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Department of Mathematics

On Quorum Queues with Vacations

A Thesis

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by

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بسم الله الرحمن الرحيم

﴿ فَتَعَالَى اللَّهُ الْمَلِكُ الْحَقُّ ۗ وَلَا تَعْجَلْ بِالْقُرْآنِ مِن قَبْلِ أَن يُقْضَىٰ إِلَيْكَ وَحْيُهُ ۖ وَقُل رَّبِّ زِدْنِي عِلْمًا ﴾

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Dedication

I dedicate my thesis to

my family and many friends.

Special gratitude to

my loving parents, whose words of encouragement

and push for tenacity ring in my ears.

My sisters and brothers have never left

my side and are incredibly special.

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List of Symbols

Symbols	**Description**
$\mathcal{F}$	$\boldsymbol{\sigma}$-algebra
Ω	Sample space
$\mathbb{P}$	Probability measure
$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space
$\mathfrak{B}(\mathbb{R})$	The Borel $\boldsymbol{\sigma}$-algebra on $\mathbb{R}$
$\mathfrak{B}(\mathbb{R}^n)$	The Borel $\boldsymbol{\sigma}$-algebra on $\mathbb{R}^{\boldsymbol{n}}$
$\mathcal{X}$	Random variable
$(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$	Measurable space on $\mathbb{R}$
$(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$	Measurable space on $\mathbb{R}^{\boldsymbol{n}}$
$f: (\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n)) \to (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$	Borel function or a measurable function
$\mathbb{P}_{\mathcal{X}}(\mathcal{B})$	Probability measure on $\boldsymbol{\mathfrak{B}}(\mathbb{R})$
$F_{\mathcal{X}}(x)$	The distribution function of the random variable $\boldsymbol{\mathcal{X}}$
$f_{\mathcal{X}}(x)$	The density of the random variable
$\mathbb{E}\mathcal{Y}$	The expectation of a random variable $\boldsymbol{\mathcal{Y}}$
$\Upsilon(t)$	Characteristic function
$\vec{\mathcal{X}}$	Random vector
$F_{\vec{\mathcal{X}}}(\vec{\mathcal{X}})$	Finite-dimensional distribution function
$f_{\vec{\mathcal{X}}}(\vec{y})$	Joint density of random vector $\vec{\boldsymbol{\mathcal{X}}}$
$\mathbb{P}(\mathcal{A}/\mathcal{B})$	Conditional probability

$\mathbb{E}\{\mathcal{X}/\mathcal{B}\}$	Conditional expectation
$\mathfrak{X}$	A set
$\mathcal{P}_{t_1,t_2,\ldots,t_n}(\mathcal{B}_1 \times \mathcal{B}_2 \times \ldots \times \mathcal{B}_n)$	Finite-dimensional distributions
$\mathfrak{T}^j$	The first passage time random variable
$\mathfrak{N}^j$	The "number of visits" random variable
$\mathfrak{F}_{ij}$	The probability for first passage
$\mathfrak{R}_{ij}$	The expected number of visits
$\boldsymbol{P}$	The Markov matrix of a Markov chain
F_t	Sub $\boldsymbol{\sigma}$-algebra of $\boldsymbol{F}(\boldsymbol{\Omega})$
$(\Omega, F(\Omega), P, F_t)$	Filtration probability space
T	Stopping time
C_t	Counting point process
M_t	The counting marked point process
(X, T)	Marked Poisson process and marked Poisson counting process with independent marking
$\beta(\theta)$	Laplace-Stieltjes transform
$\mathcal{M}(\theta)$	Moment generating function
$a(z)$	The probability generating function
Q_n	The size number of queue
ρ	Offered load
α	The index of first passage time
T_α	The first passage time
M_α	The first excess level
Δ_n	The inter-renewal times
$\beta_0(z, \theta)$	Joint transform $\boldsymbol{E}\boldsymbol{z}^{X_0}\boldsymbol{e}^{-\Delta_0\boldsymbol{\theta}}$

$\beta_1(z,\theta)$	Joint transform $\boldsymbol{Ez^{X_1}e^{-\Delta_1\theta}}$
ω_α	Joint transform $\boldsymbol{Ez_1^{\alpha}z_2^{M_{\alpha-1}}z_3^{M_\alpha}e^{-\vartheta T_{\alpha-1}-\theta T_\alpha}}$
$\omega_{\alpha(n)}(x)$	Joint transform $\boldsymbol{Ez_1^{\alpha(n)}z_2^{M_{\alpha(n)-1}}z_3^{M_{\alpha(n)}}e^{-\vartheta T_{\alpha(n)-1}-\theta T_{\alpha(n)}}}$
D_n	$\boldsymbol{D_n}$ operater
D_y^n	The inverse operator of $\boldsymbol{D^n}$
η_0	Joint transform $\boldsymbol{\beta_0}(\boldsymbol{xz_2z_3},\boldsymbol{\vartheta}+\boldsymbol{\theta})$
η	Joint transform $\boldsymbol{\beta_1}(\boldsymbol{xz_2z_3},\boldsymbol{\vartheta}+\boldsymbol{\theta})$
χ_0	Joint transform $\boldsymbol{\beta_0}(\boldsymbol{xz_3},\boldsymbol{\theta})$
χ	Joint transform $\boldsymbol{\beta_1}(\boldsymbol{xz_3},\boldsymbol{\theta})$
χ_0^1	Joint transform $\boldsymbol{\beta_0}(\boldsymbol{z_3},\boldsymbol{\theta})$
χ^1	Joint transform $\boldsymbol{\beta_1}(\boldsymbol{z_3},\boldsymbol{\theta})$
$\omega^*(x)$	Operator $\boldsymbol{D_n}$**of** $\boldsymbol{\omega_{\alpha(n)}}(\boldsymbol{x})$
ϖ_α	Joint transform $\boldsymbol{Ez_3^{M_\alpha}e^{-\theta T_\alpha}}$
$\psi(z)$	Probability generating function $\boldsymbol{Ez^{M_\alpha}}$
ψ	The expectation of $\boldsymbol{M_\alpha}$
$\psi_0^i(z)$	The expectation $\boldsymbol{Ez^{(M_\alpha-R)^+}}$

List of Abbreviations

Abbreviations	**Description**
iid	Independent and identically distributed
$\mathbf{r.v.}$	Random variable
LST	Laplace-Stieltjes Transform
pgf	Probability generating function
TPM	Transition Probability Matrix
pdf	Probability density function
mgf	Moment generating function
$\boldsymbol{a.s.}$	Almost sure

Abstract

This thesis examines quorum queueing systems with N-policy and multiple vacations by considering different kinds of arrival times and vacations. The random walk processes in these queues are analyzed using the first excess level theory. In the first understudied model, the server is idle when the queue size is less than r; otherwise, it operates R units or less. In the second model, the sever is busy doing some secondary tasks when the queue has less than r things, however, this sever returns to the system to deal with R things or less. The probability generating function of queue size derives precisely under various distributions of arrival and vacation times, such as the ordinary Poisson process and type 1 geometric distribution. The future works are given to extend the distribution of arrival times and vacations to more general distribution.

Introduction

Lots of judgments must be made within the environment of randomness. Random failures of equipment, fluctuating industrial rates, and new requirements are all components of everyday decision-making processes [1]. Several requests in the industry, such as systems of digital communication, systems of computer networks, and systems of inventory control, involve the server being idle when the system has things less than level r and returning when the stuffs in it achieves a certain tolerance. Perhaps, in a production system, the manufacturer does not start while waiting for specific basic r of units collected in the system throughout the idle period [2].

The innovative study of the bulk arrival queueing system with N-policy is created by Lee and Srinivasan [3]. Besides other studies, they accomplished a scheme to get the optimal stationary functioning strategy under an appropriate linear cost structure. Next, Lee et al. [4] deemed this kind broadly during numerous methods. In regard, some ideas of this structure are examined by Lee et al. [5], Teghem [6], Medhi [7], Kalita and Choudhury [8], Choudhury and Baruah [9], Ali and Al-Obaidi [10], and Kazem and Al-Obaidi [11].

Bailey [12] initially deliberated queueing systems with batch service. This model is developed by Chaudhry and Templeton, where the server suspensions until the number of arrival units achieves a fixed level r, and they gave the name *quorum* for this model [13]. Many investigations are performed under bulk queueing systems. For example, Dshalalow and Tadj [14] analyzed the queues with fixed accumulation levels. Moreover, Abolnikov and Dshalalow [15] examined the service with delayed queueing systems. Furthermore, Dshalalow [16] studied queueing systems under q-policy.

Due to the interdisciplinary environment of the queue system, several analyses are devoted to bulk arrival queues under numerous vacation rules. Many academics, involving Baba [17], Choudhury [18-20], Lee et al. [21,5], Rosenberg and Yechiali [22], Madan and Abu-Dayyeh [23] and Teghem [24], and others, investigated bulk arrival queue under various vacation plans.

This thesis presents the analytic results for the probability generating function (in short, pgf) of the number of units in the quorum system under the N-policy and vacation disciplines. This queue is subjected to the random walk analysis by the first excess level theory. Some cases discuss corresponding with different distributions of arrivals and vacations to find the pgf of queue size, such as the ordinary Poisson process and type 1 geometric distribution. This thesis aims to provide explicit formulas of pgf of this system under several kinds of batch arrivals.

In this thesis, we include four chapters. The first one contains essential notions in stochastic systems, such as probability concepts, conditional expectation, stochastic processes, and the ergodicity of Markov chains. The second chapter covers models of quorum queues such that we begin defining lines with bulk inputs, random walk analysis, lines with N-policy, queues under multiple vacations, and quorum queues under N-policy and multiple vacations. The third chapter is about the applications on quorum queues under the N-policy, where we study three cases considering different distributions of arrival times to obtain the probability generating function of the number of units in this queue. Moreover, we apply various distributions of vacation times and arrival times on quorum queues under N-policy and multiple vacations to find the pgf of the size of this queue. The last chapter discusses the conclusion and future work.

Publication

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Chapter One

Elementary in Stochastic Systems

Chapter One

Elementary in Stochastic Systems

This chapter includes vital concepts and theories concerning stochastic systems. The stochastic systems show a significant character in natural practices, specifically disorder and oscillations. It represents the mathematical building of the natural system whose growth is operated by probability principles. Stochastic activities provide valuable forms for investigating such issues as statistical physics, communication and control, time series analysis, population growth, and management sciences [7-11]. This chapter reviews fundamental concepts from the book [1,16, 25].

1.1. Notions of Probability Concepts

We take back at this point the conventional explanation of probability. Assume that a set Ω is the collection of actual results or sample elements of a random experiment.

Definition 1.1.1.

Let $\mathcal{F}$ be a family of subsets of the set Ω. We called it a $\boldsymbol{\sigma}$**-algebra** when it satisfies the subsequent disputes:

(i) $\Omega \in \mathcal{F}$;

(ii)For each subset A $\in \mathcal{F}$, then its complement $A^c = \Omega \backslash A \in \mathcal{F}$.

(iii) For each subset $A_i \in \mathcal{F}$, $i = 1, 2, \ldots$, the following union $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The above second dispute shows that the null set $\emptyset$ (the complement of Ω) belongs to the collection $\mathcal{F}$. The members of the collection $\mathcal{F}$ are known as **events**.

Remark 1.1.2.

The above description shows that collection $\mathcal{F}$ is closed concerning a countable number of operations on events.

Definition 1.1.3.

A function $\mathbb{P}: \mathcal{F} \to [0, 1]$ from σ-algebra $\mathcal{F}$ to the closed interval $[0, 1]$ is called a **probability measure** if it meets the next requirements:

(i) $\mathbb{P}(\Omega) = 1$;

(ii)for every countable sequence of events $\{S_n\}_{n=1}^{\infty}$with mutully disjoint condition

$(S_i \cap S_j = \emptyset \; if \; i \neq j)$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(S_i)$$

We denote the **probability space** by the triplet $(\Omega, \mathcal{F}, \mathbb{P})$. This space is all the time understood to be granted primarily.

Definition 1.1.4.

The smallest σ-algebra that includes all half-open intervals $[x, y)$ is denoted by $\mathfrak{B}(\mathbb{R})$ and known as **Borel σ-algebra** on $\mathbb{R}$. Likewise, The smallest σ-algebra that includes all sets $[x_1, y_1) \times [x_2, y_2) \times \cdots \times [x_n, y_n)$ is indicated by $\mathfrak{B}(\mathbb{R}^n)$ and known as **Borel σ-algebra** on $\mathbb{R}^n$. The pairs $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ and $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$ are called measurable spaces.

Definition 1.1.5.

The function $\mathcal{Y}$ from measurable space $(\Omega, \mathcal{F})$ to measurable space $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ is called a measurable function, and it is named a **random variable** (in short, r.v.).

The set $S = \{\omega: \mathcal{Y}(\omega) \in \Delta\}$ for any function $\mathcal{Y}: \Omega \to \mathbb{R}$ and any Borel set $\Delta \in \mathfrak{B}(\mathbb{R})$, then $S \in \mathcal{F}$. The sense causing this concept of probability is defined well for all outcomes $S = \{\omega: \mathcal{Y}(\omega) \in \Delta\}$. The reason is why ω into a note of a r.v. $\mathcal{Y} = \mathcal{Y}(\omega)$ is frequently ignored because probability theory is not considered the set Ω nor its construct.

Definition 1.1.6.

The measurable function f from measurable space $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$ to measurable space $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ is known as a **Borel function**.

Definition 1.1.7.

(i) For any random variable $\mathcal{X}$, the **probability measure** on $\mathfrak{B}(\mathbb{R})$ is given as a function with the following shape:

$$\mathbb{P}_{\mathcal{X}}(\mathcal{B}) := \mathbb{P}(\{\omega : \mathcal{X}(\omega) \in \mathcal{B}\}), \mathcal{B} \in \mathfrak{B}(\mathbb{R}).$$

(ii) The **distribution function of the random variable** $\mathcal{Y}$ is given uniquely for any Borel sets $\Delta = (-\infty, y), y \in \mathbb{R}$ with the following function:

$$F_{\mathcal{Y}}(y) := \mathbb{P}(\{ \omega : \mathcal{Y}(\omega) \in (-\infty, y)\}) =: \mathbb{P}(\mathcal{Y} < y), y \in \mathbb{R}.$$

(iii) The **density of the random variable** $\mathcal{Y}$ is given as a nonnegative measurable function $f_{\mathcal{Y}}$ such that

$$F_{\mathcal{Y}}(y) = \int_{-\infty}^{y} f_{\mathcal{Y}}(t)\, dt$$

for each $y \in \mathbb{R}$.

(iv) For any r.v. $\mathcal{Y}$ and for any measurable function $h: \mathbb{R} \to \mathbb{R}$, then $\mathcal{Z} = h(\mathcal{Y})$ is also a r.v..

(v) For any r.v. $\mathcal{Y}$, the **expectation of a random variable** $\mathcal{Y}$ is given as the following

$$\mathbb{E}\mathcal{Y} := \int_{\Omega} \mathcal{Y}(\omega)\, \mathbb{P}(d\omega) = \int_{-\infty}^{\infty} y \mathbb{P}_{\mathcal{Y}}(dy)$$

with brief notation

$$\mathbb{E}\mathcal{Y} := \int_{\Omega} \mathcal{Y} d\mathbb{P}$$

(vi) For any r.v. $\mathcal{Y}$, its expectation $\mathbb{E}\mathcal{Y}$ can be written with respect to its distribution function $F_{\mathcal{Y}}$ by what is called **Stieltjes integral**

$$\mathbb{E}\mathcal{Y} := \int_{-\infty}^{\infty} y dF_{\mathcal{Y}}(y)$$

(vii) For any measurable function h and r.v. $\mathcal{Y}$, then the expectation of this function $h(\mathcal{Y})$ is given by

$$\mathbb{E}h(\mathcal{Y}) = \int_{-\infty}^{\infty} h(y) dF_{\mathcal{Y}}(y).$$

Definition 1.1.8.

For any sequence of r.vs. $\mathcal{Y}_k$, the convergence of this sequence to r.v. $\mathcal{Y}$ **with probability one (a.s.),** and we denote it by $\mathcal{Y}_k \longrightarrow \mathcal{Y}$ is that the convergence of the numerical sequence $\mathcal{Y}_k(\omega) \longrightarrow \mathcal{Y}(\omega)$ for all points ω in sample space, Ω is satisfying probability one. In other words, this gives by following

$$\mathbb{P}(\{\omega: \mathcal{Y}_k(\omega) \to \mathcal{Y}(\omega)\}) = 1$$

Definition 1.1.9.

For any r.v. $\mathcal{Y}$, then the **characteristic function** of this r.v. is shown by the following

$$\Upsilon(t) := \mathbb{E}e^{it\mathcal{X}} = \int_{-\infty}^{\infty} e^{ity} dF_{\mathcal{Y}}(y), t \in \mathbb{R}.$$

Proposition 1.1.10.

Assume that r.v. $\mathcal{Y}$ is with $\mathbb{E}|\mathcal{Y}|^k < \infty$. In other words, the kth order of the absolute moment of this r.v. is finite. Then the kth order of moment of this r.v. is defined by the k times differentiable of the characteristic function of $\mathcal{Y}$, and it is given as the following formula for every $0 \leq m \leq k$,

$$\mathbb{E}\mathcal{Y}^m = (-i)^m \frac{d^m}{dt^m} \Upsilon(t)|_{t=0}$$

Corollary 1.1.11.

The above formula gives the **expectation** and the **variance**

$$\mathbb{E}\mathcal{Y} = -i\Upsilon'(0), \quad Var\mathcal{Y} = -\Upsilon''(0) + (\Upsilon'(0))^2$$

Definition 1.1.12.

(i) A **random vector** is a set of r.vs. $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m$ and it is denoted by the following

$\vec{\mathcal{Y}} = (\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m)$.

(ii) Let the distribution of any random vector $\vec{\mathcal{Y}}$ be denoted by $\mathbb{P}_{\vec{\mathcal{Y}}}(A), A \in \mathfrak{B}(\mathbb{R}^m)$. This distribution is a unique measure established by the finite-dimensional distributions and given

$$\mathbb{P}_{\vec{\mathcal{Y}}}(A_1 \times A_2 \times \dots \times A_m) := \mathbb{P}(\bigcap_{i=1}^{m}\{\omega : \mathcal{Y}_i(\omega) \in A_i\})$$

$$:= \mathbb{P}\,(\mathcal{Y}_1 \in A_1, \mathcal{Y}_2 \in A_2, \dots, \mathcal{Y}_m \in A_m), A_i \in \mathfrak{B}(\mathbb{R}), i = 1,2, \dots, m.$$

(iii) For any vector $\vec{\mathcal{Y}}$, its **finite-dimensional distribution function** is given as the following

$$F_{\vec{\mathcal{Y}}}(\vec{y}) := \mathbb{P}(\mathcal{Y}_1 < y_1, \mathcal{Y}_2 < y_2, \dots, \mathcal{Y}_m < y_m), \vec{y} \in \mathbb{R}^m$$

(iv) The **joint density** of r.vs. $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m$ is denoted by $f_{\vec{\mathcal{Y}}}(\vec{z}), \vec{z} \in \mathbb{R}^m$ as are nonnegative measurable functions where

$$F_{\vec{\mathcal{Y}}}(\vec{z}) = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} \dots \int_{-\infty}^{y_n} f_{\vec{\mathcal{Y}}}(\vec{t})\, dt_1 dt_2 \dots dt_m$$

for every $\vec{t} = (t_1, t_2, \dots, t_m) \in \mathbb{R}^m$.

(v) For any random vector $\vec{\mathcal{Y}} = (y_1, y_2, \dots, y_m)$, then its characteristic function is defined uniquely for its distribution and indicated by the following

$$\Upsilon_{\vec{\mathcal{Y}}}(\vec{t}) := \mathbb{E}\exp\left(i(\vec{t},\vec{\mathcal{Y}})\right), \vec{t} \in \mathbb{R}^m,$$

where

$$(\vec{t},\vec{\mathcal{Y}}) := \sum_{i=1}^{m} t_i \mathcal{Y}_i.$$

1.2. The Fundamentals of Conditional Expectations

For any event $\mathcal{C}$ with nonzero probability $\mathbb{P}(\mathcal{C}) > 0$, if this event happens, then for any other event $\mathcal{D}$, the **conditional probability** of the latter event $\mathcal{D}$ given event $\mathcal{C}$ is defined as the following formula:

$$\mathbb{P}(\mathcal{D}/\mathcal{C}) := \frac{\mathbb{P}(\mathcal{D} \cap \mathcal{C})}{\mathbb{P}(\mathcal{C})}$$

Now, we note that a probability measure is defined by the conditional probability $\mathbb{P}(\cdot/\mathcal{C})$ on the σ-algebra $\mathcal{F}$ because $\mathbb{P}(\Omega/\mathcal{C}) = 1$.

Definition 1.2.1.

(i) Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s$ be events. Then, these events are **independent** if, for any selecting indices, $1 \leq s_1 < s_2 < \cdots < s_r \leq m$,

$$\mathbb{P}\left(\mathcal{C}_{s_1} \cap \mathcal{C}_{s_2} \cap \ldots \cap \mathcal{C}_{s_r}\right) = \mathbb{P}\left(\mathcal{C}_{s_1}\right)\mathbb{P}\left(\mathcal{C}_{s_2}\right) \ldots \mathbb{P}\left(\mathcal{C}_{s_r}\right).$$

(ii) For any r.vs. $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_m$, then their **independence with respect to Borel sets** $\mathcal{C}_i, i = 1,2, \ldots, m$ is shown by

$$\mathbb{P}(\mathcal{Y}_1 \in \mathcal{C}_1, \mathcal{Y}_2 \in \mathcal{C}_2, \dots \mathcal{Y}_m \in \mathcal{C}_m) = \prod_{i=1}^{m} \mathbb{P}(\mathcal{Y}_i \in \mathcal{C}_i),$$

(iii) A crucial and satisfactory requirement of independence for components of a random vector $\vec{\mathcal{Y}} = (\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m)$ is that if its distribution function is given a

$$F_{\vec{\mathcal{Y}}}(\vec{y}) = \prod_{i=1}^{m} F_{\mathcal{Y}_i}(y_i),$$

where $F_{\mathcal{Y}_i}(y_i) = \mathbb{P}(\mathcal{Y}_i < y_i)$ is the distribution function for every r.v. $\mathcal{Y}_i, i = 1,2, \dots, m$. Moreover, when its density function (joint density) $f_{\vec{\mathcal{Y}}}$ is existing, then we can write it as the following

$$f_{\vec{\mathcal{Y}}}(\vec{y}) = \prod_{i=1}^{m} f_{\mathcal{Y}_i}(y_i), \vec{y} \in \mathbb{R}^m,$$

where $f_{\mathcal{Y}_i}(y_i)$ is the density function for every r.v. $\mathcal{Y}_i, i = 1,2, \dots, m$.

(iv) An additional essential and satisfactory requirement of independence of r.vs. $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m$ is that if the following expectation is given

$$\mathbb{E}(h_1(\mathcal{Y}_1) h_2(\mathcal{Y}_2) \dots h_m(\mathcal{Y}_m)) = \prod_{i=1}^{m} \mathbb{E} h_i(\mathcal{Y}_i)$$

where bounded Borel functions h_i is a bounded Borel function for all $i = 1,2, \dots, m$.

Definition 1.2.2.

(i) For any r.v. $\mathcal{Y}$ and an event $\mathcal{C}$ with nonzero probability, then a **conditional expectation** of this r.v. $\mathcal{Y}$ given an event $\mathcal{C}$ is identified by the next formulation

$$\mathbb{E}\{\mathcal{Y}/\mathcal{C}\} := \int_{\Omega} \mathcal{Y}(\omega)\mathbb{P}(\mathcal{Y}/\mathcal{C}) = \frac{\mathbb{E}\{\mathcal{Y}\mathbb{I}_{\mathcal{C}}\}}{\mathbb{P}(\mathcal{C})}.$$

where $\mathbb{I}_{\mathcal{C}}$ is an indicator function such that $\mathbb{I}_{\mathcal{C}}(y) = 1, y \in \mathcal{C}$ and $\mathbb{I}_{\mathcal{C}}(y) = 0, y \notin \mathcal{C}$.

(ii)If $\mathcal{X} = \mathbb{I}_{\mathcal{A}}$, the above formula of conditional probability is given as:

$$\mathbb{E}\{\mathbb{I}_{\mathcal{A}}/\mathcal{B}\} := \int_{\Omega} \mathbb{P}(\mathbb{I}_{\mathcal{A}}/\mathcal{B}) = \mathbb{P}(\mathcal{A}/\mathcal{B}) = \frac{\mathbb{P}\{\mathcal{A} \cap \mathcal{B}\}}{\mathbb{P}(\mathcal{B})} = \frac{\mathbb{E}\{\mathbb{I}_{\mathcal{A}}\mathbb{I}_{\mathcal{B}}\}}{\mathbb{P}(\mathcal{B})}$$

(iii) The conditional expectation of a random variable $\mathcal{X}$ given a σ-algebra $\mathcal{Q}$ is denoted by $\mathbb{E}\{\mathcal{X}/\mathcal{Q}\}$.

(iv) Let $\mathcal{C}_n, n = 1, \dots, k$ be disjoint sets and let the algebra generated by them be denoted as $\mathcal{Q}$ then for r.v. $\mathcal{Y}$

$$\mathbb{E}\{\mathcal{Y}/\mathcal{Q}\} := \mathbb{E}\{\mathcal{Y}/\mathcal{C}_n\} = \frac{\mathbb{E}\{\mathcal{X}\mathbb{I}_{\mathcal{C}_n}\}}{\mathbb{P}(\mathcal{C}_n)}.$$

1.3. Stochastic Processes

Let $\mathfrak{X}$ be any set and let $\mathcal{X} = \{\mathcal{X}(t, \omega), t \in \mathfrak{X}\}$ be a collection of random variables depending on some parameter t. Then $\mathcal{X}$ is named a **stochastic process**.

The variable ω is omitted in the representation of a process, along with the issue of random variables to become $\mathcal{X} = \{\mathcal{X}(t), t \in \mathfrak{X}\}$.

Definition 1.3.1.

Let $\mathcal{Y} = \{\mathcal{Y}(s), s \in \mathfrak{X}\}$ be a stochastic process; then we have the following:

(i) If $\mathfrak{X}$ is an interval included in a set of the non-negative real number $\mathbb{R}_0^+$, then $\mathcal{Y}$ is a **continuous time process** with the **time parameter** t.

(ii) The process $\mathcal{Y} = \{\mathcal{Y}(s): s \in \mathfrak{X}\}$ is called a **multiparameter** if $\mathfrak{X} \subseteq \mathbb{R}^k$.

(iii) Let $\mathbb{N}$ be a set of natural numbers, then a process $\mathcal{Y} = \{\mathcal{Y}(s): s \in \mathfrak{X}\}$ is called a **stochastic sequence** if $\mathfrak{X} = \mathbb{N}$.

(iv) Let $\mathcal{Y} = \{\mathcal{Y}(s): s \in \mathfrak{X}\}$ be any stochastic process, then a **sample path**, **trajectory**, or **realization** of this process is the mapping $s \to \mathcal{Y}(s, \omega)$ for fixed sample point $\omega \in \Omega$.

Remark 1.3.2.

The challenge of the sample space expansion in the specific stochastic process leads us to define it as a set of trajectories. That means that for every sample point, there is a trajectory corresponding to this point.

Definition 1.3.3.

(i) Let stochastic processes $\mathcal{V}(s)$ and $\mathcal{U}(s)$ have the same probability space for all $s \in \mathfrak{X}$. Then, these processes are **stochastically equivalent** if

$$\mathbb{P}\big(\mathcal{U}(s) = \mathcal{V}(s)\big) = 1, \forall s \in \mathfrak{X}$$

(ii) Let stochastic processes $\mathcal{V}(s)$ and $\mathcal{U}(s)$ have the same probability space for all $s \in \mathfrak{X}$. Then, for some zero probability set Ω_0 of sample space Ω, these processes are **equivalent** if

$$\mathcal{U}(s,\omega) = \mathcal{V}(s,\omega), \forall s \in \mathfrak{X}, \forall \omega \in \Omega \backslash \Omega_0$$

(iii) Let $\mathcal{Y}$ be a stochastic process, and then the corresponding collection of **finite-dimensional distributions** is given for all $\mathcal{C}_i \in \mathfrak{B}(\mathbb{R})$ and $s_i \in \mathfrak{X}, i = 1, \dots, n$ as the following

$$\wp_{s_1,s_2,\dots,s_n}(\mathcal{C}_1 \times \mathcal{C}_2 \times \dots \times \mathcal{C}_n) \coloneqq \mathbb{P}(\mathcal{Y}(s_1) \in \mathcal{C}_1, \mathcal{Y}(s_2) \in \mathcal{C}_2, \dots, \mathcal{Y}(s_n) \in \mathcal{C}_n)$$

Remark 1.3.4.

(i) When the collection of finite-dimensional distributions of any stochastic process $\mathcal{Y}$ is found, this stochastic process is defined well. However, this process's collection of finite-dimensional distributions is not unique because it has different sample paths. Therefore, we have to select the perfect path to define this process.

(ii) The properties of symmetry and consistency are held for the collection of finite-dimensional distributions such that every $m \geq 2, t_i \in \mathfrak{X}, \mathcal{B}_i \in \mathfrak{B}(\mathbb{R}), i = 1, \dots, m$

a. $\wp_{t_1,t_2,\dots,t_m}(\mathcal{B}_1 \times \mathcal{B}_2 \times \dots \times \mathcal{B}_m) = \wp_{t_{k_1},t_{k_2},\dots,t_{k_m}}(\mathcal{B}_{k_1} \times \mathcal{B}_{k_2} \times \dots \times \mathcal{B}_{k_m})$, where $(k_1, k_2, \dots, k_m)$ is any permutation of $(1, 2, \dots, m)$;

b. $\wp_{t_1 \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_m}(\mathcal{B}_1 \times \dots \times \mathcal{B}_{i-1} \times \mathbb{R} \times \mathcal{B}_{i+1} \times \dots \mathcal{B}_m) = \wp_{t_1 \dots, t_{i-1}, t_{i+1}, \dots, t_m}(\mathcal{B}_1 \times \dots \times \mathcal{B}_{i-1} \times \mathcal{B}_{i+1} \times \dots \mathcal{B}_m)$ for every $1 \leq i \leq m$.

Theorem 1.3.5. (Kolmogorov)

Let $\wp_{t_1,t_2,\dots,t_m}(\mathcal{B}_1 \times \mathcal{B}_2 \times \dots \times \mathcal{B}_m)$ be a collection of finite-dimensional distributions with given symmetry and consistency properties, then there are probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and stochastic process $\mathcal{Y} = \{\mathcal{Y}(s): s \in \mathfrak{X}\}$ stated on this probability space where the collection of finite-dimensional distributions of this stochastic process coincides with a given one.

Definition 1.3.6.

Let two stochastic processes be not necessarily stated on the same probability space; simultaneously, they have a similar state space. If these processes have similar finite-dimensional distributions, they are called **identical in law**.

1.4. The Ergodicity of Markov Chain

This section introduces a specific stochastic process known as a Markov chain. For instance, Markov chains have the feature that the forthcoming is separate from the history, given the current. This process is identified after the probabilist A. A. Markov.

Definition 1.4.1.

Le $\mathcal{Y} = \{\mathcal{Y}_k : k = 01,2, \dots\}$ be a stochastic process defined on discrete state space E. Then, the latter process is called the **Markov chain** if it satisfies the following property of the conditional probability for any state $j \in E$ and $m = 0,1,2, \dots$

$$\mathbb{P}\{\mathcal{Y}_{m+1} = j | \mathcal{Y}_0 = i_0, \mathcal{Y}_1 = i_1, \dots, \mathcal{Y}_m = i_m\} = \mathbb{P}\{\mathcal{Y}_{m+1} = j | \mathcal{Y}_m = i_m\}$$

Definition 1.4.2.

(i) The Markov chain $\mathcal{Y}$ is called to be with **stationary transition probabilities** if it satisfies the following condition for any states $i, i_1, \dots, i_m, j$

$$\mathbb{P}\{\mathcal{Y}_1 = j | \mathcal{Y}_0 = i\} = \mathbb{P}\{\mathcal{Y}_{m+1} = j | \mathcal{Y}_m = i\}$$

(ii) The **Markov matrix** is any nonnegative square matrix $\boldsymbol{P}$ with entries given in the following conditional probabilities corresponding to Markov chain $\mathcal{Y}$ called the **transition probabilities**

$$p_{ij} = \mathbb{P}\{\mathcal{Y}_{m+1} = j | \mathcal{Y}_m = i\}$$

such that the sum of entries into every row is on.

Definition 1.4.3.

For any Markov chain $\mathcal{Y} = \{\mathcal{Y}_m : m = 0,1,2, \dots\}$ defined on state space E, if its Markov matrix $\boldsymbol{P}$ is given, then the following conditional probabilities for any $i, j = 1,2,3, \dots$ and $m = 1,2,3, \dots$

$$p_{ij}^m = \mathbb{P}\{\mathcal{Y}_m = j | \mathcal{Y}_0 = i\}$$

is the entry i-j of the matrix $\boldsymbol{P}^n$.

Definition 1.4.4.

(i) The first time for a Markov chain $\mathcal{Y}$ to exceed a given state j is called the **first passage time,** and we indicated it by $\mathfrak{T}^j$. Moreover, it is time r.v. for the chain to pass a fixed state, and we can write it as the following formula

$$\mathfrak{T}^j = \min\{m \geq 1 : \mathcal{Y}_m = j\}$$

where this time is $+\infty$ when we have the minimum of the empty set.

(ii) The r.v. that counts the **total number of visits** for the Markov chain $\mathcal{Y}$ to a specific state j throughout its lifetime is denoted by $\mathfrak{N}^j$ and given as the following mathematical expression

$$\mathfrak{N}^j = \sum_{m=0}^{\infty} \mathbb{I}_{\{j\}}(\mathcal{Y}_m),$$

where $\mathbb{I}_{\{j\}}(\mathcal{Y}_m)$ is 1 when $\mathcal{Y}_m = j$.

(iii) The probability of ultimately accessing the Markov chain $\mathcal{Y}$ state j whenever once from sate i is called the **first passage probability** and indicated by the symbol $\mathfrak{F}_{ij}$ defined by the following formula

$$\mathfrak{F}_{ij} = \mathbb{P}\{\mathfrak{T}^j < \infty \big| \mathcal{Y}_0 = i\}.$$

(iv) Let's denote the **expected number of returns** the Markov chain $\mathcal{Y}$ to state j from state i by the symbol $\mathfrak{R}_{ij}$ and defined it by the following mathematical expression

$$\mathfrak{R}_{ij} = \mathbb{E}\left[\mathfrak{N}^j \big| \mathcal{Y}_0 = i\right].$$

Theorem 1.4.5.

Let $\mathfrak{R}_{ij}$ and $\mathfrak{F}_{ij}$ be as in the above definition (iii) and (iv), respectively. Then

$$\mathfrak{R}_{ij} = \begin{cases} \dfrac{1}{1-\mathfrak{F}_{ii}} \text{ for } i = j, \\ \dfrac{\mathfrak{F}_{ij}}{1-\mathfrak{F}_{jj}} \text{ for } i \neq j\ ; \end{cases}$$

Definition 1.4.6.

(i) Let j be a state. This state is **transient** if $\mathfrak{F}_{jj} < 1$ or $\mathfrak{R}_{ij} < \infty$.

(ii) Let j be a state. This state is **recurrent** if $\mathfrak{F}_{jj} = 1$ or $\mathfrak{R}_{ij} = \infty$.

Definition 1.4.7.

(i) Let's denote a Markov matrix of Markov chain $\mathcal{Y} = \{\mathcal{Y}_m; m = 0,1, \ldots\}$ by $\boldsymbol{P}$. For any subset C of the state space E, then this subset is a **closed set** if

$$\sum_{j \in C} p_{ij} = 1 \text{ for all } i \in C.$$

(ii) An **irreducible** set of states is a closed set with no proper subset. Such states in this closed set are also closed, called **absorbing** if their set is irreducible.

(iii) The classification of states in an irreducible set is defined the same.

Theorem 1.4.8.

Let $\mathcal{C}$ be a subset of state space, then every state in $\mathcal{C}$ is **recurrent** if the set $\mathcal{C}$ is finite irreducible.

Theorem 1.4.9.

Let E be finite state space and let Markov chain $\mathcal{Y} = \{\mathcal{Y}_m; m = 01,2, \ldots\}$ with Markov matrix $\boldsymbol{P}$ defined on this state space. If the latter space is irreducible and recurrent, and the following probability is defined as

$$\pi_j = \lim_{m \to \infty} \mathbb{P}\{\mathcal{Y}_m = j | \mathcal{Y}_0 = i\}$$

Then $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \ldots)$ is a probability vector considered as the solution to find the below system

$$\boldsymbol{\pi P} = \boldsymbol{\pi}, \qquad \sum_{j \in E} \pi_j = 1$$

Chapter Two

Models of Quorum Queues

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2.1. Preliminaries

Throughout this work, we give the following essential definitions and notations. In addition, there are some theorems, and results are displayed without proof in this content. The tools in this section are subject to various efforts related to Dshalalow[26-30], and all are seen in the same given at work by Ali [10] and Kazem [11].

Definition 2.1.1[1,25]

A collection $\{F_t: t \geq 0\}$ of sub σ-algebra of $F(\Omega)$ is called a **filtration** if this collection is nondecreasing monotone.

Definition 2.1.2 [1,25]

The stochastic process $\{X_t\}$ is called F_t**-adapted process** if for every Borel set $A \subseteq R_0^+$ such that $\{\nu: X(t,\nu) \in A\} \in F_t$.

Definition 2.1.3 [1,25]

A probability space $(\Omega, F(\Omega), P)$ with filtration $\{F_t\}$ is said to be **filtration probability space** and denoted by $(\Omega, F(\Omega), P, F_t)$.

Definition 2.1.4 [1,25]

A random variable T defining on filtration probability space $(\Omega, F(\Omega), P, F_t)$ is said to be **a stopping time** if $\{t \leq T\} \in F_t$.

Definition 2.1.5 [1,25]

A point process (arrival time) is a.s. monotone increasing sequence stopping time $\{T_n : n = 1,2,3, \dots\}$ on R_0^+.

Definition 2.1.6 [10, 11]

The sum of all stopping time points in the interval $[0, t]$ is termed as **counting point process** and given in the following formula

$$C_t = C([0, t]) = \sum_{n=1}^{\infty} \mathbf{1}_{[0,t]}(T_n) = \sum_{n=1}^{\infty} \boldsymbol{\epsilon}_{T_n}([0, t])$$

where $\mathbf{1}_{[0,t]}(T_n)$ and $\boldsymbol{\epsilon}_{T_n}([0, t])$ are called the indicator function and unit mass (Dirac) measure, respectively defined on the interval $[0, t]$ as the following

$$\mathbf{1}_{[0,t]}(T_n) = \boldsymbol{\epsilon}_{T_n}([0, t]) = \begin{cases} 1, & T_n \in [0, t] \\ 0, & T_n \notin [0, t] \end{cases}$$

Definition 2.1.7 [1,25]

A counting point process N_t has **stationary increments property** if its increments $N_{t+s} - N_t$ for every $t > 0$ has the same distribution.

Definition 2.1.8 [1,25]

A counting point process N_t has **independent increments property** if its increments $N_{t_0}, N_{t_1} - N_{t_0}, N_{t_1} - N_{N_{t_0}}, \dots, N_{t_{n+1}} - N_{t_n}$ are independent for $0 \le t_0 < t_1 < \cdots < t_{n+1} < \infty$.

Definition 2.1.9 [1,25]

The point process $T = \{t_i : i = 1,2, \dots\}$ and associated counting point process N_t is called **Poisson point process** and **Poisson counting process**, respectively, if the counting process N_t has Poisson distribution with **parameter** (**rate** or **intensity of process**) λt and its increments $N_{t+1} - N_t$ for $t > 0$ has stationary and independent distributed.

Definition 2.1.10 [10, 11]

Let $T = \{t_i : i = 1,2, \dots\}$ be a point process with a counting point process N_t and let $X = \{X_n : i = 1,2, \dots\}$ be a sequence of iid real-valued random variables with probability generating function (in short, pgf) $a(z)$. Then $(X,T) = \{(X_n, T_n) : n = 1,2, \dots\}$ is called **the marked point process**, and if X is independent of T, then (X,T) is called **the marked point process with position independent marking**. However, if the mark X_n may depend on the inter-arrival times $\varDelta_n = T_n - T_{n-1}$, then (X,T) is called **the marked point process with position dependent marking**. (note: X_n on $\varDelta_n$ is conditionally independent of X_i for $i < n$).

Definition 2.1.11 [10, 11]

The counting point process in the interval $[0, t]$ associated with the marked point process given in the above definition is called **the counting marked point process,** and it is given as the following

$$M_t = M([0, t]) = \sum_{n=1}^{\infty} X_n \epsilon_{T_n}([0, t])$$

Definition 2.1.12 [10, 11]

The marked process $(X, T) = \{(X_n, T_n): n = 1,2, \dots\}$ and its counting process M_t are called **marked Poisson process** and **marked Poisson counting process with independent marking**, respectively, if the counting process has independent and stationary increments and its pgf is a compound Poisson distribution.

Definition 2.1.13 [27,30]

(i) Let T be a nonnegative random variable; then the **Laplace-Stieltjes transform** is given as

$$\beta(\theta) = E\left[e^{-\theta T}\right].$$

(ii) The **moment generating function** $\mathcal{M}$ is given as

$$\mathcal{M}(\theta) = E\left[e^{\theta T}\right] = \beta(-\theta)$$

Proposition 2.1.14

If the random variable T is independent of Poisson counting process C_T, then

$$E\left[z^{C_T}e^{-\theta T}\right] = \beta\big(\theta + \lambda(1 - z)\big).$$

Proof:

The pgf of Poisson r.v. N_t with parameter λ is given as

$$E[z^{N_t}] = e^{\lambda t(z-1)}$$

Then we have

$$E\left[z^{C_T}e^{-\theta T}\right] = E\left[E\left[z^{C_T}e^{-\theta T}|T\right]\right] = E\left[e^{-\theta T}E[z^{C_T}|T]\right] = E\left[e^{-\theta T}e^{\lambda T(z-1)}\right]$$

$$= E\left[e^{-(\theta+\lambda(1-z))T}\right] = \beta\big(\theta + \lambda(1 - z)\big) \blacksquare$$

Proposition 2.1.15

If the random variable T is independent of the marked Poisson counting process M_T, then

$$E\left[z^{M_T}e^{-\theta T}\right] = \beta\left(\theta + \lambda\big(1 - a(z)\big)\right).$$

Proof:

The pgf of compound Poisson r.v. M_t with parameter λ is given as

$$E[z^{M_t}] = e^{\lambda t(a(z)-1)}$$

Then we have

$$E\left[z^{M_T}e^{-\theta T}\right] = E\left[E\left[z^{M_T}e^{-\theta T}|T\right]\right] = E\left[e^{-\theta T}E[z^{M_T}|T]\right] = E\left[e^{-\theta T}e^{\lambda T(a(z)-1)}\right]$$

$$= E\left[e^{-(\theta+\lambda(1-a(z)))T}\right] = \beta\left(\theta + \lambda\big(1 - a(z)\big)\right) \blacksquare$$

2.2. The Bulk Input of Queues

The marked point process characterizes the bulk input of the queue [26-30]. Consider the marked Poisson point process with position independent marking $(X,T) = \{(X_n, T_n): n = 1,2,\ldots\}$ with Poisson point process $T = \{T_n: n = 1,2,\ldots\}$ and the arriving rate is λ. Obviously, the random size of units in batches is subjected to the marks X_n for $n = 1,2,\ldots$ and these marks are iid nonnegative integer-valued r.v.'s with the pgf $a(z)$ and the expected value a. Moreover, the marks X_n and its position T_n are independent for every n. Consequently, the random size of customers in the line until the nth customer leaving at a time T_n is given as

$$Q_{n+1} = \begin{cases} Q_n + A_{n+1} - 1, & Q_n > 0 \\ X_{B_n} + A_{n+1} - 1, & Q_n = 0 \end{cases}$$

such that X_{B_n} is the first bulk entering the line after the time T_n. Accordingly to Abolnikov and Dukhovny [1], the time-homogeneous Markov chain $\{Q_n\}$ is embedded in $\{Q(t)\}$ and Δ_2-matrix. That is, the process $\{Q_n\}$ is **irreducible, aperiodic**, and **recurrent positive** (**ergodic**) because of $P_1'(1-) < 1$. To see that, we have

$$P_i(z) = E[z^{Q_1}|Q_0 = i] = \begin{cases} E\left[z^{i+A_1-1}\right], & i > 0 \\ E\left[z^{X_{B_0}+A_1-1}\right], & i = 0 \end{cases} = \begin{cases} z^{i-1}E[z^{A_1-1}], & i > 0 \\ z^{-1}E\left[z^{X_{B_0}}\right]E[z^{A_1}], & i = 0 \end{cases}$$

$$= \begin{cases} z^{i-1}\beta\left(\lambda\left(1 - a(z)\right)\right), & i > 0 \\ z^{-1}a(z)\beta\left(\lambda\left(1 - a(z)\right)\right), & i = 0 \end{cases}$$

and

$$P_1'(1-) = \beta'(0)\,(-\lambda)a'(1) = ab\lambda = \rho < 1, \;\; a = a'(1) = EX_1, b = ET_1$$
$$= -\beta'(0)$$

where ρ is called the **offered load**. Therefore, the pgf of the stationary distribution for the chain $\{Q_n\}$ is given as the following

$$P(z) = \sum_{i=0}^{\infty} p_i\, P_i(z)$$

$$= p_0 z^{-1} a(z)\beta\left(\lambda\big(1 - a(z)\big)\right) + z^{-1}\beta\left(\lambda\big(1 - a(z)\big)\right)\sum_{i=1}^{\infty} p_i\, z^i$$

and the Pollaczed-Khinchine formula of this pgf is

$$P(z) = p_0\beta\left(\lambda\big(1 - a(z)\big)\right)\frac{a(z) - 1}{z - \beta\left(\lambda\big(1 - a(z)\big)\right)}, p_0 = 1 - \rho.$$

2.3. Random Walk Analysis

Let $M = \{M_n : M_n = M_{t_n}\}$ be a random walk such that M_t is renewal delay marked counting process corresponding to the renewal delay marked point process with position dependent marking $(X, T) = \{(X_n, T_n): n = 0, 1, 2, \ldots\}$, see [26,28-30]. The delayed renewal process refers to the nature of the distribution of T is not specific, and the inter-renewal times $\{\varDelta_n = T_n - T_{n-1}: n = 0, 1, 2, \ldots\}$ are independent and identically distributed except $T_0 = \Delta_0$ has a different distribution.

To avoid clustering in the point process $T = \{T_n : n = 0,1,2,\ldots\}$, we assume that T is a nondecreasing monotone sequence. Consequently, the counting process corresponding with this point process will be continuous in probability.

Let $X = \{X_n : n = 0,1,2,\ldots\}$ be a sequence of nonnegative random variables. Then we define the following joint transformations as below

$$\beta_0(z,\theta) = Ez^{X_0}e^{-\Delta_0\theta}, |z| \le 1, Re(\theta) \ge 0$$

$$\beta_1(z,\theta) = Ez^{X_1}e^{-\Delta_1\theta}, |z| \le 1, Re(\theta) \ge 0$$

We are interested in examining the performance of the random walk M when the marked counting process M_k achieves fixed stage N. Thus, we will define the following random variables:

(i) **The index of first passage time** is

$$\alpha = \inf\{k : M_k \ge N\} = \inf\{k : X_0 + \cdots + X_k \ge N\} \tag{2.1}$$

(ii) **The first passage time** (**exit time**) is T_α and

(iii) **The first excess level** is M_α.

To do this investigation, we should derive the following joint transform

$$\omega_\alpha = \omega(z_1, z_2, z_3, \theta, \vartheta) = E z_1^\alpha z_2^{M_{\alpha-1}} z_3^{M_\alpha} e^{-\vartheta T_{\alpha-1} - \theta T_\alpha} \tag{2.2}$$

corresponding to the auxiliary family of random indices

$$\{\alpha(n) \coloneqq \inf\{k : X_0 + \cdots + X_k \geq n\}, n = 0,1,2, \ldots\} \tag{2.3}$$

and the family of functionals

$$\{\omega_{\alpha(n)} = E z_1^{\alpha(n)} z_2^{M_{\alpha(n)-1}} z_3^{M_{\alpha(n)}} e^{-\vartheta T_{\alpha(n)-1} - \theta T_{\alpha(n)}}, n = 0,1,2, \ldots\} \tag{2.4}$$

by using the D_n operator defined as

$$D_n\{g(n)\}(y) \coloneqq \sum_{n=0}^{\infty} y^n (1-y) g(n), \|y\| < 1 \tag{2.5}$$

and its inverse D^n defined as

$$D_y^n\big(D_s\{g(s)\}(y)\big) = f(n), n = 0,1,2, \ldots \tag{2.6}$$

where

$$s \longmapsto D_y^s(\varphi(y,z)) = \begin{cases} \lim\limits_{y \to 0} \dfrac{1}{s!} \dfrac{\partial^s}{\partial y^s} \left[\dfrac{1}{1-y} \varphi(y,z)\right], & s \geq 0 \\ 0 & , s < 0 \end{cases} \tag{2.7}$$

It is clear that $\alpha = \alpha(N-1)$ from equations (2.1-2.4), then applying the operator D_n on the functional $\omega_{\alpha(n)}$ and the inverse of this operator D^{N-1} on $D_n(\omega_{\alpha(n)})$ can restore $\omega_{\alpha(N-1)}$ to ω_α. The next theorem will consider the essential characteristics of the inverse operator D^n, which serve us in progress sections.

Theorem 2.3.1 [26,28,29]

Let D^n be an inverse operator defined in the equation (2.7). Then the following features are accurate:

(i) D^n is a linear functional.

(ii) $D_y^n\big(\mathbf{1}(y)\big) = 1,$ where $\mathbf{1}(y) = 1$ for all $y \in \mathbb{R}$

(iii) Let h be an analytic function at zero. Then, it holds true that

$$D_y^n\left(y^i h(y)\right) = D_y^{n-i} h(y). \tag{2.2}$$

(iv) In particular of (iii), if $i = n$, we have

$$D_y^n\big(y^n h(y)\big) = h(0). \tag{2.9}$$

(v) Let $h(y) = \sum_{j=0}^{\infty} h_j y^j$. Then,

$$D_y^n\big(h(y)\big) = \sum_{j=0}^{n} h_j \ \ and \ \ D_y^n\big(h(ay)\big) = \sum_{j=0}^{n} h_j a^j \qquad (2.10)$$

(vi) For any real number s, it holds true that

$$D_y^n\left\{\frac{1}{1-sy}\right\} = \begin{cases} \dfrac{1-s^{n+1}}{1-s}, & s \neq 1 \\ n+1, & s = 1 \end{cases} \qquad (2.11)$$

(vii) For any real number s and for a positive integer m, except for $s = m = 1$, it holds true that

$$D_y^n\left\{\frac{1}{(1-sy)^m}\right\} = \begin{cases} \displaystyle\sum_{i=0}^{n} \binom{m+i-1}{i} s^i & \text{except for } s = m = 1 \\ n+1, & s = m = 1 \end{cases} \qquad (2.12)$$

$(viii)$ For two real numbers s and r and a positive integer m, it holds true that

$$D_y^n\left\{\frac{1}{1-ry}\frac{1}{(1-sy)^m}\right\}$$

$$= \begin{cases} \dfrac{1}{1-r}\displaystyle\sum_{i=0}^{n} \binom{m+i-1}{i}\left(s^i - r^{n+1}\left(\frac{s}{r}\right)^i\right), r \neq 1 \\ \displaystyle\sum_{i=0}^{n} \binom{m+i-1}{i} s^j (n-i+1), \qquad r = 1 \end{cases} \qquad (2.13)$$

The subsequent theorem is fundamental to our further issues.

Theorem 2.3.2 (The Key of Random Walk Analysis) [26-28]

Let the next functionals be given as

$$\eta_0 \coloneqq \beta_0(xz_2z_3, \vartheta + \theta), \qquad \eta \coloneqq \beta_1(xz_2z_3, \vartheta + \theta),$$
$$\chi_0 \coloneqq \beta_0(xz_3, \theta), \qquad \chi \coloneqq \beta_1(xz_3, \theta),$$
$$\chi_0^1 \coloneqq \beta_0(z_3, \theta), \qquad \chi^1 \coloneqq \beta_1(z_3, \theta).$$

Then, it holds true that

$$\omega^*(x) = D_n(\omega_{\alpha(n)}(x)) = \chi_0^1 - \chi_0 + \frac{\eta_0 z_1}{1 - \eta z_1}(\chi^1 - \chi) \tag{2.14}$$

and the functional ω_α is given as the following

$$\begin{aligned} \omega_\alpha = \omega_\alpha(z_1, z_2, z_3, \vartheta, \theta) &= E z_1^{\alpha} z_2^{M_{\alpha-1}} z_3^{M_\alpha} e^{-\vartheta T_{\alpha-1} - \theta T_\alpha}, \\ &= D_x^{N-1}\left(\chi_0^1 - \chi_0 + \frac{\eta_0 z_1}{1 - \eta z_1}(\chi^1 - \chi)\right) \end{aligned} \tag{2.15}$$

Corollary 2.3.3 [26-28]

Let $z_1 = z_2 = 1, \vartheta = 0$, then

$$\varpi_\alpha = \varpi_\alpha(z_3, \theta) = \omega_\alpha(1,1, z_3, 0, \theta) = E z_3^{M_\alpha} e^{-\theta T_\alpha} \tag{2.16}$$

$$= \beta_0(z_3,\theta) - \big(1-\beta_1(z_3,\theta)\big)D_x^{N-1}\left(\frac{\beta_0(xz_3,\theta)}{1-\beta_1(xz_3,\theta)}\right)$$

and if $T_0 = \Delta_0 = 0$ and $X_0 = M_0 = i \geq 0$, then $\beta_0(z_3,\theta) = z_3^i$ and

$$\varpi_\alpha = \varpi_\alpha(z_3,\theta) = Ez_3^{M_\alpha}e^{-\theta T_\alpha}$$
$$= z_3^i - z_3^i\big(1-\beta_1(z_3,\theta)\big)D_x^{L-1}\left(x^i\frac{1}{1-\beta_1(xz_3,\theta)}\right) \qquad (2.17)$$

2.4. N-Policy on Queues

This system has the rule to serve the customers [26,27,29,30]. The policy is that the host stops working if no persons are in line. However, this server stays idle if the line does not contain a fixed number N of customers. Therefore, the busy time starts when the number of customers becomes N or more in the queue. This system with the N-policy is subjected to the random walk analysis when $i = 0$ and by eq. (2.17), we have

$$\varpi_\alpha = \varpi_\alpha(z,\theta) = \omega_\alpha(1,1,z,0,\theta) = Ez^{M_\alpha}e^{-\theta T_\alpha}$$
$$= 1 - \big(1-\beta_1(z,\theta)\big)D_x^{N-1}\left(\frac{1}{1-\beta_1(xz,\theta)}\right)$$

Suppose that the system is modeling such that the r.v. X_i is independent of exponential r.v. Δ_i with parameter λ for all $i = 1,2,\dots$ In other words, the marked point process with position independent marking, so we have

$$\beta_1(z,\theta) = Ez^{X_1}e^{-\Delta_1\theta} = Ez^{X_1}Ee^{-\Delta_1\theta} = a(z)\beta(\theta) = a(z)\frac{\lambda}{\lambda+\theta}$$

and the pgf of M_α is

$$\psi(z) \coloneqq \omega_\alpha(1,1,z,0,0) = Ez^{M_\alpha}$$

$$= 1 - [1-a(z)]\, D_x^{N-1}\left(\frac{1}{1-a(xz)}\right)$$

To derive Kendall's formula for this system, we begin with

$$Q_{n+1} = \begin{cases} M_\alpha + A_{n+1} - 1, & Q_n = 0 \\ Q_n + A_{n+1} - 1, & Q_n > 0 \end{cases}$$

and then we get

$$P_i(z) = E[z^{Q_1}|Q_0 = i] = \begin{cases} z^{-1}\psi(z)\beta\big(\lambda - \lambda a(z)\big), & i = 0 \\ z^{i-1}\beta\big(\lambda - \lambda a(z)\big), & i > 0 \end{cases}$$

Thus, the pgf of this probability distribution of this model

$$P(z) = \sum_{i=0}^{\infty} p_i P_i(z) = p_0\psi(z)z^{-1}\beta\big(\lambda - \lambda a(z)\big) + z^{-1}\beta\big(\lambda - \lambda a(z)\big)\sum_{i=1}^{\infty} p_i z^i$$

By doing some steps, we have

$$P(z) = p_0\beta\big(\lambda - \lambda a(z)\big)\frac{\psi(z) - 1}{z - \beta\big(\lambda - \lambda a(z)\big)}$$

and

$$p_0 = \frac{1-\lambda ab}{\psi} = \frac{1-\rho}{\psi}$$

where

$$\psi := EM_\alpha = \psi'(z)|_{z=1}$$

2.5. Queues with Multiple Vacations

In this system, the server rests with a series of vacation segments when the queue size is less than N units. However, the server ends the trips and returns to work when the line has N units or more [5,17-24]. This model with multiple vacations is governed by the random walk analysis when $i = 0$ ($X_0 = 0 \Rightarrow \beta_0(z, \theta) = 1$) and by eq. (2.17) we have

$$\varpi_\alpha = \varpi_\alpha(z, \theta) = \omega_\alpha(1,1, z, 0, \theta) = Ez^{M_\alpha} e^{-\theta T_\alpha}$$

$$= 1 - \big(1 - \beta_1(z, \theta)\big) D_x^{N-1} \left(\frac{1}{1 - \beta_1(xz, \theta)}\right)$$

Assume that the queue is modelled as the marked point process with position-dependent marking. That means the X_i is dependent on iid inter-renewal times r.v.'s $\Delta_i = T_i - T_{i-1}$ for $i = 0,1,2, \ldots$ where $T_0, T_1, \cdots$ is a renewal point process (vacation times). So, we have

$$\beta_1(z,\theta) = Ez^{X_1}e^{-\theta T_1} = \beta_1\big(\theta + \lambda - \lambda a(z)\big) \Rightarrow \ \beta_1(zx,0) = \beta_1\big(\lambda - \lambda a(z)\big)$$

and the pgf of M_α is

$$\psi(z) \coloneqq \omega_\alpha(1,1,z,0,0) = Ez^{M_\alpha}$$

$$= 1 - [1 - \beta_1(z,0)]\, D_x^{N-1}\left(\frac{1}{1 - \beta_1(xz,0)}\right)$$

Moreover, we can write the above equation by the following

$$\psi(z) \coloneqq 1 - \big[1 - \beta_1\big(\lambda - \lambda a(z)\big)\big]\, D_x^{N-1}\left(\frac{1}{1 - \beta_1\big(\lambda - \lambda a(z)\big)}\right)$$

To derive Kendall's formula for this model, we have the same steps as in the above section to get

$$P(z) = p_0\beta\big(\lambda - \lambda a(z)\big)\frac{\psi(z) - 1}{z - \beta\big(\lambda - \lambda a(z)\big)}$$

and

$$p_0 = \frac{1 - \lambda ab}{\psi} = \frac{1 - \rho}{\psi}$$

where

$$\psi\!:= EM_\alpha = \psi'(z)|_{z=1}$$

2.6. A Bulk Queue with a Quorum Policy

This system with a quorum policy happens when the single server is subject to batch service [12-16]. That means the server is busy with R units at the same time. Moreover, the input of the queue is bulk arrivals. We are interested in discussing this queue with N-policy and generalizing it to multiple vacations.

2.6.1. A Quorum Queue with N-policy

Assume the server works under the quorum condition and the input queue is bulk. Moreover, the server becomes idle when the number of units in the line is less than $r \geq 1$. Furthermore, the capacity of service is $R \geq r$, and the server resume when the buffer contains r or more units [14,16].

Let the input batch M_t be modelled as a marked Poisson with position independent marking with joint transform

$$\beta_1(z,\theta) = Ez^{X_1}e^{-\theta T_1} = Ez^{X_1}Ee^{-\theta T_1} = a(z)\frac{\lambda}{\lambda+\theta}$$

Suppose that the server rests up when the queue size $Q_n = i < r$ at T_n and resumes when the line size of arrivals reaches level r or higher. Let $X_0 = i$ at $T_0 = 0$, then

$$\beta_0(z,\theta) = Ez^{X_0}e^{-\theta T_0} = Ez^{i}e^{-\theta(0)} = z^{i}$$

The length of the idle period in $[T_n, T_n + T_\alpha]$ is the first passage of time T_α and the functional $\psi(z)$ is given as the following

$$\psi(z) := Ez^{M_\alpha} = z^i - z^i[1 - a(z)]\, D_x^{N-1-i}\left(\frac{1}{1 - a(zx)}\right), r > i$$

To find Kendall's formula for this model, we start with

$$Q_{n+1} = \begin{cases} (M_\alpha - R)^+ + A_{n+1}, & Q_n < r \\ (Q_n - R)^+ + A_{n+1}, & Q_n \geq r \end{cases}$$

and then we have

$$P_i(z) = E[z^{Q_1} | Q_0 = i]$$

$$= \begin{cases} \beta\big(\lambda - \lambda a(z)\big)Ez^{(M_\alpha - R)^+}, & i < r \\ \beta\big(\lambda - \lambda a(z)\big)Ez^{(i-R)^+}, & i \geq r \end{cases}$$

$$= \begin{cases} \beta\big(\lambda - \lambda a(z)\big)Ez^{(M_\alpha - R)^+}, & i < r \\ \beta\big(\lambda - \lambda a(z)\big)z^{(i-R)^+}, & i \geq r \end{cases}$$

To obtain that, we see

$$Ez^{(M_\alpha - R)^+} = D_x^R\big(Ex^{M_\alpha} + z^{-R}(Ez^{M_\alpha} - E(zx)^{M_\alpha})\big)$$

$$= D_x^R(Ex^{M_\alpha}) + z^{-R}D_x^R(Ez^{M_\alpha}) - z^{-R}D_x^R(E(zx)^{M_\alpha})$$

$$= D_x^R(Ex^{M_\alpha}) + z^{-R}Ez^{M_\alpha} - z^{-R}D_x^R(E(zx)^{M_\alpha})$$

$$= D_x^R\left(x^i - x^i[1-a(x)]\, D_y^{r-1-i}\left(\frac{1}{1-a(xy)}\right)\right)$$

$$+z^{-R}\left(z^i - z^i[1-a(z)]\, D_y^{r-1-i}\left(\frac{1}{1-a(zy)}\right)\right)$$

$$-z^{-R}D_x^R\left((zx)^i - (zx)^i[1-a(zx)]\, D_y^{r-1-i}\left(\frac{1}{1-a(zxy)}\right)\right)$$

$$Ez^{(M_\alpha-R)^+} = D_x^{R-i}\left(1-[1-a(x)]\, D_y^{r-1-i}\left(\frac{1}{1-a(xy)}\right)\right)$$

$$+z^{i-R}\left(1-[1-a(z)]\, D_y^{r-1-i}\left(\frac{1}{1-a(zy)}\right)\right)$$

$$-z^{i-R}D_x^{R-i}\left(1-[1-a(zx)]\, D_y^{r-1-i}\left(\frac{1}{1-a(zxy)}\right)\right)$$

To discuss the case $R=r$, we have the following

$$Ez^{(M_\alpha-R)^+} = Ez^{(M_\alpha-r)^+} = Ez^{M_\alpha-r} = z^{-r}Ez^{M_\alpha}$$

$$= z^{-r}\left(z^i - z^i[1-a(z)]\, D_x^{r-1-i}\left(\frac{1}{1-a(zx)}\right)\right)$$

$$= z^{i-r}\left(1-[1-a(z)]\, D_x^{r-1-i}\left(\frac{1}{1-a(zx)}\right)\right)$$

Let $\psi_0^i(z) := Ez^{(M_\alpha - R)^+}$, then the pgf of Q_n is

$$P(z) = \sum_{i=0}^{\infty} p_i P_i(z)$$

$$= \sum_{i=0}^{r-1} p_i \beta\big(\lambda - \lambda a(z)\big) E z^{(M_\alpha - R)^+} + \sum_{i=r}^{\infty} p_i \beta\big(\lambda - \lambda a(z)\big) z^{(i-R)^+}$$

$$= \beta\big(\lambda - \lambda a(z)\big) \sum_{i=0}^{r-1} p_i \psi_0^i(z) + \beta\big(\lambda - \lambda a(z)\big) \sum_{i=r}^{\infty} p_i z^{(i-R)^+}$$

Since $(i-R)^+ = 0$ when $i \le R$ and $r < R$, then

$$P(z) = \beta\big(\lambda - \lambda a(z)\big) \sum_{i=0}^{r-1} p_i \psi_0^i(z) + \beta\big(\lambda - \lambda a(z)\big) \sum_{i=r}^{R-1} p_i$$

$$+ \beta\big(\lambda - \lambda a(z)\big) \sum_{i=R}^{\infty} p_i z^{i-R}$$

$$= \beta\big(\lambda - \lambda a(z)\big) \left[\sum_{i=0}^{r-1} p_i \psi_0^i(z) + \sum_{i=r}^{R-1} p_i + z^{-R} \sum_{i=R}^{\infty} p_i z^i \right]$$

$$= \beta\big(\lambda - \lambda a(z)\big) \left[\sum_{i=0}^{r-1} p_i \psi_0^i(z) + \sum_{i=r}^{R-1} p_i + z^{-R} \left(P(z) - \sum_{i=0}^{R-1} p_i z^i \right) \right]$$

$$= \beta\big(\lambda - \lambda a(z)\big) \left[\sum_{i=0}^{r-1} p_i \psi_0^i(z) + \sum_{i=r}^{R-1} p_i + z^{-R} \left(P(z) - \sum_{i=0}^{r-1} p_i z^i - \sum_{i=r}^{R-1} p_i z^i \right) \right]$$

$$= \beta\big(\lambda - \lambda a(z)\big)\left[\sum_{i=0}^{r-1} p_i \psi_0^i(z) + \sum_{i=r}^{R-1} p_i + z^{-R}P(z) - z^{-R}\sum_{i=0}^{r-1} p_i z^i - z^{-R}\sum_{i=r}^{R-1} p_i z^i\right]$$

$$= \beta\big(\lambda - \lambda a(z)\big)\left[\sum_{i=0}^{r-1} p_i[\psi_0^i(z) - z^{i-R}] + \sum_{i=r}^{R-1} p_i[1 - z^{i-R}] + z^{-R}P(z)\right]$$

$$\frac{P(z)}{\beta\big(\lambda - \lambda a(z)\big)} = \sum_{i=0}^{r-1} p_i[\psi_0^i(z) - z^{i-R}] + \sum_{i=r}^{R-1} p_i[1 - z^{i-R}] + z^{-R}P(z)$$

$$\frac{P(z)}{\beta\big(\lambda - \lambda a(z)\big)} - z^{-R}P(z) = \sum_{i=0}^{r-1} p_i\big[\psi_0^i(z) - z^{i-R}\big] + \sum_{i=r}^{R-1} p_i\big[1 - z^{i-R}\big]$$

$$\frac{P(z) - z^{-R}\beta\big(\lambda - \lambda a(z)\big)P(z)}{\beta\big(\lambda - \lambda a(z)\big)} = \sum_{i=0}^{r-1} p_i[\psi_0^i(z) - z^{i-R}] + \sum_{i=r}^{R-1} p_i[1 - z^{i-R}]$$

$$P(z)\frac{z^R - \beta\big(\lambda - \lambda a(z)\big)}{\beta\big(\lambda - \lambda a(z)\big)} = \sum_{i=0}^{r-1} p_i[z^R\psi_0^i(z) - z^i] + \sum_{i=r}^{R-1} p_i[z^R - z^i]$$

Therefore, the pgf of Q_n is

$$P(z) = \frac{\beta\big(\lambda - \lambda a(z)\big)}{\big[z^R - \beta\big(\lambda - \lambda a(z)\big)\big]}\left[\sum_{i=0}^{r-1} p_i[z^R\psi_0^i(z) - z^i] + \sum_{i=r}^{R-1} p_i[z^R - z^i]\right] \qquad (2.18)$$

2.6.2. A Quorum Queue with N-policy and Multiple Vacations

In this section, the model is upgraded such that the server with the quorum condition and the bulk is subject to multiple vacation discipline. That means the server is going on a series of vacations when the number of units in the line is less than $r \geq 1$ and returns to serve when this number is more than or equal r but not more than $R \geq r$ [14].

Let the input batch M_t be modelled as a marked Poisson with position-dependent marking with joint transform

$$\beta_1(z,\theta) = Ez^{X_1}e^{-\theta T_1}$$

Suppose that the server rests up when the queue size $Q_n = i < r$ at T_n and resumes when the line size of arrivals reaches level r or higher. Let $X_0 = i$ at $T_0 = 0$, then

$$\beta_0(z,\theta) = Ez^{X_0}e^{-\theta T_0} = Ez^i e^{-\theta(0)} = z^i$$

The length of the idle period in $[T_n, T_n + T_\alpha]$ is the first passage of time T_α and the functional $\psi(z)$ is given as the following

$$\psi(z) := Ez^{M_\alpha} = z^i - z^i[1-\beta_1(z,0)]\, D_x^{N-1-i}\left(\frac{1}{1-\beta_1(zx,0)}\right), r > i$$

To find Kendall's formula for this model, we start with

$$Q_{n+1}=\begin{cases}(M_\alpha-R)^+ + A_{n+1}, & Q_n<r\\ (Q_n-R)^+ + A_{n+1}, & Q_n\ge r\end{cases}$$

and then we have

$$P_i(z)=E[z^{Q_1}|Q_0=i]=\begin{cases}\beta\big(\lambda-\lambda a(z)\big)Ez^{(M_\alpha-R)^+}, & i<r\\ \beta\big(\lambda-\lambda a(z)\big)Ez^{(i-R)^+}, & i\ge r\end{cases}$$

$$=\begin{cases}\beta\big(\lambda-\lambda a(z)\big)Ez^{(M_\alpha-R)^+}, & i<r\\ \beta\big(\lambda-\lambda a(z)\big)z^{(i-R)^+}, & i\ge r\end{cases}$$

To obtain that, we see

$$Ez^{(M_\alpha-R)^+}=D_x^R(Ex^{M_\alpha}+z^{-R}[Ez^{M_\alpha}-E(zx)^{M_\alpha}])$$

$$=D_x^R(Ex^{M_\alpha})+z^{-R}D_x^R(Ez^{M_\alpha})-z^{-R}D_x^R(E(zx)^{M_\alpha})$$

$$=D_x^R(Ex^{M_\alpha})+z^{-R}Ez^{M_\alpha}-z^{-R}D_x^R(E(zx)^{M_\alpha})$$

$$=D_x^R\left(x^i-x^i[1-\beta_1(x,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(xy,0)}\right)\right)$$

$$+z^{-R}\left(z^i-z^i[1-\beta_1(z,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(zy,0)}\right)\right)$$

$$-z^{-R}D_x^R\left((zx)^i-(zx)^i[1-\beta_1(zx,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(zxy,0)}\right)\right)$$

$$Ez^{(M_\alpha-R)^+} = D_x^{R-i}\left(1-[1-\beta_1(x,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(xy,0)}\right)\right)$$

$$+z^{i-R}\left(1-[1-\beta_1(z,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(zy,0)}\right)\right)$$

$$-z^{i-R}D_x^{R-i}\left(1-[1-\beta_1(zx,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(zxy,0)}\right)\right)$$

To discuss the case $R = r$, we have the following

$$Ez^{(M_\alpha-R)^+} = Ez^{(M_\alpha-r)^+} = Ez^{M_\alpha-r} = z^{-r}Ez^{M_\alpha}$$

$$= z^{-r}\left(z^i - z^i[1-\beta_1(z,0)]\,D_x^{r-1-i}\left(\frac{1}{1-\beta_1(zx,0)}\right)\right)$$

$$= z^{i-r}\left(1-[1-\beta_1(z,0)]\,D_x^{r-1-i}\left(\frac{1}{1-\beta_1(zx,0)}\right)\right)$$

Let $\psi_0^i(z) := Ez^{(M_\alpha-R)^+}$, then the pgf of Q_n has the same formula in equation (2.18).

Chapter Three

Applications on Quorum Queues

Chapter Three

Applications on Quorum Queues

This chapter includes many cases of quorum queues, depending on the rules of N-policy and multiple vacations, respectively. We begin discussing N-policy discipline.

3.1. Applications on a Quorum Queue with N-policy

In this section, we will be concerned about the essential objectives of our effort. We plan to obtain pgf of the size of units for this scheme when arriving batches have several formulas as the following cases:

Case I:-

When $a(z) = z$, it is not bulk input because the input is subjected to the ordinary Poisson process. Let $r = 2, R = 3.$ Now, we begin to derive

$$\psi_0^i(z) = D_x^{R-i}\left(1 - [1 - x]\, D_y^{r-1-i}\left(\frac{1}{1 - xy}\right)\right)$$

$$+ z^{i-R}\left(1 - [1 - z]\, D_y^{r-1-i}\left(\frac{1}{1 - zy}\right)\right)$$

$$- z^{i-R} D_x^{R-i}\left(1 - [1 - zx]\, D_y^{r-1-i}\left(\frac{1}{1 - zxy}\right)\right)$$

$$= D_x^{R-i}\left(1-(1-x)\ \frac{1-x^{r-i}}{(1-x)}\right)+z^{i-R}\left(1-(1-z)\ \frac{1-z^{r-i}}{(1-z)}\right)$$

$$-z^{i-R}D_x^{R-i}\left(1-(1-zx)\ \frac{1-(zx)^{r-i}}{(1-zx)}\right)$$

$$= D_x^{R-i}\left(1-1+x^{r-i}\right)+z^{i-R}\left(1-1+z^{r-i}\right)$$

$$-z^{i-R}D_x^{R-i}\left(1-1+(zx)^{r-i}\right)$$

$$= 1-D_x^{R-i}\left(x^{r-i}\right)+z^{i-R}z^{r-i}-z^{i-R}D_x^{R-i}\left((zx)^{r-i}\right)$$

$$= 1-1+z^{r-R}-z^{i-R}z^{r-i}D_x^{R-i}\left(x^{r-i}\right)$$

$$= z^{r-R}-z^{r-R}$$

$$= 0$$

Since $r=2, R=3$, then we have for all i

$$\psi_0^i(z)=0$$

and

$$P(z)=\frac{\beta(\lambda-\lambda z)}{[z^3-\beta(\lambda-\lambda z)]}\left[\sum_{i=0}^{2-1}p_i[z^3\psi_0^i(z)-z^i]+\sum_{i=2}^{3-1}p_i[z^3-z^i]\right]$$

$$=\frac{\beta(\lambda-\lambda z)}{[z^3-\beta(\lambda-\lambda z)]}\left[\sum_{i=0}^{1}p_i[z^3(0)-z^i]+\sum_{i=2}^{2}p_i[z^3-z^i]\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[\sum_{i=0}^{1} p_i[-z^i] + p_2(z^3 - z^2)\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}[-z^0 p_0 - z^1 p_1 + p_2(z^3 - z^2)]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}[-p_0 - p_1 z + p_2(z^3 - z^2)]$$

$$= \frac{\beta(\lambda - \lambda z)}{[\beta(\lambda - \lambda z) - z^3]}[p_0 + p_1 z + p_2 z^2 - p_2 z^3]$$

Since $P(1) = 1$, we have

$$1 = P(1) = \beta\big(\lambda - \lambda(1)\big)\frac{p_1 + p_2(2z - 3z^2)}{[\beta'(\lambda - \lambda z)(-\lambda) - 3z^2]}\Bigg|_{z=1}$$

$$= \beta(0)\frac{p_1 + p_2\big(2(1) - 3(1)\big)}{\big[\beta'\big(\lambda - \lambda(1)\big)(-\lambda) - 3(1)\big]} = (1)\frac{p_1 - p_2}{[\beta'(0)(-\lambda) - 3]}$$

$$= \frac{p_1 - p_2}{(\lambda b - 3)}$$

Then $p_1 - p_2 = (\lambda b - 3)$

<u>Case II:-</u>

When the arriving batch is type 1 geometrically distributed with parameter p, then $a(z) = \frac{pz}{1-qz}$, $q = 1 - p$. Let $r = 2, R = 3$. Now, we start to obtain

$$\psi_0^i(z) = D_x^{R-i}[1 - \left[1 - \frac{px}{1-qx}\right] D_y^{r-1-i}\left(\frac{1}{1-\frac{pxy}{1-qxy}}\right)] + z^{i-R}(1$$

$$- \left[1 - \frac{pz}{1-qz}\right] D_y^{r-1-i}\left(\frac{1}{1-\frac{pzy}{1-qzy}}\right)) - z^{i-R}D_x^{R-i}(1$$

$$- \left[1 - \frac{pzx}{1-qzx}\right] D_y^{r-1-i}\left(\frac{1}{1-\frac{pzxy}{1-qzxy}}\right))$$

$$= D_x^{R-i}[1 - \left[\frac{1-qx-px}{1-qx}\right] D_y^{r-1-i}\left(\frac{1}{\frac{1-qxy-pxy}{1-qxy}}\right)] + z^{i-R}(1$$

$$- \left[\frac{1-qz-pz}{1-qz}\right] D_y^{r-1-i}\left(\frac{1}{\frac{1-qzy-pzy}{1-qzy}}\right)) - z^{i-R}D_x^{R-i}(1$$

$$- \left[\frac{1-qzx-pzx}{1-qzx}\right] D_y^{r-1-i}\left(\frac{1}{\frac{1-qzxy-pzxy}{1-qzxy}}\right))$$

$$= D_x^{R-i}[1 - \left[\frac{1-(q+p)x}{1-qx}\right] D_y^{r-1-i}\left(\frac{1}{\frac{1-(q+p)xy}{1-qxy}}\right)] + z^{i-R}(1$$

$$- \left[\frac{1-(q+p)z}{1-qz}\right] D_y^{r-1-i}\left(\frac{1}{\frac{1-(q+p)zy}{1-qzy}}\right)) - z^{i-R}D_x^{R-i}(1$$

$$- \left[\frac{1-(q+p)zx}{1-qzx}\right] D_y^{r-1-i}\left(\frac{1}{\frac{1-(q+p)zxy}{1-qzxy}}\right))$$

$$= D_x^{R-i}[1 - \left[\frac{1-x}{1-qx}\right] D_y^{r-1-i}\left(\frac{1-qxy}{1-xy}\right)] + z^{i-R}(1$$

$$- \left[\frac{1-z}{1-qz}\right] D_y^{r-1-i}\left(\frac{1-qzy}{1-zy}\right)) - z^{i-R}D_x^{R-i}(1$$

$$- \left[\frac{1-zx}{1-qzx}\right] D_y^{r-1-i}\left(\frac{1-qzxy}{1-zxy}\right))$$

$$= D_x^{R-i}[1 - \left[\frac{1-x}{1-qx}\right] D_y^{r-1-i}\left(\frac{1}{1-xy}\right) + qx\left[\frac{1-x}{1-qx}\right] D_y^{r-2-i}\left(\frac{1}{1-xy}\right)]$$

$$+ z^{i-R}(1 - \left[\frac{1-z}{1-qz}\right] D_y^{r-1-i}\left(\frac{1}{1-zy}\right)$$

$$+ qz\left[\frac{1-z}{1-qz}\right] D_y^{r-2-i}\left(\frac{1}{1-zy}\right)) - z^{i-R}D_x^{R-i}(1$$

$$- \left[\frac{1-zx}{1-qzx}\right] D_y^{r-1-i}\left(\frac{1}{1-zxy}\right) + qzx\left[\frac{1-zx}{1-qzx}\right] D_y^{r-2-i}\left(\frac{1}{1-zxy}\right))$$

$$= D_x^{R-i}[1 - \left[\frac{1-x}{1-qx}\right]\frac{1-x^{r-i}}{1-x} + qx\left[\frac{1-x}{1-qx}\right]\frac{1-x^{r-i-1}}{1-x}\ ] + z^{i-R}(1$$

$$- \left[\frac{1-z}{1-qz}\right]\frac{1-z^{r-i}}{1-z} + qz\left[\frac{1-z}{1-qz}\right]\frac{1-z^{r-i-1}}{1-z}) - z^{i-R}D_x^{R-i}(1$$

$$- \left[\frac{1-zx}{1-qzx}\right]\frac{1-(zx)^{r-i}}{1-zx} + qzx\left[\frac{1-zx}{1-qzx}\right]\frac{1-(zx)^{r-i-1}}{1-zx})$$

$$= D_x^{R-i}[1 - \frac{1-x^{r-i}}{1-qx} + qx\frac{1-x^{r-i-1}}{1-qx}\ ] + z^{i-R}(1 - \frac{1-z^{r-i}}{1-qz}$$

$$+ qz\frac{1-z^{r-i-1}}{1-qz}) - z^{i-R}D_x^{R-i}(1 - \frac{1-(zx)^{r-i}}{1-qzx}$$

$$+ qzx\frac{1-(zx)^{r-i-1}}{1-qzx})$$

$$= D_x^{R-i}[1 + \frac{-1+x^{r-i}+qx-qx^{r-i}}{1-qx}\ ] + z^{i-R}(1 + \frac{-1+z^{r-i}+qz-qz^{r-i}}{1-qz})$$

$$- z^{i-R}D_x^{R-i}(1 + \frac{-1+(zx)^{r-i}+qzx-q(zx)^{r-i}}{1-qzx})$$

$$= D_x^{R-i}[1 + \frac{-(1-qx)+(1-q)x^{r-i}}{1-qx}\ ] + z^{i-R}(1$$

$$+ \frac{-(1-qz)+(1-q)z^{r-i}}{1-qz}) - z^{i-R}D_x^{R-i}(1$$

$$+ \frac{-(1-qzx)+(1-q)(zx)^{r-i}}{1-qzx})$$

$$= D_x^{R-i}[1 - 1 + \frac{(1-q)x^{r-i}}{1-qx}\ ] + z^{i-R}(1 - 1 + \frac{(1-q)z^{r-i}}{1-qz}) - z^{i-R}D_x^{R-i}(1$$

$$- 1 + \frac{(1-q)(zx)^{r-i}}{1-qzx})$$

$$= (1-q)D_x^{R-r}[\frac{1}{1-qx}\ ] + \frac{(1-q)z^{r-R}}{1-qz} - (1-q)z^{r-R}D_x^{R-r}(\frac{1}{1-qzx})$$

$$= (1-q)\frac{1-q^{R-r+1}}{1-q} + \frac{(1-q)z^{r-R}}{1-qz} - (1-q)z^{r-R}\frac{1-(qz)^{R-r+1}}{1-qz}$$

$$= 1 - q^{R-r+1} + \frac{(1-q)z^{r-R}}{1-qz} - \frac{(1-q)z^{r-R} - (1-q)q^{R-r+1}z^{r-R}z^{R-r+1}}{1-qz}$$

$$= 1 - q^{R-r+1} + \frac{(1-q)z^{r-R} - (1-q)z^{r-R} + (1-q)q^{R-r+1}z}{1-qz}$$

$$= 1 - q^{R-r+1} + \frac{pq^{R-r+1}z}{1-qz}$$

$$= 1 - q^{R-r+1}\left(1 - \frac{pz}{1-qz}\right)$$

$$= 1 - q^{R-r+1}\left(\frac{1-qz-pz}{1-qz}\right)$$

$$= 1 - q^{R-r+1}\left(\frac{1-z+pz-pz}{1-qz}\right)$$

$$= 1 - q^{R-r+1}\left(\frac{1-z}{1-qz}\right)$$

$$\psi_0^i(z) = 1 - q^{R-r+1}\left(\frac{1-z}{1-qz}\right)$$

Since we know pgf of Q_n and it is given as

$$P(z) = \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[p_0[z^3\psi_0^0(z) - 1] + p_1[z^3\psi_0^1(z) - z] + p_2[z^3 - z^2]\right]$$

where

$$\psi_0^0(z) = \psi_0^1(z) = 1 - q^{R-r+1}\left(\frac{1-z}{1-qz}\right)$$

then

$$P(z) = \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[p_0\left[z^3\left(1 - q^{R-r+1}\left(\frac{1-z}{1-qz}\right)\right) - 1\right]\right.$$

$$\left. + p_1\left[z^3\left(1 - q^{R-r+1}\left(\frac{1-z}{1-qz}\right)\right) - z\right] + p_2[z^3 - z^2]\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[p_0 z^3\left(1 - q^{R-r+1}\left(\frac{1-z}{1-qz}\right)\right) - p_0\right.$$

$$\left. + p_1 z^3\left(1 - q^{R-r+1}\left(\frac{1-z}{1-qz}\right)\right) - p_1 z + p_2 z^3 - p_2 z^2\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[(p_0 + p_1)z^3\left(1 - q^{R-r+1}\left(\frac{1 - z}{1 - qz}\right)\right) - p_0 - p_1 z + p_2 z^3 - p_2 z^2\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[(p_0 + p_1)\left(1 - q^{R-r+1}\left(\frac{1 - z}{1 - qz}\right)\right)z^3 - (p_0 + p_1 z + p_2 z^2 - p_2 z^3)\right]$$

<u>Case III:-</u>

When the arriving batch is distributed with pgf $a(z) = pz + qz^2, q = 1 - p, 0 < p < 1$. Let $r = 2, R = 3$. We will simplify the following

$$1 - a(z) = 1 - pz - qz^2 = p + q - pz - qz^2 = p(1 - z) + q(1 - z^2)$$

$$= [p + q(1 + z)](1 - z) = [p + (1 - p)(1 + z)](1 - z)$$

$$= [p + 1 + z - p - pz](1 - z) = [1 - (1 - p)z](1 - z)$$

$$= (1 - qz)(1 - z)$$

Now, we start to obtain

$$\psi_0^i(z) = D_x^{R-i}[1-[(1-qx)(1-x)]\,D_y^{r-1-i}\left(\frac{1}{(1-qxy)(1-xy)}\right)] + z^{i-R}(1$$

$$-[(1-qz)(1-z)]\,D_y^{r-1-i}\left(\frac{1}{(1-qzy)(1-zy)}\right)) - z^{i-R}D_x^{R-i}(1$$

$$-[(1-qzx)(1-zx)]\,D_y^{r-1-i}\left(\frac{1}{(1-qzxy)(1-zxy)}\right))$$

$$= D_x^{R-i}[1-[(1-qx)(1-x)]\,D_y^{r-1-i}\left(\frac{1}{(1-qxy)(1-xy)}\right)] + z^{i-R}(1$$

$$-[(1-qz)(1-z)]\,D_y^{r-1-i}\left(\frac{1}{(1-qzy)(1-zy)}\right)) - z^{i-R}D_x^{R-i}(1$$

$$-[(1-qzx)(1-zx)]\,D_y^{r-1-i}\left(\frac{1}{(1-qzxy)(1-zxy)}\right))$$

To find the following quantity, we have

$$\frac{1}{(1-qzy)(1-zy)} = \frac{A}{1-qzy} + \frac{B}{1-zy} = \frac{A(1-zy)+B(1-qzy)}{(1-qzy)(1-zy)}$$

$$= \frac{A+B-Azy-Bqzy}{(1-qzy)(1-zy)}$$

$$A+B=1, -Azy-Bqzy=0 \Rightarrow A=1-B, A+Bq=0 \Rightarrow 1-B+Bq=0$$

$$\Rightarrow B = \frac{1}{1-q} = \frac{1}{p}$$

$$\Rightarrow A = 1-\frac{1}{p} = \frac{p-1}{p} = \frac{-q}{p} \Rightarrow \frac{1}{(1-qzy)(1-zy)} = \frac{\frac{1}{p}}{1-qzy} + \frac{\frac{-q}{p}}{1-zy}$$

$$D_y^{r-1-i}\left(\frac{1}{(1-qzy)(1-zy)}\right)=D_y^{r-1-i}\left(\frac{\frac{1}{p}}{1-qzy}+\frac{\frac{-q}{p}}{1-zy}\right)$$

$$=D_y^{r-1-i}\left(\frac{\frac{1}{p}}{1-qzy}\right)+D_y^{r-1-i}\left(\frac{\frac{-q}{p}}{1-zy}\right)$$

$$=\frac{1}{p}D_y^{r-1-i}\left(\frac{1}{1-qzy}\right)-\frac{q}{p}D_y^{r-1-i}\left(\frac{1}{1-zy}\right)$$

$$=\frac{1}{p}\frac{1-(qz)^{r-i}}{1-qz}-\frac{q}{p}\frac{1-z^{r-i}}{1-z}$$

$$=\frac{\frac{1}{p}-\frac{1}{p}(qz)^{r-i}-\frac{q}{p}+\frac{q}{p}z^{r-i}}{(1-qz)(1-z)}$$

$$=\frac{\frac{1}{p}-\frac{q}{p}+\frac{q}{p}z^{r-i}(1-q^{r-i-1})}{(1-qz)(1-z)}$$

$$=\frac{1+\frac{q}{p}(1-q^{r-i-1})z^{r-i}}{(1-qz)(1-z)}$$

So, we can return to our task

$$\psi_0^i(z) = D_x^{R-i}\left[1 - \left((1-qx)(1-x)\right) \frac{1+\frac{q}{p}(1-q^{r-i-1})x^{r-i}}{(1-qx)(1-x)}\right]$$

$$+ z^{i-R}\left[1 - (1-qz)(1-z) \frac{1+\frac{q}{p}(1-q^{r-i-1})z^{r-i}}{(1-qz)(1-z)}\right]$$

$$- z^{i-R}D_x^{R-i}\left[1 - (1-qzx)(1-zx) \frac{1+\frac{q}{p}(1-q^{r-i-1})(zx)^{r-i}}{(1-qzx)(1-zx)}\right]$$

$$= D_x^{R-i}\left[1- \ 1-\frac{q}{p}(1-q^{r-i-1})x^{r-i}\right] + z^{i-R}\left[1-1-\frac{q}{p}(1-q^{r-i-1})z^{r-i}\right]$$

$$- z^{i-R}D_x^{R-i}\left[1-1-\frac{q}{p}(1-q^{r-i-1})(zx)^{r-i}\right]$$

$$= -\frac{q}{p}\left(1-q^{r-i-1}\right)D_x^{R-i}\left[x^{r-i}\right] - \frac{q}{p}\left(1-q^{r-i-1}\right)z^{r-R} + \frac{q}{p}(1$$

$$- q^{r-i-1})z^{r-R}D_x^{R-i}\left[x^{r-i}\right]$$

$$= -\frac{q}{p}\left(1-q^{r-i-1}\right) - \frac{q}{p}\left(1-q^{r-i-1}\right)z^{r-R} + \frac{q}{p}\left(1-q^{r-i-1}\right)z^{r-R}$$

$$= \frac{q}{p}\left(q^{r-i-1}-1\right)$$

So, we have

$$\psi_0^0(z) = \frac{q}{p}(q^{2-0-1} - 1) = \frac{q}{p}(q-1) = -q, \;\; \psi_0^1(z) = \frac{q}{p}(q^{2-1-1} - 1)$$

$$= \frac{q}{p}(0-1) = -\frac{q}{p}$$

Since we know pgf of Q_n and it is given as

$$P(z) = \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[p_0[-qz^3 - 1] + p_1\left[-\frac{q}{p}z^3 - z\right] + p_2[z^3 - z^2]\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[-qz^3p_0 - p_0 - \frac{q}{p}z^3p_1 - zp_1 + p_2z^3 - p_2z^2\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[(-qp_0 - \frac{q}{p}p_1)z^3 - (p_0 + zp_1 + p_2z^2 - p_2z^3)\right]$$

3.2. Applications on a Quorum Queue with N-policy and Multiple Vacations.

This section displays how can find the pgf of the quorum queue size when it is controlled by N-policy and multiple vacations simultaneously. We are concerned with testing this model by using numerous stations for the arrival and vacations times in the following cases:

<u>Case I</u>:-

When the vacation times are subjected to exponentially distributed with parameter μ and the input is exposed to the ordinary Poisson process $(a(z) = z)$. Let $r = 2, R = 3$. Now, we begin to derive

$$\beta_1(z,0) = \beta_1\big(\lambda - \lambda a(z)\big) = \frac{\mu}{\mu + \lambda - \lambda a(z)} = \frac{\mu}{\mu + \lambda - \lambda z}$$

If we have

$$1 - \beta_1(x,0) = 1 - \frac{\mu}{\mu + \lambda - \lambda x} = \frac{\mu + \lambda - \lambda x - \mu}{\mu + \lambda - \lambda x} = \frac{\lambda(1-x)}{\mu + \lambda - \lambda x}$$

and

$$\frac{1}{1 - \beta_1(xy,0)} = \frac{1}{\frac{\lambda(1-xy)}{\mu + \lambda - \lambda xy}} = \frac{\mu + \lambda - \lambda xy}{\lambda(1-xy)} = \frac{\mu}{\lambda}\frac{1}{(1-xy)} + \frac{\lambda(1-xy)}{\lambda(1-xy)}$$

$$= \frac{\mu}{\lambda}\frac{1}{(1-xy)} + 1$$

Then

$$D_y^{r-1-i}\left(\frac{1}{1 - \beta_1(xy,0)}\right) = D_y^{r-1-i}\left(\frac{\mu}{\lambda}\frac{1}{(1-xy)} + 1\right) = \frac{\mu}{\lambda}\frac{1 - x^{r-i}}{(1-x)} + 1$$

and

$$[1-\beta_1(x,0)]D_y^{r-1-i}\left(\frac{1}{1-\beta_1(xy,0)}\right)=\left[\frac{\lambda(1-x)}{\mu+\lambda-\lambda x}\right]\left[\frac{\mu}{\lambda}\frac{1-x^{r-i}}{(1-x)}+1\right]$$

$$=\frac{\lambda(1-x)}{\mu+\lambda-\lambda x}\frac{\mu}{\lambda}\frac{1-x^{r-i}}{(1-x)}+\frac{\lambda(1-x)}{\mu+\lambda-\lambda x}=\frac{\mu-\mu x^{r-i}}{\mu+\lambda-\lambda x}+\frac{\lambda(1-x)}{\mu+\lambda-\lambda x}$$

$$=\frac{\mu-\mu x^{r-i}+\lambda-\lambda x}{\mu+\lambda-\lambda x}=\frac{\mu+\lambda-\lambda x-\mu x^{r-i}}{\mu+\lambda-\lambda x}=1-\frac{\mu x^{r-i}}{\mu+\lambda-\lambda x}$$

$$=1-\frac{\mu}{\mu+\lambda}\frac{x^{r-i}}{1-\frac{\lambda}{\mu+\lambda}x}$$

So, we have

$$D_x^{R-i}\left(1-[1-\beta_1(x,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(xy,0)}\right)\right)$$

$$=D_x^{R-i}\left(1-1+\frac{\mu}{\mu+\lambda}\frac{x^{r-i}}{1-\frac{\lambda}{\mu+\lambda}x}\right)=\frac{\mu}{\mu+\lambda}D_x^{R-i}\left(\frac{x^{r-i}}{1-\frac{\lambda}{\mu+\lambda}x}\right)$$

$$=\frac{\mu}{\mu+\lambda}D_x^{R-i-r+i}\left(\frac{1}{1-\frac{\lambda}{\mu+\lambda}x}\right)=\frac{\mu}{\mu+\lambda}D_x^{R-r}\left(\frac{1}{1-\frac{\lambda}{\mu+\lambda}x}\right)$$

$$=\frac{\mu}{\mu+\lambda}\frac{1-(\frac{\lambda}{\mu+\lambda})^{R-r+1}}{1-\frac{\lambda}{\mu+\lambda}}=\frac{\mu}{\mu+\lambda}\frac{1-(\frac{\lambda}{\mu+\lambda})^{R-r+1}}{\frac{\mu+\lambda-\lambda}{\mu+\lambda}}$$

$$=\frac{\mu}{\mu+\lambda}\frac{1-(\frac{\lambda}{\mu+\lambda})^{R-r+1}}{\frac{\mu}{\mu+\lambda}}=1-(\frac{\lambda}{\mu+\lambda})^{R-r+1}$$

We need to show the following

$$\psi_0^i(z) = D_x^{R-i}\left(1-[1-\beta_1(x,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(xy,0)}\right)\right)$$

$$+z^{i-R}\left(1-[1-\beta_1(z,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(zy,0)}\right)\right)$$

$$-z^{i-R}D_x^{R-i}\left(1-[1-\beta_1(zx,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(zxy,0)}\right)\right)$$

$$=1-\left(\frac{\lambda}{\mu+\lambda}\right)^{R-r+1}+z^{i-R}\left(\frac{\mu}{\mu+\lambda}\frac{z^{r-i}}{1-\frac{\lambda}{\mu+\lambda}z}\right)$$

$$-z^{i-R}\left(1-\left(\frac{\lambda}{\mu+\lambda}z\right)^{R-r+1}\right)$$

So, we can note that

$$\psi_0^i(z)=1-\left(\frac{\lambda}{\mu+\lambda}\right)^{R-r+1}+\frac{\mu}{\mu+\lambda}\frac{z^{r-R}}{1-\frac{\lambda}{\mu+\lambda}z}-z^{i-R}+(\frac{\lambda}{\mu+\lambda})^{R-r+1}z^{i-r+1}$$

Since $r=2, R=3$, and $i=0,1$, then we find

$$\psi_0^0(z)=1-\left(\frac{\lambda}{\mu+\lambda}\right)^{3-2+1}+\frac{\mu}{\mu+\lambda}\frac{z^{2-3}}{1-\frac{\lambda}{\mu+\lambda}z}-z^{0-3}+\left(\frac{\lambda}{\mu+\lambda}\right)^{3-2+1}z^{0-2+1}$$

$$= 1 - \left(\frac{\lambda}{\mu+\lambda}\right)^2 + \frac{\mu}{\mu+\lambda}\frac{z^{-1}}{1-\frac{\lambda}{\mu+\lambda}z} - z^{-3} + \left(\frac{\lambda}{\mu+\lambda}\right)^2 z^{-1}$$

$$\psi_0^1(z) = 1 - \left(\frac{\lambda}{\mu+\lambda}\right)^{3-2+1} + \frac{\mu}{\mu+\lambda}\frac{z^{2-3}}{1-\frac{\lambda}{\mu+\lambda}z} - z^{1-3} + (\frac{\lambda}{\mu+\lambda})^{3-2+1} z^{1-2+1}$$

$$= 1 - \left(\frac{\lambda}{\mu+\lambda}\right)^2 + \frac{\mu}{\mu+\lambda}\frac{z^{-1}}{1-\frac{\lambda}{\mu+\lambda}z} - z^{-2} + \left(\frac{\lambda}{\mu+\lambda}\right)^2$$

$$= 1 + \frac{\mu}{\mu+\lambda}\frac{z^{-1}}{1-\frac{\lambda}{\mu+\lambda}z} - z^{-2}$$

Thus, we see that

$$P(z) = \frac{\beta(\lambda-\lambda z)}{[z^3-\beta(\lambda-\lambda z)]}\left[\sum_{i=0}^{2-1} p_i[z^3\psi_0^i(z) - z^i] + \sum_{i=2}^{3-1} p_i[z^3 - z^i]\right]$$

$$= \frac{\beta(\lambda-\lambda z)}{[z^3-\beta(\lambda-\lambda z)]}[p_0[z^3\psi_0^0(z) - z^0] + p_1[z^3\psi_0^1(z) - z^1] + p_2[z^3 - z^2]]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[p_0\left[z^3\left(1 - \left(\frac{\lambda}{\mu+\lambda}\right)^2 + \frac{\mu}{\mu+\lambda}\frac{z^{-1}}{1-\frac{\lambda}{\mu+\lambda}z} - z^{-3}\right.\right.\right.$$

$$\left.\left. + \left(\frac{\lambda}{\mu+\lambda}\right)^2 z^{-1}\right) - 1\right]$$

$$\left. + p_1\left[z^3\left(1 + \frac{\mu}{\mu+\lambda}\frac{z^{-1}}{1-\frac{\lambda}{\mu+\lambda}z} - z^{-2}\right) - z\right] + p_2[z^3 - z^2]\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[p_0\left[z^3 - \left(\frac{\lambda}{\mu+\lambda}\right)^2 z^3 + \frac{\mu}{\mu+\lambda}\frac{z^2}{1-\frac{\lambda}{\mu+\lambda}z} - 1\right.\right.$$

$$\left. + \left(\frac{\lambda}{\mu+\lambda}\right)^2 z^2 - 1\right] + p_1\left[z^3 + \frac{\mu}{\mu+\lambda}\frac{z^2}{1-\frac{\lambda}{\mu+\lambda}z} - z - z\right] + p_2 z^3$$

$$\left. - p_2 z^2\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]} \left[p_0 \left[\left(1 - \left(\frac{\lambda}{\mu + \lambda} \right)^2 \right) z^3 \right.\right.$$

$$\left. + \left(\frac{\mu}{\mu + \lambda} \frac{1}{1 - \frac{\lambda}{\mu + \lambda} z} + \left(\frac{\lambda}{\mu + \lambda} \right)^2 \right) z^2 - 2 \right]$$

$$\left. + p_1 \left[z^3 + \frac{\mu}{\mu + \lambda} \frac{z^2}{1 - \frac{\lambda}{\mu + \lambda} z} - 2z \right] + p_2 z^3 - p_2 z^2 \right]$$

So, we have

$$P(z) = \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]} \left[p_0 \left[(1 - K^2) z^3 + \left((1 - K) \frac{1}{1 - Kz} + K^2 \right) z^2 - 2 \right] \right.$$

$$\left. + p_1 \left[z^3 + (1 - K) \frac{z^2}{1 - Kz} - 2z \right] + p_2 z^3 - p_2 z^2 \right]$$

where

$$K = \frac{\lambda}{\mu + \lambda}, 1 - K = \frac{\mu}{\mu + \lambda}$$

We can also write it as

$$P(z) = \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\Bigg[\big((1-K^2)p_0 + p_1 + p_2\big)z^3$$

$$+ \left(\left((1-K)\frac{1}{1-Kz} + K^2\right)p_0 + \left((1-K)\frac{1}{1-Kz}\right)p_1 - p_2\right)z^2$$

$$- 2p_1 z - 2p_0\Bigg]$$

<u>Case II</u>:-

In this case, we consider the input is modelling as type 1 geometrically distributed with parameter p. Moreover, the vacation times have the same distribution in Case I (exponentially distributed with parameter μ). In addition, we assume that $r = 2, R = 3$. So, we have

$$a(z) = \frac{pz}{1-qz}, p = 1 - q$$

And

$$\beta_1(z,0) = \beta_1\big(\lambda - \lambda a(z)\big) = \frac{\mu}{\mu + \lambda - \lambda a(z)}$$

$$= \frac{\mu}{\mu + \lambda - \lambda \frac{pz}{1-qz}}$$

$$= \frac{\mu(1-qz)}{(\mu+\lambda)(1-qz) - \lambda pz}$$

$$\beta_1(z,0) = \frac{\mu(1-qz)}{\mu+\lambda-\mu qz-\lambda qz-\lambda pz}$$

$$= \frac{\mu(1-qz)}{\mu+\lambda-\mu qz-\lambda(p+q)z}$$

$$= \frac{\mu(1-qz)}{\mu+\lambda-\mu qz-\lambda z}$$

$$= \frac{\mu(1-qz)}{\mu+\lambda-(\mu q+\lambda)z}$$

Then

$$1-\beta_1(x,0) = 1-\frac{\mu(1-qx)}{\mu+\lambda-(\mu q+\lambda)x} = \frac{\mu+\lambda-(\mu q+\lambda)x-\mu+\mu qx}{\mu+\lambda-(\mu q+\lambda)z}$$

$$= \frac{\lambda-\mu qx-\lambda x+\mu qx}{\mu+\lambda-(\mu q+\lambda)x} = \frac{\lambda-\lambda x}{\mu+\lambda-(\mu q+\lambda)x} = \frac{\lambda(1-x)}{\mu+\lambda-(\mu q+\lambda)x}$$

and

$$\frac{1}{1-\beta_1(xy,0)} = \frac{1}{\frac{\lambda(1-xy)}{\mu+\lambda-(\mu q+\lambda)xy}} = \frac{\mu+\lambda-(\mu q+\lambda)xy}{\lambda(1-xy)}$$

$$= \frac{\mu+\lambda}{\lambda}\frac{1}{1-xy} - \frac{(\mu q+\lambda)}{\lambda}\frac{xy}{1-xy}$$

Now, we are ready to find the following

$$D_y^{r-1-i}\left(\frac{1}{1-\beta_1(xy,0)}\right) = D_y^{r-1-i}\left(\frac{\mu+\lambda}{\lambda}\frac{1}{1-xy} - \frac{(\mu q+\lambda)}{\lambda}\frac{xy}{1-xy}\right)$$

$$= \frac{\mu+\lambda}{\lambda} D_y^{r-1-i}\left(\frac{1}{1-xy}\right) - \frac{(\mu q+\lambda)}{\lambda} D_y^{r-1-i}\left(\frac{xy}{1-xy}\right)$$

$$= \frac{\mu+\lambda}{\lambda}\frac{1-x^{r-1-i+1}}{1-x} - \frac{(\mu q+\lambda)x}{\lambda} D_y^{r-1-i-1}\left(\frac{1}{1-xy}\right)$$

$$= \frac{\mu+\lambda}{\lambda}\frac{1-x^{r-i}}{1-x} - \frac{(\mu q+\lambda)x}{\lambda}\frac{1-x^{r-i-2+1}}{1-x}$$

$$= \frac{(\mu+\lambda)\left(1-x^{r-i}\right) - (\mu q+\lambda)\left(x-x^{r-i}\right)}{\lambda(1-x)}$$

$$= \frac{(\mu+\lambda) - (\mu+\lambda)x^{r-i} - (\mu q+\lambda)x + (\mu q+\lambda)x^{r-i}}{\lambda(1-x)}$$

$$= \frac{(\mu+\lambda) - (\mu q+\lambda)x - [(\mu+\lambda) - (\mu q+\lambda)]x^{r-i}}{\lambda(1-x)}$$

$$= \frac{(\mu+\lambda) - (\mu q+\lambda)x - [\mu+\lambda-\mu q-\lambda]x^{r-i}}{\lambda(1-x)}$$

$$= \frac{(\mu+\lambda) - (\mu q+\lambda)x - [\mu-\mu q]x^{r-i}}{\lambda(1-x)}$$

$$= \frac{(\mu+\lambda) - (\mu q+\lambda)x - \mu p x^{r-i}}{\lambda(1-x)}$$

and it is easy to note that

$$[1-\beta_1(x,0)]D_y^{r-1-i}\left(\frac{1}{1-\beta_1(xy,0)}\right)$$

$$=\frac{\lambda(1-x)}{\mu+\lambda-(\mu q+\lambda)x}\frac{(\mu+\lambda)-(\mu q+\lambda)x-\mu p x^{r-i}}{\lambda(1-x)}$$

$$=\frac{(\mu+\lambda)-(\mu q+\lambda)x-\mu p x^{r-i}}{\mu+\lambda-(\mu q+\lambda)x}$$

$$[1-\beta_1(x,0)]D_y^{r-1-i}\left(\frac{1}{1-\beta_1(xy,0)}\right)$$

$$=\frac{\lambda(1-x)}{\mu+\lambda-(\mu q+\lambda)x}\frac{(\mu+\lambda)-(\mu q+\lambda)x-\mu p x^{r-i}}{\lambda(1-x)}$$

$$=\frac{(\mu+\lambda)-(\mu q+\lambda)x-\mu p x^{r-i}}{\mu+\lambda-(\mu q+\lambda)x}$$

and then we show that

$$1-[1-\beta_1(x,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(xy,0)}\right)=1-\frac{(\mu+\lambda)-(\mu q+\lambda)x-\mu p x^{r-i}}{\mu+\lambda-(\mu q+\lambda)x}$$

$$=\frac{(\mu+\lambda)-(\mu q+\lambda)x-(\mu+\lambda)+(\mu q+\lambda)x+\mu p x^{r-i}}{\mu+\lambda-(\mu q+\lambda)x}$$

$$=\frac{\mu p x^{r-i}}{\mu+\lambda-(\mu q+\lambda)x}=\frac{\mu p}{\mu+\lambda}\frac{x^{r-i}}{1-\frac{(\mu q+\lambda)}{\mu+\lambda}x}$$

The above result leads us to find the following

$$D_x^{R-i}\left(1-[1-\beta_1(x,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(xy,0)}\right)\right)$$

$$=D_x^{R-i}\left(\frac{\mu p}{\mu+\lambda}\frac{x^{r-i}}{1-\frac{(\mu q+\lambda)}{\mu+\lambda}x}\right)$$

$$=\frac{\mu p}{\mu+\lambda}D_x^{R-i-r+i}\left(\frac{1}{1-\frac{(\mu q+\lambda)}{\mu+\lambda}x}\right)$$

$$=\frac{\mu p}{\mu+\lambda}D_x^{R-r}\left(\frac{1}{1-\frac{(\mu q+\lambda)}{\mu+\lambda}x}\right)=\frac{\mu p}{\mu+\lambda}\frac{1-\left[\frac{(\mu q+\lambda)}{\mu+\lambda}\right]^{R-r+1}}{1-\frac{(\mu q+\lambda)}{\mu+\lambda}}$$

$$=\frac{1-\left[\frac{(\mu q+\lambda)}{\mu+\lambda}\right]^{R-r+1}}{\frac{\mu+\lambda}{\mu p}-\frac{(\mu+\lambda)}{\mu p}\frac{(\mu q+\lambda)}{(\mu+\lambda)}}=\frac{1-\left[\frac{(\mu q+\lambda)}{\mu+\lambda}\right]^{R-r+1}}{\frac{\mu+\lambda-\mu q-\lambda}{\mu p}}$$

$$=\frac{1-\left[\frac{(\mu q+\lambda)}{\mu+\lambda}\right]^{R-r+1}}{\frac{\mu(1-q)}{\mu p}}=\frac{1-\left[\frac{(\mu q+\lambda)}{\mu+\lambda}\right]^{R-r+1}}{\frac{\mu p}{\mu p}}$$

$$=1-\left[\frac{(\mu q+\lambda)}{\mu+\lambda}\right]^{R-r+1}$$

Again, we return to put things together to find the following:

$$\psi_0^i(z)=D_x^{R-i}\left(1-[1-\beta_1(x,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(xy,0)}\right)\right)$$

$$+z^{i-R}\left(1-[1-\beta_1(z,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(zy,0)}\right)\right)$$

$$-z^{i-R}D_x^{R-i}\left(1-[1-\beta_1(zx,0)]\,D_y^{r-1-i}\left(\frac{1}{1-\beta_1(zxy,0)}\right)\right)$$

$$=1-\left[\frac{(\mu q+\lambda)}{\mu+\lambda}\right]^{R-r+1}+z^{i-R}\left(\frac{\mu p}{\mu+\lambda}\frac{z^{r-i}}{1-\frac{(\mu q+\lambda)}{\mu+\lambda}z}\right)$$

$$-z^{i-R}\left(1-\left[\frac{(\mu q+\lambda)}{\mu+\lambda}z\right]^{R-r+1}\right)$$

$$=1-\left[\frac{(\mu q+\lambda)}{\mu+\lambda}\right]^{R-r+1}+\frac{\mu p}{\mu+\lambda}\frac{z^{r-R}}{1-\frac{(\mu q+\lambda)}{\mu+\lambda}z}-z^{i-R}$$

$$+z^{i-R+R-r+1}\left[\frac{(\mu q+\lambda)}{\mu+\lambda}\right]^{R-r+1}$$

$$=1-\left[\frac{(\mu q+\lambda)}{\mu+\lambda}\right]^{R-r+1}+\frac{\mu p}{\mu+\lambda}\frac{z^{r-R}}{1-\frac{(\mu q+\lambda)}{\mu+\lambda}z}-z^{i-R}$$

$$+z^{i-r+1}\left[\frac{(\mu q+\lambda)}{\mu+\lambda}\right]^{R-r+1}$$

$$=1-z^{i-R}+\left(z^{i-r+1}-1\right)\left[\frac{(\mu q+\lambda)}{\mu+\lambda}\right]^{R-r+1}+\frac{\mu p}{\mu+\lambda}\frac{z^{r-R}}{1-\frac{(\mu q+\lambda)}{\mu+\lambda}z}$$

Obviously, we get

$$\psi_0^i(z) = 1 - z^{i-R} + \left(z^{i-r+1} - 1\right)\left[\frac{(\mu q + \lambda)}{\mu + \lambda}\right]^{R-r+1} + \frac{\mu p}{\mu + \lambda}\frac{z^{r-R}}{1 - \frac{(\mu q + \lambda)}{\mu + \lambda}z}$$

Therefore, we can find $\psi_0^0(z), \psi_0^1(z)$, respectively by considering $r = 2, R = 3$, as the following

$$\psi_0^0(z) = 1 - z^{0-R} + (z^{0-r+1} - 1)\left[\frac{(\mu q + \lambda)}{\mu + \lambda}\right]^{R-r+1} + \frac{\mu p}{\mu + \lambda}\frac{z^{r-R}}{1 - \frac{(\mu q + \lambda)}{\mu + \lambda}z}$$

$$= 1 - z^{-R} + (z^{-r+1} - 1)\left[\frac{(\mu q + \lambda)}{\mu + \lambda}\right]^{R-r+1} + \frac{\mu p}{\mu + \lambda}\frac{z^{r-R}}{1 - \frac{(\mu q + \lambda)}{\mu + \lambda}z}$$

and

$$\psi_0^1(z) = 1 - z^{1-R} + (z^{1-r+1} - 1)\left[\frac{(\mu q + \lambda)}{\mu + \lambda}\right]^{R-r+1} + \frac{\mu p}{\mu + \lambda}\frac{z^{r-R}}{1 - \frac{(\mu q + \lambda)}{\mu + \lambda}z}$$

$$= 1 - z^{1-R} + (z^{2-r} - 1)\left[\frac{(\mu q + \lambda)}{\mu + \lambda}\right]^{R-r+1} + \frac{\mu p}{\mu + \lambda}\frac{z^{r-R}}{1 - \frac{(\mu q + \lambda)}{\mu + \lambda}z}$$

Now, we are ready to find

$$P(z) = \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[\sum_{i=0}^{2-1} p_i[z^3\psi_0^i(z) - z^i] + \sum_{i=2}^{3-1} p_i[z^3 - z^i]\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\left[\sum_{i=0}^{2} p_i[z^3\psi_0^i(z) - z^i] + \sum_{i=2}^{2} p_i[z^3 - z^i]\right]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}[p_0[z^3\psi_0^0(z) - z^0] + p_1[z^3\psi_0^1(z) - z^1] + p_2[z^3 - z^2]]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}[p_0[z^3\psi_0^0(z) - 1] + p_1[z^3\psi_0^1(z) - z] + p_2[z^3 - z^2]]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]}\Bigg[p_0\Bigg[z^3\Bigg(1 - z^{-R} + (z^{-r+1} - 1)\left[\frac{(\mu q + \lambda)}{\mu + \lambda}\right]^{R-r+1}$$

$$+ \frac{\mu p}{\mu + \lambda}\frac{z^{r-R}}{1 - \frac{(\mu q + \lambda)}{\mu + \lambda}z}\Bigg) - 1\Bigg]$$

$$+ p_1\Bigg[z^3\Bigg(1 - z^{1-R} + (z^{2-r} - 1)\left[\frac{(\mu q + \lambda)}{\mu + \lambda}\right]^{R-r+1}$$

$$+ \frac{\mu p}{\mu + \lambda}\frac{z^{r-R}}{1 - \frac{(\mu q + \lambda)}{\mu + \lambda}z}\Bigg) - z\Bigg] + p_2[z^3 - z^2]\Bigg]$$

$$= \frac{\beta(\lambda - \lambda z)}{[z^3 - \beta(\lambda - \lambda z)]} \Bigg[p_0 \Bigg[z^3 - z^{3-R} + (z^{4-r} - z^3) \left[\frac{(\mu q + \lambda)}{\mu + \lambda}\right]^{R-r+1}$$

$$+ \frac{\mu p}{\mu + \lambda} \frac{z^{r-R+3}}{1 - \frac{(\mu q + \lambda)}{\mu + \lambda} z} - 1 \Bigg]$$

$$+ p_1 \Bigg[z^3 - z^{4-R} + (z^{5-r} - z^3) \left[\frac{(\mu q + \lambda)}{\mu + \lambda}\right]^{R-r+1}$$

$$+ \frac{\mu p}{\mu + \lambda} \frac{z^{r-R+3}}{1 - \frac{(\mu q + \lambda)}{\mu + \lambda} z} - z \Bigg] + p_2 [z^3 - z^2] \Bigg]$$

Chapter Four

Conclusions and Future Works

Chapter Four

Conclusions and Future Works

4.1. Conclusions

In this thesis, we derive an explicit general formula of pgf for queue size when the quorum queue is subjected to N-policy and multiple vacations. Moreover, we obtain the directed expression for this pgf when the input of the system has various distributions of arrivals under different times of vacations. By using the first-level access theory, these investigations build on the random walk analysis. In this occurrence, the number of units in the queue hits level r to affect the server's job. In fact, the server is active when the line has r components or more and serves R units or less. Otherwise, this server will be idle. We assume the arrival batches are ordinary Poisson processes, type 1 geometrically distributed, and other kinds of arrival batches. Under the same above conditions of N-policy, the server is subjected to multiple vacations where this server proceeds to rest in the sequence of breaks instead of being idle. We can extend our results to more general systems by adding more conditions on the server.

4.2. Future Works

In the upcoming, we intend to obtain the pgf of queue magnitude when we request to generalize the bulk input to be distributed as Poisson or Binormal distributions, and the vacation time has Erlank or phase-type distribution. It is crucial to indicate that Erlank or phase-type distribution are the generalizations of the exponential distribution.

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الخلاصة

هذه الرسالة تبحث في أنظمة طابور النخبة ذات سياسة N واستراحات من خلال النظر في توزيعات ازمنة الوصول والاستراحات المختلفة. يتم تحليل عمليات المشي العشوائية في قوائم الانتظار باستخدام نظرية المستوى الزائد الأول. حيث ان في هذا النموذج ، يكون الخادم خاملاً او يأخذ استراحات متعددة عندما يكون حجم قائمة الانتظار أقل من r ؛ وإلا فإنه يعمل بمعالجة R من الوحدات أو أقل. تم اشتقاق دالة المولدة لاحتمالية حجم الطابورعلى وجه التحديد في ظل توزيعات مختلفة لأوقات الوصول والإجازة ، و على سبيل المثال عملية Poisson العادية والتوزيع الهندسي من النوع 1. ان هذا العمل يقدم صيغًا واضحة لدوال المولدة لاحتمالات لهذا النموذج بدفعات وصول وأوقات استراحات مختلفة.

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حول طوابير النخبة ذات الاستراحات

رسالة

مقدمة إلى مجلس كلية التربية للعلوم الصرفة / جامعة بابل كجزء من متطلبات نيل درجة الماجستير في التربية / الرياضيات

من قبل

حنين حيدر محمد حسين كاظم

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1445 هـ **2023 م**