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تقدمت بها

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ESTIMATIONS FOR MONOTONE APPROXIMATION

A thesis submitted to the Department of Mathematics College of Education Babylon University as a partial in partial fulfillment of the requirement for the degree of the Master of Science in Mathematics.

By

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بسم الله الرحمن الرحيم

ألا إن أولياء الله لا خوف عليهم ولا هم يحزنون (62)

الذين آمنوا وكانوا يتقون (63) لهم البشرى في الحياة

الدنيا وفي الآخرة لا تبديل لكلمات الله ذلك هو

الفوز العظيم (64)

صدق الله العلي العظيم سورة يونس

SUPERVISOR CERTIFICATION

I certify that this thesis was prepared under our supervision at the Babylon university, College of Education as a partial fulfillment of the requirements of degree Master of Science in Mathematics.

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TO MY

FAMILY

Contents

Abstract

We wish sometimes to approximate a function *f* defined on a finite interval say [-1, 1], while preserving Certain intrinsic shape properties.To be specific, we demand that the approximation process preserves some properties this what we called monotone approximation.

In order to draw a meaningful picture in our minds for the approximation theory and prepare the background for our work and motivate our results. We understand the need to know the point of the origin for the theory of approximation by following most prominent scientists and the geometric view of best approximation for an element in a normed linear spase , recall definitions, examples, applications and some results related to the basic concepts for constrained approximation by algebric polynomials.

Our first achievement in this work is that for the function in $L_P[-1,1]$, $0 < P < 1$, We obtain a result for the rate of monotone approximation of function in terms of the second Ditzian .Totike modulus of smoothness ω_2^φ. Also a converse theorem for this direct theorem were obtained, and we characterize the monotone function in the generalized Lipschitz space through-their approximation properties. This constraint restrict very much the degree of approximation that the algebric polynomial can achieve, namely only ω_2^φ. To emphasize that our estimates

are sharp in the sense that ω_2^{φ} can not replaced by ω_3^{φ}.We introduce a negative theorem to ensure this theorem.

In[31] K.A. Kopotun proved that the monotonicity requirement in interval of length proportional to $\frac{1}{n}$ about the interior extreme of the function and near the end points, allow the polynomial to achieve a pointwise approximation rate of ω_3^{φ}.

We show in this work that no relaxing of the monotonicity requirement in sets of measure approaching Zero, allow ω_4^{φ} estimates .

In near future we hope to stutdy basic properties for the modulus of continuity and the moduls of smoothness in $L_{p,\mu}$ for $o < p < 1$, μ is measurable function.

Introduction

Our interest in approximation theory stems from its beauity, its utility and its rich history .There are also many connections that can be drawn to question in both classical and modren analysis.

In our thesis we introduce estimates for pointwise approximation .Our thesis consists of four chapters:

In chapter one, we prepare a background for our object and introduce some notes on a historical review of approximate theory, Approximation in a normed linear space, The space $L_p, 0 < p < 1$, Moduli of Smoothness, Approximation with Polynomials and Applications of the theory of best Approximation.

In chapter two we prove two theorems; The first theorem is a direct inequality for monotone approximation:

Theorem I. If f is an increasing function in $L_p[-1,1]$, $0 < p < 1$ then for each n=1,2,….., there is an increasing polynomial in $L_p(I)$ of degree (8n) satisfying

$$\|f - p_n\|_p \leq c(p)\,\omega_2^{\varphi}\left(f, \frac{1}{n}\right)_p,$$

For this direct theorem we prove the following inverse theorem

Theorem II. Let f be an increasing function in $L_p(I), 0 < p < 1,$ then

$$\omega_2^{\varphi}\left(f, n^{-1}\right)_p^p \leq c(p) E_n^{(1)}(f)_p^p + c(p) n^{-2p} \sum_{m>n} (m+1)^{p-1} E_n^{(1)}(f)_p^p.$$

More results in chapter three are:

In the first theorem we characterize the monotone function in the generalized Lipschitz space through their approximation properties:

Theorem III. If $0 < \alpha < 2,$ then an increasing function f is in Lip_{α}^{*} iff for each n=1,2,….., there is an increasing algebraic polynomial p_n of degree 8n such that $E_n(f)_p \leq \|f - p_n\|_p \leq c(p)\frac{1}{n^{\alpha}},$

Then we prove negative theorem for our direct theorem then in chapter two:

Theorem IV. For each n and $0 < p < 1,$ there is a function $f \in \Delta^1 \cap C^3(I),$ such that for every polynomial $p_n \in \Delta^1 \cap \Pi_n$ either

$$\lim_{x\to -1} \sup \frac{\|f - p_n\|_p}{\varphi^3(x)} = \infty \ or \ \lim_{x\to 1} \sup \frac{\|f - p_n\|_p}{\varphi^3(x)} = \infty, x \in I.$$

In chapter four we strengthen the theorem of Kopotun for nearly monotone approximation for increasing function in L_p, and we prove

Theorem V. for each sequence $\bar{\varepsilon} = \{\varepsilon_n\}_{n=1}^{\infty}$ of non negative numbers tending to zero, and each $0 < p < 1$, there exists an increasing function $f = f_{\bar{\varepsilon}} \in L_p[-1,1]$ such that

$$\underset{n\to\infty}{Lim}\sup \frac{E_n^{(1)}(f;\varepsilon_n)_p}{\omega_4^{\varphi}\left(f,\frac{1}{n}\right)_p} = \infty$$

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Sukeina. A. A.

2008

List of definitions of Symbols

Symbol	Definition
c	absolute constant
C(p)	absolute constant depending on p
$W_P^K[a,b]$	Sobolev space: the set of all functions from $L_p[a,b]$ such that $f^{(K-1)}$ are absolutely continuous and $f^{(K)} \in L_p[a,b]$.
C(I)	The space of all continuous functions on $I=[-1,1]$
$\Delta_n(x)$	$\Delta_n(x)=n^{-1}\sqrt{1-x^2}+n^{-2}$
$\Delta_h^r(f,x)$	The rth symmetric difference of f is given by $\Delta_h^r(f,x,[a,b]):=\Delta_h^r(f,x):=\begin{cases}\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i} f\left(x-\frac{rh}{2}+ih\right), & x\pm\frac{rh}{2}\in[a,b]\\ 0, & \text{o.w}\end{cases}$
$E_n^1(f)_p$	$E_n^1(f)_p=\inf_{p_n\in\Delta^1\cap\Pi_n}\lVert f-p_n\rVert_p$
$t_v(x)$	Chebyshev polynomials for the interval [-2 , 2] , for v > 1 let $t_v(x)=\cos v\cos^{-1}\frac{x}{2}, x\in[-2,2]$.
$K_n(t)$	Jackson kernel $K_n(t)=a_n\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^8$, where a_n is a constant depending on n
$\widetilde{K}_{r,\varphi}(f,\delta^r)_p$	Ditzian-Totik $\tilde{K}$ functional defined by $\widetilde{K}_{r,\varphi}(f,\delta^r)_P:=\inf_{p_n\in\Pi_n}\left\{\lVert f-P_n\rVert_{L_P(J)}+\delta^r\lVert\varphi^r P_n^{(r)}\rVert_{L_P(J)}\right\}$

Symbol	Definition
$L_P(I)$	space $L_p(I)$ consisting of all measurable function on I for which $\|f\|_p^p = \int_I |f(x)|^p dx < \infty$
$\|.\|_p$	$\|.\|_P = \|.\|_{Lp(I)}$ quasi-norm where $I = [-1,1]$
$\|f\|_{[a,b]}$	$\|f\|_{[a,b]} = \sup_{x\in[a,b]} |f(x)|$ uniform norm $X = [a,b]$,the space of all continuous functions $f:[a,b] \to R$
$\omega_r(f,\delta,[a,b])_p$	r^{th} usual modulus of smoothness of $f \in L_P[a,b]$ is defined by $\omega_r(f,\delta,[a,b])_p := \sup_{0<h\le\delta} \|\Delta_h^r(f,.)\|_{L_P[a,b]}, \delta \ge 0$
$\tau_r(f,\delta,[a,b])_p$	τ- mdulus (or Sendov-Popov modulus), an averaged modulus of smoothness, defined for bounded measurable functions on [*a, b*] by $\tau_r(f,\delta,[a,b])_P := \|\omega_r(f,.,\delta)\|_{L_p[a,b]}, \delta \ge 0$
$\omega_r(f,x,\delta)_p$	$\omega_r(f,x,\delta)_P := \sup\left\{\left|\Delta_h^r(f,y)\right| : y \pm \frac{rh}{2} \in \left[x - \frac{r\delta}{2}, x + \frac{r\delta}{2}\right] \cap [a,b]\right\}$] is the r^{th} local modulus of smoothness of *f*.
$\omega_r^\varphi(f,\delta,J)_p$	Ditzian-Totik modulus of moothness which is defined for such an *f* as follow $\omega_r^\varphi(f,\delta,J)_P := \sup_{0<h\le\delta} \|\Delta_{h\varphi(.)}^r(f,.)\|_{L_P(J)}$.
Π_n	Π_n the space of all polynomials of degree not exceeding n.
$L_n(f)$	The set of bounded linear transformations on f
lip_α^*	lip_α^* spaces defined as the set of all functions f such

	that ω_2^{φ}(f)=O(t^{α}), $0 < \alpha < 2$

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المستخلص

نرغب أحيانا بتقريب دالة معرفة على الفترة $[-1,1]$, بحيث يكون ذلك التقريب حافظاً لبعض الخواص الشكلية بعبارة أخرى في بعض الأحيان نحتاج إن يكون تقريب دالة ما حافظاً لبعض خواصها الشكلية, كأن تكون رتيبة وهذا ما نسميه بالتقريب الرتيب .

لرسم صورة متكاملة في أذهاننا عن نظرية التقريب . و من اجل إن نهيئ بعض المعلومات الأساسية التي نحتاجها حتى يصبح العمل أكثر وضوحا ,استذكرنا نقطة أصل نظرية التقريب بواسطة اقتفاء اثأر ابرز علماء نظرية التقريب و المعنى الهندسي لتقريب عنصر في فضاء معياري بالإضافة إلى التعريفات,الأمثلة,التطبيقات ,وبعض النتائج التي ترتبط أساسيا بالتقريب المقيد بواسطة متعددات حدود جبرية .

إن أول النتائج التي حصلنا عليها في هذا العمل هي نظرية مباشرة في التقريب الرتيب للدوال القياسية في الفضاء L_p عندما $0<p<1$ بدلالة مقياس النعومة ω_2^{φ} .

وكذلك أوجدنا النظرية العكسية للنظرية المباشرة التي حصلنا عليها وميزنا الدوال الرتيبة في الفضاء Generalized Lipscitz space من خلال خواص التقريب لتلك الدوال.

إن قيد الرتابة في النتيجة أعلاه سيحدد درجة التقريب الأفضل بواسطة المتعددات الجبرية بدلالة المقياس ω_2^{φ} فقط , ولتأكيد ذلك برهنا انه لا يمكن استبدال المقياس ω_2^{φ} للدالة f بالمقياس ω_3^{φ} ,لذلك قدمنا النظرية النقيضة للنظرية المباشرة.

العالم K.A. Kopotun برهن انه عندما يجعل متعددة الحدود الجبرية التقريبية تكون غير رتيبة مع دالة الهدف في بعض الفترات الجزئية

من الفترة $[-1,1]$ التي تحتوي على النقاط الحرجة للدالة f بالإضافة إلى الفترات الجزئية التي تحتوي نهايات الفترة $[-1,1]$ والتي تكون أطوالها مقتربة إلى الصفر يمكن إن يحصل على التقريب بدلالة مقياس النعومة ω_3^{φ}.

برهنا انه لا يمكن استبدال مقياس النعومة ω_3^{φ} بالمقياس ω_4^{φ} حتى لو كانت المتعددة الجبرية غير رتيبة مع دالة الهدف في مجموعات جزئيه من الفترة $[-1,1]$ تقترب أطوالها إلى الصفر.

في المستقبل القريب نأمل دراسة الخواص الأساسية لمقاييس الاستمرارية والنعومة في الفضاء $L_{p,\mu}$ عندما $0<p<1$, μ دالة قياسية .

Chapter One

Approximation Theory: an Overview

1.1 A Historical review of approximate theory

We can know the history of approximation theory by following the most prominent scientists:

1.1.1 P.L. Chebyshev

1821-1894 the first of major memories that contain chebyshev research on the best approximation function was presented to the academy of sciences in January 1853 ,this is soon after Chebyshev had returned from a long trip, where he visited the most important European scientific and industrial centers, and where he paid equal attention to studying factories, plants, and different kinds of interesting subjects in applied mechanics, as well as making personal contacts and conversing with famous geometric ,mainly French once .

We can summarize the problem that Chebyshev considered as: Given a contain function f defined on $[a,b]$ and a positive *n*, can we represent f by a polynomial $p(x) = \sum_{i=0}^{n} a_i x_i$ of degree at most *n,* in such a way that the maximum error at any point x in $[a,b]$ is controlled ? In particular, It is possible to construct p in such a way that the error $\max_{x\in[a,b]} |f(x) - p(x)|$ is minimized.

Chebyshev was interested in the exact computation of the number $E_n(f)$ for given *n*, and also in constructing the corresponding uniquely defined polynomial *p*, the trigonometric polynomial, can be first founded in same specific problems , in Chebyshev work of 1881,[10].

1.1.2 K. Weierstrass

The best approximation of a function originated a proposition established in 1885 by Weierstrass, head of the Berlin school of mathematics.

For any continuous function $f(x)$ on a given interval $a \le x \le b$. and for any $\varepsilon > 0$ there is a polynomial $p(x)$ such that

$$\|f - p\|_{[a,b]} \le \varepsilon$$

Weierstrass proved that the limit of uniformly converging sequences of continuous functions is also continuous functions, the property established by Weierstrass is exactly equivalent to the continuity of the function in the given interval. In other words, the set of polynomials is dense in the set C[a, b].

Weierstrass theorem was the first stone in fundamental of functional analysis and a solid basis for further development of this new direction was not yet in place.

He proved

$$\lim_{n\to\infty} E_n(f)=0$$

1.1.3 De Ia valle poussion.

In 1903, the Belgian Academy of sciences following a suggestion of its member De Ia vallee poussion , posed the following research challenge ."To present new investigations in the area of expanding real or analytics function in to series of polynomial". They posed the following precise question "Is it possible to approximate a polynomial of degree n at a rate of higher then $\frac{1}{n}$? " In other words is it possible to replace the expression $E_n(|x|)=O\left(\frac{1}{n}\right)$ by more precise $E_n(|x|)=o\left[\frac{1}{n}\right]$? Research in this direction was stared by De Ia vallee poussin himself. In a work published in 1910, [44]. He demonist rated a trick that allows him to get a lower bound on $E_n(|x|)$, where as any approximating polynomial , obviously gives an upper bound and he uses the trick to obtain the inequality:

$$E_n(|x|) > \frac{k}{n(Log\,n)^3}, k > 0.$$

Approximation of $|x|^{\frac{1}{2}}$ is also considered there.

In 1919, there appears a new monograph by De Ia vallee poussin [45], which Synthesized with weierstrass, in functional approximation theory. Not one property of this book, it is first book that precisely posed and studied (in connection to each other) two analogues problems :

Approximating a function given on an interval of the real by an algebraic polynomial of degree n and approximating a periodic function with period 2π by trigonometric polynomial of degree n . (De Ia vallee poussin called them trigonometric sums).

1.1.4 S.N.Bernstein

Let us briefly mention the most significant results of the ph.D.Dissertation of S.N.Bernstein, who is an academicia.[4]

(i) The question posed by De Ia vallee poussin answered negatively there are positive numbers A and B such that

$$\frac{A}{n} < E_n(|x|) < \frac{B}{n}$$

(ii) If $E_n(f) = O\left(\frac{1}{n^{r+\varepsilon}}\right)$, then the function f has a continuous derivative of order r .

(iii) As a tool in his proof, S.N.Bernstein used the following theorem of great independent interest .If $p_n(x)$ is a polynomial of degree n ,then the inequality

$$\max_{-1\le x\le 1}|p_n(x)|\le 1$$

Implies the inequality

$$|p'_n(x)|\le \frac{n}{\sqrt{1-x^2}},(-1<x<1).$$

(iv) An upper bound on $E_n(f)$ is established as a function of the upper bound on $\left|f^{(n+1)}_{(x)}\right|$ in a given interval .

S.N.Bernstein returned to the problem of best approximation of the simplest functions that are not infinitely differentiable, $|x^k|, k>0$ and proved in[5] an equality ,for $n\to\infty$ of the form

$$E_n\left(|x|^k\right)\approx \frac{m(k)}{n^k}.$$

Where $m(k)$ is continuous function of the variable k .Eestablished in [6] a more general relation

$$E_n\left(|x-c|^k\right)\approx \left(1-c^2\right)^{k/2}\frac{m(k)}{n^k},(-1<x<1).$$

And found a method for computing $E_n(f)$ for arbitrary function f that has only a finite number of corners on the basic interval. There for S.N.Bernstein found new direction in theory of function , which he later named constructive.

1.1.5 D.Jakson .

In 1911 the American mathematician D.Jakson, [27] presented a dissertation ,in which he proved a series of theorem that are converses to Bernstein theorems. From the positive a assumption on the differential properties of a function being approximation conclusions are drawn about the rate $E_n(f)$ of best approximation .

In 1912 a rather complete theory was formed based on the result of S.N.Bernstein and D.Jackson that become the main part of report presented by S.N.Bernstein at International Mathematics congress in Cambridge, [3]

In [28] ,D.Jackson proved that for r = 1

$$E_n(f) \le c\,\omega_r\left(f, n^{-1}\right).$$

This inequality led to beautiful theory due Nikoliskii [40]later investigation by Timan[10] ,Freud [26], Dzjadyk[25] and Brudny [8] ,established

(i)There exists polynomial $p_n \in \Pi_n$

$$|f(x) - p_n(x)| \le c(r)\omega_r(f, \Delta_n(x)).$$

For all $x \in (-1,1), where\ \Delta_n(x) = n^{-1}\sqrt{1-x^2} + n^{-2}$

(ii) If there exists for every n .A polynomial $p_n \in \Pi_n$, such that

$$|f(x) - p_n(x)| \le \omega(\Delta_n(x)).$$

For some increasing function $\omega(x)$ .then

$$w_r(f,\delta)\, c\, \delta^r \le \sum_{o \le n < \frac{1}{\delta}} (n+1)^{r+1} w\left(n^{-1}\right),$$

1.2 Approximation in a normed linear space

1.2.1 The normed linear space

The norm is a real valued function $\|x\|$ that is defined for $x \in X$,where X is metric space .Its properties are such that the function .

$$d(x,y)=\|x-y\|, \text{ [43]}$$

Is suitable as a distance function ,we may deduce the triangle inequality

$$\|x+y\| \le \|x\| + \|y\|.$$

Moreover ,the norm must satisfy the homogeneity condition

$$\|\lambda x\| = |\lambda|\|x\|$$

For all $x \in X$ and for scalar λ

There are many norms we might consider on R^n.Of particular interest are the L_p- norms in the cases when p=1,2 and ∞ .

(i) Consider $X = R^2$ under the norm $\|(x,y)\| = \max\{|x|,|y|\}, or\ X = R^n$ the scale of norms

$$\left\|(x_i)_{i=1}^n\right\|_{L_p} = \begin{cases} \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} & ,1 \le p < \infty \\ \max\limits_{1\le i\le n}|x_i| & ,p=\infty \end{cases}$$

(ii) consider $X = [a,b]$,the space of all continuous functions $f:[a,b] \to R$ under the uniform norm

$$\|f\|_{[a,b]} = \sup_{x\in[a,b]} |f(x)|$$

Chebyshev's problem is perhaps best understood by rephrasing it in modern terms. What we have is a problem of linear approximation in a normed linear space .

The abstract version of Chebshe'v problem can be restated. Given a subset (or sub space)Y of X and a point $x \in X$, is there an element $y \in Y$ which is nearest to x, that is we can fined a vector $y \in Y$ such that

$$\|x - y\| = \inf_{z\in Y} \|x - z\|$$

If there is such a best approximation $Z \in Y$ to x from elements of Y.Is it unique ?

Collecting the results obtained in [11], one can make the following statements:

If a set Y in a normed linear space X is also linear space and of finite dimensional, then for each element $x \in X$, there is an element, y in Y that provides a best approximation :if Y is ompact and strictly convex set in a normed linear space X , then for all $x \in X$, there is just one best approximation from Y.

1.2.2 Examples

(i) Existence of the best approximation

The best approximation for an element is not always existingLet Y be the set of all polynomials on $\left[0,\frac{1}{2}\right]$ of any degree, considered as a sub space of $C\left[o,\frac{1}{2}\right]$then $\dim Y=\infty$.

Let $x(t)=(1-t)^{-1}$

Then for every $\in>0$ there is N such that

$Y_n(t)=1+t+...........+t^n$,

We have

$$\|x-Y_n\|<\in \text{ For all } n>N$$

Hence $\delta(x,Y)=\inf\|x-Y_n\|=0$.

However, since x is not a polynomial, we see that there is no $Y_o\in Y$

Satisfying $\delta=\delta(x,Y)=\|x-Y_o\|=0$

But $x(t)=(1-t)^{-1}$

$Y=span\{1,t,t^2,.......\}=\text{P}$, $where\, span\{t^i\}_{i=0}^{\infty}$ is the set of all finite linear combinations of the elements of the system $\{t^i\}_{i=0}^{\infty}$.

(ii)Uniqueness of the best approximation

It is of practical interest, too, since for given x and Y there may more than one best approximation. Consider $X=R^2$ and let us take the point $x(1,-1)$ and the sub space Y that is, $Y=\{y=(n,n)\ n\ real\}$

Then for all $y\in Y$ we clearly have

$$\|x-y\|_1=|1-n|+|-1-n|\geq 2.$$

The distance from x to Y is $\delta = (x,Y) = 2$, and all $|n| \le 1$ are best approximation to x out of Y. This illustrates that even in such a simple space for given x and Y we may have several best approximation even infinity many of them.

1.2.3 The geometric view of best approximation

In most approximation examples there exists a normed linear space X for that contains both f and the set of approximation A, When we require a best approximation from A to f, it is sometimes helpful to think. Of the balls of different radii whose centers are at f .The ball of radius r is defined to be the set

$$N(f,r) \equiv \{g : \|g - f\| \le r, g \in X\},\ [43].$$

We know that, if A is a finite dimensional linear space, then the equation

$$r^* = \inf \|f - a\| = \|f - a^*\| \ .$$

Is obtained for a point a^* in A.

For example, suppose that X is two-dimensional Euclidean space R^2 and that we are using the 2-norm .Let f be the point whose components are (2,1), and let A be the linear space of vectors

$$A = \{(\lambda, \lambda); -\infty < \lambda < \infty\},$$

Where λ is a real parameter. Figure (1.2.3) shows the set A and three balls centre f, whose radii are $\frac{1}{2}, \sqrt{\frac{1}{2}}\ and\ 1$,we see that the best approximation of f is the point where the ball of radius $\sqrt{\frac{1}{2}}$ touch A.

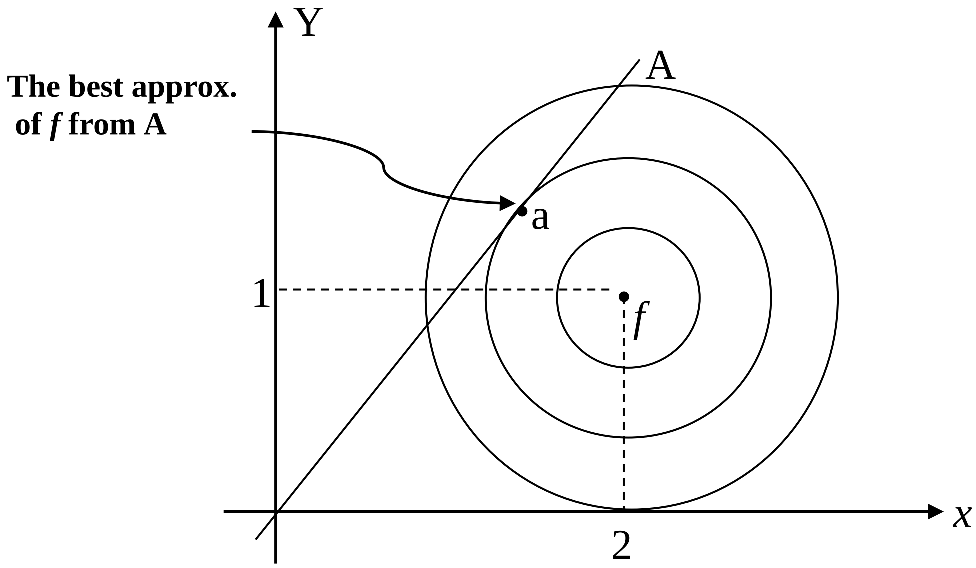

Figure 1.2.3 an approx .problem in R^2

1.3 The space $L_p, 0<p<1$

We are going to study the degree of constrained approximation of function f in the $L_p[-1,1], 0<p<1$ quasi-norm .The degree of approximation will be measured by the approximation quasi-norm which we denote by $\|.\|_p = \|.\|_{Lp(I)}$ where $I=[-1,1]$ since we need the L_p quasi-norm on other intervals we will in all cases on an interval $J \neq I$, indicate that by writing $\|.\|_{Lp(J)}$, Also we refer to the uniform norm on I by $\|.\|$ and on interval J by $\|.\|_J$. $\|.\|_p$ is a norm for $1 \le p \le \infty$. Characteristic for $L_p, p \ge 1$ are the inequalities of Holder and Mkowski. The dual space of $L_p, 1 \le p < \infty$, is spaces L_q with conjugate exponent q of p. Thus the spaces L_p with $1 \le p < \infty$ are reflexive.

The different structure of the spaces $L_p, 0 < p < 1$ and the numerous questions by others lead us to understand the need for the following few facts about $L_p \ for, 0 < p < 1$.

We consider the space $L_p(I)$,consisting of all measurable function on I for which $\|f\|_p^p = \int_I |f(x)|^p dx < \infty$

Recall that $\|f\|_p \le 2^{\frac{1}{p}-1} \|f\|_1 \ \ thatis \ L_1 \subset L_p$.

As we will see shortly ,the L_p norm is not actually a norm for p<1 nevertheless, it is not hard to see that $L_p(I)$ is complete metric space .

It is easy to prove the following theorem .

theoremA.[7]

$$\|f+g\|_p \le \left(\|f\|_p^p + \|g\|_p^p\right)^{\frac{1}{p}} \le 2^{\frac{1}{p}} \left(\|f\|_p + \|g\|_p\right) \ for\, any\, f, g \in L_p[I]$$

thus $d(f,g) = \|f-g\|_p^p$ defines a translation invariant metric on L_p.It is a complete metric because convergence (respectively

Cauchy)in $Lp(I)$ implies convergence (respectively Cauchy)in measure since

$$\varepsilon^p meas\{t:|f(t)|>\varepsilon\}\le \int_{-1}^{1}|f(t)|^p\,dt$$

What this means to us is that

(i) A linear map on Lp is continuous if and only if it is bounded (continuous at zero)

(ii) The open mapping and closed graph theorem still apply.[33]

(iii) The Hahn Banach theorem may fail

Indeed as we will see shortly, Lp is not locally convex. In fact it is impossible to define a norm on Lp which gives the same topology as the usual metric there are several ways to see that $Lp(I)$ is not normable for 0<p<1 .Most useful from our point of view is

Theorem B. [15] $L_p(I), 0<p<1$ has a trivial dual

Proof . Suppose that $T:L_p\to R$ is continuous linear and non zero then we can find $f\in L_p(I)\, such\ that\ T(f)=1$ .Now since $|f|^p$ is intergable so that the map $x\to f\chi[-1,x]$ is continuous since.

$$\| f\chi[-1,x]-f\chi[-1,y]\|_p^p=\int_x^y|f(t)|^p\,dt \text{ for } x<y$$

Where $\chi_J(t)=\begin{cases}1, & t\in J \quad , for\ any\ interval\ J\\ 0, & o.w\end{cases}$

Thus we may choose $-1\le x\le 1$ such that

$$T(f\chi[-1,x]) = T(f\chi[x,1]) = \frac{1}{2}.$$

Next, notice that $g = f\chi[-1,x]$ *and* $h = f\chi[x,1]$,satisfy

$$\|f\|_p^p = \int_{-1}^{x} |f(t)|^p \, dt + \int_{x}^{1} |f(t)|^p \, dt = \|g\|_p^p + \|h\|_p^p$$

Thus at least one of $\|g\|_p^p$ *or* $\|h\|_p^p$ *is* $\leq \frac{1}{2}\|f\|_p^p$ .Let us say that

$\|g\|_p^p \leq \frac{1}{2}\|f\|_p^p$ .Then $f_1 = 2g$ satisfies

$$T(f_n) = 1 \text{ and } \quad \|f_n\|_p^p \leq 2^{n(p-1)}\|f\|_p^p$$

Since $p-1<0,$ so that $f_n \to 0$ *in* $L_p(I)$ *while* $T(f_n) = 1.$

No good! thus T=0 is the only continuous linear functional .

The Hahn Banach theorem allows a much fancier sounding version of this result :

Corollary C. [7] There are no non zero continuous linear may from L_p into any normed space.

In any event ,it should now be clear that there can be no norm on L_p which gives the same topology as the usual metric and that the Hanach Banach theorem evidently fails in $L_p(I)$ *for* $0<p<1$ .the fact that L_p has a trivial dual with the theorem from, [32] states that "There exists anon zero continuous linear function on a linear space X if and only if there is at least one convex set that is not all of X.",imply another rather strange result that would be hard to believe other wise .

Corollary D. [7] If U is any neighborhood of zero in $L_p(I)$,then

$$L_p(I) = conv\, U.$$

In particular

$$L_p(I) = conv\left\{ f : \|f\|_p^p < 1 \right\},$$

Where conv U is a convex neighborhood of zero contained in U. Now we will settle the question of which $L_P(I)$, 0< P <1 embed into L_q for q ≥1. Or which subspaces of $Lp(I)$ on which all of the various $Lq(I)$ quasi norms for 0< $p < q$ are equivalent. the key in this article is due to kadec and pelzynyski from [30]:

For $0 < \varepsilon < 1$ *and* $0 < p < \infty$ consider the following sub set of $Lp(I)$

$$\mathrm{M}(p,\varepsilon) = \left\{ f \in Lp(I) : meas\left\{ x : |f(x)| \geq \varepsilon \|f\|_p \right\} \geq \varepsilon \right\}$$

,We mean by meas the measure of the set.

Notice that if $\varepsilon_1 < \varepsilon_2$, *then* $\mathrm{M}(p,\varepsilon_1) \supset \mathrm{M}(p,\varepsilon_2)$. *Also*

$\underset{\varepsilon>0}{U}\, \mathrm{M}(p,\varepsilon) = Lp(I)$, since for any non zero $f \in Lp(I)$ we have

$meas\{|f| \geq \varepsilon\} \to meas\{f \neq o\}\ as\ \varepsilon \to 0$. In fact any finite subset of $Lp(I)$ is contained in an $M(P,\varepsilon)$ for some $\varepsilon > 0$. finally note that

$meas\left\{ |f| \geq \|f\|_p \right\} \geq 1\ \ implies\ \ |f| = \|f\|_p$ almost every where the following theorem puts this observation to good use

Theorem E. [12] For a subset S of $Lp(I)$, the following are equivalent

(i) $S \subset \mathrm{M}(p,\varepsilon)$ *for some* $\varepsilon > 0$

(ii)for each 0< p < q there exists a constaint $c(q) < \infty$ such that

$$\|f\|_q \leq \|f\|_p \leq c(q)\|f\|_q, \quad for\, all\ f \in S .$$

1.4 Moduli of Smoothness

Moduli of smoothness are intended for mathematicians working in approximation theory, numerical analysis and real analysis. Measuring the smoothness of a function by differentiability is too crude for many purposes in approximation theory. More subtle measurement are provided by the moduli of smoothness The modulus of a function *f* can be defined when *f* is a given on any metric space X, but we will restrict ourselves to *X:*=[*a,b*]. We will use moduli of smoothness which are connected with difference of higher orders .

The r th symmetric difference of *f* is given by

$$\Delta_h^r(f, x, [a,b]) := \Delta_h^r(f, x) := \begin{cases} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \, f\left(x - \frac{rh}{2} + ih\right), & x \pm \frac{rh}{2} \in [a,b], [7] \\ 0, & \text{o.w} \end{cases}$$

Then the r th usual modulus of smoothness of $f \in L_P[a,b]$ is defined by

$$\omega_r(f, \delta, [a,b])_p := \sup_{0<h\leq\delta} \left\|\Delta_h^r(f, .)\right\|_{L_P[a,b]}, \delta \geq 0, [7]$$

The so called τ- mdulus (or Sendov-Popov modulus), an averaged modulus of smoothness, defined for bounded measurable functions on [*a, b*] by

$\tau_r(f,\delta,[a,b])_P := \|\omega_r(f,.,\delta)\|_{L_P[a,b]}$, $\delta \geq 0$,[7]

where

$\omega_r(f,x,\delta)_P := \sup\left\{\left|\Delta_h^r(f,y)\right| : y \pm \frac{rh}{2} \in \left[x-\frac{r\delta}{2}, x+\frac{r\delta}{2}\right] \cap [a,b]\right\}$,[7]

is the rth local modulus of smoothness of *f*. From the definition one can easily see

$\tau_r(f,\delta,[a,b])_\infty := \omega_r(f,\delta,[a,b])_\infty$;[7]

The following relationship between the ω and τ moduli holds for any $f \in W_p^1[a,b]$ and $1 \leq p \leq \infty$

$$\tau_r(f,\delta,[a,b])_p \leq c(r)\delta\omega_{r-1}(f',\delta,[a,b])_p$$

where $W_P^K[a,b]$ the set of all functions from $L_p[a,b]$ such that $f^{(K-1)}$ are absolutely continuous and $f^{(K)} \in L_P[a,b]$. If the interval $I := [-1,1]$ is used in any of the above notations, it will be omitted for the sake of simplicity, for example

$\omega_r(f,\delta)_P := \omega_r(f.\delta,[-1,1])_P$,

and we will also denote

$\omega_r(f,\delta) := \omega_r(f.\delta,[-1,1])_\infty$.

A new way of measuring smoothness was introduced by Ditzian and Totik in [20]. The need for this new concepts arises from the failure of the classical moduli of smoothness to solve some basic problems, such as the inquality

$\omega_r(f,\delta)_p \le c\delta\,\omega_{r-1}(f',\delta)_p$ not satisfies, the relation between $\tau_r(f,n^{-1})_p$ and $\omega_r^{\varphi}(f,n^{-1})_p$ not known, the estimate $E_n(f) \le c n^{-1}\,\omega_r(f,\delta)_p$ is not true in general for any r $\in$ N $\cup$ {0}, And there are several possible definition of Sobolev space which are all equivalent if $1 \le p \le \infty$ and if 0<p<1 they need not by equivalent any more, and these are no good news. characterizing the behavior of the best polynomial approximation in $L_p(I)$.

The **Ditzian-Totik modulus of moothness** which is defined for such an *f* as follow

$$\omega_r^{\varphi}(f,\delta,J)_p := \sup_{0<h\le\delta} \left\| \Delta_{h\varphi(.)}^r(f,.) \right\|_{L_P(J)} . [7]$$

In the applications the φ usually used

$$\varphi(x) = \left(1-x^2\right)^{1/2} \; for \; J := [-1,1],$$
$$\varphi(x) = \left(x\left(1-x^2\right)\right)^{1/2} \; and \; \varphi(x) = \sqrt{x}(1-x) \; for \; J := [0,1],$$
$$\varphi(x) = \sqrt{x}, \varphi(x) = (x(1+x))^{1/2} \; and \; \varphi(x) = x \; for \; J := (0,\infty) \; ,$$

We have

$$\omega_r^{\varphi}(f,\delta)_P \le \omega_r(f,\delta)_P \le \tau_r(f,\delta)_P, \qquad 1 \le P \le \infty$$

and

$$\omega_r^{\varphi}(f,\delta)_P \le \omega_r(f,\delta)_P, \qquad 0 < P < 1$$

But Lemma 2.2.5 [7] shows that moduli ω_r^{φ} and ω_r for an *f* defined on $J := [a,b] \subset [-1,1]$, are equivalent if $|J| \approx |\Delta_n(a)|$

With $\Delta_n(a) = n^{-1}\sqrt{1-a^2} + n^{-2}$. Namely we prove the following result

Theorem 1.[7] Let $[a,b]\subset[-1,1]$, be such that $b-a\le c_1\,\Delta_n(a)$, where $c_1 \ge 1$ is an absolute constant. Then for any nonnegative integer *r* there exists a constant c(*r*) such that

$$\omega_r^{\varphi}\left(f,n^{-1},[a,b]\right)_P \ge c(r)\,\omega_r\left(f,\Delta_n(a),[a,b]\right)_P .$$

Now let us compare between the classical modulus of smoothness and its extension to ω_r^{φ} by the following basic properties: The following most basic fact about $\omega_r(f,\delta,J)_P$, satisfied for the two moduli

(a) $\lim\limits_{\delta\to 0}\omega_r(f,\delta,J)_P=0$ for all $f\in L_P(J)$, $0<P\le\infty$.

The second property of $\omega_r(f,\delta,J)_P$ is

(b) $\omega_r\left(f,\delta,J\right)_p$ *is* nondecreasing function of δ.

Examining the definition , we have

(b') $\omega_r^{\varphi}(f,\delta,J)_P$ is non decreasing function of δ .

Another, important property of $\omega_r(f,\delta,J)_P$ is the inequality

(c) $\omega_r\left(f,\lambda\delta,J\right)_P \le C\lambda^r\,\omega_r\left(f,\delta,J\right)_P$ *for* $\lambda\ \ge 1$

It turns out (c) caries over to $\omega_r^{\varphi}(f,\delta,J)_P$. While the combinatorial proof that is used in most texts to prove (c). via $\omega_r(f,n\delta,J)_P \le n^r\,\omega_r(f,\delta,J)_P$ is not valid for $\omega_r^{\varphi}(f,\delta,J)_P$.

The difficulty is that for *r*=2 the identity $\Delta_h^2(f,x)=\Delta_h^1\left(\Delta_h^1(f,x)\right)$ does not generalize as

$\Delta^{2}_{h\sqrt{x}}(f,x)= f\left(x-h\sqrt{x}\right)-2f(x)+ f\left(x+h\sqrt{x}\right)$ $for\ \varphi(x)=\sqrt{x}$, and

$$\Delta^{1}_{h\varphi} f\left(x+\frac{h}{2}\sqrt{2}\right)-\Delta^{1}_{h\varphi} f\left(x-\frac{h}{2}\sqrt{x}\right)$$

$$\Delta^{1}_{h}\varphi\left(\Delta^{1}_{h}\varphi(f,x)\right)=\Delta^{1}_{h}\varphi\left(f\left(x+\frac{h}{2}\sqrt{x}\right)\right)-f\left(x-\frac{h}{2}\sqrt{x}\right)$$

$$= f\left(x+\frac{h}{2}\sqrt{x}+\frac{h}{2}\sqrt{x+\frac{h}{2}\sqrt{x}}\right)-f\left(x+\frac{h}{2}\sqrt{x}-\frac{h}{2}\sqrt{x+\frac{h}{2}\sqrt{x}}\right)$$

$$-f\left(x-\frac{h}{2}\sqrt{x}+\frac{h}{2}\sqrt{x-\frac{h}{2}\sqrt{x}}\right)+f\left(x-\frac{h}{2}\sqrt{x}-\frac{h}{2}\sqrt{x-\frac{h}{2}\sqrt{x}}\right)$$

Where $\Delta^{1}_{h\varphi} f(z)= f\left(z+\frac{h}{2}\varphi(z)\right)-f\left(z-\frac{h}{2}\varphi(z)\right)$ are different and the second expression is quite complicated. However, we do have the analog of (c) using the following lemma from [20] and [21].

Lemma F. [7] For a function $f\in L_P(J), 0<P\le\infty$, we have

$$\omega^{\varphi}_{r}(f,\delta,J)_P\approx\tilde{K}_{r,\varphi}\left(f,\delta^{r}\right)_P,$$

where $\tilde{K}_{r,\varphi}$ is the Ditzian-Totik $\tilde{K}$ functional defined by

$$\tilde{K}_{r,\varphi}\left(f,\delta^{r}\right)_P:=\inf_{\substack{P_n\in II_n\\ n=\left[\frac{1}{\delta}\right]}}\left\{\left\|f-P_n\right\|_{L_P(J)}+\delta^{r}\left\|\varphi^{r}P_n^{(r)}\right\|_{L_P(J)}\right\}$$,[7]

where $[a]$ denote the largest integer not exceeding a . So we can generalize (c) by the following theorem

Theorem 2. [7]For $0 < P \le \infty$ we have for $\lambda \ge 1$

$\omega_r^{\varphi}(f, \lambda\delta, J)_P \le c\,\lambda^r\,\omega_r^{\varphi}(f, \delta, J)_P$.

Proof. In view of Lemma F we have

$$\omega_r^{\varphi}(f, \lambda\delta, J)_P \le c\,\tilde{K}_{r,\varphi}\left(f, (\lambda\delta)^r\right)_P$$

$$\le c\,\lambda^r\,\tilde{K}_{r,\varphi}\left(f, \delta^r\right)_P$$

$$\le c\lambda^r\,\omega_r^{\varphi}(f, \delta, J)_P$$

Another basic property of the classical modulus of smoothness is

(d) $\omega_{r+1}(f, \delta, J)_P \le 2\omega_r(f, \delta, J)_P$

which is generalized, in the inequality from [20]

(d') $\omega_{r+1}^{\varphi}(f, \delta, J)_P \le c\,\omega_r^{\varphi}(f, \delta, J)_P$.

Another inequality about the classical moduli of smoothness is

(e) $\omega_{k+r}(f, \delta, J)_P \le c\,\delta^r\,\omega_k\left(f^{(r)}, \delta, J\right)_P, P \ge 1$

which is valid if $f^{(r)} \in L_P(J)$ and $f^{(r-1)}$ is absolutely continuous in every closed interval $J= [a, b]$. The inequality (e) is not true if $0 < p < 1$. But the inequality [20]

$$\omega_{K+r}^{\varphi}(f, \delta, J)_P \le c\delta^r\,\omega_K^{\varphi}\left(f^{(r)}, \delta, J\right)_P,$$

Where c depends on P if $0<P < 1$, is true if $f \in W_P^r[a,b]$.

1.5 Approximation with Polynomials

In many applications of mathematics, we face functions which are far more complicated than the standard functions from classical analysis. Some of these functions can not be expressed in closed form via the standard functions, and some are only known implicitly or via their graph. Think, for example, of an electric circuit, where we are measuring the current at a certain point as a function of time: the outcome might be quite complicated, and best described via a graph .

An engineer measuring the current in an electric circuit will speak about having a signal ; for a mathematician , this just means that the output of the measurement is a function, to be called f , for which $f(x)$ equals the current at the time x. We will not give an .exact definition of what is meant by a signal: for our purpose, it is sufficient to think about a signal as a manifestation of a physical "event in terms of a function, or, in cases to be a sequence of numbers.

Signals are usually not given directly in terms of a function; for example , they often appear via a measurement . This makes it difficult or impossible to extract exact information on the signal from the function f describing it, especially if we need to perform some calculations on f . In such cases, it is important to be able to approximate f with a simpler function .

1.1.5. Approximation of a function on an interval

Our starting point must be a precise definition of what it means that a function is approximated by another function. As we will see soon , there are in. fact several different ways of defining this, and the correct definition depends on the situation at hand. Let us give a concrete example where approximation theory is needed:

Example 5.1.1 Assume that we want to calculate the integral

$$\int_0^1 e^{-x^2}\,dx \qquad (1.1)$$

It is well known that there does not exist a formula for the integral $\int e^{-x^2}dx$ in terms of the elementary functions; that is, we can not find (1.1) simply by integrating the function e^{-x^2} and then inserting the limits $x = 0$, $x = 1$.

Thus, we have to find another way to estimate (1.1). This is the point where approximation theory enters the picture: in this concrete case, our program suggests that we search for a function g for which

$\int_0^1 g(x)dx$ can be calculated, and g *(x)* is close to e^{-x^2} for $x \in [0,1]$, in the sense that we can control how much $\int_0^1 g(x)dx$ deviates from $\int_0^1 e^{-x^2}dx$, One way of doing so is to find a positive function g which we can integrate, and for which, for some $\varepsilon > 0$,

$$-\varepsilon \leq e^{-x^2} - g(x) \leq \varepsilon, \forall x \in [0,1],$$

or

$$-\varepsilon + g(x) \leq e^{-x^2} \leq \varepsilon + g(x), \forall x \in [0,1],$$

This, in turn, implies that

$$\int_0^1 (-\varepsilon + g(x))dx \leq \int_0^1 e^{-x^2} dx \leq \int_0^1 (\varepsilon + g(x))dx,$$

and therefore

$$-\varepsilon + \int_0^1 g(x)dx \leq \int_0^1 e^{-x^2} dx \leq \varepsilon + \int_0^1 g(x)dx,$$

Thus, $\int_0^1 g(x)dx$ gives us an approximate value for the desired integral.

Now we come back to the question about how to chose g in example 5.1.1

In Taylor's theorem we will approximate the function e^{-x^2} via polynomial p_n, n∈N, Of the from

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots\ldots\ldots\ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$= \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n . [13]$$

We return to the question of estimation of $\int_0^1 e^{-x^2} dx$

When $x \in [0,1]$, we have that $-x^2 \in [-1,0]$; thus, we get

$$\left| e^{-x^2} - \sum_{n=0}^{8} \frac{\left(-x^2\right)^n}{n!} \right| \leq 0.015 \quad \text{for all } x \in [0,1]$$

$$-0.015+\int_0^1 \sum_{n=0}^{8} \frac{\left(-x^2\right)^n}{n!} dx \le \int_0^1 e^{-x^2} dx \le 0.015+\int_0^1 \sum_{n=0}^{8} \frac{\left(-x^2\right)^n}{n!} dx$$

Now,

$$\int_0^1 \sum_{n=0}^{8} \frac{(-x^2)^n}{n!} dx = \left[\sum_{n=0}^{8} (-1)^n \frac{x^{2n+1}}{(2n+1)n!} \right]_0^1$$

$$= \sum_{n=0}^{8} (-1)^n \frac{1}{(2n+1)n!}$$

$$= 0.747$$

Then

$$\int_0^1 e^{-x^2} dx = 0.747 \pm 0.015$$

The obtained result is actually much closer to the exact value of the integral than the above tolerance indicates. For $N = 8$,

1.6 Applications of the theory of best Approximation

It is necessary to note applications are understood by Chebshev. They are not limited to the area of technical sciences , but are rather related to very different forms of human activity or serve the internal needs of mathematics it self . Let us new consider the applications that Chebyshev mention .

(i) Kinematics of mechanisms .

(ii) Solving algebraic equations .

(iii) Interpolation (Remainder estimate) .

(iv) A rule for finding approximately distance on the surface of the Earth .

(v) Approximate quadratures .

(vi) Constructing geographic maps .

Recently approximation theory have also many application both numerical and analytical the most prominent of these have been to

(i) Image processing [39] .

(ii) Statistical estimation [22] , [23] .

(iii) Numerical and analytic treatment of differential equations [14],[15] .

Chapter Two

Inverse and direct theorems for monotone approximation

Abstract

We prove that if f is increasing function on $[-1,1]$ then for each $n = 1,2,\ldots,$ there is an increasing algebraic polynomial p_n of degree $8n$ such that $\|f - p_n\|_p \le c(p)\omega_2^{\varphi}\left(f, \frac{1}{n}\right)_p$,

Where ω_2^{φ} is the second order Ditizian – Totik modulus of smoothness. Also a converse theorem for this direct theorem were obtained. These results complement the classical pointwise estimates of the same type for unconstrained polynomial approximation.

2.1 Introduction and Main Results

Several results show that in some sense monotone approximation by algebraic polynomials performs as well as unconstrained approximation. For example Lorentz and Zeller, [36] have shown that for each increasing function f in $C(I)$ (the space of all continuous functions on $I = [-1,1]$) there is an increasing polynomial p_n of degree n that satisfies

$$\|f - p_n\| \le c\ \omega\left(f, \frac{1}{n}\right), \quad n = 1,2,... \qquad (2.1.1)$$

where ω is the modulus of continuity of f.

A general result for (2.1.1) for any $k = 0,1,\ldots$ there are increasing p_n that satisfies

$$\|f - p_n\| \le c\, n^{-k}\ \omega\left(f^{(k)}, \frac{1}{n}\right), \quad n = 1,2,... \qquad (2.1.2)$$

this result of Lorentz [38], where as the general case was proved by DeVore [18], the cases $k = 0,1$ are much easier to prove than the general cases. Since they can be proved using linear method, in contrast, the proof in [18] uses rather involved non linear techniques. It is well known that for unconstrained approximation much improvement can be made in estimates of the form (2.1.1) where x is near the end points of I.

In this thesis, we are interested in pointwise estimates for monotone approximation, the only result of this type that we know of is by Beatson [2]. He proved that the estimate

$$|f(x) - p_n(x)| \le c\omega\ (f, \Delta_n(x)), x \in I, n = 1,2,...$$

$\Delta_n(x) = \sqrt{\frac{1-x^2}{n} + \frac{1}{n^2}}$, holds for suitable increasing polynomials p_n whenever f is increasing. Devore and Yn have shown that if f is increasing function on $I = [-1,1]$, then for each $n = 1,2,...$ there is an increasing polynomial p_n of degree n such that

$$|f(x)-p_n(x)| \le c\omega_2\left(f,\sqrt{\frac{1-x^2}{n}}\right),$$

where ω_2 is the second order moduli of smoothness. Among other things, we shall show that this can be improved to allow second order modulus of smoothness for the spaces $L_p, 0<p<1$.

Theorem I: If f is an increasing function in $L_p(I), 0<p<1$ then for each $n=1,2,\ldots$ there is an increasing polynomial in $L_p(I)$, of degree (8n) satisfying

$$\|f-p_n\|_p \le c(p)\omega_2^{\varphi}\left(f,\frac{1}{n}\right)_p. \tag{2.1.3}$$

Using this theorem we can obtain our second Inverse inequality:

Theorem II: Let f be an increasing function in $L_p(I), 0<p<1$, then

$$\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p \le c(p)E_n^1(f)_p^p + c(p)n^{-2p}\sum_{m>n}(m+1)^{p-1}E_n^1(f)_p^p.$$

2. 2 Auxiliary Lemmas

Before we prove our theorems we need the following notions and lemmas. Our proof is based on a two stage approximation. We first approximate f by an increasing piecewise linear function S_n. We then approximate S_n by an increasing algebraic polynomial . S_n is the piecewise linear function that interpolates f at $\xi_k, k=-n,\ldots,n$, if we let s_j be the slops

$$s_j = \frac{f(\xi_{j+1}) - f(\xi_j)}{\xi_{j+1} - \xi_j}, \quad j=-n,\ldots,n-1 \quad . \tag{2.2.1}$$

Then S_n can be represented by using the function $\Phi_j(x) = \max\{(x-\xi_j),0\}$ as

$$S_n(x) = f(-1) + s_{-n}(x+1) + \sum_{j=n+1}^{n-1} (s_j - s_{j-1}) \Phi_j(x), \text{ [20]} \tag{2.2.2}$$

It is clear that S_n is increasing if f is also.

We shall now contract a polynomial R_j, $j=-n,\ldots,n-1$, as in [20] that approximate the function $\Phi_j(x)$. The construction of R_j begins with trigonometric polynomial $T_j, j=1,\ldots,2n$ with $t_j = \frac{j\pi}{2n}, \quad j=0,1,\ldots,2n$. Let K_n denote the Jackson kernel

$$K_n(t) = a_n \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^8 , \tag{2.2.3}$$

where a_n is a constant depending on n chosen so that

$$\int_{-\pi}^{\pi} K_n(t)dt = 1 .$$

Here and throughout $c(p)$, c, denote absolute constants depending on p and $c(p), c$'s, values may vary with each occurrence on the same line.

Define

$$T_j(t) = \int_{t-t_j}^{t+t_j} K_n(u)du, j = 0,1,\ldots,2n ,[20]$$

and define

$$d_j(t) = \max(n\ dist\ (t, \{-t_j, t_j\}), 1), [20] \qquad (2.2.4)$$

Now let

$$r_j(x) = T_{m-j}(t), \qquad x = \cos t .$$

And for $x \in [-1,1]$ define

$$R_j(x) = \int_{-1}^{x} r_j(u)du, j = -n,\ldots,n ,[20] \qquad (2.2.5)$$

In particular $R_{-n}(x) = x + 1 = \Phi(x)$ and $R_n(x) = 0$, the points ξ_j are defined by the equations

$1 - \xi_j = R_j(1), j = -n,\ldots,n$. If $f \in L_p(I)$ we define

$$L_n(f) = f(-1) + s_{-n}R_{-n} + \sum_{-n+1}^{n-1}(s_j - s_{j-1})R_j . \qquad (2.2.6)$$

with s_j defined by (2.2.1) if f is increasing, the $s_j \geq 0$, $j = -n,\ldots,n-1$ and since we can also write

$$L_n(f) = f(-1) + \sum_{-n}^{n-1} s_j(R_j - R_{j+1}).$$

Now from the definition of the polynomials T_j we have $T_{n-j} - T_{n-(j+1)} \geq 0$, hence $r_j - r_{j+1} \geq 0$, and therefore $R_j - R_{j+1}$ is increasing , it follows that $L_n(f)$ is increasing.

We now estimate

$$E(x) = S_n(x) - L_n(f,x) = \sum_j (s_j - s_{j-1})(\Phi_j(x) - R_j(x)). \qquad (2.2.7)$$

Now for $j = -n,\ldots,n-1, x = \cos t$ with $0 \leq t \leq \pi$, we have

$$\left|\Phi_j(x) - R_j(x)\right| \leq cn^{-1}\left(\sin t_{n-j} + \left|t - t_{n-j}\right|\right)\left(d_{n-j}(t)\right)^{-5} \text{ ,[20]} \qquad (2.2.8)$$

Lemma 2.2.9. $\|L_n(f)\|_p \leq c(p)\|f\|_p$

Proof: We have

$$S_n(x) - L_n(f,x) = \sum_{j=-n+1}^{n-1} (s_j - s_{j-1})(\Phi_j(x) - R_j(x)).$$

Then

$$|L_n(f,x)| = \left|S_n(x) - \sum_{j=-n+1}^{n-1} (s_j - s_{j-1})(\Phi_j(x) - R_j(x))\right|$$

$$\leq |S_n(x)| + \left|\sum_{j=-n+1}^{n-1} (s_j - s_{j-1})(\Phi_j(x) - R_j(x))\right|.$$

Definition of S_n implies

$$|S_n(x)| \leq |f(-1)| + |s_{-n}(x+1)| + \sum_{j=-n+1}^{n-1} |s_j - s_{j-1}| |\Phi_j(x)|.$$

Thus

$$|S_n| \leq c|f(x)|.$$

And

$$|L_n(f,x)| \leq c|f(x)| + \sum_{j=-n+1}^{n-1} |s_j - s_{j-1}| |\Phi_j - R_j|.$$

Then (2.2.1) implies

$$\left|s_j\right| \le \frac{c\left|f(x)\right|}{\delta_j} \le \frac{c\left|f(x)\right|}{\delta_{j+1}} \quad \text{where } \delta_j = \xi_{j+1} - \xi_j .$$

Then

$$\left|L_n(f,x)\right| \le c\left|f(x)\right| + c\left|f(x)\right| \sum_{j=-n+1}^{n-1} \delta_j^{-1}\left|\Phi_j - R_j\right|,$$

and by $\delta_j = \xi_{j+1} - \xi_j = \frac{c}{n}$, we have $\delta_j^{-1} \le cn$.

So

$$\left|L_n(f,x)\right| \le c\left|f(x)\right| + c\left|f(x)\right| \sum_{j=-n+1}^{n-1} \left(1 + n\left|t - t_{n-j}\right|\right)\left(d_{n-j}(t)\right)^{-5}$$

$$\le c\left|f(x)\right| + c\left|f(x)\right| \sum_{j=-n+1}^{n-1} \frac{1}{n^4}$$

Then by the following [20]

Lemma 2.2.10 If g' is absolutely continuous and $\left|g''\right| \le M$ almost every where on I. Then for each $n = 1,2,..$ and each $x \in I$, we have

$$\left|g(x) - L_n(g,x)\right| \le cM\left(\frac{\sqrt{1-x^2}}{n}\right)^2.$$

We can prove

Lemma 2.2.11 If g' is absolutely continuous and $\left|g''\right| \le M$ almost every where on I, then for each $n = 1,2,..,$ and each $x \in I$ we have

$$\left\|g - L_n(g)\right\|_p \le \frac{c(p)}{n^2}.$$

2.3 proof of theorem I

Firstly let us introduce the so called Ditzian Totik functional definition as

$$\widetilde{K}_{2,\varphi}\left(f,\frac{1}{n^2}\right)_p=\inf_g\left(\|f-g\|_p+\frac{1}{n^2}\|\varphi^2 g''\|_p\right),$$

for $f\in L_p(I).0<p\le\infty$.

We have

$$\omega_2^{\varphi}\left(f,n^{-1}\right)_p\approx\widetilde{K}_{2,\varphi}\left(f,\frac{1}{n^2}\right)_p \text{ [7].}$$

Given $x\in I$, then from the result above there is a g satisfies

$$\|f-g\|_p\le c(p)\omega_2^{\varphi}\left(f,n^{-1}\right)_p$$

and

$$\frac{1}{n^2}\|\varphi^2 g''\|\le c(p)\omega_2^{\varphi}\left(f,n^{-1}\right)_p \qquad (2.3.1)$$

$$\|f-L_n(f)\|_p^p\le\|f-g\|_p^p+\|g-L_n(g)\|_p^p+\|L_n(g)-L_n(f)\|_p^p.$$

Then by the linearity, and the boundedness of $L_n(f)$, we obtain

$$\|f-L_n(f)\|_p^p\le\|f-g\|_p^p+\|g-L_n(g)\|_p^p+\|L_n\|_p^p\|f-g\|_p^p$$

$$\le\left(1+\|L_n\|_p^p\right)\|f-g\|_p^p+\|g-L_n(g)\|_p^p.$$

Lemma (2.2.11) implies $\|g-L_n(g)\|_p^p\le\frac{c(p)}{n^2}$.

Using (2.3.1), Lemma(2. 2.11) and the linearity of $L_n(f)$, we have

$$\|f-L_n(f)\|_p^p\le\left(1+\|L_n\|_p^p\right)c(p)\,\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p+\frac{c(p)}{n^2}\|\varphi^2 g''\|_p^p$$

$$\le\left(1+\|L_n\|_p^p\right)c(p)\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p+c(p)\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p.$$

By virtue of Lemma (2. 2.9) we have

$$\|f-L_n(f)\|_p^p \le c(p)\left(\left(1+\|f\|_p^p\right)\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p+\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p\right).$$

$$\le c(p)\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p$$

Since $L_n(f)$ is an increasing polynomial of degree ≤ 8n we have proved theorem I ♣

2.4 proof of Theorem II

For ℓ given by $\ell=\max\{i:2^i<n\}, 2^\ell=n$, we expand $p_n(x)$ by

$$p_n(x)-p_0(x)=\left(p_n(x)-p_{2^\ell}(x)\right)+\left(p_{2^\ell}(x)-p_{2^{\ell-1}}(x)\right)+\ldots+\left(p_1(x)-p_0(x)\right).$$

We recall that for $m<n$ $\quad \|p_n-p_m\|_p^p \le c(p)E_m^1(f)_p^p$

$$\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p \le c(p)\|f-p_n\|_p^p+c(p)n^{-p}\ \omega_1^{\varphi}\left(p_n,n^{-1}\right)_p^p$$

$$\le c(p)E_n^1(f)_p^p+c(p)n^{-2p}\|p_n'\|_p^p$$

$$\le c(p)E_n^1(f)_p^p+c(p)n^{-2p}\left\|\sum_{i=1}^{\ell} p_{2^i}'\right\|_p^p,$$

where p_{2^i} is an algebraic polynomial of best monotone approximation of degree not greater than or equal 2^ℓ it mean

$$\|f-p_{2^i}\|_p^p=E_{2^i}^1(f)_p^p \ . \tag{2.4.1}$$

Then

$$\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p \le c(p)E_n^1(f)_p^p+c(p)n^{-2p}\left\|\sum_{i=1}^{\ell} p_{2^i}'-p_{2^{i-1}}'\right\|_p^p$$

$$\le c(p)E_n^1(f)_p^p+c(p)n^{-2p}\sum_{i=1}^{\ell}\left\|p_{2^i}'-p_{2^{i-1}}'\right\|_p^p.$$

Then by Bernstein inequality we have

$$\omega_2^{\varphi}\left(f, n^{-1}\right)_p^p \le c(p) E_n^1(f)_p^p + c(p) n^{-2p} \sum_{i=1}^{\infty} \left(2^i n\right)^p \; E_{2^i n}^1(f)_p^p .$$

Applying the inequality

$$2^{ivp} \le c(p, v) \sum_{m=2^{i-1}+1} (m+1)^{vp-1}, v \in N \quad [24]$$

We get

$$\omega_2^{\varphi}\left(f, n^{-1}\right)_p^p \le c(p) E_n^1(f)_p^p + c(p) n^{-2p} \sum_{m>n}^{\infty} (m+1)^{p-1} E_m^1(f)_p^p \quad ♣$$

Chapter Three

A Saturation theorem and negative theorem for monotone approximation

Abstract.

In chapter two we introduce inverse and direct theorems for monotone approximation in $L_p, p < 1$, spaces. To complete the idea and characterize the monotone function in the generalized Lipschitz space through their approximation properties we introduce a theorem concerning the saturation of the increasing polynomial for this monotone approximation. One may hope that for a continuous increasing target function it would be possible to obtain estimate involving higher moduli of smoothness of this target function and this give better rate of monotone approximation. In our second result of this chapter we show that it is not possible to obtain direct theorem with higher moduli of smoothness when we assume the target function increasing with three continuous derivatives.

3.1 Introduction and main results

A major problem of the theory of approximation of function is concerned with the connection between the structural properties of a function and its degree of the approximation. The objective is to relate the smoothness of the function to the rate of decrease of the degree of approximation to zero .We are interested in our paper in examining these questions for algebraic polynomial approximation . these are then the most classical settings where the results are the most penetrating and satisfying.

In many applications, it is desirable that the mathematical model preserves certain geometric properties of the data such as monotionicity .This is the subject that the so called monotone approximation.

Recently there have been more attention given to approximation with constrains. The appearance of constraints can make if more difficult to obtain direct estimates. still the general lines of attack developed for the non constrained problem can be very useful. We want to indicate what modifications are necessary push the approach through. We do this for monotone approximation.

In monotone approximation. We are given a monotone function f and we wish to approximate f by a monotone polynomials. The question then is: does the monotonic cost us any thing or can we achieve the same degree of approximation as in the non constrained case ? The main purpose of this chapter is to provide answer to this question .

Interest in the subject began in the 1960's with work on monotone approximation by shishe [42] and by Lorentz and Zeller [36] , [37]. It gained momentum in the 197'0s and 1980's with work on monotone approximation of Devore, Beatson, Leviatan, Yu, Kopotun, Shevchuk and others[17],[18],[1],[20],[34],[35],[41],
[29],and 1990's and 2000's with works on coapproximation in [7].

Let us recall the Ditzan –Totik modulus of smoothness which defined for such an f as follows

$$\omega_r^{\varphi}(f,\delta,I)=\sup_{0<h\le\delta}\left\|\Delta^r_{h\varphi(.)}(f,.)\right\|_p,\quad \delta\ge 0\ [20]$$

In the applications the φ usually used

$\varphi(x)=\left(1-x^2\right)^{1/2}$ for $x\in I=[-1,1]$

Where the r^{th} symmetric difference of f is given by

$$\Delta_h^r(f,x,[-1,1])=\Delta_h^r(f,x)=\begin{cases}\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}f[x-\frac{rh}{2}+ih] & ,x\pm\frac{rh}{2}\in[a,b]\\ 0 & o.w\end{cases}$$

In our work the approximation will be carried out by polynomial $p_n \in \Pi_n$,the space of polynomials of degree not exceeding n. Note that 1-monotone functions are just nondecreasing functions, respectively . we denote the class of all 1-monotone functions by Δ^1.

Now for $f \in \Delta^1 \cap L_{p(I)}$, we denote by

$$E_n^1(f)_p = \inf_{p_n \in \Delta^1 \cap \Pi_n} \|f - p_n\|_p ,$$

where the infimum is taken over all polynomials $p_n \in \Pi_n$.

Firstly in our negative theorem ,we show that ω_2^φ can not replaced by ω_3^φ,we may hope that for continuous function f, it would be possible to obtain estimates involving higher moduli of smoothness of f and this way have better rate of monotone approximation.

In this theorem we show that it is not possible to obtain a direct estimate with higher moduli of smoothness when we assume that

$f \in \Delta^1 \cap C^3$(I). Namely, We prove :-

Theorem I: If $0 < \alpha < 2$, then an increasing function f is in Lip_α^* iff for each *n=1,2,…*, there is an increasing algebraic polynomial p_n of degree *8n* such that

$$E_n^1(f)_p \le \|f - p_n\|_p \le c(p)\frac{1}{n^\alpha} .$$

Theorem II: **For each n and0< p <1, there is a function f $\in \Delta^1 \cap C^3$(I), such that for every polynomial $p_n \in \Delta^1 \cap \Pi n$,**

either $\lim_{x \to -1} \sup \frac{\|f - p_n\|_p}{\varphi^3(x)} = \infty$ **(3.1.1)**

or $\lim_{x \to 1} \sup \frac{\|f - p_n\|_p}{\varphi^3(x)} = \infty$, $x \in$ **I** .

Now with inverse and direct theorems for approximation by algebraic polynomials .We have the following result, which characterizes the lip^*_α spaces, which are defined as the set of all f such that

ω_2^φ(f)=O(t^α), $0 < \alpha < 2$.

3.2 Proof of theorem I

let f increasing function in lip^*_α from the direct theorem we have

$$\|f - p_n\|_p \le \omega_2^\varphi\left(f, \frac{1}{n}\right)_p \le c(p)\left(\frac{1}{n^\alpha}\right)$$

Inverse theorem implies

$$\omega_2^\varphi\left(f, n^{-1}\right)_p \le E_n^1(f)_p + c(p)n^{-2} \sum_{i=1}^{n} E_{i-1}(f)_p$$

$$\le c(p)\frac{1}{n^\alpha} + c(p)n^{-2} \sum_{i=2}^{n} \frac{1}{(i-1)^\alpha}$$

$$\le c(p)\frac{1}{n^\alpha} + c(p)\frac{1}{n^\alpha} \sum_{i=1}^{\infty} \frac{1}{i^2} \quad \text{(conversing series)}$$

$$\le c(p)\frac{1}{n^\alpha} \quad ♣$$

3.3 Proof of theorem II

let $f(x)=\begin{cases}\frac{1}{3!}\left[\frac{1}{n^6}-\left[\frac{1}{n^2}-x-1\right]^3\right], & -1\le x\le -1+\frac{1}{n^2}\\ \frac{1}{3!n^6}, & -1+\frac{1}{n^2}\le x\le 1\end{cases}$, f is non decreasing function in

$L_p\ [-1,1]$. In fact :

$$f'(x)=\begin{cases}\frac{\left(\frac{1}{n^2}-x-1\right)^2}{2}, & -1\le x\le -1+\frac{1}{n^2}\\ 0, & -1+\frac{1}{n^2}\le x\le 1\end{cases},$$

when $x=-1+\frac{1}{n^4}$

$$f'\left(-1+\frac{1}{n^4}\right)=\left(\frac{1}{n^2}-\left(-1+\frac{1}{n^4}\right)-1\right)^2\Big/2=\left(\frac{1}{n^2}-\frac{1}{n^4}\right)^2\Big/2\ge 0.$$

suppose that there is a non decreasing polynomial p_n for which (3.1.1) fails .

Then for that polynomial and some constant B, we have $\|f-p_n\|_p\le B\|\varphi^3\|_p$ and $\|f-p_n\|_\infty=\sup_{x\in[-1,1]}|f(x)-p_{n(x)}|$

Then by using the inequality: $|p_n(x)|\le c(p)\,n^{\frac{p}{2}}\|p_n\|_p$ [7]

$$|f(x)-p_n(x)|\le c(p)n^{\frac{p}{2}}\ \|f-p_n\|_p\le c(p)Bn^{\frac{p}{2}}\|\varphi^3\|_p, |f(-1)-p_n(-1)|\le c(p)Bn^{\frac{p}{2}}\|\varphi^3(-1)\|_p$$

so $|f(-1)-p_n(-1)|=0$ and $f(-1)=p_n(-1)=0$ (3.3.1)

$f(-1)=\frac{1}{3!}\left(\frac{1}{n^6}-\frac{1}{n^6}\right)=0$ (3.3.2)

and

$f'(x)=\left(n^{-2}-x-1\right)^2\Big/2$, $-1\le x\le -1+\frac{1}{n^2}$

$$f'(-1)=\frac{n^{-4}}{2}$$

thus by (3.3.2) we have

$$f'(-1)=p'_n(-1)=\frac{n^{-4}}{2}.$$

Then applying the inequality $|p'_n|\le c(p)\ n^{2+\frac{2}{p}}\ \|p_n\|_p$,[17]

$$f(1)=\frac{1}{3!n^6}\le\frac{n^{-4}}{2}=p'_n(-1)\le c(p)n^{2+\frac{2}{p}}\|p_n\|_p\le n^4\|p_n\|_p$$

Now $n^4\|p_n\|_p=n^4\left(\int_{-1}^{1}|p_n|^p\right)^{\frac{1}{p}}$

$$\le n^4\left(2|p_n(1)|^p\right)^{\frac{1}{p}}$$

$$=n^4 2^{\frac{1}{p}}p_n(1)$$

$$\le n^5 p_n(1).$$

And $n^{-6}/3!\le n^{-4}/2\le n^5 p_n(1)$, and $f(1)=n^{-6}/3!\le n^5 p_n(1)$.

Thus $f(1)\ne p_n(1)$ and (3.1.1)satisfied ♣

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Chapter Four

Negative theorem of nearly monotone approximation in $L_p, o < p < 1$

Abstract

In our direct theorem we see that , when we approximate an increasing function $f \ in \ L\rho[-1,1]$, we wish sometimes that the approximating polynomials be increasing also. However, this constraint, restricts very much the degree of approximation, that the polynomials can achieve, namely ; only the rate of ω_2^φ.

In[31] Kopotun, Leviatan and Prymak proved that relaxing the monotonicity requirement in intervals of measure zero near the end points allows the polynomials to achieve the rate of ω_3^φ.

On the other hand we show in this chapter , that even when we relax the requirement of monotinicity of the polynomials on sets of measure approachg zero, ω_4^φ is not reachable .

4.1 Introduction and Main result

Let $f \in L\rho, 0<\rho<1$ be increasing function on $I=[-1,1]$ *in* chapter two we proved that there exists an increasing polynomials such that

$$\|f-\rho_n\|_{L\rho(I)} \le C(\rho)\omega_2^{\varphi}\left(f,\frac{1}{n}\right)_{\rho}, \qquad (4.1.1)$$

Where $C(\rho)$ is an absolute constant depending on ρ , and $\omega_2^{\varphi}(f;o)$ denote the Ditzain Totik modulus of smoothness of order two of f, In [17] Devore proved that if $f \in C[-1,1]$ be non decreasing on $I=[-1,1]$ there exist non decreasing polynomials such that

$$\|f-\rho_n\|_{c(I)} \le C\,\omega_2\left(f,\frac{1}{n}\right)_{\rho}. \qquad (4.1.2)$$

However , even this improvement comes to ahalt; it can not be extended to w_4^{φ} ; and thus not to ω_k for any K > 3 .

In order to state our theorem we need some notation .
Given ε > 0 and a increasing function $f \in L\rho, 0<\rho<1$,We denote

$$E_n^{(1)}(f,\varepsilon)_{\rho} = \inf_{p_n \in \Pi_n \cap \Delta^1(I)} \|f-\rho_n\|_{L\rho(I)}.$$

Where the infimum is taken over all polynomials ρ_n of degree not exceeding n satisfing

$$meas\left(\{x:\rho'_n(x)\ge 0\}\cap I\right) \ge 2-\varepsilon$$

Then our main result in this chapter is :

Theorem I: **for each sequence** $\bar{\varepsilon} = \{\varepsilon_n\}_{n=1}^{\infty}$ **, of non negative numbers tending to Zero , and each** $0 < \rho < 1$ **, there exists an increasing function** $f = f_{\overline{\in}} \in L_\rho(I)$ **such that**

$$\lim_{n\to\infty} \sup \frac{E_n^{(1)}(f;\varepsilon_n)_\rho}{\omega_4^{\varphi}\left(f,\frac{1}{n}\right)_\rho} = \infty \qquad (4.1.3)$$

4.2 A counter example (proof of theorem I)

In this article we have used C as an absolute constant which may differ on different occurrences , in this section we will have to keep track of the constants , there fore we denote them by C_1, C_2 , …… ., We begin by recalling some simple properties of the Chebyshev polynomials for the interval [-2 , 2] , for v > 1 let $t_v(x) = \cos v \cos^{-1}\frac{x}{2}, x \in [-2,2]$.

Denote to the Chebyshev polynomial and let $Zj = 2\cos\frac{i\pi}{v}$, $j = 0,\dots,v$, be its extrema , In fact :

$$t_v(x)=\cos\left(v\cos^{-1}\frac{x}{2}\right)$$

$$t'_v(x)=-\sin\left(v\cos^{-1}\frac{x}{2}\right)\left(v\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\left(\frac{1}{2}\right)\right)$$

$$=\frac{v\sin\left(v\cos^{-1}\frac{x}{2}\right)}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

$$t'_v(x)=0$$

$$v\sin\left(v\cos^{-1}\frac{x}{2}\right)=0$$

$$\sin\left(v\cos^{-1}\frac{x}{2}\right)=0$$

$$v\cos^{-1}\frac{x}{2}=\sin^{-1}0$$

$$v\cos^{-1}\frac{x}{2}=j\pi \quad ,j=0,\mp 1,\mp 2,\mp 3,......$$

$$\cos^{-1}\frac{x}{2}=\frac{j\pi}{v},\frac{x}{2}=\cos\frac{j\pi}{v}$$

$$Zj=2\cos\frac{j\pi}{v} \quad ,j=0,\mp 1,\mp 2,.......$$

Given $0<b<\frac{1}{2}$, we take two points on both sides of $Zj,\ j=1,......,v-1,$ namely ,we set

$$Z_{j,l}=2\cos\left(\frac{(j+b)\pi}{v}\right)\ and\ Z_{j,r}=2\cos\left(\frac{(j-b)\pi}{v}\right),$$

$$\left|t_v\left(Z_{j,l}\right)\right|=\left|t_v\left(Z_{j,r}\right)\right|=\cos\pi b.$$

and

$$Z_{j,r}-Z_{j,l}=2\cos\left(\frac{j\pi}{v}-\frac{b\pi}{v}\right)-2\cos\left(\frac{j\pi}{v}+\frac{b\pi}{v}\right)$$

$$=4\sin\frac{j\pi}{v}\sin\frac{b\pi}{v}$$

Then since $\sin u\le u$, $u\in[0,\pi]$, so that

$$Z_{j,r}-Z_{j,l}<4\pi\frac{b}{v} \qquad (4.2.1)$$

We truncate the Chebyshev polynomial by setting

$$t_v^*(x)=t_{v,b}^*(x)=\begin{cases}\cos\pi b & t_v(x)>\cos\pi b\\ -\cos\pi b & t_v(x)<-\cos\pi b\\ t_v(x) & O.W\end{cases}$$

$$\text{since} \qquad 1-\cos\pi b=2\sin^2\frac{b\pi}{2}<5b^2 \qquad (4.2.2)$$

For any $x\in I$,it follows by the monotonicity of the areas as we go away from the origin , and the alternation in sign of these areas ,that

$$\left| \int_{Z_{\left[\frac{v}{2}\right],l}}^{Z_{\left[\frac{v}{2}\right],r}} \left(t_v(u) - t_v^*(u)\right) du \right| \le \int_{Z_{\left[\frac{v}{2}\right],l}}^{Z_{\left[\frac{v}{2}\right],r}} \left|t_v(u) - t_v^*(u)\right| du$$

$$\le \left| Z_{\left[\frac{v}{2}\right],r} - Z_{\left[\frac{v}{2}\right],l} \right| \sup \left| t_v(u) - t_v^*(u) \right|$$

$$\le 4\pi \frac{b}{v} \left(t_v(u) - t_v^*(u)\right)$$

$$\le 4\pi \frac{b}{v} (1 - \cos \pi b)$$

$$< 4\pi \frac{b}{v} 5b^2$$

$$= c_1 \frac{b^3}{v}, \qquad (4.2.3)$$

Now $\sin ce[0,x] \subset \left[Z_{\left[\frac{v}{2}\right],l}, Z_{\left[\frac{v}{2}\right],r} \right] \subset I = [-1,1]$

Then

$$\left| \int_0^x \left(t_v(u) - t_v^*(u)\right) du \right| \le \left| \int_{Z_{\left[\frac{v}{2}\right],l}}^{Z_{\left[\frac{v}{2}\right],r}} \left(t_v(u) - t_v^*(u)\right) du \right| < 4\pi \frac{b}{v} 5b^2 = c_1 \frac{b^3}{v} \qquad (4.2.4)$$

Where we applied (4.2.1).

Now , given $n \ge 1$ *and* $0 < b < \frac{1}{2}$ *, let* $v = \left[b_n^{\frac{3}{4}} \right] + 2$*, where* $[a]$

denotes the largest integer not exceeding a .put

$t_{v,b}=t_v+\cos\pi b$, *and* $t_v^{\sim}=t_{v,b}^{*}+\cos\pi b$.

Finally

$$T_{v,b}(x)=\int_0^x t_{v,b}(u)du \quad and \quad f_{n,b}(x)=\int_0^x t_{v,b}^{\sim}(u)du \quad , x\in I$$

let $x_1<x_2$

$$f_{n,b}(x_1)=\int_0^{x_1} t_{v,b}(u)du < \int_0^{x_2} t_{v,b}(u)du=f_{n,b}(x_2).$$

Obviously $f_{n,b}$ is a increasing function on I and it readily by (4.2.4) that

$$\|f_{n,b}-T_{v,b}\|_p=\left(\int_{-1}^{1}\left|\int_0^x t_{v,b}^{\sim}(u)du-\int_0^x t_{v,b}(u)du\right|^p du\right)^{\frac{1}{p}}$$

Now

$$|f_{n,b}-T_{v,b}|=\left|\int_0^x \left(t_{v,b}^{\sim}(u)-t_{v,b}(u)\right)du\right|$$

$$=\left|\int_0^x \left(t_{v,b}^{*}(u)+\cos\pi b-t_{v,b}(u)-\cos\pi b\right)du\right|$$

$$=\left|\int_0^x \left(t_{v,b}^{*}(u)-t_{v,b}(u)\right)du\right|$$

$$\le C_1\,\frac{b^3}{v}=\frac{C_1 b^3}{\left[b_n^{\frac{3}{4}}\right]+2}\le C_1\,\frac{b^3}{b_n^{\frac{3}{4}}}=C_1\,\frac{b^{\frac{9}{4}}}{n},$$

And

$$\left\| f_{n,b} - T_{v,b} \right\|_p \le \left(\int_{-1}^{1} \left| C_1 \frac{b^{\frac{9}{4}}}{n} \right|^p du \right)^{\frac{1}{p}}$$

$$\le C_1 \frac{b^{\frac{9}{4}}}{n} \left(\int_{-1}^{1} du \right)^{\frac{1}{p}} \le C_1 2^{\frac{1}{p}} \frac{b^{\frac{9}{4}}}{n} \;,$$

Then

$$\left\| f_{n,b} - T_{v,b} \right\|_p \le C_1 \, 2^{\frac{1}{p}} \frac{b^3}{v} = C_1 \, 2^{\frac{1}{p}} \frac{b^3}{\left\lceil b^{\frac{3}{4}} n \right\rceil + 2} \le C_1 \, 2^{\frac{1}{p}} \frac{b^{\frac{9}{4}}}{n} \qquad (4.2.5)$$

Where we denote by $\|.\|_{Lp(J)}$ the quas – norm taken on the interval J , and when the norm is on J . We suppress the subscript ,

If we set $\tilde{Z}_{j,L} = 2\cos\left(\frac{\left(j + \frac{b}{2} \right)\pi}{v} \right)$, and

$$\tilde{Z}_{j,r} = 2\cos\left(\frac{\left(j - \frac{b}{2} \right)\pi}{v} \right),$$

Then we have for all j for which $Z_j \in I$

$$\tilde{Z}_{j,r} - Z_j = 2\cos\left(\frac{\left(j-\frac{b}{2}\right)\pi}{v}\right) - 2\cos\frac{j\pi}{v}$$

$$= 2\cos\left(\frac{j\pi}{v} - \frac{b\pi}{2v}\right) - 2\cos\frac{j\pi}{v}$$

$$= 2\cos\left(\frac{j\pi}{v} - \frac{b\pi}{4v} - \frac{b\pi}{4v}\right) - 2\cos\left(\frac{j\pi}{v} - \frac{b\pi}{4v} + \frac{b\pi}{4v}\right)$$

$$= 2\cos\left(\frac{\left(j-\frac{b}{4}\right)\pi}{v} - \frac{b\pi}{4v}\right) - 2\cos\left(\frac{\left(j-\frac{b}{4}\right)\pi}{v} + \frac{b\pi}{4v}\right)$$

$$= 4\sin\left(\frac{\left(j-\frac{b}{4}\right)\pi}{v}\right)\sin\frac{b\pi}{4v}$$

Then since $(\sin b\pi/4 > 3b/4)$ for b satisfying

$b\pi/4 < \pi/6 \;\; if \;\; 0 < b < \frac{1}{2} \;\; and \;\; \sin u \geq \frac{2}{\pi} u, \;\; u \in \left[0, \frac{\pi}{2}\right]$

$$\tilde{Z}_{j,r} - Z_j = 4\sin\frac{\left(j-\frac{b}{4}\right)\pi}{v}\sin\frac{b\pi}{4v}$$

$$\geq 4\frac{2}{\pi}\left(\frac{\left(j-\frac{b}{4}\right)\pi}{v}\right)\frac{3b}{4} = \frac{6\left(j-\frac{b}{4}\right)b}{v} > \frac{b}{v} \qquad (4.2.6)$$

And

$$Z_j-\tilde{Z}_{j,l}=2\cos\frac{j\pi}{v}-2\cos\left(\frac{\left(j+\frac{b}{2}\right)\pi}{v}\right)$$

$$=2\cos\left(\frac{j\pi}{v}+\frac{b\pi}{4v}-\frac{b\pi}{4v}\right)-2\cos\left(\frac{j\pi}{v}+\frac{b\pi}{4v}+\frac{b\pi}{4v}\right)$$

$$=2\cos\left(\frac{\left(j+\frac{b}{4}\right)\pi}{v}-\frac{b\pi}{4v}\right)-2\cos\left(\frac{\left(j+\frac{b}{4}\right)\pi}{v}+\frac{b\pi}{4v}\right)$$

$$=4\sin\frac{\left(j+\frac{b}{4}\right)\pi}{v}\sin\frac{b\pi}{4v}$$

$$\geq 4\frac{2}{\pi}\left(\frac{\left(j+\frac{b}{4}\right)\pi}{v}\,\frac{3b}{4}=\frac{6\left(j+\frac{b}{4}\right)b}{v}\right)>\frac{b}{v} \qquad (4.2.6')$$

Let j be odd and since $\sin b\pi/4>3b/4$

For b satisfying $b\pi/4<\pi/6$, we have

$$T'_{v,b}(x)=t_{v,b}(x)\leq-\cos b\pi/2+\cos b\pi\ ,x\in\left[\tilde{Z}_{j,l},\tilde{Z}_{j,r}\right].$$

For example

$$t_{v,b}(x)=t_v(x)+\cos\pi b\le t_v\left(\tilde{Z}_{j,l}\right)+\cos\pi b=\cos v\cos^{-1}\frac{2\cos\left(\frac{j\pi}{v}+\frac{b\pi}{2v}\right)}{2}+\cos\pi b$$

$$=\cos v\left(\frac{j\pi}{v}+\frac{b\pi}{2v}\right)+\cos\pi b$$

$$=\cos\left(j\pi+\frac{b\pi}{2}\right)+\cos\pi b$$

$$=-\cos\frac{b\pi}{2}+\cos\pi b$$

And

$-\cos\pi b/2+\cos\pi b=-\sin b\pi/4\sin 3b\pi/4$

In fact,

$$-\cos\pi b/2+\cos\pi b=-\cos\left(\frac{b\pi}{4}-\frac{3b\pi}{4}\right)+\cos\left(\frac{b\pi}{4}+\frac{3b\pi}{4}\right)$$

$$=-2\sin\frac{b\pi}{4}\sin\frac{3b\pi}{4}$$

$$<-2\frac{3b}{4}\frac{3b}{2}=-\frac{9b^2}{4}\quad ,x\in\left[\tilde{Z}_{j,l},\tilde{Z}_{j,r}\right]$$

Then

$$T'_{v,b}(x)=t_{v,b}(x)\le-\cos b\pi/2+\cos b\pi=-2\sin b\pi/4\sin 3b\pi/4$$

$$<-2\frac{3b}{4}\frac{3b}{2}=-\frac{9b^2}{4}\quad ,x\in\left[\tilde{Z}_{j,l},\tilde{Z}_{j,r}\right], \tag{4.2.7}$$

Then since $I\subset[-2,2]$, it follows by the Bernstein inequality

$$\left(\left\| p_n' \right\|_{L_{P[-2,2]}} \leq \frac{cn}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \left\| p_n \right\|_{Lp[-2,2]} \right) \text{that}$$

$$\left\| T_{v,b}^{(4)} \right\|_{lp[-2,2]} = \left\| t_{v,b}^{(3)} \right\| L_{lp[-2,2]} = \left\| t_v^{(3)} \right\|_{Lp[-2,2]} \leq \frac{C_2 \; V^3}{\left(1-\left(\frac{1}{2}\right)^2\right)^{\frac{3}{2}}} \left\| t_v \right\|_{Lp[-2,2]} = C_3 \, v^3,$$

In fact

$$\left\| T_{v,b}^{(4)} \right\|_{Lp[-2,2]} = \left\| \frac{d}{dx} \left(\int_0^x t_{v,b}^{(3)}(u)\, du \right) \right\|_{L_{p[-2,2]}} = \left\| t_{v,b}^{(3)} \right\|_{Lp[-2,2]}$$

$$= \left\| t_v^{(3)}(u) \right\|_{Lp[-2,2]} \leq C_2 \; V^3 \left\| \frac{1}{\left(\sqrt{1-\left(\frac{x}{2}\right)^2} \right)^3} t_v \right\|_{Lp[-2,2]}$$

$$\leq \frac{C_2 \, V^3}{\left(\sqrt{1-\left(\frac{1}{2}\right)^2} \right)^3} \left\| t_v \right\|_{Lp[-2,2]}$$

$$= \frac{C_2 \, V^3}{\left(\sqrt{1-\left(\frac{1}{2}\right)^2} \right)^3} \left(\int_{-2}^{2} \left| \cos v \cos^{-1} \frac{x}{2} \right|^p dx \right)^{\frac{1}{p}}$$

$$\leq C_3 \, V^3 \, 4^{\frac{1}{P}}$$

Hence by (4.2.5)

$$\begin{aligned}
\omega_4^{\varphi}\left(f_{n,b},\frac{1}{n}\right)_p &= \omega_4^{\varphi}\left(f_{n,b}-T_{v,b}+T_{v,b},\frac{1}{n}\right)_p \\
&\le \omega_4^{\varphi}\left(f_{n,b}-T_{v,b},\frac{1}{n}\right)_p+\omega_4^{\varphi}\left(T_{v,b},\frac{1}{n}\right)_p \\
&\le C(p)\left\|f_{n,b}-T_{v,b}\right\|_p+\frac{C(p)}{n^4}\left\|T_{v,b}^{(4)}\right\|_p \\
&\le C(p)\frac{b^{\frac{9}{4}}}{n}+\frac{C(p)v^3}{n^4} \\
&= C(p)\frac{b^{\frac{9}{4}}}{n}+C(p)\frac{\left(\left[b^{\frac{3}{4}}n\right]+2\right)^3}{n^4} \\
&\le C(p)\frac{b^{\frac{9}{4}}}{n}+C(p)\frac{\left(b^{\frac{3}{4}}n\right)^3}{n^4} \\
&= C(p)\frac{b^{\frac{9}{4}}}{n}+C(p)\frac{b^{\frac{9}{4}}n^3}{n^4} \\
&= C(p)\frac{b^{\frac{9}{4}}}{n}+C(p)\frac{b^{\frac{9}{4}}}{n} \\
&= C(p)\frac{b^{\frac{9}{4}}}{n}=C_1(p)\frac{b^{\frac{9}{4}}}{n},
\end{aligned} \tag{4.2.8}$$

Next we need a simple lemma

Lemma (4.2.9) there exists a constant C_4 such that

For any interval $J \subseteq I$, We have the following .

For any measurable sets $E \subseteq I$, if

$$p'_n(x) \geq 0,\ x \in J \setminus E, \tag{4.2.10}$$

Then

$$\left\| f_{n,b} - p_n \right\|_{L_{p(J)}} > \frac{b^2|J|}{n} - \frac{C_4}{n}\left(b^{\frac{9}{4}} + b|E| + \frac{b^{\frac{5}{4}}}{n} \right). \tag{4.2.11}$$

Proof : let J_o denote the middle third of J . We consider two cases . First we assume that J_o contains at most one of the $Z'_j s$. Then by the definition of v we get

$$|J| < C_5 \frac{1}{v} < C_5 \frac{b^{\frac{-3}{4}}}{n} \tag{4.2.12}$$

and

$$\frac{b^2|J|}{n} < C_5 \frac{b^2 b^{\frac{-3}{4}}}{n^2} = C_5 \frac{b^{\frac{5}{4}}}{n^2}$$

$$\frac{b^2|J|}{n} - C_5 \frac{b^{\frac{5}{4}}}{n^2} < 0$$

Then by (4.2.11) we have

$$\left\| f_{n,b} - p_n \right\|_{lp(J)} \geq 0 > \frac{b^2|J|}{n} - C_5 \frac{b^{\frac{5}{4}}}{n^2}, \tag{4.2.13}$$

On the other hand , if J_o contains at least two extremes of Z_j , then it contains at least $2C_6 v|J|$ extrema , for some constant C_6 ,these extremes satisfy (4.2.6) and $(4.2.6')$ and about half of them (and at least one) have odd indices , then together with (4.2.6) we conclude that

$$meas \left(J_o \cap \left\{ x : T'_{v,b}(x) < -\frac{9b^2}{4} \right\} \right) \geq \frac{1}{2} \frac{2b}{v} C_6 \, v|J| = C_6 \, b|J| \,, \qquad (4.2.14)$$

Now ,if $C_6 b|J| \leq |E|$, then

$$\left\| f_{n,b} - p_n \right\|_{L_{p(J)}} \geq 0 > \frac{b^2|J|}{n} - \frac{b|E|}{nC_6}, \qquad (4.2.15)$$

Other wise , $C_6 \, b|J| > |E|$.

Then by (4.2.14) there is a point $x_o \in J_o \setminus E$, for which

$$T'_{v,b}(x_o) < -\frac{9b^2}{4}$$

Hence ,(4.2.10) yields ,

$$\frac{9b^2}{4} \leq -T'_{v,b}(x_o) < p'_n(x_o) - T'_{v,b}(x_o) \leq \frac{2}{|J|^{\frac{1}{p}}} \frac{n}{\sqrt{1-\left(\frac{1}{3}\right)^2}} \left\| p_n - T_{v,b} \right\|_{Lp(J)}$$

$$\leq \frac{2}{|J|} \frac{n}{\sqrt{1-\left(\frac{1}{3}\right)^2}} \left\| p_n - T_{v,b} \right\|_{Lp(J)}$$

Where we used the Bernstein inequality

Therefore from (4.2.6)

$$\frac{9b^2}{4} \leq \frac{2n}{|J|\sqrt{1-\left(\frac{1}{3}\right)^2}} \left\| P_n - T_{v,b} \right\|_{Lp(J)} = \frac{2n}{|J|\frac{2\sqrt{2}}{3}} \left\| p_n - T_{v,b} \right\|_{Lp(J)}$$

$$\frac{|J|\sqrt{2}}{3n} \cdot \frac{9b^2}{4} \leq \left\| p_n - T_{v,b} \right\|_{Lp(J)}$$

$$\frac{3\sqrt{2}}{4} \frac{b^2|J|}{n} \leq \left\| p_n - T_{v,b} \right\|_{Lp(J)}$$

Now

$$\frac{b^2|J|}{n} \le \frac{3\sqrt{2}}{4}\frac{b^2|J|}{n} \le \|p_n - T_{v,b}\|_{Lp(J)}$$
$$\le \|P_n - f_{n,b}\|_{Lp(J)} + \|f_{n,b} - T_{V,b}\|_{LP(J)}$$
$$\le \|P_n - f_{n,b}\|_{Lp(J)} + C_1(p)\frac{b^{\frac{9}{4}}}{n}.$$

Then

$$\|f_{n,b} - p_n\|_{Lp(J)} \ge \frac{b^2|J|}{n} - C_1(p)\frac{b^{\frac{9}{4}}}{n}, \tag{4.2.16}$$

Taking $C_4 = \max\left\{C_5, \frac{1}{C_6}, C_1(\rho)\right\}$,(4.2.11) now follows by combining

(4.2.13),(4.2.15)*and*(4.2.16) ♣

We are now in a position to define $f_{\bar{\varepsilon}} = f$, for a given sequence

$\bar{\varepsilon} = \{\varepsilon_n\}$.

Let $b_n = \left(\max\left\{\varepsilon_n^2, \frac{1}{n}\right\}\right)^{\frac{2}{5}}$, and set $d_0 = 1$ and

$$d_j = \frac{bn_j^{\frac{9}{4}}}{n_j} d_{j-1} = \prod_{v=1}^{j} \frac{bn_v^{\frac{9}{4}}}{n_v}, \ j \ge 1,$$

Where the sequence $\{n_v\}$ is defined by induction as follows .First ,we choose n_1 so large that $b_{n_1}^{\frac{1}{8}} < \frac{1}{12}$ (as needed in (4.2.18) below) and $J_o = I$. suppose that $\{n_1, n_2, \ldots\ldots\ldots, n_{\sigma-1}\}$ and

$J_{\sigma-2} \subseteq J_{\sigma-3} \subseteq .... \subseteq J_0, \sigma \geq 2,$ have been defined

Then put

$F_{\sigma-1} = \sum_{j=1}^{\sigma-1} d_{j-1} f_{nj}, b_{nj},$

And let $J_{\sigma-1}$ be an interval such that $J_{\sigma-1} \subseteq J_{\sigma-2}$ and

$$F'_{\sigma-1}(x) = 0, x \in J_{\sigma-1} \qquad (4.2.17)$$

Let $N_{1.\sigma}$ be such that

$$|J_{\sigma-1}| \geq b_n^{\frac{1}{8}}, n \geq N_{1,\sigma} \qquad (4.2.18)$$

And let

$$N_{2,\sigma} = \left(\frac{\|F_{\sigma-1}^{(2)}\| L\rho(J_{\sigma-1})}{d_{\sigma-1}} \right)^{10} \qquad (4.2.19)$$

Finally ,we take

$$n_\sigma > \max\{n_{\sigma-1}, N_{1,\sigma}, N_{2,\sigma}\}$$

So big that the function $f'_{n_\sigma, b_{n_\sigma}}$ Oscillates a few times inside the interval ($J_{\sigma-1}$) and since it vanishes on some interval in each

Oscillation , that is , inside $J_{\sigma-1}$, there exists an interval $J_\sigma \subset J_{\sigma-1}$ as required in (4.2.17)

Now denote

$\Phi_\sigma = \sum_{j=\sigma}^{\infty} d_{j-1} f_{nj}, b_{nj},$

Where the convergence of the series is justified by the definition of the d_j's and

$$\left|f_{n,b_n}\right| = \left|\int_0^x t_{v,bn}(u)du\right|$$

$$\leq \int_0^x \left|t_{v,b_n}(u)du\right|$$

$$= \int_o^x \left|t_v(u) + \cos \pi b\right| du$$

$$\leq 2\int_0^x du \leq 2\int_0^1 du = 2$$

Now

$$\|\Phi_\sigma\|_{l_p(J_o)} \leq 8d_{\sigma-1} \qquad (4.2.20)$$

In fact

$$\|\Phi_\sigma\|^p{}_{L\rho(J_o)} = \left\| \sum_{j=\sigma}^{\infty} d_{j-1} f_{nj}, {}_{bnj} \right\|_{L\rho(J_o)}$$

$$\leq 2\sup \left| \sum_{j=\sigma}^{\infty} d_{j-1} f_{nj}, {}_{b_{nj}} \right|$$

$$\leq 2\sup \sum_{j=\sigma}^{\infty} |d_{j-1}| \| f_{nj}, {}_{bnj} |$$

$$\leq \left(d_{\sigma-1} \left(1 + \frac{bn_\sigma{}^{9/4}}{n_\sigma} + \frac{bn_\sigma{}^{9/4}}{n_\sigma} \frac{bn_{\sigma+1}^{9/4}}{n_{\sigma+1}} + \dots\dots\dots \right) \right) (2^2)$$

$$= d_{\sigma-1} \sum_{j=0}^{\infty} 2^{-j} \left(2^2 \right)$$

$$= 8 d_{\sigma-1}$$

So we define

$$f = f_{\bar{\varepsilon}} = \sum_{j=1}^{\infty} d_{j-1} f_{nj}, {}_{bnj}$$

And we prove

Lemma(4.2.21) For each $\sigma \geq 1$ we have

$$\omega_4^\varphi \left(f, \frac{1}{n_\sigma} \right)_\rho \leq C_2(\rho) d_\sigma \qquad (4.2.22)$$

Proof : First , by (4.2.20)

$$\omega_4^\varphi \left(\Phi_{\sigma+1}, \frac{1}{n_\sigma} \right)_\rho \leq C_3(\rho) \| \Phi_{\sigma+1} \|_{Lp(J_0)} \leq C_3(\rho) d_\sigma \qquad (4.2.23)$$

At the same time , (4.2.8) yields

$$\omega_4^\varphi \left(d_{\sigma-1} f_{n\sigma}, {}_{b_{n_\sigma}}, \frac{1}{n_\sigma} \right)_\rho \leq d_{\sigma-1} C_4 \frac{bn_\sigma^{9/4}}{n_\sigma} = C_4 d_\sigma \qquad (4.2.24)$$

Finally

$$\omega_4^{\varphi}\left(F_{\sigma-1},\frac{1}{n_\sigma}\right)_\rho \le C_4(\rho)\omega_2^{\varphi}\left(F_{\sigma-1},\frac{1}{n_\sigma}\right)_\rho$$

$$\le \frac{C_5(\rho)}{n_\sigma^2}\left\|F_{\sigma-1}^{(2)}\right\|_{L\rho(J_0)}$$

$$= C_5(\rho)\frac{\left\|F_{\sigma-1}^{(2)}\right\|_{L\rho(J_0)}}{d_{\sigma-1}} n_\sigma{}^{-1/10}\left(\frac{1}{n_\sigma{}^{2/5} \quad bn_\sigma}\right) d_\sigma{}^{9/4}$$

$$= C_5(\rho) N_{2,\sigma}^{\frac{1}{10}} n_\sigma^{\frac{-1}{10}}\left(\frac{1}{n_\sigma{}^{2/5} \quad bn_\sigma}\right)^{9/4} d_\sigma$$

$$< C_5(\rho) n_\sigma^{\frac{1}{10}} n_\sigma^{\frac{-1}{10}}\left(\frac{1}{n_\sigma{}^{2/5} \quad bn_\sigma}\right)^{9/4} d_\sigma$$

$$\le C_5(\rho)\left(\frac{1}{\frac{1}{b_{n_\sigma}} \quad bn_\sigma}\right)^{9/4} d_\sigma$$

$$= C_5(\rho) d_\sigma$$

(4.2.25)

By virtue of (4.2.19) and the definition of $d_{n\sigma}, d_\sigma$ and n_σ .

Now $\omega_4^{\varphi}\left(f,\frac{1}{n_\sigma}\right)_\rho \le \omega_4^{\varphi}\left(\Phi_{\sigma+1},\frac{1}{n_\sigma}\right)_\rho + \omega_4^{\varphi}\left(d_{\sigma-1} f_{n\sigma}, b_{n_\sigma} ,\frac{1}{n_\sigma}\right)_\rho + \omega_4^{\varphi}\left(F_{\sigma-1},\frac{1}{n_\sigma}\right)_\rho$

$$\le C_3(\rho) d_\sigma + C_4 d_\sigma + C_5(\rho) d_\sigma$$

Then Lemma (4.2.21) follows by combining (4.2.23) , (4.2.24) and (4.2.25) ♣

The last Lemma that we need is

Lemma (4.2.26) There is an absolute constant C_7 such that whenever $E \subset I$ is a measurable set satisfying

$$|E| \leq \varepsilon_{n\sigma}, \tag{4.2.27}$$

And ρ_{n_σ} is a polynomial satisfying

$$\rho'_{n_\sigma}(x) \geq 0, x \in I \setminus E, \tag{4.2.28}$$

Then

$$\left\| f - \rho_{n_\sigma} \right\|_{L\rho(I \setminus E)} \geq \left(b_{n_\sigma}^{-1/8} - C_7 \right) d_\sigma, \tag{4.2.29}$$

Proof : by (4.2.14) we have $F_{\sigma-1}$ is constant on $J_{\sigma-1}$, we may write

$$f(x) = d_{\sigma-1} \; f_{n_\sigma}, b_{n_\sigma}(x) + \Phi_{\sigma+1}(x) + M, x \in J_{\sigma-1} \tag{4.2.30}$$

Let $Q_{n_\sigma} = \frac{1}{d_{\sigma-1}} \left(\rho_{n_\sigma} - M \right)$

Then it follows from (4.2.28)

$$Q'_{n_\sigma}(x) \geq 0, x \in J_{\sigma-1} \setminus E$$

Thus by virtue of Lemma (4.2.9)

$$\left\| Q_{n\sigma} - f_{n_\sigma, bn_\sigma} \right\|_{L\rho(J_{\sigma-1})} \geq \frac{b_{n\sigma}^2 |J_{\sigma-1}|}{n_\sigma} - \frac{C_4}{n_\sigma} \left(\left(b^{9/4}_{n\sigma} + b_{n\sigma} |E| \right) + \frac{b_{n\sigma}^{5/4}}{n_\sigma} \right)$$

The definition of n_6 and (4.2.17)

$$b_{n_\sigma}^2 |J_{\sigma-1}| = b_{n_\sigma}^{17/8} \left(\frac{|J_{\sigma-1}|}{b_{n_\sigma}^{1/8}} \right) \geq b_{n_\sigma}^{17/8}$$

On the other hand , (4.2.24) and the definition

Of b_{n_σ} imply $\varepsilon_{n_\sigma} \le b_{n_\sigma}^{5/4}$

and

$$b_{n_\sigma} |E| \le b_{n_\sigma} \varepsilon_{n_\sigma} \le b_{n_\sigma}^{9/4}$$

Since $b_{n_\sigma} \ge \left(\frac{1}{n_\sigma}\right)^{2/5}$

Then

$$\frac{b_{n_\sigma}^{5/4}}{n_\sigma} \le b_{n\sigma}^{15/4} < b_{n\sigma}^{9/4}$$

Hence (4.2.28) implies

$$\left\| Q_{n\sigma} - f_{n\sigma}, b_{n\sigma} \right\|_{L\rho(J_{\sigma-1})} \ge \frac{1}{n_\sigma}\left(b_{n\sigma}^{17/8} - 3C_4 b_{n\sigma}^{9/4} \right) = \frac{b_{n\sigma}^{9/4}}{n_\sigma}\left(b_{n\sigma}^{-1/8} - 3C_4 \right)$$

In other words .

$$\left\| \rho_{n_\sigma} - M - d_{\sigma-1} f_{n_\sigma}, b_{n_\sigma} \right\|_{L\rho(J_{\sigma-1})} \ge d_{\sigma-1} \frac{b_{n_\sigma}^{9/4}}{n_\sigma}\left(b_{n_\sigma}^{-1/8} - 3C_4 \right)$$
$$= d_\sigma\left(b_{n_\sigma}^{-1/8} - 3C_4 \right),$$

In view of (4.2.30) , follows from (4.2.20) that

$$\left\| f - \rho_{n_\sigma} \right\|_{L\rho(I)} \ge \left\| f - \rho_{n_\sigma} \right\|_{L\rho(J_{\sigma-1})} \ge \left\| \rho_{n_\sigma} - M - d_{\sigma-1} f_{n_\sigma}, b_{n_\sigma} \right\|_{L\rho(J_{\sigma-1})} - \left\| \Phi_{\sigma+1} \right\|_{L\rho(J_{\sigma-1})}$$
$$\ge \left(\left(b_{n_\sigma}^{-1/8} - (3C_4 + 8) \right) d_\sigma \right),$$

And Lemma (4.2.26) is proved with $C_7 = 3C_4 + 8$

The proof of (4.1.3) now follows from Lemmas (4.2.21)and(4.2.26) since

$$\underset{n\to\infty}{Lim}\, su\rho \frac{E_n^{(1)}(f,\varepsilon_n)_\rho}{\omega_4^\varphi\left(f,\frac{1}{n}\right)_\rho} \geq \underset{n_\sigma\to\infty}{\lim}\, su\rho \frac{E_{n\sigma}^{(1)}\left(f,\varepsilon_{n_\sigma}\right)_\rho}{\omega_4^\varphi\left(f,\frac{1}{n_\sigma}\right)_\rho} \geq \underset{n_\sigma\to\infty}{\lim}\, su\rho \frac{\left\| f-\rho_{n_\sigma}\right\|_{L\rho}}{C_2(\rho)d_\sigma}$$

$$\geq \underset{n_\sigma\to\infty}{Lim} \frac{1}{C_2(\rho)}\left(b_{n_\sigma}^{-1/8}-C_7\right)=\infty$$

Future works

In a paper introduced to Almustansyria Magazin for Scienses, entited

Moduls of continuity and the Moduls of smoothness.

S. K. Jassim and E. H. Muhamed, introduced basic properties for the modulus of continuity and the moduls of smoothness in $L_{p,\mu}$ for $p \geq 1$.

We can improve these properties for the spase $L_{p,\mu}, 0 < p < 1$.

Where $L_{p,\mu}$[a,b]= $\left\{ f : \left(\int_a^b |f(x)|^p d_\mu x \right)^{\frac{1}{p}} < \infty \right\}$, where μ is measurable function.