ for every $x \in X$. As $x \in \mathcal{V}_x$, the family $V = \{\mathcal{V}_x\}_{x\in X}$ is a nano open cover of X. Let $x_0$be a point of X and to an element of (x_0); it follows from (1) that if $x_0 \in \mathcal{V}_x$, then to $t_0 \in \tau_R(x)$, and thus $\mathcal{V}_x \subset U_{a\in\beta}$. Hence we have $N_{\mathcal{PO}}St\,(x_0, V) \subset U_{a(t_0)}$ which shows that V is a barycentric $N_{\mathcal{PO}}$ refinement of $\mathfrak{U}$.

Remark 2.5.8

The same proof shows that if a nano locally finite open cover of a NTS has aclosed nano locally finite $N_{\mathcal{PO}}$refinement then it has also a nano locally finite barycentric $N_{\mathcal{PO}}$ refinement; indeed, if the cover $\mathfrak{U}$ is a nano locally finite, since all $N_{\mathcal{PO}}$ sets of the form (1) is a nano locally finite barycentric $N_{\mathcal{PO}}$ refinement of $\mathfrak{U}$.

Lemma 2.5.9

If a cover $\mathcal{V} = \{V_a\}_{a\in\beta}$ of a set X is a barycentric $N_{\mathcal{PO}}$ refinement of a cover $\mathcal{M} = \{M_t\}_{t\in T}$ of X, and $\mathcal{M}$is a barycentric $N_{\mathcal{PO}}$refinement of a cover $\mathcal{C} = \{C_b\}_{b\in B}$ of the same set, then $\mathcal{V}$ is a star $N_{\mathcal{PO}}$ refinement of $\mathcal{C}$.

Proof:

Let us take an $a_0 \in \beta$ and for every $x \in V_{a0}$, let us choose a $t(x) \in T$ such that

$$N_{\mathcal{PO}}St(x,\mathcal{V}) \subset \mathcal{M}_{t(x)} \dots\dots\dots (2)$$

Thus we have

$$N_{\mathcal{PO}}St(V_{a0},\mathcal{V}) = \bigcup_{x\in V_{a0}} N_{\mathcal{PO}}St\,(x,\mathcal{V}) \subset \bigcup_{x\in V_{a0}} \mathcal{M}_{t(x)} \dots\dots (3)$$

Let x_0 be a fixed element of V_{a0}; from (2) it follows that $x_0 \in \mathcal{M}_{t(x)}$ for every $x \in \mathcal{M}_{t(x)}$, so that

$$\bigcup_{x\in V_{a0}} \mathcal{M}_{t(x)} \subset N_{\mathcal{PO}}St(x_0,\mathcal{V})\,.$$

Since $N_{\mathcal{PO}}St\,(x_0,\mathcal{V}) \subset \mathcal{C}_x$ for a $w \in W$, the last inclusion, along with (3),implies that $\mathcal{V}$ is $N_{\mathcal{PO}}$star $N_{\mathcal{PO}}$ refinement of $\mathcal{C}$.

Lemma 2.5.10.

If every open cover of a NTS X has a $N_{\mathcal{PO}}$ star $N_{\mathcal{PO}}$ refinement, we get any open cover of X has also an open $\sigma -$ nano discrete $N_{\mathcal{PO}}$refinement.

Proof:

Consider an open cover $\mathfrak{U} = \{U_a\}_{a\in \beta}$ of the nano space X. Let $\mathfrak{U}_0 = \mathfrak{U}$and denoted by $\mathfrak{U}_1, \mathfrak{U}_2, \ldots$a sequence of open cover of X such that

$\mathfrak{U}_{i+1}$ is a $N_{\mathcal{PO}}$ star $N_{\mathcal{PO}}$refinement $\mathfrak{U}_i$ for $i = 0,1, \ldots \ldots$(1)

For every $a \in \beta$ and $i = 1, 2, \ldots$ take the nano open set

$U_{a,i} = \{x \in X : x \approx$ has a nano pre-neighbourhood $\mathcal{V}$ such that $N_{\mathcal{PO}}St\ (\mathcal{V}, \mathfrak{U}_i) \subset U_a\}$.

The family $\{U_{a,i}\}_{a\in \beta}$ is a $N_{\mathcal{PO}}$ star $N_{\mathcal{PO}}$ refinement of $\mathfrak{U}$ for $i = 1, 2, \ldots$ Let us observe that

if $x \in U_{a,i}$and $y \notin U_{a,i+1}$,then there is no $U \in \mathfrak{U}_{i+1}$such that $x, y \in U$…(2)

Indeed, it follows from (1) that for every $U \in \mathfrak{U}_{i+1}$ there exists a $\mathcal{W} \in \mathfrak{U}_i$; such that $N_{\mathcal{PO}}St\ (U, \mathfrak{U}_{i+1}) \subset \mathcal{W}$; therefore if $x\ U \cap U_{a,i}, i$, then $\mathcal{W} \subset N_{\mathcal{PO}}St\ (x, \mathfrak{U}_i) \subset U_a$, which implie $N_{\mathcal{PO}}\ St\ (U, \mathfrak{U}_{i+1}) \subset U_a$, and $U \subset U_{a,i+1}$.

Take a well-ordering relation $<$ on the $N_{\mathcal{PO}}$ set M and let

$$\boldsymbol{\mathcal{V}_{a_0,i} = U_{a_0,i} \backslash Cl(\bigcup_{a<a_0} U_{a,i+1})} \ldots \ldots . . (\mathbf{3})$$

For every pair a_1, a_2 of distinct elements of $\mathcal{M}$ we have either $a_1 < a_2$ or $a_2 < a_1$; depending on which part of the alternative holds , by virtue of (3) we have either $\boldsymbol{\mathcal{V}_{a_2,i}} \subset \boldsymbol{X \setminus U_{a_1,i+1}}$ **or** $\boldsymbol{\mathcal{V}_{a_1,i}} \subset \boldsymbol{X \setminus U_{a_2,i+1}}$.

Hence, it follows from (2) that if $x \in \mathcal{V}_{a_1,i}$ and $y \in \mathcal{V}_{a_2,i}$, where $a_1 \neq a_2$,then there is $U \in \mathfrak{U}_{i+1}$such that $x, y \in U$. Thus the family of nano open sets $\{\mathcal{V}_{a,i}\}_{a\in \beta}$ is discrete for $i = 1,2, \ldots$

To conclude the proof it enought to prove that $\{\mathcal{V}_{a,i}\}_{i=1\ ,a\in \beta}^{\infty}$is a cover of X.

Let x a point of X; denote by $a(x)$ the smallest element in β such that $x \in U_{a(x),i}$ for somepositive integer $i -$ the existence of $a(x)$ follows from the fact that for $i = 1,2, \ldots$ the family $\{U_{a,i}\}_{a\in \beta}$ is a cover of X. Since $x \notin U_{a,i+2}$ for $a < a(x)$, it follows from (3) that

$$\boldsymbol{N_{\mathcal{PO}}St(x, \mathfrak{U}_{i+2})} \cap \bigcup_{\boldsymbol{a<a(x)}} \boldsymbol{U_{a,i+1}} = \emptyset \ ,$$

and this show that $x \in \mathcal{V}_{a(x),i}$.

The next theorem contains yet three characterizations of $N_{\mathcal{PO}}$ −Paracompactness ; it will be deduced directly from Lemmas 2.5.7, 2.5.9, and 2.5.10 that are stated and proved below.

<u>Theorem 2.5.11</u>

For every $N_{\mathcal{PO}} - T_1\ space\ X$ the following conditions are equivalent:

(1) The nano topological space X is $N_{\mathcal{PO}}$ −Paracompact space.

(2) Every open cover $\mathcal{V}$ of the nano space X has a barycentric $N_{\mathcal{PO}}$refinement.

(3) Every open cover $\mathcal{V}$ of the nano space X has a $N_{\mathcal{PO}}$star $N_{\mathcal{PO}}$refinement.

(4) The nano space X is a $N_{\mathcal{PO}}$ −regular space and every open cover $\mathcal{V}$ of X has a σ −nano discrete $N_{\mathcal{PO}}$ refinement.

Proof:

By virtue of the last three lemmas and by Theorem 2.3.4, is suffices to show that every nano T_1 $-$space X satisfying (3) is $N_{\mathcal{P}O}$ $-$regular. Consider a point $x \in X$ and a closed set $\mathcal{F} \subset X$ such that $x \notin \mathcal{F}$ and take a $N_{\mathcal{P}O}$ star $N_{\mathcal{P}O}$refinement $\mathfrak{U}$ of the open cover $\{X \backslash \mathcal{F}, X \backslash \{x\}\}$ of the space X.

Let U be a member of $\mathfrak{U}$ that contains x . As $N_{\mathcal{P}O}St\ (U, \mathfrak{U}) \subset X \backslash \mathcal{F}$, we have $N\ Cl(U) \cap \mathcal{F} = \emptyset$, so that the nano space X is $N_{\mathcal{P}O}$ $-$regular.

2.6 Collectionwise $N_{\mathcal{PO}}$ −normal

Definition 2.6.1

Let $(U, \tau_R(X))$ be a NTS is called a **collectionwise $N_{\mathcal{PO}}$ −normal** if it is $N_{\mathcal{PO}} - T_1$ space and for every discrete family $\{\mathcal{F}_s\}_{s\in S}$ of $N_{\mathcal{PC}}$ subset of X there exists a discrete family $\{\mathcal{V}_s\}_{s\in S}$ of $N_{\mathcal{PO}}$ subset of X such that $\mathcal{F}_s \subset \mathcal{V}_s$ for every $s \in S$.

Remark 2.6.2

Clearly, every collectionwise $N_{\mathcal{PO}}$ −normal spaceis $N_{\mathcal{PO}}$ −normal.

Theorem 2.6.3

Every $N_{\mathcal{PO}}$ −Paracompact spaceis a collectionwise $N_{\mathcal{PO}}$ −normal.

Proof:

Let $\{\mathcal{F}_s\}_{s\in S}$be a discrete family of $N_{\mathcal{PC}}$ subsets of a $N_{\mathcal{PO}}$ −Paracompact space.For every $x \in X$ choose a nano pre-neighbourhood $\mathcal{H}_x$ of the point x whose closure meets at most one set $\mathcal{F}_s$, consider an open nano locally finite $N_{\mathcal{PO}}$ refinement W of the cover $\{\mathcal{H}_x\}_{x\in X}$, and for every $s \in S$,let $U_s = X \setminus \bigcup\{N\ Cl\ (\mathcal{W}) : \mathcal{W} \in W \ and\ N\ Cl\ (\mathcal{W}) \cap \mathcal{F}_s = \emptyset\}$. Clearly $\mathcal{F}_s \subset U_s$, so that to conclude the proof it suffices to show that every $\mathcal{W} \in W$ meets at most one element of the family $\{U_s\}_{s\in S}$. This, however, follows from the fact that $N\ Cl\ (\mathcal{W})$ meets at most one set $\mathcal{F}_s$.

Definition2.6.4

Let $(U, \tau_R(X))$ be a NTS is called a **$N_{\mathcal{PO}}$ −Lindelof space**, if every open cover $\mathcal{V}$ of X has a $N_{\mathcal{PO}}$ −countable subcover .

Lemma 2.6.5

Every nano locally finite family of non-empty $N_{\mathcal{PO}}$subsets of a $N_{\mathcal{PO}}$ −Lindelöf space is $N_{\mathcal{PO}}$ −countable.

Proof:

Let A be a nano locally finite family of non-empty $N_{\mathcal{PO}}$ subsets of a $N_{\mathcal{PO}}$ −Lindelof space X.

For every $x \in X$ choose a nano pre-neighbourhood $\mathcal{V}_x$ of the point x which meets only finitely many or members of A and

take a $N_{\mathcal{PO}}$ −countable subcover $\mathfrak{U}$ of the cover $\{\mathcal{V}_x\}_{x \in X}$ of X. Since every member of A meets a $\mathcal{V} \in \mathfrak{U}$, it follows that $|A| \leq \mathcal{N}_0$.

Introduction

In this chapter, new types of the Nano topological space by using the Nano pre-open subsets are constructed, investigated and studied, such as the countable Nano pre-open paracompact and weakly ,strongly $N_{\mathcal{PO}}$ −Paracompact, and the relation between them.

3.1 Countable $N_{\mathcal{PO}}$ −Paracompact.

Definition 3.1.1

Let $(U, \tau_R(X))$ be a NTS is called **countable $N_{\mathcal{PO}}$ −Paracompact** ($C\ N_{\mathcal{PO}}$ −Paracompact) if it is $N_{\mathcal{PO}} - T_2$ and any countable open cover of $\mathcal{V}$ has a nano local finite $N_{\mathcal{PO}}$refinement cover $\mathcal{M}$.

Remark3.1.2

Any $N_{\mathcal{PO}}$ −Paracompact space is countably $N_{\mathcal{PO}}$ −Paracompact.

Theorem3.1.3

For every $N_{\mathcal{PO}} - T_2$space, the following equivalent:

(i) U is $C\ N_{\mathcal{PO}}$ −Paracompact.

(ii) For every countable open cover $\{\mathcal{A}_i\}_{i=1}^{\infty}$ of the nano topological space U there exists a nano local finite $N_{\mathcal{PO}}$cover $\{\beta_i\}_{i=1}^{\infty}$of Usuch that $\beta_i \subset \mathcal{A}_i$ for $i = 1,2,..$

(iii) For every increasing sequence $\mathcal{W}_1 \subset \mathcal{W}_2 \subset \ldots$ in $N_{\mathcal{PO}}$ subset of U satisfying $\bigcup_{i=1}^{\infty} \mathcal{W}_i = U$ there exists a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ in U such that $\mathcal{F}_i \subset \mathcal{W}_i$ for $i = 1,2,\ldots$ and $\bigcup_{i=1}^{\infty} N\ Int\ (\mathcal{F}_i) = U$

(iv) For every decreasing sequence $\mathcal{F}_1 \supset \mathcal{F}_2 \ldots$ in $N_{\mathcal{PC}}$ with $\bigcap_{i=1}^{\infty} \mathcal{F}_i = \emptyset$,there exists a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ in $N_{\mathcal{PO}}$ such that $\mathcal{F}_i \subset \mathcal{W}_i$,for $i = 1,2,\ldots$ and $\bigcap_{i=1}^{\infty} N\ Cl\ (\mathcal{W}_i) = \emptyset$.

Proof:

To show that (i) ⇒ (ii)

It suffices to take a nano locally finite $N_{\mathcal{PO}}$ refinement $\mathcal{M}$ of the cover $\{\mathcal{A}_i\}_{i=1}^{\infty}$ for every $\beta \in \mathcal{M}$ choose a natural number β_i such that $\beta \subset \mathcal{A}_{i(\beta)}$, and let $\beta_j = \cup \{\beta : \beta_i = i\}$.

We shall show that (ii) ⇒ (iii).

Put $\{\mathcal{W}_{\mathrm{i}}\}_{\mathrm{i}=1}^{\infty}$ be an open cover which is a countable open cover of X, there exists a $N_{\mathcal{PO}}$ locally finite cover $\{\beta_i\}_{i=1}^{\infty}$ of X , such that $\beta_i \subset \mathcal{W}_i$; for $i = 1,2, \ldots$ The sets $\mathcal{F}_i = X \setminus \bigcup_{j>i} \beta_j \subset \bigcup_{j\le i} \beta_j$, are $N_{\mathcal{PC}}$ and since $\bigcup_{j\le i} \beta_j \subset \bigcup_{j\le i} \mathcal{W}_i = \mathcal{W}_i$ we have $\mathcal{F}_i \subset \mathcal{W}_i$ for $i = 1, 2, \ldots$ The family $\{\beta_i\}_{i=1}^{\infty}$, being a nano locally finite, every point $x \in X$ has a nano pre-neighbourhood that contained in some $\mathcal{F}_i$, hence $\bigcup_{i=1}^{\infty} N\ Int\ (\mathcal{F}_i) = X$.

It follows easily that (iii) and (iv) be equivalent (From De Morgan's laws); hence, it suffices prove that (iii)⇒(i). Let $\{\mathcal{A}_i\}_{i=1}^{\infty}$ be countable open cover of X.

Let $\mathcal{W}_1 \subset \mathcal{W}_2 \subset \cdots$in $N_{\mathcal{PO}}$be increasing sequence where $\mathcal{W}_i = \bigcup_{j\le i} \mathcal{A}_j$, as $\bigcup_{i=1}^{\infty} \mathcal{W}_i = X$, there exists a sequence $\mathcal{F}_1 , \mathcal{F}_2$ ... in $N_{\mathcal{PC}}(U, X)$ such that $\mathcal{F}_i \subset \mathcal{W}_i$ and $\bigcup_{i=1}^{\infty} N\ Int\ (\mathcal{F}_i) = X$. The set $\beta_i = \mathcal{A}_i \setminus \bigcup_{j<i} \mathcal{F}_j \subset \mathcal{A}_i$ is a $N_{\mathcal{PO}}$ for $i = 1, 2, \ldots$; since $\bigcup_{j<i} \mathcal{F}_j \subset \bigcup_{j<i} \mathcal{W}_j \subset \bigcup_{j<i} \mathcal{A}_j$,we have $\mathcal{A}_i \setminus \bigcup_{j<i} \mathcal{A}_j \subset \beta_i$ which implies C that the family $\{\beta_i\}_{i=1}^{\infty}$is a cover of X.

Every point $x \in X$ has a nano pre-neighbourhood of the form $N\ Int\ (\mathcal{F}_j)$; this nano pre-neighbourhood is disjoint from all sets β_i for $i > j$, so that the cover $\{\beta_i\}_{i=1}^{\infty}$is a nano locally finite.

To show that (i) ⇒ (iv)

Suppose that X is countably $N_{\mathcal{PO}}$ −Paracompact. Suppose that $\{\mathcal{F}_i : i = 1,2,\dots\}$ decreasing sequence of $N_{\mathcal{PC}}$ subsets of X as in the condition in the theorem. Then $\mathcal{M} = \{X - \mathcal{F}_i : i = 1,2,\dots\}$ is an open cover of X. Let $\mathcal{V}$ be a nano locally finite $N_{\mathcal{PO}}$refinement of $\mathcal{M}$. For each $i = 1,2,\dots$, define the following:

$$\mathcal{W}_i = \cup \{V \in \mathcal{V} : V \cap \mathcal{F}_i \neq \emptyset\} .$$

It is clear that $\mathcal{F}_i \subset \mathcal{W}_i$ for each i. The $N_{\mathcal{PO}}$sets $\mathcal{W}_i$ are decreasing, hence $\mathcal{W}_1 \supset \mathcal{W}_2 \supset \cdots$ since the $N_{\mathcal{PC}}$ sets $\mathcal{F}_i$ are decreasing. To show that $\bigcap_{i=1}^{\infty} N\ Cl\ (\mathcal{W}_i) = \emptyset$, let $z \in X$. The goal is to find $\mathcal{W}_j$ such that $z \notin N\ Cl\ (\mathcal{W}_j)$. Once $\mathcal{W}_j$ is found, we will obtain a $N_{\mathcal{PO}}$set V such that $z \in V$ and V contains no points of $\mathcal{W}_j$.

Since $\mathcal{V}$ is nano locally finite, there exists a $N_{\mathcal{PO}}$ set V such that $z \in V$ and V meets only finitely many sets in $\mathcal{V}$. Suppose that these finitely many $N_{\mathcal{PO}}$ sets in $\mathcal{V}$ are $V_1, V_2, \dots, V_m$. Observe that for each $i = 1,2,\dots,m$, there is some $j(i)$ such that $V_i \cap \mathcal{F}_{j(i)} = \emptyset$ (Hence, $V_i \subset X - \mathcal{F}_{j(i)}$). This follows from the fact that $\mathcal{V}$ is a $N_{\mathcal{PO}}$refinement $\mathcal{M}$. Let j be the maximum of all $j(i)$ where $i = 1,2,\dots,m$. Then $V_i \cap \mathcal{F}_j = \emptyset$ for all $i = 1,2,\dots,m$. It follows that the $N_{\mathcal{PO}}$set V contains no points of $\mathcal{F}_j$. Thus $z \in N\ Cl\ (\mathcal{W}_j)$.

For the other direction, suppose that the nano topological space X satisfies the condition given in the theorem. Let $\mathfrak{U} = \{U_i : i = 1,2,\dots\}$ be a $N_{\mathcal{PO}}$ cover of X. For each i, define $\mathcal{W}_i$ as follows:

$$\mathcal{F}_i = X - U_1 \cup U_2 \cup \dots \cup U_i .$$

Then the $N_{\mathcal{PC}}$ sets $\mathcal{F}_i$ form a decreasing sequence of $N_{\mathcal{PC}}$ sets with empty intersection. Let $\mathcal{W}_i$ be decreasing $N_{\mathcal{PO}}$ such that $\bigcap_{i=1}^{\infty} N\ Cl\ (\mathcal{W}_i) = \emptyset$ and $\mathcal{F}_i \subset \mathcal{W}_i$ for each i. Let $\mathcal{C}_i = X - \mathcal{W}_i$ for each i. Then $\mathcal{C}_i \subset \bigcup_{j=1}^{i} U_j$. Define $V_1 = U_1$. For each $i \geq 2$, define $V_i = U_i - \bigcup_{j=1}^{i-1} \mathcal{C}_j$. Clearly each V_i is $N_{\mathcal{PO}}$and $V_i \subset U_i$. It is straightforward to verify that $\mathcal{V} = \{V_i : i = 1,2, \dots\}$is a cover of X.

We claim that $\mathcal{V}$ is nano locally finite in X. Let $z \in X$. Choose the least i such that $z \notin N\,Cl\,(\mathcal{W}_i)$. Choose a$N_{\mathcal{PO}}$ set δ such that $z \in \delta$ and $\delta \cap N\,Cl\,(\mathcal{W}_i) = \emptyset$. Then $\delta \cap \mathcal{W}_i = \emptyset$ and $\delta \subset \mathcal{C}_i$. This means that $\delta \cap V_k = \emptyset$ for all $k \geq i+1$. Thus the $N_{\mathcal{PO}}$ cover $\mathcal{V}$ is a nano locally finite $N_{\mathcal{PO}}$ refinement of $\mathfrak{U}$.

Corollary3.1.4

A $N_{\mathcal{PO}}$ −normal space X is countably $N_{\mathcal{PO}}$ −Paracompact if and only if for every decreasing sequence $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \dots$in $N_{\mathcal{PO}}(U,X)$with $\bigcap_{i=1}^{\infty} \mathcal{F}_i = \emptyset$, there exists a sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ in $N_{\mathcal{PO}}(U,X)$ such that $\mathcal{F}_i \subset \mathcal{W}_i$ for $i = 1, 2, \dots$ .... and $\bigcap_{i=1}^{\infty} \mathcal{W}_i = \emptyset$.

It is very clear and does not require proof.

Theorem3.1.5

For every $N_{\mathcal{PO}} - T_1$ space X the following be equivalent:

(i) X is $N_{\mathcal{PO}}$ −normal and countably $N_{\mathcal{PO}}$ −Paracompact.

(ii) For every countable open cover $\{\mathcal{A}_i\}_{i=1}^{\infty}$ of the nano space X there exists a nano locally finite $N_{\mathcal{PO}}$ cover $\{\beta_i\}_{i=1}^{\infty}$ of X such that $N\,Cl\,(\beta_i) \subset \mathcal{A}_i$ for $i = 1,2,\dots$

(iii) For every countable open cover $\{\mathcal{A}_i\}_{i=1}^{\infty}$ of the nano space X there exists a $N_{\mathcal{PC}}$ cover $\{\mathcal{F}_i\}_{i=1}^{\infty}$ such that $\mathcal{F}_i \subset \mathcal{A}_i$for $i = 1,2,\dots$

Proof:

To prove that (i) ⇒ (ii)

It suffices to observe that, by virtue of Theorem 3.1.3, there exists a nano locally finite $N_{\mathcal{PO}}$ cover $\{\mathcal{W}_i\}_{i=1}^{\infty}$ of X such that $\mathcal{W}_i \subset \mathcal{A}_i$; for $i = 1, 2, ...$

(ii) ⇒ (iii) is obvious; (iii)⇒ (i). Consider a $N_{\mathcal{PO}} - T_1$ space X which satisfies condition (iii).

Let us observe first that X is $N_{\mathcal{PO}}$ −normal; indeed, if for a pair $\mathcal{A}, \beta \in N_{\mathcal{PO}}(U, X)$ such that $\mathcal{A} \cup \beta = X$ we let $\mathcal{A}_1 = \mathcal{A}$, $\mathcal{A}_2 = \beta$, $\mathcal{A}_3 = \mathcal{A}_4 = \cdots = \emptyset$ then (iii) yields closed subsets $\mathcal{F}_1, \mathcal{F}_2, ...$in $N_{\mathcal{PC}}(U, X)$ such that $\mathcal{F}_1 \subset \mathcal{A}$, $\mathcal{F}_2 \subset \beta$ and $\mathcal{F}_1 \cup \mathcal{F}_2 = X$. Now, it follows from De Morgan's laws that, for every sequence $\mathcal{F}_1, \mathcal{F}_2, ...$in $N_{\mathcal{PC}}(U, X)$ satisfying $\bigcap_{i=1}^{\infty} \mathcal{F}_i = \emptyset$, there exists a sequence $\mathcal{W}_1, \mathcal{W}_2, ...$in $N_{\mathcal{PO}}(U, X)$ such that $\mathcal{F}_i \subset \mathcal{W}_i$; for $i = 1, 2, ...$ and $\bigcap_{i=1}^{\infty} \mathcal{W}_i = \emptyset$; hence, X is countably $N_{\mathcal{PO}}$ −Paracompact by Corollary 3.1.4.

<u>Theorem 3.1.6</u>

A NTS is $N_{\mathcal{PO}}$ − normal and C $N_{\mathcal{PO}}$ −Paracompactif and only if the Cartesian product $X \times I$ of X and the $N_{\mathcal{PO}}$ unit interval I is $N_{\mathcal{PO}}$ −normal.

Proof:

The above lemma implies that for every C $N_{\mathcal{PO}}$ −Paracompact− normal space X the Cartesian product $X \times I$ *is* $N_{\mathcal{PO}}$ − normal consider now a NTS X such that $X \times I$ is a $N_{\mathcal{PO}}$ − normal space. Since X is homeomorphic to the nano closed subspace $X \times \{0\}$ *of* $X \times I$, the nano space X *is* $N_{\mathcal{PO}}$ − normal. Now we show that X satisfies (iv) in Theorem

3.1.3. Let $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$in $N\mathcal{P}Cl\ (U, X)$ such that $\bigcap_{i=1}^{\infty} \mathcal{F}_i = \emptyset$. The $N_{\mathcal{PO}}$ sets.

$$\mathcal{A} = \bigcup_{i=1}^{\infty}(\mathcal{F}_i \times \{1/i\}) \text{ and } \beta = X \times \{0\}$$

are disjoint and$N_{\mathcal{PC}}$ in $X \times Y$. Therefore, there exists $\mathcal{H}, \mathcal{G}$ in $N_{\mathcal{PO}}\ (U, X) \subset X \times Y$ such that $\mathcal{A} \subset \mathcal{H}, \beta \subset \mathcal{G}$ and $\mathcal{H} \cap \mathcal{G} = \emptyset$. The sets $\mathcal{W}_i = \left\{x \in X\colon \left(x, \left\{\frac{1}{i}\right\}\right) \in \mathcal{H}\right\}$ are $N_{\mathcal{PO}}$ for $i = 1,2, \ldots$, and $\bigcap_{i=1}^{\infty} N\ Cl\ (\mathcal{W}_i) = \emptyset$ because $N\ Cl\ (\mathcal{H}) \cap \beta = \emptyset$. To conclude the proof, it suffices to note that $\mathcal{F}_i \subset \mathcal{W}_i$for $i = 1,2, \ldots .$

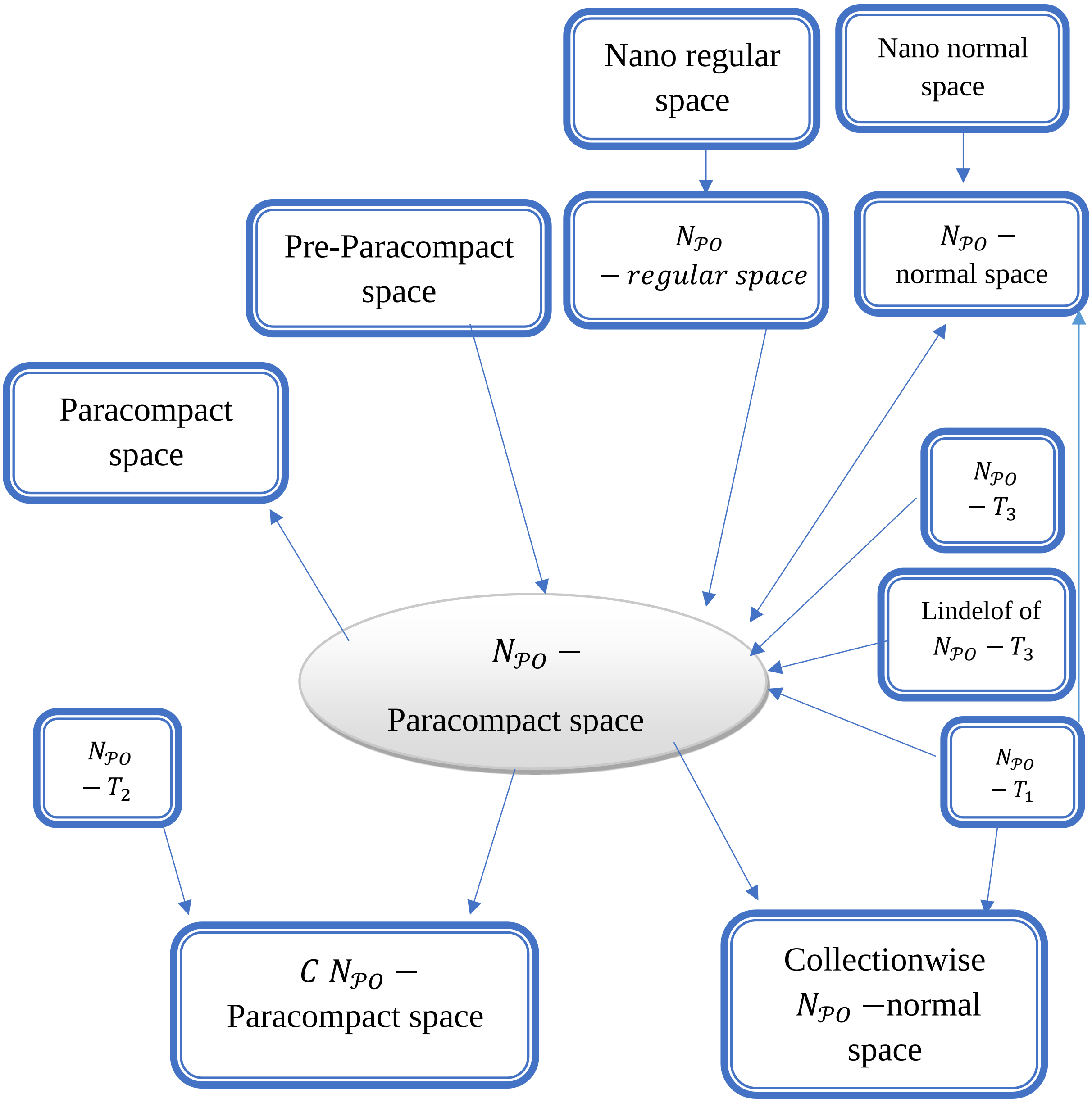


Figure 3.1 : The relation of $N_{\mathcal{PO}} -$ Paracompact space.

3.2 Weakly and Strongly $N_{\mathcal{PO}}-$ Paracompact spaces. ($W\ N_{\mathcal{PO}}-$ Paracompact space, $S\ N_{\mathcal{PO}}-$ Paracompact space)

Definition 3.2.1

A NTS is called **weakly $N_{\mathcal{PO}}-$ Paracompact spaces**($W\ N_{\mathcal{PO}}-$ Paracompactspace) if X is a $N_{\mathcal{PO}}$Hausdorff ($N_{\mathcal{PO}}-T_2$) and any open cover of X has a point- finite $N_{\mathcal{PO}}$ refinement.

Remark 3.2.2

1.Every $N_{\mathcal{PO}}-$ Paracompacts pace is ($W\ N_{\mathcal{PO}}-$ Paracompact spaces).

2. Every weakly $N_{\mathcal{PO}}-$Paracompact $N_{\mathcal{PO}}$ normal space is countable $N_{\mathcal{PO}}-$ Paracompact.

Theorem 3.2.3

Every $W\ N_{\mathcal{PO}}-$ Paracompact space collectionwise $N_{\mathcal{PO}}-$normal space is $N_{\mathcal{PO}}-$Paracompact.

Proof:

By theorem 2.3.4 (part 2), it enought to show that every point-finit nano open cover of a collectionwise $N_{\mathcal{PO}}-$normal space X has a $\sigma-$nano locally finite $N_{\mathcal{PO}}$refinement.

We shall define inductively for $i=0,1,2,...$a discrete family $\mu_i = \{M_\tau\}_{\tau\in T_i}$, of $N_{\mathcal{PO}}$subsets of X such that every member of μ_i is contained in a $\mathcal{V}_s$, and the $N_{\mathcal{PO}}$sets $\mathcal{W}_i = \cup\,\mu_i$satisfy the condition

if $|\{s\in S : x\in\mathcal{V}_s\}| \le i$, then $x\in\cup_{j=0}^{i}\mathcal{W}_i$ . ... (1)

Let $\mu_0=\{\emptyset\}$ μ_i satisfies (1) are already defined for $i\le\mathcal{K}$.

Denote by $T_{\mathcal{K}+1}$ the family of all $N_{\mathcal{PO}}$subsets of S that have exactly $\mathcal{K}+1$ elements, and for every $\tau \in T_{\mathcal{K}+1}$let

$\mathcal{A}_\tau = (X \setminus \bigcup_{j=0}^{\mathcal{K}} \mathcal{W}_j) \cap (X \setminus \bigcup_{s \notin \tau} \mathcal{V}_s)$. …(2)

Note that

$\mathcal{A}_\tau \subset \bigcap_{s\in\tau} \mathcal{V}_s$ for every $\tau \in T_{\mathcal{K}+1}$….. (3)

Indeed, if we had $x \notin \mathcal{V}_{s_0}$ for an $x \in \mathcal{A}_\tau$ and an $s_0 \in \tau$, then by virtue of (2) the point x would belong to at most $\mathcal{K}$ members of $\mathfrak{U}$ and this would contradict (1).

We shall now show that every point $x \in X$ has a nano pre-neighbourhood $U(x)$ which meets at most one member of $\{\mathcal{A}_\tau\}_{\tau \in T_{\mathcal{K}+1}}$. If x belongs to $\mathcal{K}+2$ members of $\mathfrak{U}$, say $\mathcal{V}_{s_1}, \mathcal{V}_{s_2}, \dots, \mathcal{V}_{s_{\mathcal{K}+2}}$, then letting $U(x) = \bigcap_{iq=1}^{\mathcal{K}+2} \mathcal{V}_s$, we have, by virtue of (2), $U(x) \cap \mathcal{A}_\tau = \emptyset$ for every $\tau \in T_{\mathcal{K}+1}$. If x belongs only to $i < \mathcal{K}$ members of $\mathfrak{U}$, then by virtue of (1) the set $U(x) = \bigcup_{j=0}^{\mathcal{K}} \mathcal{W}_j$ is a nano pre-neighbourhood of x disjoint from all sets $\mathcal{A}_\tau$. Finally, if x belongs to A exactly $\mathcal{K}+1$ members of $\mathfrak{U}$, say $\mathcal{V}_{s_1}, \mathcal{V}_{s_2}, \dots, \mathcal{V}_{s_{\mathcal{K}+2}}$, then the nano pre-neighbourhood $U(x) = \bigcap_{i=1}^{\mathcal{K}+1} \mathcal{V}_s$ of the point x intersect at most one member of $\{\mathcal{A}_\tau\}_{\tau \in T_{\mathcal{K}+1}}$, viz., the set $\mathcal{A}_{\tau_0}$, where $\tau_0 = \{s_1, s_2, \dots, s_{\mathcal{K}+1}\}$.

Hence $\{\mathcal{A}_\tau\}_{\tau \in T_{\mathcal{K}+1}}$is a discrete family of $N_{\mathcal{PC}}$subsets of X; let $\{\mathcal{G}_\tau\}_{\tau \in T_{\mathcal{K}+1}}$be a discrete family of $N_{\mathcal{PO}}$ subsets of X such that $\mathcal{A}_\tau \subset \mathcal{G}_\tau$ for every$\tau \in T_{\mathcal{K}+1}$. We shall show that the family $\mu_{\mathcal{K}+1} = \{M_\tau\}_{\tau \in T_{\mathcal{K}+1}}$,where

$M_\tau = \mathcal{G}_\tau \cap \bigcap_{s\in\tau} \mathcal{V}_s$ ……. (4)

has all the required properties.

Clearly the family $\mu_{\mathcal{K}+1}$is discrete, consists of $N_{\mathcal{PO}}$ subsets of X and every member of $\mu_{\mathcal{K}+1}$ is contained in a $\mathcal{V}_s$,. Consider a point X which belongs to at most $\mathcal{K}+1$ members of $\mathfrak{U}$; there exists then a $\tau \in T_{\mathcal{K}+1}$such that $x \in X \setminus \bigcup_{s\notin\tau} \mathcal{V}_s$, and by (2) we have

$$x \in X \setminus \bigcup_{s\notin\tau} \mathcal{V}_s = [(X \setminus \bigcup_{j=0}^{\mathcal{K}} \mathcal{W}_j) \cup \bigcup_{j=0}^{\mathcal{K}} \mathcal{W}_j] \cap (X \setminus \bigcup_{s\notin\tau} \mathcal{V}_s)$$

$$\subset \mathcal{A}_\tau \cup \bigcup_{j=0}^{\mathcal{K}} \mathcal{W}_j$$

The last formula, along with $(3), (4)$ and the inclusion $\mathcal{A}_\tau \subset \mathcal{G}_\tau$, implies that $x \in \bigcup_{j=0}^{\mathcal{K}+1} \mathcal{W}_j$.

Since the cover $\mathfrak{U}$ is point-finite, it follows from (1) that $\bigcup_{i=1}^{\infty} U_i$ is a σ –nano locally finite $N_{\mathcal{PO}}$refinement of $\mathfrak{U}$.

Lemma 3.2.4

For every open cover $\{\mathcal{A}_s\}_{s\in S}$of W $N_{\mathcal{PO}}$ – Paracompact space X point-finite $N_{\mathcal{PO}}$ cover $\{\beta_s\}_{s\in S} of$ X such that $\beta_s \subset \mathcal{A}_s$ for $s \in S$.

Proof:

Suffices to take point finite $N_{\mathcal{PO}}$ refinement $\mathcal{M}$ of the cover $\{\mathcal{A}_s\}_{s\in S}$ for every $\beta \in \mathcal{M}$ choose a natural number $s\ (\beta)$ such that $\beta \subset \mathcal{A}_{s(\beta)}$, and let $\beta_s = \cup \{\beta : s(\beta) = s\}$.

Definition 3.2.5

A NTS is called a **strong $N_{\mathcal{P}O}$ −Parecompact** ($S\ N_{\mathcal{P}O}$ −Parecompact) if it is $N_{\mathcal{P}O} - T_2$ (or $N_{\mathcal{P}O}$ −Hausdroff) space and each open cover $\mathcal{V}$ of X has a star- finite $N_{\mathcal{P}O}$refinement.

Remark 3.2.6

Every $S\ N_{\mathcal{P}O}$ −Paracompact space is $N_{\mathcal{P}O}$ −Paracompact.

Theorem 3.2.7

For every $N_{\mathcal{P}O}$ − regular space the following conditions are equivalent:

(i) The nano space X is $S\ N_{\mathcal{P}O}$ −Parecompact.

(ii) Every open cover of the nano space X has a $N_{\mathcal{P}C}$ refineme which is both a nano locally finite and star- finite.

Proof:

We shall show that (i) ⇒ (ii). Let $\mathfrak{U}$ be an open cover of the nano space X and $\mathcal{V} = \{V_s\}_{s\in S}$ a star- finite $N_{\mathcal{P}O}$ refinement of $\mathfrak{U}$ .Since the nano space X is $N_{\mathcal{P}O}$ − normal and the cover $\mathcal{V}$ is a nano locally finite, by virtue of Theorem 2.2.2, there exists a $N_{\mathcal{P}C}$ cover $\mathcal{F} = \{F_s\}_{s\in S}$ of the nano space X such that $F_s \subset V_s$ for every $s \in S$.

Clearly, $\mathcal{F}$ $N_{\mathcal{P}O}$ refines $\mathfrak{U}$ and is both a nano locally finite and a $s \in S$ star- finite.

3.3 Point $N_{\mathcal{PO}}$ −regular ($PN_{\mathcal{PO}}$ −regular)

Definition 3.3.1

We say that a nano base β for a NTS X is a **point $N_{\mathcal{PO}}$ −regular** ($PN_{\mathcal{PO}}$ −regular) if for every point $x \in X$ and any nano pre-neighbourhood $\mathfrak{U}$ of x the $N_{\mathcal{PO}}$ set of all members of β that contain x and meet $X \backslash \mathfrak{U}$ is finite.

Lemma 3.3.2

If ρ is a $PN_{\mathcal{PO}}$ −regular nano base for a nano space X, then the family $\rho^m \subset \rho$ is a point-finite (a nano locally finite) cover of X.

Proof:

First, we show $\cup \rho^m = X$. For each $x \in X$ there exists a $\mathfrak{U}_0 \in \rho$ that contains x; assuming that x is not contained in any member of ρ^m, we can define an infinite sequence $\mathfrak{U}_0 \subset \mathfrak{U}_1 \subset \mathfrak{U}_2 \subset ..$. of members of $\rho, \ni \mathfrak{U}_i \neq \mathfrak{U}_{i+1}$ for $i = 1, 2, \ldots$, hence, an infinite subfamily $\{\mathfrak{U}_i\}_{i=1}^{\infty}$, of ρ whose members contain x and meet $X \setminus \mathfrak{U}_0$, which is impossible. Hence, $x \in \cup \rho^m$ and $\cup \rho^m = X$.

Now, we shall show that if the nano base ρ is $PN_{\mathcal{PO}}$ −regular, then the cover $\rho^m$is a point-finite (nano locally finite). Take a point $x \in X$ and a set $\mathfrak{U} \in \rho^m$ which contains$x$. The $N_{\mathcal{PO}}$ set of all members of ρ that meet both $X \backslash \mathfrak{U}$ and $\{x\}$ (and a nano pre-neighbourhood $\mathcal{V}$ of x) is finite. However, every $\mathfrak{U}' \in \rho^m \backslash \{\mathfrak{U}\}$ that contains x (meets $\mathcal{V}$) also meets $X \backslash \mathfrak{U}$, so that only finitely many members of ρ^m contain x (meet $\mathcal{V}$), which shows that $\rho^m$is a point-finite (a nano locally finite) cover of X.

Lemma3.3.3

If ρ is a nano base for a $N_{\mathcal{P}O} - T_1$ space, then for every point-finite cover $\rho' \subset \rho$ the family $\rho'' = (\rho \setminus \rho') \cup I(X)$ is a nano base for X.Morcover, if the nano base $\rho$is $PN_{\mathcal{P}O}$ −regular,then the nano base ρ''also is a $PN_{\mathcal{P}O}$ −regular.

Proof:

Let $z \in X, \mathfrak{U}$ a nano pre-neighbourhood of z. If z is an isolated point, then $\{z\} \in I(X)$ and $z \in \{z\} \subset \mathfrak{U}$; if the point z is not isolated, then the intersection $\cap \{\mathcal{W} \in \rho' : z \in \mathcal{W}\}$ contains a point $k \neq z$, and any nano pre-neighbourhood $\mathcal{V} \in \rho$ of the point z satisfying $\mathcal{V} \subset \mathfrak{U} \setminus \{k\}$ belongs to $\rho \setminus \rho'$. The second part of the lemma is obvious.

Theorem3.3.4

Every $N_{\mathcal{P}O} - T_2$space of X has a $PN_{\mathcal{P}O}$ −regular nano base.

Proof:

Let $\mathfrak{U}$ be an open cover of a $N_{\mathcal{P}O} - T_2$ spac X which has a $PN_{\mathcal{P}O}$ −regular nano base ρ. The family ρ_0 of all members of ρ that are contained in a member of $\mathfrak{U}$ clearly form a $PN_{\mathcal{P}O}$ −regular nano base for the nano space X, so that by Lemma 3.3.2, the family ρ_0^m is a point-finite $N_{\mathcal{P}O}$ refinement of $\mathfrak{U}$. Hence X is a $WN_{\mathcal{P}O}$ − Paracompact space.

3.4 Perfectly $N_{\mathcal{PO}}$ −normal space

Definition3.4.1

A NTS $\left(U, \tau_R(X)\right)$ is said to be a $\boldsymbol{N_{\mathcal{PO}}} - \boldsymbol{T_4}$ **space** if and only if U is $N_{\mathcal{PO}}$ −normal space and $N_{\mathcal{PO}} - T_1$ space.

Example 3.4.2

Let $U = \{\alpha, \beta, \delta\}$ and $U \backslash R = \{\{\alpha\}, \{\beta\}, \{\beta, \delta\}\}$, $X = \{\alpha, \delta\}$

$\tau_R(X) = \{\emptyset, U, \{\alpha\}, \{\beta, \delta\}\}$

The nano closed sets $F = \{U, \emptyset, \{\beta, \delta\}, \{\alpha\}\}$.

$N_{\mathcal{PO}}(U, X) = \left\{\emptyset, U, \{\alpha\}, \{\beta\}, \{\delta\}, \{\alpha, \beta\}, \{\alpha, \delta\}, \{\beta, \delta\}\right\}$.

$N_{\mathcal{PC}}(U, X) = \left\{U, \emptyset, \{\beta, \delta\}, \{\alpha, \delta\}, \{\alpha, \beta\}, \{\delta\}, \{\beta\}, \{\alpha\}\right\}$.

$\{\alpha\} \subseteq \{\alpha\}$ and $\{\beta, \delta\} \subseteq \{\beta, \delta\}$, $\{\alpha\} \cap \{\beta, \delta\} = \emptyset$.

$\{\beta\} \subseteq \{\beta\}$ and $\{\alpha\} \subseteq \{\alpha\}$, $\{\beta\} \cap \{\alpha\} = \emptyset$.

$\{\delta\} \subseteq \{\delta\}$ and $\{\beta\} \subseteq \{\beta\}$, $\{\delta\} \cap \{\beta\} = \emptyset$.

$\therefore$ $(U, \tau_R(X))$ is a $N_{\mathcal{PO}}$ − normal space .

$\alpha \neq \beta \Rightarrow \alpha \in \{\alpha\} \land \beta \notin \{\alpha\}$ and $\alpha \notin \{\beta\} \land \beta \in \{\beta\}$.

$\alpha \neq \delta \Rightarrow \alpha \in \{\alpha\} \land \delta \notin \{\alpha\}$ and $\alpha \notin \{\delta\} \land \delta \in \{\delta\}$.

$\beta \neq \delta \Rightarrow \beta \in \{\beta\} \land \delta \notin \{\beta\}$ and $\beta \notin \{\delta\} \land \delta \in \{\delta\}$.

$\therefore$ $(U, \tau_R(X))$ is a $N_{\mathcal{PO}} - T_1$ space.

$\therefore$ $(U, \tau_R(X))$ is a $N_{\mathcal{PO}} - T_4$space.

Definition 3.4.3

Let $(U, \tau_R(X))$ be a NTS is a **perfectly $N_{\mathcal{PO}}$ −normal space** if for each $N_{\mathcal{PO}}$ subset C, there is a continuous nonnegative real valued function on X for which C is the set of its zeros.

Lemma 3.4.4

Every $N_{\mathcal{PO}}$ −regular space which has a σ −nano locally finite base is perfectly $N_{\mathcal{PO}}$ −normal.

Proof:

Let $\beta = \bigcup_{i=1}^{\infty} \beta_i$, where the families β_i are nano locally finite, be a nano base for a $N_{\mathcal{PO}}$ −regular space X.Consider an arbitrary $N_{\mathcal{PO}}$ set $\mathcal{W} \subset X$. For every $x \in \mathcal{W}$ there exists a natural number $i(x)$ and a $N_{\mathcal{PO}}$ set $U(x) \in \beta_i(x)$ such that $x \in U(x) \subset N\,Cl\,(U(x)) \subset \mathcal{W}$.

Letting $\mathcal{W}_i = \cup \{U(x): i(x) = i\}$ we obtain a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of $N_{\mathcal{PO}}$ subsets of X such that $\mathcal{W} = \bigcup_{i=1}^{\infty} \mathcal{W}_i$ and − by Theorem 2.2.2 $Cl(\mathcal{W}_i) \subset \mathcal{W}$ for $i = 1, 2, \ldots$ Normality of X now follows i from Lemma 2.2.17; since $\mathcal{W} = \bigcup_{i=1}^{\infty} \mathcal{W}_i$ the nano space X is perfectly $N_{\mathcal{PO}}$ −normal.

Corollary 3.4.5

Every perfectly $N_{\mathcal{PO}}$ −normal space X is $C\,N_{\mathcal{PO}}$ −Paracompact.

Proof:

Let $\{\mathfrak{U}_i |\, i = 1,2, \ldots\}$ be an increasing $N_{\mathcal{PO}}$ covering of X. Since X is perfectly $N_{\mathcal{PO}}$ −normal, for each i there are $N_{\mathcal{PC}}$ sets $\mathcal{F}_{i\mathcal{C}}, \mathcal{C} = 1, 2, \ldots$, $\ni$

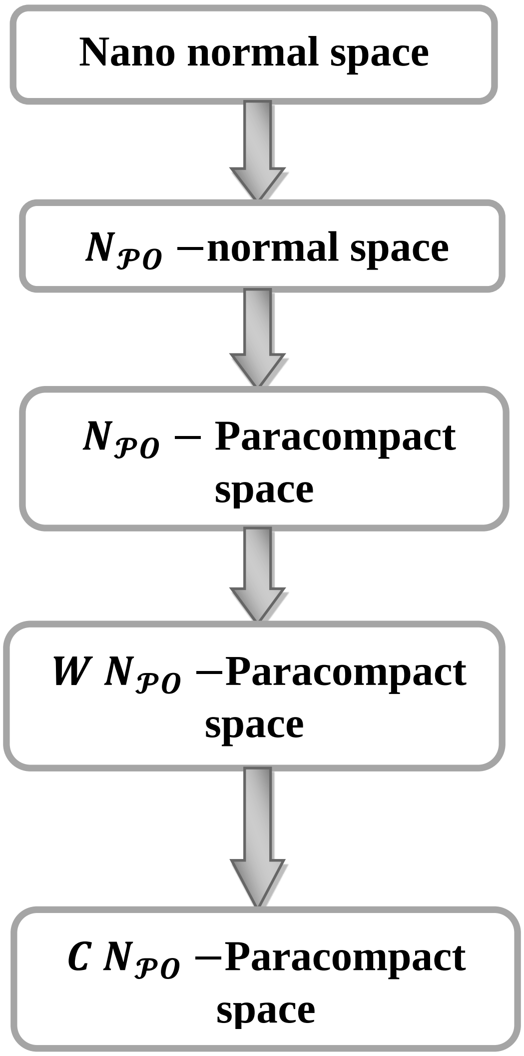
$\mathfrak{U}_i = \bigcup_{s=1}^{\infty} \mathcal{F}_{i\mathcal{C}}$ and $\mathcal{F}_{i\mathcal{C}} \subset \mathcal{F}_{i\mathcal{C}+1}$.

Put

$$\mathcal{G}_i = \bigcup_{j=1}^{i} \mathcal{F}_{ji} .$$

Then $\mathcal{G}_i \subset \mathfrak{U}_i$ and $\{ \mathcal{G}_i | \ i = 1,2, \dots \}$ is a $N_{\mathcal{PC}}$ covering of X. Hence by (iv) of the theorem, X is $C\ N_{\mathcal{PO}}$ −Paracompact.

Without assuming normality, we can characterize countably $N_{\mathcal{PO}}$ –Paracompact spaces as follows.



Figures 3.2 : The relation of $C\ N_{\mathcal{PO}}$ −Paracompact space.

4.1 Conclusions :

In the present thesis, we constructed and defined a new class of Nano topological spaces and we proved that this class possesses the property of Paracompact, which is a very important property as a property of nano topological spaces using Pre-open sets.

This work has many new result that can be summarized in the following facts :

1. Every $N_{\mathcal{PO}} -$ Paracompact space is Paracompact.
2. Every closed nano subspace of a $N_{\mathcal{PO}}$ −Paracompact space is a nano Paracompact.
3. Every $N_{\mathcal{PO}} -$ Paracompact space is $N_{\mathcal{PO}}$ −normal.
4. Let $(\mathrm{U}, \tau_R(\mathrm{X}))$be a NTS.If $N_{\mathcal{PO}}$ $(U, X)$is $\mathcal{PO}$ −Paracompact space, then $(U, \tau_R(X))$is a $N_{\mathcal{PO}}$ −Paracompact space.
5. Every open σ −nano locally finite cover $\mathscr{V}$ of a NTS X has a nano locally finite $N_{\mathcal{PO}}$refinement cover $\mathscr{M}$.
6. For every$N_{\mathcal{PO}}$ −regular space$X$ the following conditions are equivalent:

   (1) The nano space X is $N_{\mathcal{PO}}$ −Paracompactspace.

   (2) Every open cover of the nano space X has an open σ −nano locally finite$N_{\mathcal{PO}}$refinement.

   (3) Every open cover of the nano space X has a nano locally finite $N_{\mathcal{PO}}$refinement.

   (4) Every open cover of the nano space X has a closed nano locally finite $N_{\mathcal{PO}}$refinement.
7. If X is a $N_{\mathcal{PO}} - T_3$, the following are equivalent

   (1) X is a $N_{\mathcal{PO}}$ −Paracompact.

(2) Each open cover of X has an open σ −nano locally finite$N_{\mathcal{P}O}$ refinement.

(3) Each open cover has a nano locally finite$N_{\mathcal{P}O}$ refinement (not necessarily open).

(4) Each open cover of X has a closed nano locally finite $N_{\mathcal{P}O}$ refinement.

8. A barycentric $N_{\mathcal{P}O}$ refinement z of a barycentric $N_{\mathcal{P}O}$ refinement ν of μ.
9. For every $N_{\mathcal{P}O} - T_1\ space\ X$ the following conditions are equivalent:

   (1) The nano space X is $N_{\mathcal{P}O}$ −Paracompact space.

   (2) Every open cover $\mathcal{V}$ of the nano space X has a barycentric $N_{\mathcal{P}O}$refinement.

   (3) Every open cover $\mathcal{V}$ of the nano space X has a $N_{\mathcal{P}O}$star refinement.

   (4) The nano space X is a$N_{\mathcal{P}O}$ −regular space and every open cover $\mathcal{V}$ of X has a σ −nano discrete $N_{\mathcal{P}O}$ refinement.
10. If an open cover U of a $NTSX$ has a closed nano locally finite $N_{\mathcal{P}O}$refinement, then U has also an open barycentric $N_{\mathcal{P}O}$ refinement.
11. Every $N_{\mathcal{P}O}$ −Paracompact space is a collectionwise $N_{\mathcal{P}O}$ −normal.
12. Every $W\ N_{\mathcal{P}O}$ − Paracompact space collectionwise $N_{\mathcal{P}O}$ −normal space is $N_{\mathcal{P}O}$ −Paracompact.
13. Every $S\ N_{\mathcal{P}O}$ −Parecompactspace is $N_{\mathcal{P}O}$ −Parecompact.
14. Every $N_{\mathcal{P}O}$ −regular space which has a σ −nano locally finite base is perfectly $N_{\mathcal{P}O}$ −normal.
15. Every perfectly $N_{\mathcal{P}O}$ −normal space X is $C\ N_{\mathcal{P}O}$ −Paracompact.

4.2 Future works:

1. One can define new classes of paracompactness in nano topological spaces for example,

The class of countable $N_{\mathcal{P}O}$ –Paracompact spaces and $N_{\mathcal{P}O}$ –normal spaces using the following facts>

(i) Every countable open cover of the nano space X has a nano locally finite closed $N_{\mathcal{P}O}$ refinement**.**

(ii)Every countable open cover of the nano space X has aσ −nano locally finite closed $N_{\mathcal{P}O}$ refinement.

(iii) Every countable open cover of the nano space X has a σ −nano discrete closed $N_{\mathcal{P}O}$ refinement.

(x) Every countable open cover of the nano space X has a countable $N_{\mathcal{P}O}$ star $N_{\mathcal{P}O}$ refinement.

(ix) Every countable open cover of the nano space X has a $N_{\mathcal{P}O}$ star $N_{\mathcal{P}O}$ refinement.

2. One can study the Cartesian product $U_1 \times U_2$ of a $N_{\mathcal{P}O}$ −Paracompactspace$U_1$and U_2 to prove that the following conditions are equivalent:

(1) The Cartesian product $U_1 \times U_2$ is $N_{\mathcal{P}O}$ −Paracompact.

(2) The Cartesian product $U_1 \times U_2$ is collectionwise $N_{\mathcal{P}O}$ −normal.

(3) The Cartesian product $U_1 \times U_2$ is $N_{\mathcal{P}O}$ −normal.

(4) The Cartesian product $U_1 \times U_2$ is $C\ N_{\mathcal{P}O}$ −Paracompact.

3. In the end, this work is merely the start of many structures, and we have only looked at a few concepts. Further theoretical study in the practical area will therefore need to be done as a crucial first step happing to get new results caapplicable in medicine and life in the future.

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الملخص

في هذا العمل تم تعريف فئة جديدة من الفضاءات التبولوجية النانوية وأثبتنا ان هذه الفئة تمتلك خاصية Paracompact وهي خاصية مهمة جداً كخاصية التبولوجية النانوية وهذا يقودنا الى دراسة فئات وصنوف جديدة اكثر عموماً من الفضاءات التبولوجية النانوية.

تضمن هذا العمل التعريفات الأساسية في الفضاءات التبولوجية والنانو التبولوجية ومجموعة Pre-open تم تعريف فئة جديدة من Paracompact في الفضاءات التبولوجية النانوية باستخدام تعريف مجموعات Pre-open حيث تم بناء فضاء تبولوجي نانوي باستخدام هذه المجموعة وتم دراسة خاصية paracompact لهذه الفضاءات.

تم تطوير الفضاءات المنتظمة او العادية او الطبيعية Nano Pre-regular, Nano Pre-normal وتم ذكر العديد من النظريات الجديدة واثباتها. وتم تعريف فئة جديدة من الفضاءات التبولوجية النانوية باستخدام تعريف مجموعة Pre-open حيث تم بناء فضاء تبولوجي نانوي باستخدام هذه المجموعة وتم دراسة خاصية Star لهذه الفضاءات. وتم تعريف فئة جديدة من الفضاءات التبولوجية النانوية باستخدام مجموعة Pre-open وتم دراسة خاصية Collectionwise لهذه الفضاءات.

وتم تعريف فئة جديدة من الفضاءات التبولوجية النانوية باستخدام مجموعة Pre-open مثل فضاء Paracompact القابل للعد والفضاء Paracompact القوي والضعيف وتم تعريف نوع اخر من الفضاءات التبولوجية النانوية النقطية في الفضاء التبولوجي النانوي باستخدام مجموعة Pre-open للفضاء المنتظم او العادي. وتم تعريف نوع اخر من الفضاءات التبولوجية النانوية التامة للفضاءات التبولوجية النانوية باستخدام مجموعة Pre-open للفضاء الطبيعي.

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كجزء من متطلبات نيل درجة الماجستير في
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من قبل
غفران علي حسين عبد الرزاق

بإشراف
أ.د. زاهر دبيس عزاوي النافعي

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