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Some Conceptions of Chaotic Properties in Fuzzy Dynamical Systems

A Thesis

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By

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بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

***(يَرْفَعِ اللّهُ الّذِينَ آمَنُوا مِنْكُم وَالّذِينَ أوتُوا الْعِلْم
دَرَجَاتٍ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ)***

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Contents

Dedication

I dedicate the fruit of my effort to

My mother…

My father…

My brothers and sisters…

My friends…

Acknowledgments

First I would like to thank Allah Almighty and the Prophet Muhammad , peace be upon him and his family, and the Ahl al-Bayt, peace be upon

I would like to thank my supervisor Dr. May.A.A.AL-Yaseen for helping me at every step to complete this work. Deep appreciation and love to all my family especially my mother and my father for supporting me and encourage to complete my studies.

Abstract

In this thesis, we will rely on the definition given by George and Veeramani in 1994, which is a modification of the concept of fuzzy metric space introduced by Kramosil and Michalek in 1975.

This thesis involves the study of some chaotic properties in fuzzy dynamical systems and introduced new definitions (fuzzy positive expansive, fuzzy Lipschitz, fuzzy stable topology, fuzzy L-shadowing property, fuzzy two-sided limit shadowing property, fuzzy negative limit shadowing, fuzzy h-shadow).

Through this study, we prove the properties (fuzzy L-shadowing property, fuzzy two –sided limit shadowing) are invariant under fuzzy topological conjugacy.

Our result indicates that two fuzzy dynamical systems on two fuzzy metric spaces achieve the properties (fuzzy expansive, fuzzy Lipschitz, fuzzy L-shadowing property, fuzzy two-sided limit shadowing property, fuzzy positive expansive, fuzzy h-shadow) if and only if the product up to a special t-norm achieves the above mentioned properties.

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List of Symbols

Symbols	*Description*
$\boldsymbol{f}$	**Fuzzy homeomorphism**
$*$	***Continuous t-norm***
$\boldsymbol{X}$	**non-empty set**
$(\boldsymbol{X}, \boldsymbol{M}, *)$	**Fuzzy metric space**
$(\boldsymbol{x_k})$	**Infinite sequence**
$(\boldsymbol{x_m})$	**Finite sequence**
$\boldsymbol{f^{-1}}$	**Invers function of $\boldsymbol{f}$**
$\mathbb{N}$	**The set of natural numbers**
$\boldsymbol{h}$	**Conjugate map**
$\mathbb{Z}$	**The set of integer numbers**
$\boldsymbol{\mathcal{L}}$	**Lipschitz constant**
$\boldsymbol{I_X}$	**Identity map on $\boldsymbol{X}$**
$\boldsymbol{t}$	**Time**
T	$= \boldsymbol{f} \times \boldsymbol{g}$ **product map**
$\mathbb{N}_\circ$	$\mathbb{N} \cup \{\boldsymbol{0}\}$
$\lvert k \rvert$	**The absolute value of $\boldsymbol{k}$**
$-\mathbb{N}$	**The set of negative natural numbers**
$\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{t})$	**The fuzzy open ball**

Introduction

In 1965, Zadeh introduced the theory of fuzzy sets. Later, in 1975, Kramosil defined fuzzy metric spaces, and Grabiecin 1988 studied fixed points in the fuzzy metric space [10]. In 1994, Greori and Veeramani modification of the concept of Kramosil and studied some results in the fuzzy metric space [8]. Roman-Flores and Chalca-Cano in 2008 studied it Some properties of chaos in the fuzzy dynamical systems [23]. In 2011, Kupka expanded the conjugacy between fuzzy dynamical systems [18]. In 2014 he studied some chaotic and mixing properties of fuzzy dynamical systems [19].

In 2012 Ahmedi Seyyed and Molaei Mohammad defined shadowing property, limit shadowing property for fuzzy dynamical systems and prove that this property is invariant under fuzzy topological conjugacy, by using of dynamical systems with the shadowing property they presented a method to construct fuzzy dynamical systems with the fuzzy shadowing property [1]. Mehdi Fatehi Nia defined topological transitivity on fuzzy dynamical systems in 2018 [22]. Taixiang Sun a, Caihong Han and etc defined periodic points in fuzzy metric in 2021 [26].

Some implementations of fuzzy metric spaces a fast impulsive noise color image filter using fuzzy metrics [21]. Colour image smoothing through a soft-switching mechanism using a graph model [15].

This work is divided into two chapters. Chapter one consists of two sections. Some basic definitions and some concepts of chaos in fuzzy dynamical systems in section one. We have demonstrated some fundamental dynamical properties in the fuzzy systems in section two.

Chapter two consists two sections, we have demonstrated some properties under fuzzy topological conjugation and some shadowing properties in fuzzy dynamical systems in section one. We studied properties (fuzzy expansive, fuzzy positive expansive, fuzzy Lipschitz, fuzzy L-shadowing property, fuzzy two-sided limit-shadowing) for the product of fuzzy dynamical systems in section two.

List of Publications Arising from This Thesis

- S. E. A. Askar and M. A. A. AL-Yaseen, Shadowing in Fuzzy Dynamical Systems, Journal of Kufa for Mathematics and Computer, http://dx.doi.org/10.31642/JoKMC/2018/100224.

Chapter One

Preliminaries

1.1.Basic Definitions

In this section we recall some fundamental definitions. We introduce the notions(fuzzy positive expansive , fuzzy Lipschitz , fuzzy L-shadowing property, fuzzy h-shadow , fuzzy two-sided limit shadowing property, fuzzy simple two-sided limit shadowing property , fuzzy negative limit shadowing , fuzzy topological stability) for fuzzy dynamical systems.

For this purpose we assume that $(X, M, f, *)$ a fuzzy dynamical system, i.e.,$(X, M, *)$ is a fuzzy metric space and $f : X \longrightarrow X$ is fuzzy homeomorphism, X is a non-empty set and $*$ is a continuous $t - norm$.

Definition 1.1.1: [26]

A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous triangular norm ($\boldsymbol{t - norm}$) if ([0,1],$*$) topological monoid with unit 1 such is a that $a * b \leq c * d$ whenever $a \leq c \; and \; b \leq d$ $(a, b, c, d \in$ [0,1]).

Definition 1.1.2:[8]

The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5. $M(x, y, \cdot) : (0, \infty) \longrightarrow [0,1]$ is a continuous,

$x, y, z \in X \; and \; t, s > 0.$

Example 1.1.3:[8]

Let $X = \mathbb{R}$. Define $a * b = ab$ and

$$M(x, y, t) = \left[exp \left(\frac{|x - y|}{t} \right) \right]$$

for all $x, y \in X$ and $t \in (0, \infty)$. Then $(X, M, *)$ is a fuzzy metric space.

Proof. (1) Clearly $M(x, y, t) = 1$ if and only if $x = y$.

(2) $M(x, y, t) = M(y, x, t)$.

(3) To prove $M(x, y, t)\, M(y, z, s) \leq M(x, z, t + s)$,

we know that

$$|x - z| \leq \left(\frac{t + s}{t} \right) |x - y| + \left(\frac{t + s}{s} \right) |y - z|,$$

i.e.

$$\frac{|x - z|}{t + s} \leq \frac{|x - y|}{t} + \frac{|y - z|}{s}.$$

Therefor

$$exp \left(\frac{|x - z|}{t + s} \right) \leq exp \left(\frac{|x - y|}{t} \right) exp \left(\frac{|y - z|}{s} \right).$$

Thus

$$M(x, y, t)\, M(y, z, s) \leq M(x, z, t + s).$$

(4) $M(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1]$ is a continuous.

Hence $(X, M, *)$ is fuzzy metric space.

Definition 1.1.4: [8]

Let $(X, M,*)$ be a fuzzy metric space. We define **fuzzy open ball** $B(x, r, t)$ *with center* $x \in X$ *and radius*, $0 < r < 1$ as

$$B(x,r,t) = \{ y \in X;\ M(x,y,t) > 1 - r\}.$$

Result 1.1.5:[8]

Every open ball is an open set.

Definition 1.1.6:[10]

A sequence (x_k) in a *fuzzy* metric space $(X, M,*)$ is **Cauchy** if $\lim_{k\to\infty} M(x_{k+p}, x_k, t) = 1$ *for each* $t > 0$ *and* $p > 0$. A sequence (x_k) in X is **convergent** to $x \in X$ if $\lim_{k\to\infty} M(x_k, x, t) = 1$ for each $t > 0$. Notation : $\lim_{k\to\infty} x_k = x$ (Since $*$ is continuous**,** it follows $M(x,y,t) * M(y,z,s) \leq M(x,z,t + s)$ the limit is uniquely determined).

A fuzzy metric space in which every Cauchy sequence is convergent is complete. It is called **compact** if every sequence contains a convergent subsequence.

Definition 1.1.7:[9]

A fuzzy map $f: X \to X$ is **fuzzy continuous at** x_0, for each $0 < \varepsilon < 1$ *and* $t > 0$ there is $0 < \delta < 1$ so that for each $x \in X$ with $M(x_0, y, t) > 1 - \delta$, we deduce $M(f(x_0), f(y), t) > 1 - \varepsilon$.

Definition 1.1.8:[1]

We say that two fuzzy maps $f : X \to X$ and $g : Y \to Y$ on fuzzy metric spaces $(X, M,*)$ and $(Y, M',*')$ are **topologically conjugate** if there is a fuzzy homeomorphism $h : X \to Y$ (a fuzzy continuous bijection map with fuzzy continuous inverse) so that $h \circ f = g \circ h$.

Definition 1.1.9:[3]

Let (X, d) be compact metric space, and $f: X \to X$ a homeomorphism . We say that f has **L-shadowing property**, if for every $\varepsilon > 0$, there exists $\delta > 0$, such that for every sequence $(x_k)_{k\in\mathbb{Z}} \subset X$ satisfying $d(f\ (x_k), x_{k+1}) \leq \delta$, for every $k \in \mathbb{Z}$ and $d(f\ (x_k), x_{k+1}) \to 0$ when $|k| \to \infty$, there is $p \in X$ satisfying $d(f^k(p), x_k) \leq \varepsilon$ for every

$k \in \mathbb{Z}$, and $d(f^k(p), x_k) \to 0,\ when\ |k| \to \infty$.

Definition1.1.10:[21]

Let (X, d) be compact metric space and $f: X \longrightarrow X$ be continuous .We say that f has **h-shadow** if and only if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every finite $\delta - pseudo\ orbit$ $\{x_0, x_1, \dots, x_m\} \subseteq X$ there is $p \in X$ such that $d(f^k(p), x_k) \leq \varepsilon$ for every $k, m \in \mathbb{Z}, k < m$ and $f^m(p) = x_m$.

Definition 1.1.11:[5]

Let (X, d) be compact metric space and $f: X \to X$ a homeomorphism. The **limit shadowing property** for f^{-1} means the following: for every sequence

$(x_k)_{k\in\mathbb{N}_0} \subset X$ satisfying

$$d\big(f^{-1}(x_k), (x_{k+1})\big) \to 0, k \to \infty.$$

There exists $p \in X$ satisfying

$$d(f^{-k}(p),(x_k)) \to 0, k \to \infty.$$

During the text we will need the following property: for every sequence$(x_k)_{k\in -\mathbb{N}_0} \subset X$ ($-\mathbb{N}_0$denotes the set of non-positive integers) satisfying

$$\mathrm{d}(f^{-1}(x_k),(x_{k-1})) \to 0, k \to \infty.$$

There exists $p \in X$

$$\mathrm{d}(f^k(p),(x_k)) \to 0, k \to \infty.$$

Definition 1.1.12:[6]

This property will be called **negative limit shadowing** and the sequence $(x_k)_{k\in -\mathbb{N}_0}$ will be called a negative limit pseudo-orbit for f. We will say that p limit shadows $(x_k)_{k\in -\mathbb{N}_0}$in the past and that $(x_k)_{k\in -\mathbb{N}_0}$ is limit shadowed in the past by p.

Definition 1.1.13:[3]

Let(X,d) be compact metric space and $f: X \to X$ be a homeomorphism. We say that f has **the two-sided limit shadowing property** if every two-sided limit pseudo-orbit is two-sided limit shadowed.

A sequence $(x_k)_{k\in\mathbb{Z}}$ is two-sided limit pseudo-orbit if it satisfies $d(f((x_k), x_{k+1}) \to 0, |k| \to \infty$.

The sequence$(x_k)_{k\in\mathbb{Z}}$, is two-sided limit shadowed if there exists $p \in X$ satisfying $d(f^k(p), x_k) \to 0, |k| \to \infty$.

Definition 1.1.14:[5]

For $p, z \in X$ and we consider the sequence$(x_k)_{k\in\mathbb{Z}}$ defined by

$$x_k = \begin{cases} f^k(z), & k \geq 0; \\ f^k(p), & k < 0, \end{cases}$$

This sequence consists of the past orbit of x and the future orbit of z. Sequence of this type will be called simple two-sided limit pseudo-orbits. We say that f has **the simple two-sided limit shadowing property** when every simple two-sided limit pseudo-orbit is two-sided limit shadowed.

Definition 1.1.15:[1]

Let $(X, M,*)$ be a fuzzy metric space. Let $f : X \to X$ be a fuzzy homeomorphism on X.We say that f has the fuzzy pseudo-orbit tracing property on X, if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $0 < \delta < 1$ so that for a given sequence $(x_k)_{k\in\mathbb{Z}} \subset X$ with $M(f(x_k), x_{k+1}, t) > 1 - \delta$ for every $k \in \mathbb{Z}$ (called δ-$pseudo\ orbit$) there exists a point $p \in X$ $such\ that$ $M(f^k(p), x_k, t) > 1 - \varepsilon$ for every $k \in \mathbb{Z}$.

Definition 1.1.16:[1]

Let$(X, M,*)$be a fuzzy metric space. Let $f : X \to X$ be a fuzzy homeomorphism on X.We say that f has the **fuzzy limit shadowing property** on X, if for each $t > 0$ and each sequence $(x_k)_{k\in\mathbb{Z}} \subset X$ with $\lim_{k\to\infty} M(f(x_k), x_{k+1}, t) = 1$ there exists a point $p \in X$ so that $\lim_{k\to\infty} M(f^k(p), x_k, t) = 1$.

Definition 1.1.17:[4]

Let $(X, M,*)$ be a fuzzy compact metric space. A fuzzy homeomorphism $f : X \to X$ is $t -$ **fuzzy expansive,** if there is $0 < e < 1$

such that for given $x \neq y$ and $t > 0$, there exists $k \in \mathbb{Z}$ such that $M(f^k(x), f^k(y), t) \leq 1 - e$.

We will define some definitions in the fuzzy dynamical systems according to the concept of George and Veeramani, which is a modification of the concept of fuzzy metric space introduced by Kramosil and Michalek. Let $(X, M,*)$ be a fuzzy compact metric space and let $f : X \to X$ fuzzy map.

Definition 1.1.18

Let $(X, M,*)$ be a fuzzy compact metric space and let $f: X \to X$ be a fuzzy homeomorphism on X. We say that f is **fuzzy topological stable** if for every $0 < \varepsilon < 1$ *there exists* $0 < \delta < 1$ such that for every fuzzy homeomorphism g $\delta - fuzzy\ close$ to f there is a conjugation

$h : X \to X$ *between* f and g, such that $M(h, I_X) \geq 1 - \varepsilon$. We denote by I_X the identity map on X.

Definition 1.1.19

Let $(X, M,*)$ be a fuzzy compact metric space. A fuzzy continuous $f: X \longrightarrow X$ is **fuzzy positive expansive** if there is $0 < c < 1$ such that for given $x = y$ and $t > 0$, there exists $k \in \mathbb{Z}$ such that

$$M(f^k (x), f^k(y), t) \geq 1 - c.$$

Definition 1.1.20

Let $(X, M,*)$ be fuzzy compact metric space. A fuzzy continuous $f : X \to X$ has **fuzzy Lipschitz** if there exists a constant $0 < \mathcal{L} < 1$ and $t > 0$ such that $M(f(x), f(y), t) > 1 - \mathcal{L}\ M(x, y, t)$ for all $x, y \in X$.

Definition1.1.21

Let $(X, M, *)$ be a fuzzy compact metric space and $f: X \longrightarrow X$ be continuous map. We say that f has **fuzzy h-shadow** if and only if every

$0 < \varepsilon < 1$, there is $0 < \delta < 1$and $t > 0$ such that for every finite δ −pseudo orbit $\{x_0, x_1, \dots, x_m\} \subseteq X$ there is $p \in X$ *such that* $M(f^k(p), x_k, t) > 1 - \varepsilon$ for every $k, m \in \mathbb{Z}, k < m$ and $f^m(p) = x_m$.

Definition 1.1.22

Let $(X, M, *)$ be a fuzzy compact metric space. Let $f : X \to X$ a fuzzy homeomorphism map. We say that f has **fuzzy L-shadowing property** if for every $0 < \varepsilon < 1$, there exists $0 < \delta < 1$, such that for every sequence$(x_k)_{k \in \mathbb{Z}}$ satisfying $M(f(x_k), x_{k+1}, t) > 1 - \delta$ *for every* $k \in \mathbb{Z}$ and $\lim_{k \to \infty} M(f(x_k), x_{k+1}, t) = 1$, there is $p \in X$ satisfying $M(f^k(p), x_k, t) > 1 - \varepsilon$ for every $k \in \mathbb{Z}$, $\lim_{k \to \infty} M(f^k(p), x_k, t) = 1$.

Definition 1.1.23

We say that f has the **fuzzy negative limit shadowing property** on X, if for each $t > 0$ and each sequence $(x_k)_{k \in -\mathbb{N}} \subset X$ with $\lim_{k \to -\infty} M(f(x_k), x_{k+1}, t) = 1$ there exists a point $p \in X$ so that $\lim_{k \to -\infty} M(f^k(p), x_k, t) = 1$.

Definition 1.1.24

Let $(X, M, *)$ be a fuzzy compact metric space and $f: X \to X$ a fuzzy homeomorphism. The **fuzzy limit shadowing property** for f^{-1} means the following: for every sequence $(x_k)_{k \in \mathbb{N}} \subset X$ satisfying

$$\lim_{k \to \infty} M(f^{-1}(x_k), (x_{k+1}), t) = 1.$$

There exists $p \in X$ satisfying

$$\lim_{k\to\infty} M(f^{-k}(p), (x_k), t) = 1.$$

Definition 1.1.25

Let $(X, M,*)$ be a fuzzy compact metric space. Let $f: X \longrightarrow X$ a homeomorphism. We say that f has **the fuzzy two-sided limit shadowing property** if every fuzzy two-sided limit pseudo-orbit is fuzzy two-sided limit shadow.

A sequence $(x_k)_{k\in\mathbb{N}}$ of point in X is fuzzy two-sided limit pseudo-orbit if it satisfies $\lim_{k\to\infty} M(f(x_k), x_{k+1}, t) = 1$

A fuzzy two-sided limit pseudo-orbit is fuzzy two-sided limit shadowed if there is a point $p \in X$ such that $\lim_{k\to\infty} M(f^k(p), x_k, t) = 1$.

Definition 1.1.26

Let $(X, M,*)$ be a fuzzy compact metric space and $f: X \to X$ a fuzzy homeomorphism map. For $p, z \in X$ and we consider the sequence$(x_k)_{k\in\mathbb{Z}}$ defined by

$$x_k = \begin{cases} f^k(z), & k \geq 0; \\ f^k(p), & k < 0, \end{cases}$$

This sequence consists of the past orbit of p and the future orbit of z. Sequence of this type will be called f fuzzy simple one $-$ sided limit pseudo-orbits. We say that f has the fuzzy simple one $-$ sided limit shadowing property when every fuzzy simple one $-$ sided limit pseudo-orbit is fuzzy one-sided limit shadowed.

1.2.General Properties in Fuzzy Metric Space

In this section we will study the properties in [4] when the parameter t is constant. We prove that fuzzy expansive is invariant under fuzzy topological conjugation and fuzzy expansive is invariant under invers.

Proposition 1.2.1

Let $(X, M,*)$ and $(Y, M',*')$ be two fuzzy compact metric spaces and $h : X \to Y$ be a fuzzy homeomorphism. For each $0 < \varepsilon < 1$ there exists $0 < \delta < 1$with the property $M(x, y, t) > 1 - \delta$,

then

$$M'(h(x), h(y), t) > 1 - \varepsilon.$$

Proof:

Suppose that there is a pair of sequence (x_k) , (y_k) in X with property that $M(x_k, y_k, t) > 1 - \frac{1}{k}$ and a sequence (t_k) in $(0, +\infty)$ such that $M'(h(x_k), h(y_k), t_k) < 1 - \varepsilon$.

By fuzzy compact we can assume that $x_k \overset{M}{\to} x$ and $y_k \overset{M}{\to} y$

$[x \in B(x_k, \eta, t), y \in B(y_k, \eta, t)]$ therefor $M(x, x_k, t) > (1 - \eta)$, $M(y, y_k, t) > (1 - \eta)$ with $x, y \in X$. Since given $t > 0$ and each $0 < \eta < 1$ for $k \in \mathbb{N}$ we have

$$M(x, y, , t) \geq M\left(x, x_k, \frac{t}{3}\right) * M\left(x_k, y_k, \frac{t}{3}\right) * M\left(y_k, y, \frac{t}{3}\right)$$

$$> (1 - \eta) * \left(1 - \frac{1}{k}\right) * (1 - \eta).$$

Then $x = y$. Let $m \in \mathbb{N}$ such that

$\left(1-\frac{1}{m}\right) *' \left(1-\frac{1}{m}\right) > 1-\varepsilon$ and then by continuity of h in x holds

$$M'(h(x_k), h(y_k), t_k) \geq M'\left(h(x_k), h(x), \frac{t_k}{2}\right) *' M'\left(h(y), h(y_k), \frac{t_k}{2}\right),$$

$$> \left(1-\frac{1}{m}\right) *_2 \left(1-\frac{1}{m}\right),$$

$$> 1-\varepsilon.$$

This is contradiction. □

Proposition 1.2.2

Let $(X, M,*)$ be a fuzzy compact metric space and $f: X \to X$ be a fuzzy expansive homeomorphism with expansive constant $0 < e < 1$. Given $0 < \varepsilon < 1$, there exists $N \geq 1$ such that if

$$M(f^k(x), f^k(y), t) \geq 1 - e, for\ every\ |k| \leq N, then$$

$$M(x, y, t) > 1 - \varepsilon.$$

<u>Proof :</u>

Let $0 < \varepsilon < 1$ be given then to each $N \geq 1$ there exists two sequences $(x_N), (y_N)$ in Xsuch that for all $|j| \leq N$

$$M\left(f^j(x_N), f^j(y_N), t\right) \geq 1 - e$$

and there is a sequence(t_N) in $(0, +\infty)$ with the property that $M(x_N, y_N, t_N) \leq 1 - \varepsilon$. By fuzzy compactness we can assume that $x_N \overset{M}{\to} x$ and $y_N \overset{M}{\to} y$ with $x, y \in X$. Let $n \in \mathbb{Z}\ and\ t > 0$. For every $N \in \mathbb{N}$such that $|n| \leq N$ we have

$M(f^n(x), f^n(y), t)$

$$\geq M\left(f^n(x), f^n(x_N), \frac{t}{3}\right) * M\left(f^n(x_N), f^n(y_N), \frac{t}{3}\right) *$$

$$M\left(f^n(y_N), f^n(y), \frac{t}{3}\right)$$

$$\geq M\left(f^n(x), f^n(x_N), \frac{t}{3}\right) * (1 - e) * M(f^n(y_N), f^n(y), \frac{t}{3}).$$

By continuity we obtain that

$M(f^n(x), f^n(y), t) \geq 1 - e$, then $x = y$. □

Proposition 1.2.3

Let $(X, M, *)$ be a fuzzy metric space and $f : X \to X$ a fuzzy expansive homeomorphism with the fuzzy shadowing property. There are $0 < \varepsilon < 1$ and $0 < \delta < 1$ with the following property every δ-fuzzy pseudo orbit $(x_k)_{k \in \mathbb{Z}}$ can be ε-fuzzy traced in f by a unique point in X.

Proof:

Let $0 < e < 1$ be a fuzzy expansive constant of f .Take $0 < \varepsilon < 1$ satisfying $(1 - \varepsilon) * (1 - \varepsilon) \geq (1 - e)$ and choose $0 < \delta < 1$ corresponding to ε from the fuzzy shadowing property. Suppose that there exists a δ-fuzzy pseudo orbit $(x_k)_{k \in \mathbb{Z}}$ for f that can be ε-fuzzy traced by x and y. Given $k \in \mathbb{Z}$ and $t > 0$

We have

$$M(f^k(x), f^k(y), t) > M\left(f^k(x), x_k, \frac{t}{2}\right) * M\left(x_k, f^k(y), \frac{t}{2}\right)$$

$$> (1 - \varepsilon) *_1 (1 - \varepsilon) \geq 1 - e.$$

Since f is a fuzzy expansive we have $x = y$. □

Proposition.1.2.4

If $f: X \to X$ is a fuzzy homeomorphism map then f has a fuzzy expansive if and only if f^{-1} has a fuzzy expansive.

<u>**Proof:**</u>

Let f be a fuzzy expansive then there exists $0 < e < 1$ so that for each $t > 0$, $M(f^k(x_1), f^k(y_1), t) > 1 - e$.

For all $k \in \mathbb{Z}$, $x_1 = y_1 \in X$. We choose $0 < \varepsilon < 1$ and $t > 0$ so that the inequality $M(x, y, t) > 1 - \varepsilon$ implies $M(f(x), f(y), t) > 1 - e$.

Let $g = f^{-1}$, for all $k \in \mathbb{Z}$, $x_2, y_2 \in X$ satisfies the inequality

$M(g^k(x_2), g^k(y_2), t) > 1 - \varepsilon$ then

$$M(f^{-k}(x_2), f^{-k}(y_2), t) = M\left(f\left(f^{-k}(x_2)\right), f\left(f^{-k}(y_2)\right) t\right)$$

$$> 1 - e$$

f^{-1} is t- fuzzy expansive .

Conversely, let f^{-1} is a fuzzy expansive, then there exists $0 < e < 1$ so that for each $t > 0$ then

$$M(f^{-k}(x_2), f^{-k}(y_2), t) > 1 - e.$$

For all $k \in \mathbb{Z}, x_2 = y_2 \in X$. We choose $0 < \varepsilon < 1$ and $t > 0$ so that the inequality $M(f(x_1), f(y_1), t) > 1 - \varepsilon$ implies $M(f^{-1}(x_1), f^{-1}(y_1), t) > 1 - e$.

Let $g = f$, for all $k \in \mathbb{Z}$, $x_1 = y_1 \in X$ satisfies the inequality

$$M(g^k(x_1), g^k(y_1), t) > 1 - \varepsilon$$

then $M(f^k(x_1), f^k(y_1), t) =$

$$M\left(f^{-1}\left(f^{k}(x_1)\right), f^{-1}\left(f^{k}(y_1)\right), t\right) > 1 - e.$$

f is t- fuzzy expansive . □

Proposition 1.2.5

Let $(X, M, *)$ and $(Y, M', *')$ be two fuzzy compact metric spaces. Let $f : X \to X$ *and* $g : Y \to Y$ be two fuzzy homeomorphisms and

$h : X \to Y$ be a conjugation between f and g. Then g is fuzzy expansive if and only if f is fuzzy expansive.

Proof:

We suppose that $0 < e < 1$ is an expansive constant of f .Let $x, y \in X, x = y$.

$$M(f^{k}(x), f^{k}(y), t) > 1 - e.$$

Let $0 < \varepsilon < 1$ and $t > 0$ so that the inequality $M(x, y, t) > 1 - e$ implies $M'(h(x), h(y), t) > 1 - \varepsilon$. Given $w, q \in Y$, $x = h^{-1}(q), y = h^{-1}(w)$, then for every $k \in \mathbb{Z}$ it follows that

$$M(f^{k}(x), f^{k}(y), t) = M\left(f^{k}\left(h^{-1}(q)\right), f^{k}\left(h^{-1}(w)\right), t\right)$$

$$= M'(h\left(f^{k}\left(h^{-1}(q)\right), h\left(f^{k}(h^{-1}(w)\right), t\right)$$

$$= M'\left(g^{k}\left(h\left(h^{-1}(q)\right)\right), g^{k}\left(h\left(h^{-1}(w)\right)\right), t\right)$$

$$= M'(g^{k}(q), g^{k}(w), t) > 1 - \varepsilon.$$

We conclude that g is fuzzy expansive map.

Conversely, we suppose that $0 < \varepsilon < 1$ is expansive constant of g

$$M'(g^{k}(q), g^{k}(w), t) > 1 - \varepsilon.$$

Let $0 < e < 1$ and $t > 0$ so that the inequality $M'(x, y, t) > 1 - \varepsilon$ implies $M(h^{-1}(x), h^{-1}(y), t) > 1 - e$. Given $q, w \in Y, q = h(x), w = h(y)$, then for every $k \in \mathbb{Z}$ it follows that

$$M'(g^k(q), g^k(w), t) = M\left(h^{-1}\left(g^k\big(h(x)\big)\right), h^{-1}\left(g^k\big(h(y)\big)\right), t\right)$$

$$M\left(h^{-1}\left(h\left(f^k(x)\right)\right), h^{-1}\left(h\left(f^k(y)\right)\right), t\right)$$

$$= M(f^k(x), f^k(y), t) > 1 - e.$$

We conclude that f is fuzzy expansive map. □

Before we prove Walter's theorem according to the definition of the expansive homeomorphism in [4], we need to define the following functions

Let$(X, M,*)$ be a $fuzzy\, metric$ space and $f, g : X \to X$ be fuzzy continuous maps, we define $\mathcal{M}(f, g) = M(f(x), g(x), t)$.

Given a fuzzy continuous map $g : X \to X$ defined on the fuzzy metric space $(X, M,*)$ and $0 < \delta < 1$ we say that g is δ-$fuzzy\, close$ to f whenever $\mathcal{M}(f, g) > 1 - \delta$.

Theorem 1.2.6 ($Walters - Fuzzy\, version$**)**

Every fuzzy expansive homeomorphism with the fuzzy shadowing property and $fuzzy\ compact$ metric space is fuzzy topological stable.

Proof:

Let $0 < e < 1$ be an expansivity constant of f.Take $0 < \varepsilon < 1$ and $0 < \delta < 1$ from proposition1.2.3. We can suppose that

$(1 - \varepsilon) * (1 - \varepsilon) * (1 - \varepsilon) > (1 - e)$.

Let $g: X \rightarrow X$ be an homeomorphism such that $\mathcal{M}(f,g) > 1-\delta$ and $x \in X$. We claim that the orbit $(g^k(x))_{k\in\mathbb{Z}}$ is a δ-$fuzzy\, pseudo\, orbit$ for f.Indeed, by definition of $\mathcal{M}(f,g)$, for every $k \in \mathbb{Z}$ and $t > 0$

We have

$$M\left(f\left(g^k(x)\right), g^{k+1}(x), t\right) = M\left(f\left(g^k(x)\right), g\left(g^k(x)\right), t\right)$$

$$\geq \mathcal{M}(f,g) > 1 - \delta.$$

By proposition 1.2.3, there is a unique $h(x) \in X$ such that $(g^k(x))_{k\in\mathbb{Z}}$ can be $\varepsilon - fuzzy\, traced$ by $h(x)$ for f .This defines a map $h: X \rightarrow X$ that satisfies

$$M\left(f^k\left(h(x)\right), g^k(x), t\right) > 1 - \varepsilon, \forall k \in \mathbb{Z}.$$

Next, we show that h is a semi-conjugation between f and g.

$M(f^k(h(g(x))), g^{k+1}(x), t) > 1 - \varepsilon, \forall k \in \mathbb{Z}$,

and for every $k \in \mathbb{Z}$,

$$M\left(f^k\left(f\left(h(x)\right)\right), g^{k+1}(x), t\right) = M\left(f^{k+1}\left(h(x)\right), g^{k+1}(x), t\right)$$

$$> 1 - \varepsilon,$$

We have that $h(g(x))$ and $f(h(x))$ are points in X that ε-fuzzy trace the δ-fuzzy pseudo orbit$(g^{k+1}(x_1))_{k\in\mathbb{Z}}$. By proposition 1.2.2, it holds that

$$g \circ h = h \circ f.$$

Lastly, we show that h is fuzzy continuous.

Furthermore, by proposition 1.2.1 we choose $\eta > 0$, such that

$M(x, y, t) > 1 - \eta$ implies that

$$M(g^k(x), g^k(y), t) \geq 1 - \varepsilon, \forall |k| \leq N.$$

For x , $y \in X$ such that $M(x, y, t) > 1 - \eta$ and $\xi > 0$

We have

$$M(f^k(h(x)), f^k(h(y)), \xi) = M(h(g^k(x)), h(g^k(y)), \xi)$$

$$\geq M\left(h\left(g^k(x)\right), g^k(x), \frac{\xi}{3}\right) * M\left(g^k(x), g^k(y), \frac{\xi}{3}\right) *$$

$$M\left(g^k(y), h\left(g^k(y)\right), \frac{\xi}{3}\right)$$

$$\geq \mathcal{M}(h, I_X) * M\left(g^k(x), g^k(y), \frac{\xi}{3}\right) * \mathcal{M}(h, I_X)$$

$$> (1 - \varepsilon) * (1 - \varepsilon) * (1 - \varepsilon)$$

$$> 1 - e$$

For all $|k| < N$. Therefore $M(x, y, t) > 1 - \eta$ implies that

$$M(h(x), h(y), t) > 1 - \partial,$$

Thus the continuity of h is proved. □

Chapter Two

Some Chaotic Properties of Product in Fuzzy Dynamical Systems and Topological Conjugacy

2.1. Shadowing in Fuzzy Dynamical Systems

In this is section we prove that properties (fuzzy L-shadowing property, fuzzy two – sided limit shadowing) are invariant under fuzzy topological conjugation and some shadowing properties in fuzzy dynamical systems.

Proposition 2.1.1

A fuzzy homeomorphism f on a fuzzy compact metric space X has the fuzzy negative limit shadowing property if and only if f^{-1} has the fuzzy limit shadowing property.

Proof:

Suppose that f has the fuzzy negative limit shadowing property.

Let $(x_k)_{k\in\mathbb{N}}$ be a fuzzy negative limit pseudo-orbit for f satisfying

$$\lim_{k\to-\infty} M(f(x_k), x_{k+1}, t) = 1$$

There exists $p \in X$ such that

$$\lim_{k\to-\infty} M(f^k(p), x_k, t) = 1$$

Let $(y_k)_{k\in\mathbb{N}}$ be a fuzzy limit pseudo-orbit for f^{-1} , $(x_k) = y_{-k}$ such that

$$\lim_{k\to-\infty} M(f^{-1}(x_k), x_{k+1}, t) = \lim_{k\to-\infty} M(f^{-1}(y_{-k}), y_{-k-1}, t) = 1$$

Which implies that

$$\lim_{k\to\infty} M(f^{-k}(p), y_k, t) = 1$$

It follows that $(y_k)_{k\in\mathbb{N}}$ is fuzzy limit shadowed for f^{-1} and f^{-1} that has the fuzzy limit shadowing property.

Conversely, let f^{-1} has the fuzzy limit shadowing property. Let $(y_k)_{k\in -\mathbb{N}}$ be a fuzzy negative limit pseudo-orbit for f satisfying

$$\lim_{k\to\infty} M(f^{-1}(y_k), y_{k+1}, t) = 1$$

There exists $p \in X$ such that

$$\lim_{k\to\infty} M(f^{-k}(p), y_k, t) = 1$$

The sequence $(x_k)_{k\in\mathbb{N}}$ is a fuzzy limit pseudo-orbit for f , $y_k = x_{-k}$

$$\lim_{k\to\infty} M(f^{-1}(y_k), y_{k+1}, t) = \lim_{k\to\infty} M(f^{-1}(x_{-k}), x_{-k-1}, t) = 1.$$

Which implies that

$$\lim_{k\to-\infty} M(f^{k}(p), x_k, t) = 1$$

Hence f has the fuzzy negative limit shadowing property. □

Proposition 2.1.2

Let $(X, M,*)$ and $(Y, M',*')$ be two fuzzy compact metric spaces. Let $f : X \to X$ *and* $g: Y \to Y$ be two fuzzy homeomorphisms and $h : X \to Y$ be a conjugation between f and g. Then f is fuzzy L-shadowing property if and only if g is fuzzy L-shadowing property

Proof:

Let f be a fuzzy L-shadowing property. If $0 < \varepsilon < 1$ and $t > 0$ there is $0 < \varepsilon_1 < 1$ so that the inequality $M(x, y, t) > 1 - \varepsilon_1$ implies $M'(h(x), h(y), t) > 1 - \varepsilon$. From the fuzzy L-shadowing property, if every $0 < \varepsilon_1 < 1$, there is $0 < \delta_1 < 1$ such that for every sequence$(x_k)_{k\in\mathbb{Z}} \subset X$

$$M(f(x_k), x_{k+1}, t) > 1 - \delta_1$$

and

$$\lim_{k\to\infty} M(f(x_k), x_{k+1}, t) = 1$$

There exists $p \in X$

$$M(f^k(p), x_k, t) > 1 - \varepsilon_1$$

and

$$\lim_{k\to\infty} M(f^k(p), x_k, t) = 1$$

holds.

We choose $0 < \delta < 1$ so that the inequality $M'(x, y, t) > 1 - \delta$ implies $M(h^{-1}(x), h^{-1}(y), t) > 1 - \delta_1$. Given $(y_k)_{k\in\mathbb{Z}} \subset Y$. To do this put $x_k = h^{-1}(y_k)$ for $k \in \mathbb{Z}$,we have

$$M(f(x_k), x_{k+1}, t) = M\big(f\big(h^{-1}(y_k)\big), h^{-1}(y_{k+1}), t\big)$$
$$= M\big(h^{-1}\big(g(y_k)\big), h^{-1}(y_{k+1}), t\big) > 1 - \delta_1.$$

Thus (x_k) satisfies the relation

$$M(f(x_k), x_{k+1}, t) > 1 - \delta_1.$$

Hence

$$M'(g(y_k), y_{k+1}, t) > 1 - \delta$$

and

$$\lim_{k\to\infty} M'\,(g(y_k), y_{k+1}, t) = 1.$$

There exist $p \in X$

$$M(f^k(p), x_k, t) > 1 - \varepsilon_1$$

therefore it follows that

$$M(f^k(p), x_k, t) = M'\left(h\left(f^k(p)\right), h(x_k), t\right)$$

$$= M'(g^k(h(p), y_k, t) > 1 - \varepsilon$$

and

$$\lim_{k\to\infty} M'\,(g^k(q), y_k, t) = 1$$

Hence g has the fuzzy L-shadowing property.

Conversely, let g be a fuzzy L-shadowing property. If $0 < \varepsilon_1 < 1$ and $t > 0$ there is $0 < \varepsilon < 1$ so that the inequality $M(x, y, t) > 1 - \varepsilon$ implies $M'(h(x), h(y), t) > 1 - \varepsilon_1$.

From the fuzzy L-shadowing property if every $0 < \delta < 1$ there is $0 < \varepsilon < 1$ such that for every sequence$(y_k)_{k\in\mathbb{Z}} \subset Y$

$$M'(g(y_k), y_{k+1}, t) > 1 - \delta$$

and

$$\lim_{k\to\infty} M'\,(g(y_k), y_{k+1}, t) = 1.$$

There exists $q \in Y$

$$M'(g^k(q), y_k, t) > 1 - \varepsilon$$

and

$$\lim_{k\to\infty} M'\,(g^k(q), y_k, t) = 1$$

holds.

We choose $0 < \delta_1 < 1$ so that the inequality $M'(x, y, t) > 1 - \delta_1$ implies $M(h^{-1}(x), h^{-1}(y), t) > 1 - \delta$. Given $(x_k)_{k\in\mathbb{Z}} \subset X$. To do this put $y_k = h(x_k)$ for $k \in \mathbb{Z}$.

We have

$$M'(g(y_k), y_{k+1}, t) = M'\big(g\big(h(x_k)\big), h(x_{k+1}), t\big)$$
$$= M\big(h\big(f(x_k)\big), h(x_{k+1}), t\big) > 1 - \delta.$$

Thus (y_k) satisfies the relation

$$M'(g(y_k), y_{k+1}, t) > 1 - \delta.$$

Hence

$$M(f(x_k), x_{k+1}, t) > 1 - \delta_1$$

and

$$\lim_{k \to \infty} M(f(x_k), x_{k+1}, t) = 1.$$

There exists $q \in Y$

$$M'(g^k(q), y_k, t) > 1 - \varepsilon.$$

Therefore it follows that

$$M'(g^k(q), y_k, t) = M'(g^k(q), h(x_k), t)$$

$$= M\left(h^{-1}\left(g^k(q)\right), h^{-1}\big(h(x_k)\big), t\right)$$

$$= M(f^k(h^{-1}(q), x_k, t) > 1 - \varepsilon_1$$

and

$$\lim_{k \to \infty} M(f^k(p), x_k, t) = 1.$$

Hence f has fuzzy L −shadowing property. □

Proposition 2.1.3

Let $(X, M,*)$ and $(Y, M',*')$ be two fuzzy compact metric spaces. Let $f: X \to X$ and $g: Y \to Y$ be two fuzzy homeomorphisms and $h: X \to Y$ be a conjugation between f and g. Then f is fuzzy two –sided limit shadowing property if and only if g is fuzzy two –sided limit shadowing property.

Proof:

Let f be a fuzzy two –sided limit shadowing property. If $0 < \varepsilon < 1$ there is $0 < \varepsilon_1 < 1$ so that the inequality $M(x, y, t) > 1 - \varepsilon_1$ implies $M'(h(x), h(y), t) > 1 - \varepsilon$.

Since f has fuzzy two –sided limit shadowing property. The sequence $(x_k)_{k\in\mathbb{Z}} \subset X$ such that

$$\lim_{k\to\infty} M(f(x_k), x_{k+1}, t) = 1$$

There exists a point $p \in X$ such that

$$\lim_{k\to\infty} M(f^k(p), x_k, t) = 1.$$

We choose $0 < \delta < 1$ so that the inequality $M'(x, y, t) > 1 - \delta$ implies $M(h^{-1}(x), h^{-1}(y), t) > 1 - \delta_1$.

To do this put $x_k = h^{-1}(y_k)$ for $k \geq k_0$

We have

$$M(f(x_k), x_{k+1}, t) = M\big(f\big(h^{-1}(y_k)\big), h^{-1}(y_{k+1}), t\big)$$

$$= M\big(h^{-1}\big(g(y_k)\big), h^{-1}(y_{k+1}), t\big)$$

$$= M'(g(y_k), y_{k+1}, t) \geq 1 - \delta, \quad k \geq k_0 \ .$$

Hence there is $k_0 \in \mathbb{N}$

Thus

$$\lim_{k\to\infty} M'(g(y_k), y_{k+1}, t) = 1.$$

Therefore there is $k_1 \in \mathbb{N}$ so that for

$$M(f^k(p), x_k, t) \geq 1 - \varepsilon_1 \ , k \geq k_1.$$

Thus

$$M(f^k(p), x_k, t) = M'\left(h\left(f^k(p)\right), h(x_k), t\right)$$

$$= M'(g^k(h(p)), y_k, t) \geq 1 - \varepsilon \ \text{ for } k \geq k_1.$$

Hence

$$\lim_{k\to\infty} M'(g^k(h(p)), y_k, t) = 1.$$

So g has fuzzy two –sided limit shadowing property.

Conversely, let g be a fuzzy two –sided limit shadowing property. If $0 < \varepsilon_1 < 1$ there is $0 < \varepsilon < 1$ such that the inequality $M(x, y, t) > 1 - \varepsilon$ implies $M'(h(x), h(y), t) > 1 - \varepsilon_1$.

Since g has fuzzy two –sided limit shadowing property .The sequence $(y_k)_{k\in\mathbb{Z}} \subset Y$ such that

$$\lim_{k\to\infty} M'(g(y_k), y_{k+1}, t) = 1$$

there exists $q \in Y$ such that

$$\lim_{k\to\infty} M'(g^k(q), y_k, t) = 1.$$

We choose $0 < \delta_1 < 1$ so that the inequality $M'(x, y, t) > 1 - \delta_1$ implies $M\left(h^{-1}(x), h^{-1}\left(y_y\right), t\right) > 1 - \delta.$

To do this put $y_k = h(x_k)$ for $k \geq k_0$, the sequence $(y_k)_{k\in\mathbb{Z}}$.

We have

$$M'(g(y_k), y_{k+1}, t) = M'\big(g\big(h(x_k)\big), h(x_{k+1}), t\big)$$
$$= M'(h(f(x_k), h(x_{k+1}), t)$$

$$= M(f(x_k), x_{k+1}, t) \geq 1 - \delta_1, k \;\geq\; k_0.$$

There is $k_0 \in \mathbb{N}$.

Thus

$$\lim_{k\to\infty} M(f(x_k), x_{k+1}, t) = 1$$

Therefore there is $k_1 \;\in\; \mathbb{N}$ so that

$$M'(g^k(q), y_k, t) \;\; \geq 1 - \varepsilon, k \;\geq\; k_1.$$

Thus

$$M'(g^k(q), y_k, t) = M\left(h^{-1}\left(g^k(q)\right), h^{-1}(x_k), t\right)$$

$$= M'(f^k(h^{-1}(q)), x_k, t) \geq 1 - \varepsilon_1 \;\; for\; k \;\geq\; k_1.$$

Hence

$$\lim_{k\to\infty} M(f^k(h^{-1}(q), x_k, t) = 1$$

So f has fuzzy two –sided limit shadowing property. □

Proposition 2.1.4

Let $(X, M,*)$ be a fuzzy compact metric space and $f: X \longrightarrow X$ be continuous. If f is positively expansive with shadowing, then f has h-shadowing.

Proof:

Let $0 < \varepsilon < 1, 0 < c < 1, 1 - \varepsilon < 1 - c$ and $0 < \delta < 1, t > 0$ be provided by shadowing for ε .let δ −pseudo orbit $\{x_0, x_1, \dots, x_m\}$ and extend it to the finite δ −pseudo orbit

$$x_0, x_1, \dots, x_m, f(x_m), f^2(x_m), \dots$$

If p is a point with ε −shadow the above pseudo orbit, then

$$M\left(f^{j+m}(p), f^j(x_m), t\right) < 1 - c \text{ for all } j \geq 0$$

Which implies that $f^m(p) = x_m$. □

Proposition 2.1.5

Every fuzzy expansive homeomorphism $f: X \longrightarrow X$ with fuzzy shadowing property has the fuzzy limit shadowing property.

Proof:

Since f is expansive there exists $0 < e < 1$ such that

$$M(f^k(x), f^k(y), t) \leq 1 - \text{e}.$$

For all $x \neq y$. Choose $0 < \delta < 1$ such that every δ −pseudo orbit is ε-shadowed.

Let $(x_k)_{k \in \mathbb{N}}$ be limit pseudo orbit such that

$$M(f(x_k), x_{k+1}, t) < 1 - \delta.$$

The shadowing property assure the existence of points $p \in X$ such that

$$M(f^k(p), x_k, t) < 1 - \delta.$$

We claim that $(x_k)_{k \in \mathbb{N}}$ is limit shadowed by y_1.Indeed

$M(f^k(y_1), f^k(p), t) \leq M\left(f^k(y_1), x_k, \frac{t}{2}\right) * M\left(x_k, f^k(p), \frac{t}{2}\right)$

$< (1-\delta) * (1-\delta) < 1-\varepsilon$.

By choice of ε we obtain

$$\lim_{k\to\infty} M(f^k(y_1), f^k(p), t) = 1 .$$

This imply that $\lim_{k\to\infty} M(f^k(y_1), x_k, t) = 1$.

Thus every fuzzy limit pseudo-orbit is fuzzy limit shadow and f has the fuzzy limit shadowing property. □

Proposition 2.1.6

A fuzzy homeomorphism on a fuzzy compact metric space has the fuzzy two-sided limit shadowing property if and only if it has the fuzzy limit shadowing property, the fuzzy negative limit shadowing property and the fuzzy simple one-sided limit shadowing property.

Proof:

It is obvious that the fuzzy two-sided limit shadowing property implies the fuzzy limit shadowing property, the fuzzy negative limit shadowing property and the fuzzy simple one-sided limit shadowing property. It suffices to prove the converse statement. The fuzzy limit shadowing property and the fuzzy negative limit shadowing property assure the existence of points $p_1, p_2 \in X$, $(x_k)_{k\in\mathbb{Z}}$ satisfying

$$\lim_{k\to\infty} M(f^k(p_1), x_k, t) = 1$$

and

$$\lim_{k\to\infty} M(f^k(p_2), x_k, t) = 1.$$

Thus the sequence

$$y_k = \begin{cases} f^k(p_2), k \geq 0; \\ f^k(p_1), k < 0, \end{cases}$$

is a fuzzy simple one-sided limit pseudo-orbit. The fuzzy simple one-sided limit shadowing property assures the existence of a point $p \in X$ that fuzzy two- sided limit shadows$(y_k)_{k\in\mathbb{Z}}$. This point also fuzzy one-sided limit shadowing $(x_k)_{k\in\mathbb{Z}}$. □

Remark 2.1.7

- From Definition (1.1.21), Definition (1.1.22) and Definition(1.1.15) and from Proposition 1.2.6 find that Figure(2.1)
- From the Definition(1.1.25) and Definition (1.1.26)we find Figure (2.2)
- From Proposition 2.1.2 Figure(2.3)
- From Proposition2.1.5 Figure(2.4)

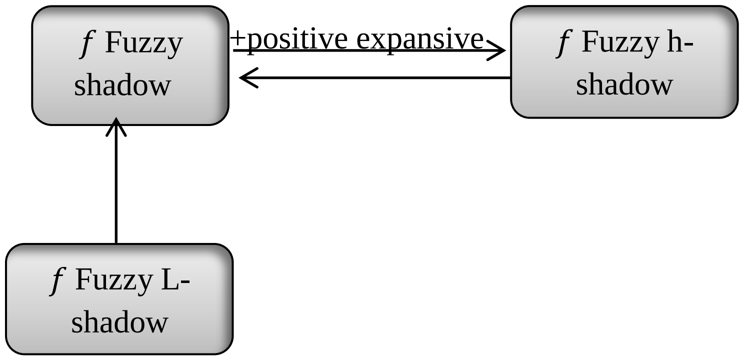

Figure (2.1): The relation fuzzy shadow, fuzzy h-shadow and fuzzy L-shadow.

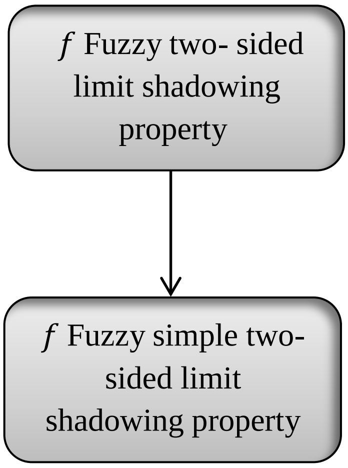

Figure (2.2): The relation between fuzzy two- sided limit shadowing property and fuzzy simple two-sided limit shadowing property.

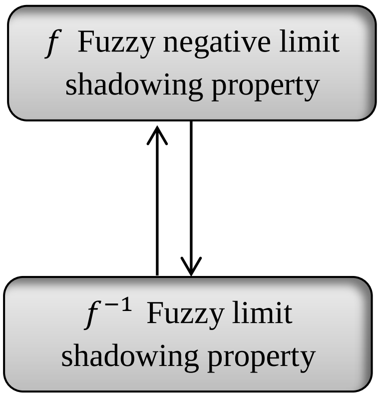

Figure (2.3): The relation between fuzzy negative limit shadowing property and fuzzy limit shadowing property.

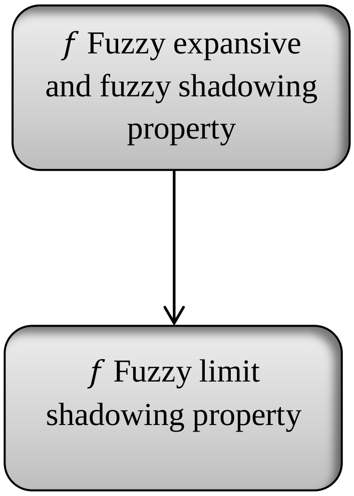

Figure (2.4): The relation between fuzzy expansive and fuzzy shadowing property and fuzzy limit shadowing property.

2.2.Product Some Properties of Chaos in Fuzzy Dynamical Systems.

In this section we prove that two fuzzy dynamical systems on two fuzzy metric spaces achieve the properties (fuzzy expansive, fuzzy Lipschitz, fuzzy L-shadowing property, fuzzy two-sided limit shadowing property, fuzzy positive expansive, fuzzy h-shadow) if and only if the product up to a special $t-norm$ achieves the above mentioned properties.

For this purpose we assume that $(X, M, f, *)$ and $(Y, M', g, *')$ be fuzzy dynamical systems, i.e.,$(X, M, *)$ and $(Y, M', *')$ two fuzzy compact metric spaces and let $f : X \longrightarrow X$ and $g: Y \longrightarrow Y$ be fuzzy continuous.

To prove the propositions, we first recall the following Theorems.

Theorem 2.2.1:[25]

If $(X, M, *)$ is a fuzzy metric space then:

i) For given $a, b \in [0,1], a * b = 1$ implies $a = b = 1$;

ii) The inequality $t \leq s$ implies $M(x, y, t) \leq M(x, y, s)$,

Where $x, y \in X$.

From the proof of this Theory it is found that when

$$s = t + r, \text{ then } M(x, y, s) = M(x, y, t + r) \geq M(x, y, t) * M(y, y, r)$$

$$= M(x, y, t) * 1$$

$$= M(x, y, t).$$

Theorem2.2.2:[25]

If $*,*'$ *are two* $t-norms$, then $*_m$: $[0,1]\times[0,1]\to[0,1]$, such that

$(a,b)\mapsto min\{a*b, a*' b\}$ is a $t-norm$.

Theorem 2.2.3:[25]

Let$(X,M,*)$ and $(Y,M',*')$ be two fuzzy metric spaces. Assume that the $t-norm$ $*$ has the following additional property:

A: $a*a>0$ *for all* $0<a$.

Then $X\times Y$with the mapping

$M^{\star}:(X\times Y)\times(X\times Y)\times(0,\infty)\to[0,1]$, such that

$((x_1,y_1),(x_2,y_2),t)\longmapsto M(x_1,x_2,t) *_m M'(y_1,y_2,t)$

is a fuzzy metric space.

Remark 2.2.4

If $(X,M,f,*)$ and $(Y,M',g,*')$ are two fuzzy dynamical systems,
then $(X\times Y,M^{\star},T,*_m)$ is a fuzzy dynamical system, where

$$T=f\times g:X\times Y\to X\times Y$$

is the map defined

$$f\times g(x,y)=\big(f(x),g(y)\big) for\ (x,y)\in X\times Y.$$

Proposition 2.2.5

If $(X,M,f,*)$ and $(Y,M',g,*')$ are two fuzzy dynamical systems then $(X,M,f,*)$ and $(Y,M',g,*')$ are t- expansive fuzzy dynamical systems if and only if $(X\times Y,M^{\star},T,*_m)$ is a t- expansive fuzzy dynamical system.

Proof:

Let $(X, M, f, *)$ and $(Y, M', g, *')$ be t- expansive fuzzy dynamical systems and $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $0 < e_2 < 1, 0 < e_1 < 1, 0 < e < 1, t > 0,$ $such\ that\ x_1 \neq x_2, 1 - e_1 * 1 - e_2 \leq 1 - e.$

Then by Definition 1.1.17

$$M\ (f^k(x_1), f^k(x_2), t) \leq 1 - e_1 \qquad (2.1)$$

$$M'\ (g^k(y_1), g^k(y_2), t) \leq 1 - e_2\ . \qquad (2.2)$$

Let $(x_1, y_1) \in X \times Y$, $(x_2, y_2) \in X \times Y$, we need to prove that

$M^{\star}(T^k(x_1, y_1), T^k(x_2, y_2), t) \leq 1 - e.$

Now

$$M^{\star}(T^k(x_1, y_1), T^k(x_2, y_2), t) =$$

$$M^{\star}\left(\left(f^k(x_1), g^k(y_1)\right), \left(f^k(x_2), g^k(y_2)\right), t\right) =$$

$$M\ (f^k(x_1), f^k(x_2), t) *_m M'(g^k(y_1), g^k(y_2), t) =$$

$$\min\begin{Bmatrix} M\ (f^k(x_1), f^k(x_2), t) * M'(g^k(y_1), g^k(y_2), t), \\ M\ (f^k(x_1), f^k(x_2), t) *' M'(g^k(y_1), g^k(y_2), t) \end{Bmatrix}.$$

By (2.1), (2.2) we have

$$M\ (f^k(x_1), f^k(x_2), t) * M'(g^k(y_1), g^k(y_2), t) \leq 1 - e_1 * 1 - e_2$$

$$\leq 1 - e.$$

Hence

$$M^{\star}(T^k(x_1, y_1), T^k(x_2, y_2), t) \leq 1 - e.$$

Thus, $(X \times Y, M^{\star}, T, *_m)$ is t- expansive fuzzy dynamical system .

Conversely, let $(X \times Y, M^{\star}, T, *_m)$ be a t −expansive fuzzy dynamical system. If $x_1, x_2 \in X, y_1, y_2 \in Y$ and $x_1 \neq x_2$ satisfying

$$M\ (f^k(x_1), f^k(x_2), t) \leq 1 - e_1 \qquad (2.3)$$

$$M'\ (g^k(y_1), g^k(y_2), t) \leq 1 - e_2. \qquad (2.4)$$

Then

$$M^{\star}(T^k(x_1\ , y_1), T^k(x_2\ , y_2), t) \leq 1 - e.$$

Hence

$$M\ (f^k(x_1), f^k(x_2), t) \leq 1 - e_1 \qquad (2.5)$$

$$M'\ (g^k(y_1), g^k(y_2), t) \leq 1 - e_2. \qquad (2.6)$$

So $(X, M, f, *)$ and $(Y, M', g, *')$ are a t −expansive fuzzy dynamical systems. □

Proposition 2.2.6

If $(X, M, f, *)$ and $(Y, M', g, *')$ are two fuzzy dynamical systems, then $(X, M, f, *)$ and $(Y, M', g, *')$ be L- shadowing fuzzy dynamical systems if and only if $(X \times Y, M^{\star}, T, *_m)$ is L-shadowing fuzzy dynamical system.

Proof:

Let $(X, M, f, *)$ and $(Y, M', g, *')$ be fuzzy L-shadowing property, let$(x_k)_{k \in \mathbb{Z}} \subset X$ and $(y_k)_{k \in \mathbb{Z}} \subset Y$. Given $0 < \varepsilon_1, \varepsilon_2 < 1, 0 < \delta_1, \delta_2 < 1,$

$t > 0, 1 - \delta_1 * 1 - \delta_2 > 1 - \delta.$

Then by Definition 1.1.22

$$M(f(x_k), x_{k+1}, t) > 1 - \delta_1,\ \ for\ \ k \in \mathbb{Z} \qquad (2.7)$$

and

$$\lim_{k\to\infty} M\,(f(x_k), x_{k+1}, t) = 1. \qquad (2.8)$$

There is $p \in X$ such that

$$M(f^k(p), x_k, t) > 1 - \varepsilon_1 \qquad (2.9)$$

and

$$\lim_{k\to\infty} M(f^k(p), x_k, t) = 1 \qquad (2.10)$$

$$M'(g(y_k), y_{k+1}, t) > 1 - \delta_2\ , \qquad for\ \ k \in \mathbb{Z} \qquad (2.11)$$

and

$$\lim_{k\to\infty} M'\,(g(y_k), y_{k+1}, t) = 1. \qquad (2.12)$$

There is $q \in Y$ such that

$$M'(g^k(q), y_k, t) > 1 - \varepsilon_2 \qquad (2.13)$$

and

$$\lim_{k\to\infty} M'(g^k(q), y_k, t) = 1. \qquad (2.14)$$

We want to prove that $(X \times\ Y, M^\star, T, *_m)$ has fuzzy L-shadowing property.

Let $(x_k, y_k) \subset X \times Y, 0 < \varepsilon < 1\ , 0 < \delta < 1.$

$$M^\star(T(x_k, y_k), (x_{k+1}, y_{k+1}), t) > 1 - \delta$$

and

$$\lim_{k\to\infty} M^\star\,(T(x_k, y_k), (x_{k+1}, y_{k+1}), t) = 1.$$

$$M^\star(T(x_k, y_k)\ , (x_{k+1}, y_{k+1}, t) = M^\star\big((f(x_k), g(y_k)\big), (x_{k+1}, y_{k+1}), t))$$

$$= M(f(x_k), x_{k+1}, t) *_m\ M'(g(y_k), y_{k+1}, t)$$

$$\min\begin{Bmatrix} M(f(x_k), x_{k+1}, t) *\ M'(g(y_k), y_{k+1}, t), \\ M(f(x_k), x_{k+1}, t)\ *'\ M'(g(y_k), y_{k+1}, t) \end{Bmatrix}.$$

By (2.7), (2.9)

we have

$M(f(x_k), x_{k+1}, t) \ *' \ M'(g(y_k), y_{k+1}, t) = 1 - \delta_1 * \ 1 - \delta_2 \ > 1 - \delta.$

Hence

$$M^\star(T(x_k, y_k), (x_{k+1}, y_{k+1}), t) > 1 - \delta$$

and

$$\lim_{k\to\infty} M^\star \, (T(x_k, y_k), (x_{k+1}, y_{k+1}) \, , t) = 1.$$

Thus there exists $(p, q) \in X \times Y$.

$$M^\star(T^k(p, q), (x_k, y_k), t) > 1 - \varepsilon$$

and

$$\lim_{k\to\infty} M^\star(T^k \, (p, q), (x_k, y_k), t) = 1.$$

Thus, $(X \times \ Y, M^\star, T, *_m)$ has fuzzy L-shadowing property.

Conversely, let$(X \times \ Y, M^\star, T, *_m)$ be L-shadowing fuzzy dynamical system. If$(x_k)_{k\in\mathbb{Z}} \subset X, \ (y_k)_{k\in\mathbb{Z}} \subset Y$satisfying

$$M(f(x_k), x_{k+1}, t) > 1 - \delta_1 \, , \qquad \text{for } \ k \in \mathbb{Z}. \qquad (2.15)$$

And

$$\lim_{k\to\infty} M \, (f(x_k), x_{k+1}, t) = 1. \qquad (2.16)$$

$$M'(g(y_k), y_{k+1}, t) > 1 - \delta_2, \quad \text{for } \ k \in \mathbb{Z}. \qquad (2.17)$$

And

$$\lim_{k\to\infty} M' \, (g(y_k), y_{k+1}, t) = 1. \qquad (2.18)$$

Then

$$M^{\star}(T(x_k, y_k), (x_{k+1}, y_{k+1}), t) > 1 - \delta$$

and

$$\lim_{k\to\infty} M^{\star}\,(T(x_k, y_k), (x_{k+1}, y_{k+1}), t) = 1.$$

Thus there exists $(p, q) \in X \times Y$, so that

$$M^{\star}(T^k(p, q), (x_k, y_k), t) > 1 - \varepsilon$$

and

$$\lim_{k\to\infty} M^{\star}(T^k(p, q), (x_k, y_k), t) = 1.$$

So

$$\lim_{k\to\infty} M(f^k(p), x_k, t) = 1. \qquad (2.19)$$

And

$$\lim_{k\to\infty} M'(g^k(q), y_k, t) = 1. \qquad (2.20)$$

So $(X, M, f, *)$ *and* $(Y, M', g, *')$ are L−shadowing fuzzy dynamical systems. □

Proposition 2.2.7

If $(X, M, f, *)$ and $(Y, M', g, *')$ are two fuzzy dynamical systems, then$(X, M, f, *)$ and $(Y, M', g, *')$ be the two-sided limit shadowing property fuzzy dynamical systems if and only if $(X \times Y, M^{\star}, T, *_m)$ is the two-sided limit shadowing property fuzzy dynamical system.

Proof:

Let $(X, M, f, *)$ *and* $(Y, M', g, *')$ be fuzzy two-sided-limit shadowing property. Let $(x_k)_{k\in\mathbb{N}} \subset X$, $(y_k)_{k\in\mathbb{N}} \subset Y$.

Then by Definition 1.1.25

$$\lim_{k\to\infty} M(f(x_k), x_{k+1}, t) = 1 \qquad (2.21)$$

$$\lim_{k\to\infty} M'(g(y_k), y_{k+1}, t) = 1. \qquad (2.22)$$

Thus there is a point $p \in X, q \in Y$, such that

$$\lim_{k\to\infty} M(f^k(p), x_k, t) = 1 \qquad (2.23)$$

$$\lim_{k\to\infty} M'(g^k(q), y_k, t) = 1 \qquad (2.24)$$

We want to prove that $(X \times Y, M^\star, T, *_m)$ is fuzzy two-sided limit shadowing property. If $(x_k, y_k) \subset (X \times Y)$ satisfying the relation

$$\lim_{k\to\infty} M^\star\,(T(x_k, y_k), (x_{k+1}, y_{k+1}), t) = 1.$$

Hence

$$\lim_{k\to\infty} M(f(x_k), x_{k+1}, t) = 1 \qquad (2.25)$$

$$\lim_{k\to\infty} M'(g(y_k), y_{k+1}, t) = 1. \qquad (2.26)$$

Hence there exist $p \in X$ and $q \in Y$ so that

$$\lim_{k\to\infty} M(f^k(p), x_k, t) = 1 \qquad (2.27)$$

$$\lim_{k\to\infty} M'(g^k(q), y_k, t) = 1. \qquad (2.28)$$

Thus there exists $(p, q) \in X \times Y$ so that

$$\lim_{k\to\infty} M^\star\,(T^k(p, q), (x_k, y_k), t) = 1.$$

Thus, $(X \times Y, M^\star, T, *_m)$ is the two-sided limit shadowing property fuzzy dynamical system.

Conversely, let $(X \times Y, M^{\star}, T, *_m)$ be the two-sided limit shadowing property fuzzy dynamical system. If $(x_k)_{k\in\mathbb{N}} \in X$, $(y_k)_{k\in\mathbb{N}} \in Y$ satisfying

$$\lim_{k\to\infty} M(f(x_k), x_{k+1}, t) = 1 \qquad (2.29)$$

$$\lim_{k\to\infty} M'(g(y_k), y_{k+1}, t) = 1. \qquad (2.30)$$

Then

$$\lim_{k\to\infty} M^{\star}(T(x_k, y_k), (x_{k+1}, y_{k+1}), t) = 1.$$

Thus there exists $(p, q) \in X \times Y$ so that

$$\lim_{k\to\infty} M^{\star}\,(T^k(p, q), (x_k, y_k), t) = 1.$$

So

$$\lim_{k\to\infty} M(f^k(p), x_k, t) = 1 \qquad (2.31)$$

$$\lim_{k\to\infty} M'(g^k(q), y_k, t) = 1. \qquad (2.32)$$

So$(X, M, f, *)$ and $(Y, M', g, *')$ are the two−sided limit shadowing property fuzzy dynamical systems. □

Proposition 2.2.8

If $(X, M, f, *)$ and $(Y, M', g, *')$ are two fuzzy dynamical systems, then $(X, M, f, *)$ and $(Y, M', g, *')$ be positive expansive fuzzy dynamical systems if and only if $(X \times Y, M^{\star}, T, *_m)$is a positive expansive fuzzy dynamical system.

<u>Proof:</u>

Let $(X, M, f, *)$ *and* $(Y, M', g, *')$ are positive expansive fuzzy dynamical systems, and $x_1, x_2 \in X$, $y_1, y_2 \in Y$ *and* $0 < c < 1, t > 0$,

$x_1 \neq x_2, 1 - c_1 * 1 - c_2 > 1 - c.$

Then by Definition 1.1.19

$$M\ (f^k\ (x_1), f^k\ (x_2), t) > 1 - c_1 \qquad (2.33)$$

$$M'(\ g^k(x_2), g^k(y_2), t) > 1 - c_2. \qquad (2.34)$$

Let $(x_1, y_1) \in X \times Y, (x_2, y_2) \in X \times Y.$

We need to prove that.

$M^\star(T^k\ (x_1\ , y_1), T^k\ (x_2, y_2), t) > 1 - c.$

$M^\star(\ T^k\ (x_1, y_1), T^k\ (x_2\ , y_2), t) =$

$$M^\star\left(\left(f^k\ (x_1), g^k(y_1)\right), \left(\ f^k(x_2), g^k(y_2\ )\right), t\right) =$$

$$M\ (\ f^k(x_1), f^k(x_2), t) *_m M'(g^k(y_1), g^k(y_2), t) =$$

$$\min\left\{\begin{matrix} M\ (\ f^k(x_1), f^k(x_2), t) * M'(g^k(y_1), g^k(y_2), t), \\ M\ (\ f^k(x_1), f^k(x_2), t) *' \ M'(g^k(y_1), g^k(y_2), t) \end{matrix}\right\}.$$

By (2.33), (2.34) we have

$$M(\ f^k(x_1), f^k(x_2), t) * M'(g^k(y_1), g^k(y_2), t)$$

$$> 1 - c_1 * 1 - c_2 > \ 1 - c.$$

Hence

$$M^\star(T^k\ (x_1, x_2), T^k(y_1, y_2), t) > 1 - c.$$

Thus, $(X \times\ Y, M^\star, T, *_m)$ is a positive expansive fuzzy dynamical system.

Conversely, let $(X \times\ Y, M^\star, T, *_m)$ be a positive expansive fuzzy dynamical system. If $x_1, x_2 \in\ X\ ,\ y_1, y_2 \in\ Y$ satisfying

$$M\ (f^k\ (x_1), f^k\ (x_2), t) > 1 - c_1 \qquad (2.35)$$

$$M'(\ g^k(x_2), g^k(y_2), t) > 1 - c_2. \qquad (2.36)$$

Then

$$M^\star\ (T^k(x_1\ , y_1), T^k(x_2, y_2), t) > 1 - c.$$

Hence

$$M\ (f^k\ (x_1), f^k\ (x_2), t) > 1 - c_1 \qquad (2.37)$$

$$M'(\ g^k(x_2), g^k(y_2), t) > 1 - c_2. \qquad (2.38)$$

So $(X, M, f, *)$ and $(Y, M', g, *')$ are a positive expansive fuzzy dynamical systems. □

Proposition 2.2.9

If $(X, M, f, *)$ and $(Y, M', g, *')$ are two fuzzy dynamical system, then $(X, M, f, *)$ and $(Y, M', g, *')$ be $Lipschitz$ fuzzy dynamical systems if and only if $(X \times\ Y, M^\star, T, *_m)$ is $Lipschitz$ fuzzy dynamical system.

Proof:

Let $(X, M, f, *)$ and $(Y, M', g, *')$ be $Lipschitz$ fuzzy dynamical system, let $x_1, x_2 \in\ X\ \ , y_1, y_2 \in\ Y, 0 < \mathcal{L}_1, \mathcal{L}_2 < 1, 0 < \mathcal{L} < 1, t > 0, x_1 \neq x_2,$

Then by Definition 1.1.20

$$M(f(x_1), f(x_2), t) > 1 - \mathcal{L}_1\ \ M(x_1, x_2, t) \qquad (2.39)$$

$$M'(g(y_1), g(y_2), t) > 1 - \mathcal{L}_2 M'\ (y_1, y_2, t)\ . \qquad (2.40)$$

We need to prove that

$$M^\star(T(x_1, y_1), T(x_2, y_2), t) > 1 - \mathcal{L}\ M^\star\big((x_1, y_1), (x_2, y_2), t\big)$$

Where

$$1-\mathcal{L}_1\ \ M(x_1,x_2,t)*1-\mathcal{L}_2 M'\ (y_1,y_2,t)$$

$$>1-\mathcal{L}\ M^{\star}\big((x_1,y_1),(y_2,y_2),t\big)$$

$$M^{\star}(T(x_1,y_1),T(x_2,y_2),t)=M^{\star}\Big(\big(f(x_1),g(y_1)\big),\big(f(x_2),g(y_2)\big),t\Big)$$

$$=M(f(x_1),f(x_2),t)*_m\ M'(g(y_1),g(y_2),t)$$

$$=\min\begin{Bmatrix}M(f(x_1),f(x_2),t)*\ M'(g(y_1),g(y_2),t),\\ M(f(x_1),f(x_2),t)*'\ M'(g(y_1),g(y_2),t)\end{Bmatrix}$$

Since

$$M(f(x_1),f(x_2),t)>1-\mathcal{L}_1\ \ M(x_1,x_2,t)$$

$$M'(g(y_1),g(y_2),t)>1-\mathcal{L}_2 M'\ (y_1,y_2,t)$$

$$M(f(x_1),f(x_2),t)*M'(g(y_1),g(y_2),t)$$

$$>1-\mathcal{L}_1\ \ M(x_1,x_2,t)*1-\mathcal{L}_2 M'\ (y_1,y_2,t)$$

$$>1-\mathcal{L}\ M^{\star}\big((x_1,y_1),(x_2,y_2),t\big).$$

Hence

$$M^{\star}(T(x_1,y_1),T(x_2,y_2),t)>1-\mathcal{L}\ M^{\star}\big((x_1,y),(y_2,y_2),t\big)$$

Thus, $(X\times\ Y,M^{\star},T,*_m)$ is *Lipschitz* fuzzy dynamical system.

Conversely, let $(X\times\ Y,M^{\star},T,*_m)$ be a *Lipschitz* fuzzy dynamical system. If $x_1,x_2\in\ X$, $y_1,y_2\in\ Y$ satisfying

$$M(f(x_1),f(x_2),t)>1-\mathcal{L}_1\ \ M(x_1,x_2,t)\qquad(2.41)$$

$$M'(g(y_1),g(y_2),t)>1-\mathcal{L}_2 M'\ (y_1,y_2,t).\qquad(2.42)$$

Then

$$M^{\star}(T(x_1, y_1), T(x_2, y_2), t) > 1 - \mathcal{L}\, M^{\star}\big((x_1, y_1), (x_2, y_2), t\big).$$

Hence

$$M(f(x_1), f(x_2), t) > 1 - \mathcal{L}_1\ M(x_1, x_2, t) \qquad (2.43)$$

$$M'(g(y_1), g(y_2), t) > 1 - \mathcal{L}_2 M'\ (y_1, y_2, t). \qquad (2.44)$$

So $(X, M, f,*)$ and $(Y, M', g,*')$ are $Lipschitz$ fuzzy dynamical systems. □

Proposition 2.2.10

If $(X, M, f,*)$ and $(Y, M', g,*')$ are two fuzzy dynamical systems, then$(X, M, f,*)$ and $(Y, M', g,*')$ be h- shadowing fuzzy dynamical systems if and only if $(X \times\ Y, M^{\star}, T,*_m)$ is h-shadowing fuzzy dynamical system.

Proof:

Let $(X, M, f,*)$ and $(Y, M', g,*')$ be fuzzy h-shadowing property, let$(x_m)_{k\in\mathbb{N}} \subset X, (y_m)_{k\in\mathbb{N}} \subset Y$. Given $0 < \varepsilon_1, \varepsilon_2 < 1, 0 < \delta_1, \delta_2 < 1$,

$t > 0, 1 - \varepsilon_2 *\ 1 -\ \varepsilon_2 > 1 - \varepsilon$.

Then by Definition 1.1.21

$M(f^k(p), x_k, t) > 1 - \varepsilon_1$ for every $k < m$ and $f^m(p) = x_m$. (2.45)

$M'(g^k(q), y_k, t) > 1 - \varepsilon_2$ for every $k < m$ and $g^m(q) = y_m$. (2.46)

Let for every $k < m, T^m(p, q) = (x_m, y_m)$.

We need to prove that

There is $(p, q) \in X \times Y$

$$M^{\star}(T^k(p, q), (x_m y_m), t) > 1 - \varepsilon$$

$$M^{\star}\left(T^k(p,q),(x_m y_m),t\right) =$$

$$M^{\star}\left(\left(f^k(p),g^k(q)\right),(x_m,y_m),t\right) =$$

$$M(f^k(p),x_m,t) *_m \; M'(g^k(q),y_m,t) =$$

$$= min\begin{Bmatrix} M(f^k(p),x_m,t) * \; M'(g^k(q),y_m,t), \\ M(f^k(p),x_m,t) *' \; M'(g^k(q),y_m,t) \end{Bmatrix}.$$

$M(f^k(p),x_k,t) > 1-\varepsilon_1$ *for every* $k < m$ *and* $f^m(p) = x_m$.

$M'(g^k(q),y_k,t) > 1-\varepsilon_2$ *for every* $k < m$ *and* $g^m(q) = y_m$.

$M(f^k(p),x_k,t) * M'(g^k(q),y_k,t) > 1-\varepsilon_1 * 1-\varepsilon_2 > 1-\varepsilon$

for every $k < m$ and $T^m(p,q) = (x_m,y_m)$.

Thus, $(X\times Y, M^{\star}, T, *_m)$ is h−shadowing fuzzy dynamical system.

Conversely, let $(X\times Y, M^{\star}, T, *_m)$be h−shadowing fuzzy dynamical system.

If $(x_m)_{m\in\mathbb{N}} \subset X, (y_m)_{m\in\mathbb{N}} \subset Y$ satisfying

$M(f^k(p),x_k,t) > 1-\varepsilon_1$ for every $k < m$ and $f^m(p) = x_m$. (2.47)

$M'(g^k(q),y_k,t) > 1-\varepsilon_2$ for every $k < m$ and $g^m(q) = y_m$. (2.48)

Then

$$M^{\star}(T^k(p,q),(x_m y_m),t) > 1-\varepsilon$$

for every $k < m, T^m(p,q) = (x_m,y_m)$.

Hence

$M(f^k(p),x_k,t) > 1-\varepsilon_1$ for every $k < m$ and $f^m(p) = x_m$. (2.49)

$$M'(g^k(q), y_k, t) > 1 - \varepsilon_2 \text{ for every } k < m \text{ and } g^m(q) = y_m. \qquad (2.50)$$

So $(X, M, f,*)$ and $(Y, M', g,*')$ be h$-shadowing$ fuzzy dynamical systems. □

Conclusion and Future Work

Conclusions

- We introduced the new (the L-shadowing property, two-sided limit shadowing property, negative limit shadowing, positive expansive, h-shadow and Lipschitz) in fuzzy dynamical systems.
- We found that properties (fuzzy L-shadowing property, fuzzy two –sided limit shadowing) are invariant under fuzzy topological conjugacy.
- We studied the product of the (expansive, positive expansive, L-shadowing property, two-sided limit shadowing property, h-shadow and Lipschitz) of fuzzy dynamical systems.

Future Works

The following one suggestions in 2-fuzzy metric space for further studied

1. Define the properties (the L-shadowing property, two-sided limit shadowing property, negative limit shadowing, positive expansive, h-shadow and Lipschitz.
2. Study the topological conjugacy for the above mentioned properties.
3. Product study for the above mentioned properties.

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المستخلص

سنعتمد في هذه الأطروحة على التعريف الذي قدمه جورج وفيراماني عام 1994، وهو تعديل لمفهوم الفضاء المتري الغامض الذي قدمه كراموسيل وميشاليك عام 1975.

تتضمن هذه الأطروحة دراسة بعض الخصائص الفوضوية في الأنظمة الديناميكية الضبابية وإدخال تعريفات جديدة (التمدد الإيجابي الضبابي، ليبتشيتز الضبابي، الطوبولوجيا المستقرة الضبابية، خاصية التظليل L- الغامض، خاصية التظليل الحدي الضبابي ثنائي الجوانب، تظليل الحد السلبي الضبابي، خاصية التظليل h- الضبابي.

من خلال هذه الدراسة، تم إثبات ثبات الخصائص (خاصية التظليل L-، والتظليل الحدي الضبابي على الوجهين) في ظل الاقتران الطوبولوجي الضبابي.

تشير نتائجنا إلى أن نظامين ديناميكيين غامضين على مساحتين متريتين غامضتين يحققان الخصائص (التمدد الضبابي، وخاصية ليبشيتز الضبابية، وخاصية التظليل L-، وخاصية التظليل الحدي الضبابي على الوجهين، والظل الموجب الضبابي الموسع، وظل h- الضبابي) إذا وفقط إذا المنتج الذي يصل إلى معيار t خاص يحقق الخصائص المذكورة أعلاه.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

بعض مفاهيم الخواص الفوضوية في الانظمة الديناميكية الضبابية

رسالة

مقدمة الى مجلس كلية التربية للعلوم الصرفة / جامعة بابل كجزء من متطلبات نيل

درجة الماجستير في التربية / الرياضيات

من قبل

صفا ازغير علي عسكر

بإشراف

أ.م.د.مي علاء عبدالخالق الياسين

1445هـ 2023م