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University of Babylon

College of Education for Pure Sciences

On Paracompact Maps

A Dissertation
Submitted to The Council of The College of Education for Pure Sciences in University of Babylon in Partial Fulfillment of The Requirements for the Degree of Doctor of Philosophy in Education / Mathematics

By

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Supervised by

Prof. Dr. Hiyam Hassan Kadhem

2023 A.D. 1445 A.H

بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

۝ اللَّهُ نُورُ السَّمَاوَاتِ وَالْأَرْضِ ۚ مَثَلُ نُورِهِ كَمِشْكَاةٍ فِيهَا
مِصْبَاحٌ ۖ الْمِصْبَاحُ فِي زُجَاجَةٍ ۖ الزُّجَاجَةُ كَأَنَّهَا كَوْكَبٌ دُرِّيٌّ
يُوقَدُ مِن شَجَرَةٍ مُّبَارَكَةٍ زَيْتُونَةٍ لَّا شَرْقِيَّةٍ وَلَا غَرْبِيَّةٍ يَكَادُ زَيْتُهَا
يُضِيءُ وَلَوْ لَمْ تَمْسَسْهُ نَارٌ ۚ نُّورٌ عَلَىٰ نُورٍ ۗ يَهْدِي اللَّهُ لِنُورِهِ مَن
يَشَاءُ ۚ وَيَضْرِبُ اللَّهُ الْأَمْثَالَ لِلنَّاسِ ۗ وَاللَّهُ بِكُلِّ شَيْءٍ عَلِيمٌ(35).

صدق الله العلي العظيم

سورة النور الاية 35

Supervisor's Certification

I certify that this dissertation "***On Paracompact Maps***" by student "***Saad Mahdi Jaber Muter***" was prepared under my supervision at the University of Babylon, Faculty of Education for Pure Sciences, in a partial fulfillment of the requirements for the degree of Doctor of Philosophy in Education / Mathematics.

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In view of the available recommendations, I forward this dissertation for debate by the examining committee.

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Contents

Dedication

FOR ALLAH

With whole obedience and thanking

To the great prophet of God " Mohammed"

To my sirs and lords "imam Ali till imam Mahdi" (peace be upon them) ,

To my big home ((IRAQ))

To the spirits of my father and mother , my God have mercy on them,

To my family dearly ,

To my friends considerably,

and

To my colleagues faithfully .

Acknowledgments

Praise be to God, Lord of the worlds, and prayers and peace be upon the most honorable of the prophets and messengers, our Master Muhammad, his family, his companions, and those who followed them with kindness until the Day of Judgment, and after...

I thank God Almighty for his bounty for allowing me to accomplish this work thanks to Him. Praise be to Him first and foremost.

I would like to extend my sincere appreciation to my mentor and my supervisor Prof. Dr. Hiyam Hassan Kadhem, for her guidance and encouragement during the preparation of this work. Her expertise, insightful comments and useful advice have decisively contributed to my work. The words, really, are not enough to express my gratitude for all that he has done for me.

I would like to express my sincere thanks to everyone who helped me in one way or another, particularly, the head of the Mathematic dept feat in Ccollege Education for Pure Sciences / University of Babylon and its staff members.

Finally, I must express my appreciation and affection to my family and my friends.

Saad Mahdi

Abstract

This study is about a paracompact map by relying on the paracompact space and trying to address some real-life problems. The use of the relationships between a paracompact map with a Bourbaki proper map and a closed map helps to satisfy certain basic characteristics of this map. The paracompact map has been generalized to other new types of maps and has been linked to these new maps. Accordingly, these new types of maps have been separated into two classes. The first class is called a strong form, which implies a paracompact map under certain conditions. The other class is called a weaker form of a paracompact map, where the paracompact map implies them. Finally, the composition operations of paracompact maps are studied. A motivational utilization of G-space leads to defining a paracompact G-space and proving that paracompactness implies a G-space under certain conditions. Different types of paracompact actions are presented, such as nearly paracompact, compact, fully normal, and metacompact. In addition, a fuzzy paracompact map is introduced, and some of its properties are proposed. The final results depict the relationships between the fuzzy paracompact map and important maps such as closed maps and proper maps. Finally, the concept of the fuzzy paracompact map is generalized to fuzzy topographic topological mapping (FTTM).

List of Figures

List of Symbols

Symbol	Description
$\overline{\mathrm{A}}$	The closure of A
$\cong$	Homeomorphic to
$\mathrm{X}\times\mathrm{Y}$	The Product space of X and Y
$\mathbb{G}.x$	The orbit of x under $\mathbb{G}$
$\mathbb{G}_x$	The stability subgroup of $\mathbb{G}$ at x
FTTM	Fuzzy Topographic Topological mapping
F^c	The complement of the set F
$\mathrm{I}_{\mathbb{E}}$	The identity mapping of $\mathbb{E}$
R	Equivalence relation
$\mathbb{G}R(R)$	The graph of R
$\mathbb{P}\mathbb{r}$	The projection map
$I^{\mathbb{W}}$	The collection of all fuzzy sets in $\mathbb{W}$
$\mathcal{L}\Delta\mathcal{J}$	The diagonal map

Publications

Six papers from this dissertation were accepted and published:

i- "Certain Strong and Weak Types of Paracompact Map", accepted for publication in Volume (65) Issue (2) and will be published on (February) 2024 in the Iraqi Journal of Science (IJS). EISSN: 2312-1637 ISSN: 0067-2904.

ii- "Paracompactly closed map", Wasit Journal for Pure Science Vol. (1) No. (3), P(81-89), 2023.

iii- "Characterizations of Paracompact Map", accepted in the first international conference for physics and mathematics (1st ICPM 2022) held for the period 19-20/ October /2022. Will be published at the IEEE Publisher conference.

iv- "Paracompact Action", accepted in the first international conference for physics and mathematics (1st ICPM 2022) held for the period 19-20/October/2022. Will be published in the IEEE Publisher conference

v- "Certain Types of Paracompact Actions", accepted in the 4th international scientific conference of Alkafeel University ISCKU 2022, in Al-Najaf Al-Ashraf, Iraq, December 20^{th}-21^{st}. Will be published at the AIP Publisher conference.

vi- "On the Fuzzy Paracompact Map in Fuzzy Topographic Topological Mapping", accepted in the 5th International Conference on Information Technology, Applied Mathematics and Statistics, ICITAMS 2023, Diwaniyah, Iraq, 20-22 March. Will be published at the IEEE Publisher conference.


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Acceptance Letter

To: Saad Mahdi Jaber[1], Hiyam Hassan Kadhem[2]
[1]Department of Mathematics, Faculty of Education for Pure Science, University of Babylon
[2]Department of Computer Science, Faculty of Education, University of Kufa

Title:

Certain Strong and Weak Types of Paracompact Maps

Dear Author:

We are pleased to inform you that your manuscript is accepted for publication in **Volume (65) Issue (2)** and will be published on **(February) 2024** in **Iraqi Journal of Science (IJS)**.

*Thank you for submitting your work to the **Iraqi Journal of Science (IJS)**.*

Your Sincerely

Prof. Ali H. Ad'hiah (Ph.D.)
Editor in Chief
Iraqi Journal of Science

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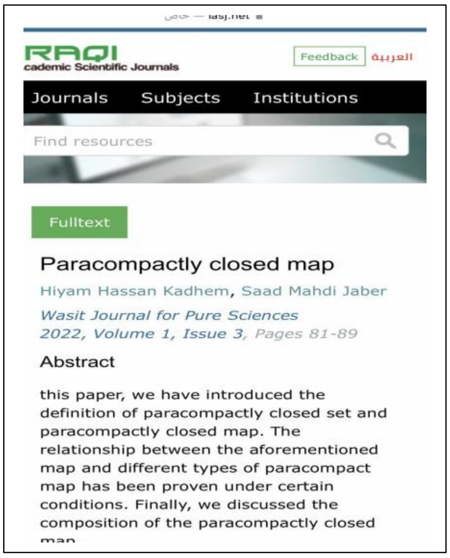

Paracompactly closed map

Hiyam Hassan Kadhem, Saad Mahdi Jaber

Wasit Journal for Pure Sciences
2022, Volume 1, Issue 3, Pages 81-89

Abstract

this paper, we have introduced the definition of paracompactly closed set and paracompactly closed map. The relationship between the aforementioned map and different types of paracompact map has been proven under certain conditions. Finally, we discussed the composition of the paracompactly closed


UNIVERSITY OF ANBAR
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CERTIFICATE OF PARTICIPATING

THIS CERTIFICATE IS AWARDED TO

Saad Mahdi Jaber Abu-Ragheef and Hiyam Hassan Kadhem

FOR ATTENDING AND PARTICIPATING IN THE "FIRST INTERNATIONAL CONFERENCE FOR PHYSICS AND MATHEMATICS (1st ICPM 2022) HELD FOR THE PERIOD 19-20/ OCTOBER /2022" BY THE PAPER ENTITLED:

Characterizations of Paracompact Map

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Prof. Dr. Abdul Rahman S. Juma
Dean of the College of Education for Pure Sciences

Asst. Prof. Dr. Sabiha F. Jawad
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Acceptance Letter

Author(s) :Saad Mahdi Jaber [1],Hiyam Hassan Kadhem [2]
Affiliation : Faculty of Education for Pure Science University of Babylon [1],Faculty of Education, University of Kufa[2]
Email of Corresponding Author : saad.jaber.pure247@student.uobabylon.edu.iq
Dear (author) We are pleased to inform you that your manuscript
Certain Types of Paracompact Actions
after being peer- reviewed , has been accepted for participating in the

4th INTERNATIONAL SCIENTIFIC CONFERENCE OF ALKAFEEL UNIVERSITY , ISCKU 2022

in Al-Najaf Al-Ashraf, Iraq ,December 20th -21st , 2022 . Given that your manuscript will be published after passing the conference Scopus-indexed journals requirements

With Best Regards

Prof. Dr. N. Al-Dahan
Rector of Alkafeel University

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The 5th International Conference on Information Technology, Applied Mathematics and Statistics
ICITAMS 2023
Dewaniyah, Iraq

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FORMAL ACCEPTANCE AND INVITATION LETTER
From
International Conference on Information Technology, Applied Mathematics and Statistics 2023

Paper ID: 1570886943
Date: 10th March 2023

Dear respected author(s):

Saad Mahdi Abu-Ragheef
University of Babylon, Iraq

Hiyam Hassan Kadhem
University of Kufa, Iraq

We are delighted and pleased to inform you that your research paper entitled (***On the Fuzzy Paracompact Map in Fuzzy Topographic Topological Mapping***) has been reviewed and accepted by ICITAMS 2023 committee for oral presentation at ICITAMS 2023 conference 20-22 March 2023 and going to be submitted to the IEEE Xplore digital library.

We would thank you for your submission and participation.

Asst Prof. Dr. Dhiah Al-Shammary
University of Al-Qadisiyah
Conference Chair

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The 5th International Conference on Information Technology, Applied Mathematics and Statistics

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Introduction

Paracompactness is a very useful property in topology because it implies many important results, such as the existence of partitions of unity and the extension of continuous functions from closed subsets to the whole space. On another side, maps between topological spaces provide a powerful tool for studying the properties of topological spaces and their behavior under continuous transformations. By analyzing the maps between different spaces, we can gain insight into their properties and understand the structure of topological spaces more deeply.

The motivation of compactness into topology was beginning to generalize the properties of the bounded and closed subset $\mathbb{R}^n$. In 1944, Dieudonné [1] introduced a wider class of compact spaces, namely paracompact spaces. In 1951, Dowker [2] had given generalization of paracompact spaces by introducing the class of countably paracompact spaces. Through the use of α-open, pre-open, semi-open, regular open, and β-open sets, new generalizations of paracompact spaces were given. Nearly paracompact space was defined by Singal and Arya [3] using the regular open. In 2006, Al-Zoubi [4] introduced the notion of S-paracompact space using a semi-open set. Demir and Ozbakir [5] defined 2013 β -paracompact spaces, by replacing the open cover with a β-open cover in the definition of paracompact space.

The concept of the fuzzy set was introduced in 1965 by Zadeh [6]. Fuzzy topology is a kind of topology developed on fuzzy sets, for the first time Chang [7] defined fuzzy topological spaces ($\mathbb{W}, T$) in the framework of fuzzy sets in 1968. Lowen [8] in 1976, has given another definition for a fuzzy 'topology by including all constant functions instead of just 0 and 1 (where 0 and 1 are fuzzy sets which take every $x \in \mathbb{W}$ to 0 and every

$x \in \mathbb{W}$ to 1 respectively) of Chang's definition. In this work, we are following Lowen's definition.

In 1981, fuzzy paracompact was initiated by S. R. Malghan and S. S. Benchalli [9]. Also In 1985, Hu Chang-ming introduces the concept of paracompact in fuzzy topological space [10]. In 1992, Al-Munsef introduced the concept of fuzzy paracompact and *fuzzy paracompact. In 2005, Qutaiba Ead Hassanin in introduced characterizations of fuzzy paracompactness [11]. In 2018, Munir Abdul Khalik introduced and studied the fuzzy b-paracompact space and its relation of it to fuzzy paracompact spaces [12].

On the other hand, there are maps known as parallel advanced spaces. In 1947, Halfar [13] introduced the concept of a compact map in a metric space. Garg and Goel [14] 1993 initiated a countably compact map and then Buhagiar [15] 1997 introduced the notion of a paracompact map. After that, in 2003 a countably paracompact map was defined by AL-Zoubi and Hdeib [16].

The theory of dynamical systems was developed to describe trajectory behaviors, within this theory has been dealt with ($\mathbb{R}-space$ or $\mathbb{Z}-space$), as a special case of general theory is said to be topological transformation group ($\mathbb{G}$-space theory). The properties of the general topological transformation group investigated in several studies have been presented with these types of functions to clarify. Most of the basic concepts of $\mathbb{G}$-space were discussed by Palais [17], which is significantly taken with $\mathbb{G}$ as a compact (Lie) group and $\mathbb{W}$ as a completely regular topological space. On the other hand, Bourbaki [18], introduced the concept of proper $\mathbb{G}$-space for any topological group and general topological space. After Palais and Bourbaki's work, many studies had arisen to investigate conditions on the topic. Saddam had studied a new type of group action namely Strongly Bourbaki proper actions [19].

Feebly Bourbaki's proper action in traduced introduced in 2005 by AL-Badairy [20]. Then, δ-proper Action was presented by Saad [21].

Fuzzy Topographic Topological Mapping (FTTM) is a model which was built to solve neuromagnetic inverse problems to determine the cerebral current sources. In 1999, a fuzzy modeling research group (FMRG) at UTM led by Prof. Dr. Tahir Ahmad developed software for determining the location of epileptic foci in epilepsy disorder patients. The first version (FTTM1) of FTTM was introduced by Yun [22] in 2001. FTTM1 consists of three algorithms that link among the four components of the model. The four components of FTTM1 are the magnetic contour plane (MC), base magnetic plane (BM), fuzzy magnetic field (FM), and topographic magnetic field (TM) [24]. FTTM2 is the extended version of FTTM1which is specifically designed to solve the inverse problem of the multi-current source. Similar to FTTM1, the model is comprised of four components. They are Magnetic Image Plane (MI), Base Magnetic Image Plane (BMI), Fuzzy Magnetic Image Field (FMI), and Topographic Magnetic Image Field (TMI). The four components are linked by three different equations [25]. In 2005, Tahir et al. [26] proved that the components of FTTM1 are topological spaces and they verified the homeomorphisms between the components of FTTM1. Notice that Yun [24] confirmed that the components of FTTM2 are also topological spaces Additionally, FTTM1, as well as FTTM2, are specially designed to have equivalent topological structures between their components, whereas the homeomorphisms between the components of FTTM2 were proved by Yun [24].

Added to this, the sequence of n versions of FTTM was defined by Tahir et al.[27] in 2010 as FTTM1, FTTM2, FTTM3,…, FTTMn for $n \in \mathbb{Z}^+$.

A sequence of n versions of FTTM is defined as FTTM1, FTTM2, FTTM3,…, FTTMn such that MC $_{v}\cong$ MC $_{v+1}$, BM $_{v}\cong$ BM $_{v+1}$, FM $_{v}\cong$ FM $_{v+1}$, and TM $_{v}\cong$ TM $_{v+1}$, for v=1,2,..,(n-1), and n $\in\mathbb{Z}^{+}$ [22].

This thesis aims to study more broadly the paracompact map and it contains six chapters, starting with the mathematical background. In the first chapter that includes the basic definitions and theories for the rest chapters as paracompactness as a space in various types with G-space and the fuzzy topological space.

The next four chapters contain the major contributions of the study. In Chapter 2, a paracompact map is initiated. The properties of this map are investigated. In addition, the relationships between a paracompact map with Bourbaki proper map and a closed map are discussed. The concept of Pa-closed space is introduced, and its properties are investigated. The paracompactly closed set and paracompactly closed map are given and the composition of the paracompactly closed maps is discussed.

Chapter 3 presents a paracompact map under certain conditions. The chapter describes new weaker and strong forms of maps by using the concept of paracompactness and investigating their composition in various cases.

Chapter 4 focuses on introducing a paracompact action map. Some theorems relating to revised properties of paracompact action map and paracompact G-space are derived. Indeed, various types of paracompact action are presented, which are nearly paracompact, compact, fully normal, and metacompact.

Chapter five is mainly devoted to introducing a fuzzy paracompact map and applying it to a mathematical model of fuzzy topographic topological mapping ($FTTM$). Some more theorems are derived and proven that the fuzzy map between the fuzzy component FM in $FTTM1$ and the corresponding component FMI in $FTTM2$ is a fuzzy paracompact map. These theorems are based on the properties of the components and maps in fuzzy topographic topological mapping. The map between the components of $FTTM$ proves that paracompact, countably paracompact, compact, countably compact, metacompact, countably metacompact, nearly paracompact, fully T4, and fully normal.

Chapter six focuses on the conclusions and future work of the research. The major contributions and findings of the research and future recommendations are given. Lastly, the cited references are given at the end of the thesis.

Chapter One

MATHIMATICAL BACKGROUND

1.1 Basic Definition on Paracompact Spaces

Definition 1.1.2 [28]

A subset A of a space $\mathbb{W}$ is called α-open if $A \subseteq int(cl(int(A)))$.

Definition 1.1.3 [29]

A subset A of a space $\mathbb{W}$ is called preopen if $A \subseteq int(cl(A))$.

Definition 1.1.4 [30]

A subset A of a space $\mathbb{W}$ is called semi open if $A \subseteq cl(int\,(A))$.

Definition 1.1.5 [31]

A subset A of a space $\mathbb{W}$ is called β-open if $A \subseteq cl(int(cl(A)))$.

Definition 1.1.6 [32]

A subset A of a space $\mathbb{W}$ is called regular open if $A = int(cl(A))$.

Definition 1.1.7 [33]

A G_δ set is a countable intersection of open set.

Definition 1.1.8 [34].

A map $\mathcal{L}$ from $\mathbb{W}$ into $\mathbb{M}$ is called paracompact, if for every $y \in \mathbb{M}$ and every family $\mathbb{U} = \{U_\alpha\}_{\alpha \in I}$ of open subsets of $\mathbb{W}$ satisfying $\mathcal{L}^{-1}(\{y\}) \subseteq \bigcup \mathbb{U}$, there exists a neighborhood V_y of y such that $\mathcal{L}^{-1}(\{y\})$ is covered by $\mathbb{U}$ and $\{U_\alpha \cap \mathcal{L}^{-1}(V_y)\}_{\alpha \in I}$ has an open refinement $\mathbb{V}$ such that $\mathbb{V}$ is locally finite at $\mathcal{L}^{-1}(\{y\})$.

Definition 1.1.9 [33]

A space $\mathbb{W}$ is said to be compact if every open cover of it has a finite subcover.

Definition 1.1.10 [33]

Let $\mathbb{S}$ be a cover for a space $\mathbb{W}$ with a topology τ and $\mathcal{V}$ is an open cover for $\mathbb{W}$, then $\mathcal{V}$ is called open refinement of $\mathbb{S}$ if $\forall\, V \in \mathcal{V}: \exists U \in \mathbb{S}: V \subseteq U$ and $\mathcal{V} \subseteq \tau$.

Definition 1.1.11 [1]

A space $\mathbb{W}$ is said to be paracompact if any open cover of it has a locally-finite open refinement.

Definition 1.1.12 [33]

A collection of subsets $\mathbb{S} = \{S_i\}_{i\in I}$ of a space $\mathbb{W}$ is said to be;

1) Discrete if each point $x \in \mathbb{W}$ has a neighborhood intersecting at most one of the sets in $\mathbb{S}$.
2) Locally-finite if each point $x \in \mathbb{W}$ has a neighborhood intersecting at most finitely many of the sets in $\mathbb{S}$.
3) Point-finite if each point $x \in \mathbb{W}$ is contained in at most finitely many of the sets in $\mathbb{S}$.

Definition 1.1.13 [35]

Let $\mathbb{W}$ be a set and let $\mathbb{S}$ be a covering of $\mathbb{W}$, that is $\mathbb{W} = \bigcup \{ U : U \in \mathbb{S} \}$ and $\mathcal{V}$ is a refinement of $\mathbb{S}$. Given a subset S of $\mathbb{W}$, the star of S with respect to $\mathbb{S}$ is the union of all the sets $U \in \mathbb{S}$ that intersect S, that is st(S, $\mathbb{S}$)= $\bigcup \{ U \in \mathbb{S}: S \cap U \neq \emptyset \}$. Given a point $x\epsilon\mathbb{W}$, we write st(x, $\mathbb{S}$) instead of st($\{x\}$, $\mathbb{S}$).The covering $\mathcal{V}$ is called a star refinement of $\mathbb{S}$ if for every $U \in \mathbb{S}$ the star st(U, $\mathbb{S}$) is contained in some $V\in\mathcal{V}$.

Definition 1.1.14 [36]

Let $\mathbb{W}$ be a space and let x be any arbitrary point in a space $\mathbb{W}$. A class B_x of open sets containing x is called a local base at x if, for each open set U containing x, there is $U_x \in B_x$ with the property $x \in U_x \subseteq U$.

Definition 1.1.15 [37]

A space $\mathbb{W}$ is said to be:

(i) Separable if it has a countable dense subset.
(ii) Second countable (or completely separable or perfectly separable) if it has a countable basis.
(iii) First-countable if it has a countable local basis.

Definition 1.1.16 [33]

A space $\mathbb{W}$ is said to be a countably compact if every countable open cover of it has a finite subcover.

Definition 1.1.17 [33]

A space $\mathbb{W}$ is said to be a Lindelöf space if every open cover of $\mathbb{W}$ has a countable subcover.

Definition 1.1.18 [13]

Let $\mathbb{W}$ and $\mathbb{M}$ be two spaces. A map $\mathcal{L}: \mathbb{W}\to\mathbb{M}$ is known as compact providing the pre-image of any compact set in $\mathbb{M}$ is compact in $\mathbb{W}$.

Definition 1.1.19 [35]

A space $\mathbb{W}$ is called extremally disconnected (briefly e.d.) if the closure of every open set in $\mathbb{W}$ is open.

Definition 1.1.20 [35]

A space $\mathbb{W}$ is called completely extremally disconnected if it is extremally disconnected and $\bar{A} \cap \bar{B} = \emptyset$ for each $A, B \subseteq \mathbb{W}$.

Definition 1.1.21 [2]

A space $\mathbb{W}$ is called countably paracompact (some time called binormal) if every countable open covering has a locally finite open refinement.

Definition 1.1.22 [4]

A space $\mathbb{W}$ is said to be S-paracompact if any open cover of it has a locally-finite semi-open refinement.

Definition 1.1.23 [38]

A space $\mathbb{W}$ is said to be nearly paracompact if every regularly open covering admits a locally finite open refinement.

Definition 1.1.24 [39]

A space $\mathbb{W}$ is metacompact if every open cover of $\mathbb{W}$ has a point finite open refinement.

Definition 1.1.25 [40]

A space $\mathbb{W}$ is countably metacompact if every countable open cover of $\mathbb{W}$ has a point finite open refinement.

Definition 1.1.26 [5]

A space $\mathbb{W}$ is said to be β-paracompact if every open cover of $\mathbb{W}$ has a β-locally finite β-open refinement.

Definition 1.1.27 [33]

A space $\mathbb{W}$ is said to be fully T_4 if every open cover of $\mathbb{W}$ has a star refinement.

Definition 1.1.28 [41]

A space $\mathbb{W}$ is said to be fully normal if every open cover of $\mathbb{W}$ has star refinement and all points of $\mathbb{W}$ are closed .

Definition 1.1.29 [42]

A topological space $\mathbb{W}$ is called submaximal if each dense subset of $\mathbb{W}$ is open in $\mathbb{W}$.

Definition 1.1.30 [43]

A topological space $\mathbb{W}$ is said to be almost countably compact space, if for every countable open cover $\{\mathrm{U}_n : n \in \mathbb{N}\}$ of $\mathbb{W}$, there exist a finite subset $\{\mathrm{U}_{n_i}\}_{i=1}^{m}$, where $m \in \mathbb{N}$ provided that $\mathbb{W} = \bigcup_{i=1}^{m} cl(\mathrm{U}_{n_i})$.

Definition 1.1.31 [44]

A space $\mathbb{W}$ is known a semi-Hausdorff if any two distinct points x and y in $\mathbb{W}$ have their own semi-open sets $SO(x)$ and $SO(y)$ such that $SO(x) \cap SO(y) = \emptyset$.

Definition 1.1.32 [43]

A space $\mathbb{W}$ is said to be as almost compact providing every open covering of $\mathbb{W}$ has a finite subcollection, the closures of whose members cover $\mathbb{W}$.

Definition 1.1.33 [3]

A space $\mathbb{W}$ is said to be almost regular, if for each regular closed set $\mathbb{G}$ and each point $x \in \mathbb{W} - \mathbb{G}$, there exist disjoint open sets U and V such that $x \in \mathrm{U}$ and $\mathbb{G} \subseteq \mathrm{V}$.

Definition 1.1.34 [44]

A space $\mathbb{W}$ is said to be pre-regular space if every closed set $\mathbb{G}$ and every point $x \notin \mathbb{G}$, there exists pre-open sets U and V such that $x \in U, \mathbb{G} \subset V$ and $U \cap V = \emptyset$.

Definition 1.1.35 [45]

Let $\mathbb{W}$ and $\mathbb{M}$ be two spaces. Then:

1- A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is known as a countably compact providing the pre-image of any closed and countably compact set in $\mathbb{M}$ is countably compact in $\mathbb{W}$.

2- A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is known as an Almost countably compact providing the pre-image of any closed and Almost countably compact set in $\mathbb{M}$ is Almost countably compact in $\mathbb{W}$.

3- A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is said to be weakly countably compact providing the pre-image of any closed and weakly countably compact set in $\mathbb{M}$ is weakly countably compact in $\mathbb{W}$.

4- A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is called sequentially compact providing the pre-image of any closed and sequentially compact set in $\mathbb{M}$ is sequentially compact in $\mathbb{W}$.

5- A map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is called pseudo-compact providing the pre-image of any closed and pseudo-compact set in $\mathbb{M}$ is pseudo-compact in $\mathbb{W}$.

6- A map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is said to be a feebly compact map providing the pre-image of any closed and feebly compact set in $\mathbb{M}$ is feebly compact set in $\mathbb{W}$.

7- A map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is known as a semi-compact map providing the pre-image of any open and semi-compact set in $\mathbb{M}$ is semi-compact set in $\mathbb{W}$.

8- A map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is called is a nearly compact map providing the pre-image of any nearly compact and $\alpha - open\ set$ in $\mathbb{M}$ is nearly compact set in $\mathbb{W}$.

9- A map $\mathcal{L}:\mathbb{W} \rightarrow \mathbb{M}$ is known as a po-compact map providing the pre-image of any po-compact and $\alpha - open\ set$ in $\mathbb{M}$ is po-compact set in $\mathbb{W}$.

10- A map $\mathcal{L}:\mathbb{W} \rightarrow \mathbb{M}$ is called is an almost compact map providing the pre-image of any open and almost compact set in $\mathbb{M}$ is an almost compact set in $\mathbb{W}$.

<u>Definition 1.1.36 [18]</u>

Let $\mathbb{W}$ and $\mathbb{M}$ be spaces. Then, a map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is said to be proper if:

- $\mathcal{L}$ is a continuous map.
- $\mathcal{L} \times I_{\mathbb{E}}: \mathbb{W} \times \mathbb{E} \rightarrow \mathbb{M} \times \mathbb{E}$ is closed, for any space $\mathbb{E}$.

<u>Definition 1.1.37 [46]</u>

A space $\mathbb{W}$ is said to be P-space if any G_δ set in $\mathbb{W}$ is open.

<u>Definition 1.1.38 [33]</u>

A space $\mathbb{W}$ is called a sequentially compact if every sequence has a convergent subsequence.

<u>Definition 1.1.39 [33]</u>

A space $\mathbb{W}$ is known as pseudo-compact in any case that any real-valued map on $\mathbb{W}$ is bounded.

<u>Definition 1.1.40 [35]</u>

A net in a set $\mathbb{W}$ is a mapping $\chi : D \rightarrow \mathbb{W}$, where D is a directed set . The point $\chi(d)$ is usually denoted by χd .

1.2 Basic Theorems on Paracompact Spaces

Theorem 1.2.1 [47]

Semi-regularization is inherited by open subspaces and dense subspaces.

Theorem 1.2.2 [48]

Let $\mathbb{W}$ be a Hausdorff space. $\mathbb{W}$ is hereditarily extremally disconnected if and only if it is a completely extremally disconnected.

Theorem 1.2.3 [49]

Every subspace K of a submaximal space $\mathbb{W}$ is submaximal.

Theorem 1.2.4 [36]

A Hausdorffness is a hereditary property.

Theorem 1.2.5 [47]

Every subspace of regular space is regular.

Theorem 1.2.6 [47]

Every closed subspace of normal space is normal.

Theorem 1.2.7 [47]

A closed subset of a Lindelöf space, is a Lindelöf subspace.

Theorem 1.2.8 [33]

Every paracompact space is a countably paracompact.

Theorem 1.2.9 [33]

Every countably compact space is a countably paracompact space.

Theorem 1.2.10 [2]

Every closed subspace of compact (res. paracompact, countably compact, countably paracompact) space is compact (res. paracompact, countably compact, countably paracompact).

Theorem 1.2.11 [50]

A Hausdorff space implies semi-Hausdorff space.

Theorem 1.2.12 [51]

Any Hausdorff space is a pre-regular space.

Theorem 1.2.13 [52]

A topological space $\mathbb{W}$ is semi-regular and almost regular if and only if it is regular.

Theorem 1.2.14 [4]

Every extremally disconnected S-paracompact T_2-space is Paracompact.

Theorem 1.2.15 [2]

Every compact space is paracompact.

Theorem 1.2.16 [53]

Every compact subset of Hausdorff space is closed.

Theorem 1.2.17 [33]

i) Every compact space is a Lindelof.

ii) Every compact is countably compact space.

Theorem 1.2.18 [54]

A continuous image of a compact space is compact.

Theorem 1.2.19 [33]

If a space $\mathbb{W}$ is a countably compact and paracompact, then it is compact.

Theorem 1.2.20 [33]

Every countably paracompact and Lindelöf space is paracompact.

Theorem 1.2.21 [33]

Every countably paracompact (or binormal) space is normal.

Theorem 1.2.22 [33]

Every countably paracompact space is countably metacompact.

Theorem 1.2.23 [55]

Every normal metacompact space is countably paracompact space.

Theorem 1.2.24 [33]

Every metacompact space is a countably metacompact.

Theorem 1.2.25 [33]

Every paracompact space is a metacompact.

Theorem 1.2.26 [33]

Every metacompact countably compact space is compact space.

Theorem 1.2.27 [38]

Every Lindelöf countably metacompact space is metacompact space.

Theorem 1.2.28 [4]

Every paracompact space is S-paracompact.

Theorem 1.2.29 [38]

Every paracompact space is nearly paracompact.

Theorem 1.2.30 [38]

Nearly paracompactness and paracompactness are equivalent on semi-regularization space.

Theorem 1.2.31 [56]

Every regular space is semiregular.

Theorem 1.2.32 [5]

Every S-paracompact space is β-paracompact.

Theorem 1.2.33 [5]

Every paracompact space is a β-paracompact.

Theorem 1.2.34 [5]

Let $\mathbb{W}$ be an e.d. submaximal space. If $\mathbb{W}$ is a β-paracompact space then $\mathbb{W}$ is paracompact space.

Theorem 1.2.35 [33]

Every fully T_4 and Hausdorff space is paracompact.

Theorem 1.2.36 [33]

A space $\mathbb{W}$ is fully normal if it is fully T_4 and T_1-space.

Theorem 1.2.37 [33]

Every fully normal space is fully T_4.

Theorem 1.2.38 [57]

Every fully normal and T_1- space is paracompact.

Theorem 1.2.39 [58]

Hausdorff and paracompact space is fully normal.

Theorem 1.2.40 [59]

Any countably compact space is almost countably compact space

Theorem 1.2.41 [60]

Every regularly closed subset of an almost countably compact space is almost countably compact.

Theorem 1.2.42 [33]

Every paracompact space is countably paracompact.

Theorem 1.2.43 [35]

Let $\mathbb{W}$ and $\mathbb{M}$ be two spaces. A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is continuous if $\mathcal{L}^{-1}(\mathrm{K})$ is closed in $\mathbb{W}$ for each closed set K in $\mathbb{M}$.

Theorem 1.2.44 [45]

1- Every compact map onto a Lindelöf space is a countably compact map.

2- Every compact map onto a Hausdorff and Lindelöf space is a weakly countably compact map

3- Every compact map onto a regular and Lindelöf space is an almost countably compact map.

4- Every compact map onto a Lindelöf and regular space is a pseudo-compact map.

5- Every compact map of a first-countable space onto a Lindelöf and regular space is a sequentially compact map.

6- Every compact map onto a Lindelöf and regular space is a feebly compact map.

7- Every compact map onto a semi-Hausdorff space is a semi-compact map.

8- Every compact map onto a Lindelöf and pre-regular space is a nearly compact map.

9- Every compact map onto a Lindelöf and pre-regular space is a po-compact map.

10- Every compact map onto a Hausdorff and almost regular space is an almost compact map.

Theorem 1.2.45 [56]

Let $\mathbb{W}$ be a space and A be a subset of $\mathbb{W}$, $x \in \mathbb{W}$. Then $x \in \overline{\mathrm{A}}$ if and only if there exists a net in A which converges to x.

Theorem 1.2.46 [18]

Let $\mathbb{W}$ and $\mathbb{M}$ be spaces and $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is a continuous map. Then, the following statements are equivalent:

(i) $\mathcal{L}$ is a proper map.

(ii) $\mathcal{L}$ is a closed map and $\mathcal{L}^{-1}(\{y\})$ is compact for each $y \in \mathbb{M}$.

(iii) If (χ_d) are a net in $\mathbb{W}$ and $y \in \mathbb{M}$ is a cluster point of the net $\mathcal{L}(\chi_d)$, then there is a cluster point $x \in \mathbb{W}$ of (χ_d), such that $\mathcal{L}(x) = y$.

Proposition 1.2.47 [18]

Let $\mathcal{L}_1: \mathbb{W}_1 \rightarrow \mathbb{M}_1$ and $\mathcal{L}_2: \mathbb{W}_2 \rightarrow \mathbb{M}_2$ maps, then $\mathcal{L}_1 \times \mathcal{L}_2: \mathbb{W}_1 \times \mathbb{W}_2 \rightarrow \mathbb{M}_1 \times \mathbb{M}_2$ is proper if and only if $\mathcal{L}_1$ and $\mathcal{L}_2$ are proper.

Proposition 1.2.48 [36]

A continuous map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is closed if and only if for every $F \subseteq \mathbb{M}$ and every open set $U \subseteq \mathbb{W}$ which contains $\mathcal{L}^{-1}(F)$, there exists an open set $A \subseteq \mathbb{M}$ containing F such that $\mathcal{L}^{-1}(A) \subseteq U$.

Proposition 1.2.49 [39]

A continuous map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is closed if and only if for every point $y \subseteq \mathbb{M}$ and every open set $U \subseteq \mathbb{W}$ which contains $\mathcal{L}^{-1}(\{y\})$, there exists neighborhood set V of y in a space $\mathbb{M}$ such that $\mathcal{L}^{-1}(V) \subseteq U$.

Proposition 1.2.50 [18]

Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ be a proper map. If H is a subset of $\mathbb{M}$, then $\mathcal{L}_H: \mathcal{L}^{-1}(H) \rightarrow H$ is proper.

Proposition 1.2.51 [18]

If $\mathcal{J} o \mathcal{L}$ is proper and $\mathcal{J}$ is injective open, then a continuous map $\mathcal{L}$ is proper.

Proposition 1.2.52 [5]

The topological product $\mathbb{W} \times \mathbb{M}$ of compact space $\mathbb{W}$ and a paracompact space $\mathbb{M}$ is paracompact.

Proposition 1.2.53 [18]

Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M} = \{y\}$ be a map and $y \notin \mathbb{W}$, then $\mathcal{L}$ is proper if and only if $\mathbb{W}$ is compact set.

Proposition 1.2.54[18]

If $\mathbb{W}$ is any compact space and $\mathbb{M}$ is topological space, then the projection $\mathbb{P}r_2: \mathbb{W} \times \mathbb{M} \rightarrow \mathbb{M}$ is proper.

Proposition 1.2.55 [18]

Every continuous map $\mathcal{L}$ of a compact space $\mathbb{W}$ into a Hausdorff space $\mathbb{M}$ is proper.

Proposition 1.2.56 [61]

Every second countable space is Lindelöf space.

Proposition 1.2.57 [62]

A Lindelöf subspace of Hausdorff P-space is closed.

Proposition 1.2.58 [1]

The intersection of a compact set with a closed set is compact.

Theorem 1.2.59 [57]

Every metrizable space is paracompact.

Theorem 1.2.60 [57]

A metrizable is topological property.

Theorem 1.2.61 [63]

The component M_1 of FTTM1 is compact, countably compact, Lindelöf, T_0, T_1, T_2, regular, T_3, normal, T_4 and metrizable.

Theorem 1.2.62 [63]

Each component in the sequence of n FTTM are compact, countably compact, Lindelöf, T_0, T_1, T_2, regular, T_3, normal, T_4 and metrizable., for $n \in \mathbb{Z}^+$.

Theorem 1.2.63 [18]

Let $\mathbb{W}$, $\mathbb{M}$ be two topological spaces. In order that a mapping $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ should be continuous it is necessary and sufficient that the mapping $\mathcal{J}: x \to (x, \mathcal{L}(x))$ is a homeomorphism of $\mathbb{W}$ onto the graph $\mathbb{G}$ of $\mathcal{L}$.

Theorem 1.2.64 [18]

If $\mathcal{L}$ is a continuous mapping a topological space $\mathbb{W}$ into a Hausdorff space $\mathbb{M}$, then the graph of $\mathcal{L}$ is closed in $\mathbb{W} \times \mathbb{M}$.

The following figure illustrates the relationships between certain types of paracompact spaces and compact spaces that are needed in our work.

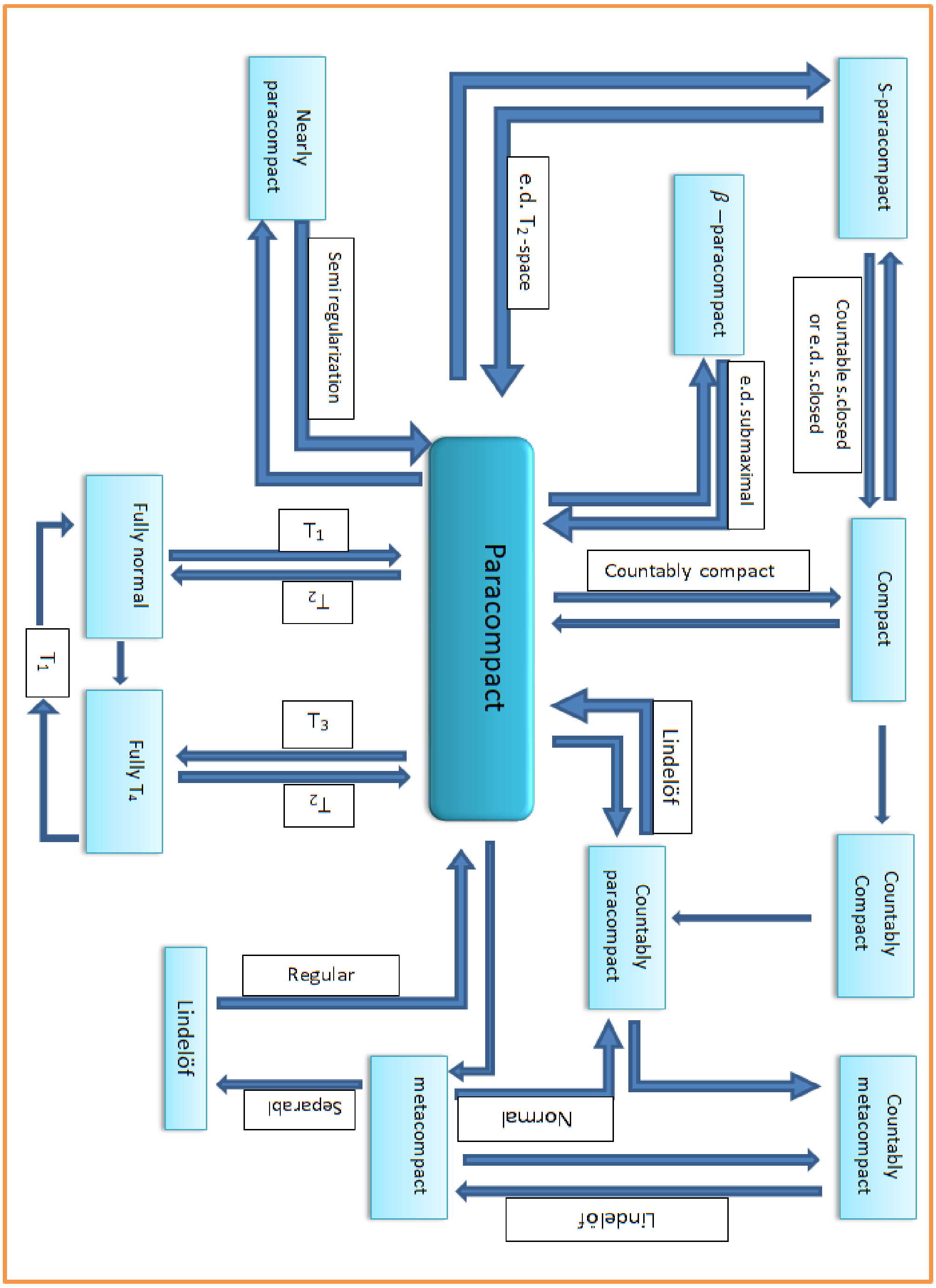

Figure 1.1 Relationships between Certain Types of Paracompact Spaces

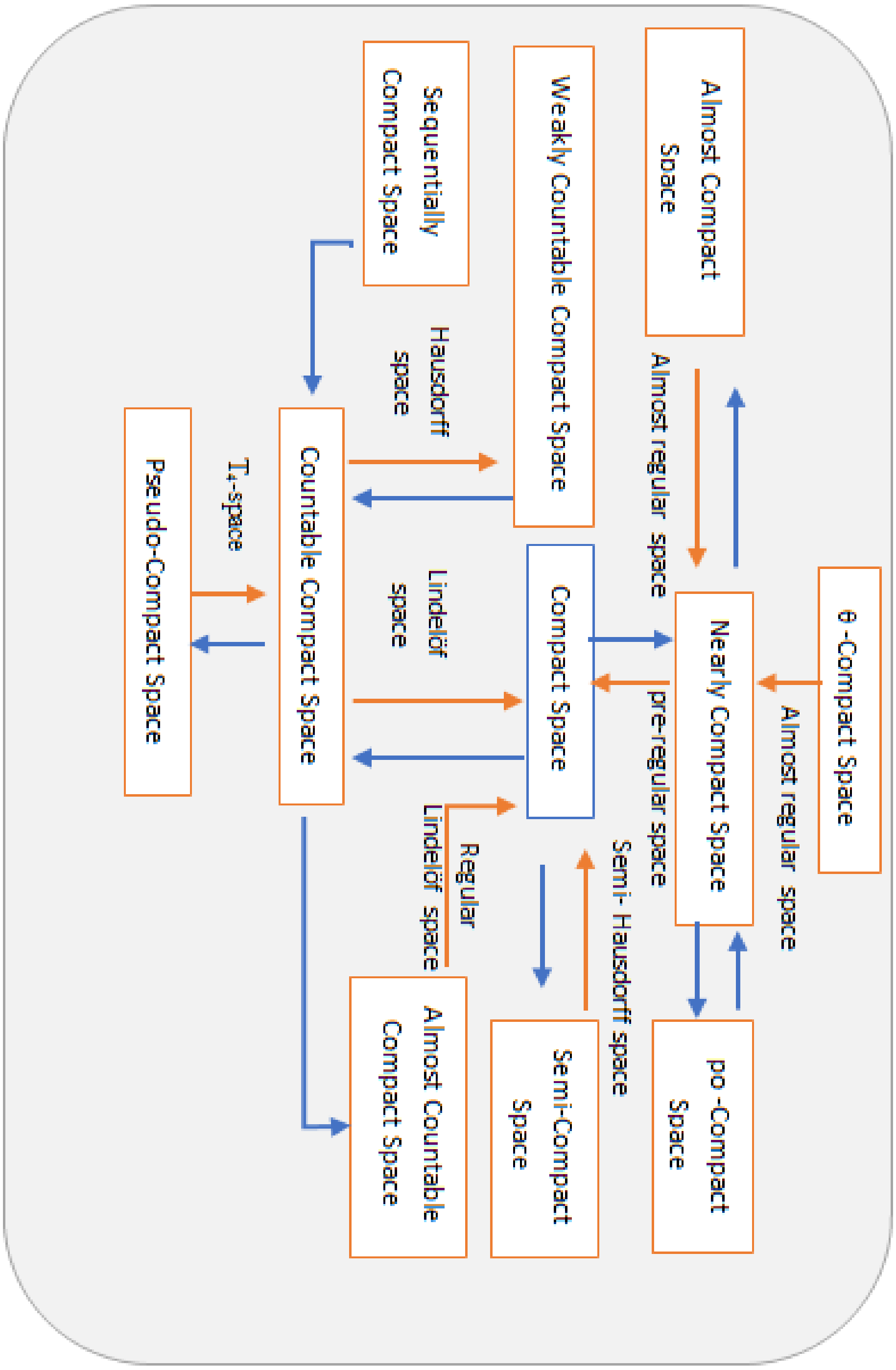

Figure 1.2 [26] Relationships between Certain Types of Compact Spaces.

1.3 Important Definitions and Concepts of G-space

Definition 1.3.1 [64]

A topological group is a set $\mathbb{G}$ with two structures:

(i) $\mathbb{G}$ is a group.

(ii) $\mathbb{G}$ is a space. such that the two structures are compatible, i.e, the multiplication map $\Omega: \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ which is defined by $\Omega(g_1, g_2) = g_1 . g_2$, $\forall (g_1, g_2) \in \mathbb{G} \times \mathbb{G}$ and the inversion map $v: \mathbb{G} \longrightarrow \mathbb{G}$ which is defined by $v(g) = g^{-1}$ for all $g \in \mathbb{G}$, are both continuous maps. In other words, the map $\theta: \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ which is defined by $\theta(g_1, g_2) = g_1 . {g_2}^{-1}$, $\forall\ (g_1, g_2) \in \mathbb{G} \times \mathbb{G}$, is a continuous map.

Definition 1.3.2 [64]

A topological subgroup H of a topological group $\mathbb{G}$, is a subgroup of the group $\mathbb{G}$ with the relative topology from the space $\mathbb{G}$.

Definition 1.3.3 [64]

A group action is defined as a continuous map $\varphi: \mathbb{G} \times \mathbb{W} \to \mathbb{W}$ [$\mathbb{G}$ is a topological group and $\mathbb{W}$ is any space] such that $\varphi(e, x) = x$, for all $x \in \mathbb{W}$ and $\varphi\big(g_1, \varphi(g_2, x)\big) = \varphi(g_1 g_2, x)$ for all $x \in \mathbb{W}, g_1, g_2 \in \mathbb{G}$**,** The triple $(\mathbb{G}, \mathbb{W}, \varphi)$ is called a (left) topological transformation group and $\mathbb{W}$ is called a $\mathbb{G}$–space on $\mathbb{G}$**.**

Definition 1.3.4 [64]

A subgroup action is defined as a continuous map $\varphi: H \times B \to B$ [H is a topological subgroup of $\mathbb{G}$ and B is any subset of $\mathbb{W}$ which is a $\mathbb{G}$–space on $\mathbb{G}$] such that $\varphi(e, x) = x$, for all $x \in B$ and $\varphi\big(g_1, \varphi(g_2, x)\big) = \varphi(g_1 g_2, x)$ for all $x \in B, g_1, g_2 \in H$**,** The triple (H, B, φ) is called a (left) topological transformation group and B is called H–space on H**.**

Proposition 1.3.5 [65]

Let $\mathbb{G}$ be a topological group and let H be a normal subgroup of group $\mathbb{G}$. Then, $\mathbb{G}/H$ is a topological group. $\mathbb{G}/H = \{gH : g \in \mathbb{G}\}$ and $g.H = \{gh | h \in H\}$.

Remark 1.3.6 [66]

Let $\mathbb{W}$ be a $\mathbb{G}$-space. If $H \subseteq \mathbb{G}$ and $A \subseteq \mathbb{W}$, then A is said to be invariant under H if and only if $H.A \subseteq A$.

Definition 1.3.7 [65]

Let $\mathbb{W}$ be a $\mathbb{G}$-space and $\in \mathbb{W}$, then:

(1) The set $\mathbb{G}.x = \{g.x: g \in \mathbb{G}\}$ is said to be the orbit of x under $\mathbb{G}$. The set $\mathbb{G}_x = \{g \in G: g.x = x\}$ is said to be the stability subgroup of $\mathbb{G}$ at x.

(2) The set $Ker(\varphi) = \{g \in G: g.x = e$ for all $x \in \mathbb{W}\}$ is said to be the kernel of the action φ. Notice that $Ker(\varphi) = \bigcap_{x\in\mathbb{W}} \mathbb{G}_x$.

(3) Let R be a relation on $\mathbb{G}$ defined as follows:
$\forall\, x, y \in \mathbb{W}, xRy \Leftrightarrow \exists g \in \mathbb{G}$, such that $g.x = y$. Then, R is an equivalence relation and the equivalence class of x is $\mathbb{G}\,.x$ the orbit of x under $\mathbb{G}$, and $\mathbb{W}/\mathbb{G}$ with the quotient topology is called quotient space or the orbit space induced by the equivalence relation R which is defined by $\mathbb{G}$ on $\mathbb{W}$, and the map $\pi: \mathbb{W} \longrightarrow \mathbb{W}/\mathbb{G}$, such that $\pi(x) = \mathbb{G}.x$, for all $x \in \mathbb{W}$ is called natural projection map onto the orbit space.
Recall, an action of $\mathbb{G}$ on $\mathbb{W}$ is said to be Free if $\mathbb{G}_x = \{e\}$ for each $x \in \mathbb{W}$.

Remarks 1.3.8 [67]

(i) If $\mathbb{G}$ is freely on $\mathbb{W}$ and R equivalence relation R which is defined by $\mathbb{G}$ on $\mathbb{W}$, and $\mathbb{G}R(R) \subseteq \mathbb{W} \times \mathbb{W}$ is the graph of R, then $\Phi: \mathbb{G}R(R) \longrightarrow \mathbb{G}$ such that $\Phi(x, y) = g$,[where $g \in \mathbb{G}$ and $y = g.x$] is a surjective map.

(ii) Let $\mathbb{W}$ be a $\mathbb{G}$-space. If $\mathbb{G}$ is freely on $\mathbb{W}$, then the map $\ell: \mathbb{G} \times \mathbb{W} \to \mathbb{W} \times \mathbb{W}$ which is defined by $\ell(g, x) = (x, g.x)$ for each $(g, x) \in \mathbb{G} \times \mathbb{W}$, is an injective map.

Definition 1.3.9[66]

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ and $\mathcal{J}: \mathbb{W} \to \mathbb{E}$ be two maps. Then, $\mathcal{L}\Delta\mathcal{J}: \mathbb{W} \to \mathbb{M} \times \mathbb{E}$ is called the diagonal map, which is defined by $(\mathcal{L}\Delta\mathcal{J})(x) = \big(\mathcal{L}(x), \mathcal{J}(x)\big)$, for each $x \in \mathbb{W}$.

Lemma 1.3.10 [68]

If $\mathbb{G}$ is a topological group and H is a normal subgroup, then $\mathbb{G}/H$ is Hausdorff if and only if H is closed.

1.4 Basic concepts of fuzzy topological space

Remarks 1.4.1 [69]

1- For any set $\mathbb{W}$, $I^{\mathbb{W}}$denotes the collection of all functions on $\mathbb{W}$ to $I = [0,1]$. $I^{\mathbb{W}} = \{\lambda : \lambda \in \mathbb{W} \times \mathrm{I}\}$ where $\lambda = \{(x, f_\lambda(x)) | f_\lambda : \mathbb{W} \to [0,1]\}$ .A member λ of $I^{\mathbb{W}}$ is called a fuzzy set of $\mathbb{W}$.

2- The union $\vee \lambda_\alpha$(the intersection $\wedge \lambda_\alpha$) of a family $\{\lambda_\alpha\}$ of fuzzy sets of $\mathbb{W}$ is defined to be the fuzzy set $\sup\lambda_\alpha$ ($\inf \lambda_\alpha$).

3- For any two members λ and μ of $I^{\mathbb{W}}$; $\lambda \geq \mu$ if and only if $\lambda(x) \geq \mu(x)$, for each $x \in \mathbb{W}$, and in this case, λ is said to contain μ or μ is said to be contained in λ .

4- The constant maps 0 and 1 take the whole of 0 and 1, respectively.

5- The complement $\lambda^c$of a fuzzy set λ of $\mathbb{W}$ is $1 - \lambda$, defined by $(1 - \lambda)(x) = 1 - \lambda(x)$, for each $x \in \mathbb{W}$.

Definition 1.4.2 [70]

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a map on $\mathbb{W}$ to $\mathbb{M}$. It is a fuzzy set of $\mathbb{W}$, $\mathcal{L}(\lambda)$ is defined as follows:

$$(\mathcal{L}(\lambda))(y) = \begin{cases} \sup\limits_{x \in \mathcal{L}^{-1}(y)} \lambda(x) & if \ \mathcal{L}^{-1}(y) \neq \emptyset; \\ 0 & otherwise, \end{cases}$$

For each $y \in \mathbb{M}$; and if μ is a fuzzy set in $\mathbb{M}$, $\mathcal{L}^{-1}(\mu)$ is defined as follows: $\mathcal{L}^{-1}(\mu)(x) = (\mu \circ \mathcal{L})(x)$, for each $x \in \mathbb{W}$.

Definition 1.4.3 [70]

A fuzzy point x_μ in $\mathbb{W}$ is a fuzzy set defined as follows:

$$x_\mu(y) = \begin{cases} \mu & if \ y = x \\ 0 & if \ y \neq x \end{cases}$$

Where $0 < \mu \leq 1$; μ is called its value and x in support of x_μ.

The set of all fuzzy points in $\mathbb{W}$ will be denoted by $FP(\mathbb{W})$.

Definition 1.4.4 [8]

$T \subset I^{\mathbb{W}}$ is a fuzzy topology on $\mathbb{W}$ iff:

(1) $\forall \, \sigma$ constant, $\sigma \in T$;

(2) $\forall \, \lambda, \mu \in T \Longrightarrow \lambda \wedge \mu \in T$;

(3) $\forall \, (\mu_i)_{i \in \Gamma} \subset T \Longrightarrow \sup_{i \in \Gamma} \mu_i \in T$.

The pair $(\mathbb{W}, T)$ is called a fuzzy topological space (FTS, for short). A fuzzy set $\mu \in T$ is called a fuzzy open. μ^c is called a fuzzy closed set, where μ is a fuzzy open set.

Definition 1.4.5 [69]

A fuzzy set μ in $\mathbb{W}$ is called quasi–coincident with a fuzzy set λ in $\mathbb{W}$ denoted by $\mu q \lambda$ if and only if $\mu(x) + \lambda(x)$>1, for some $x \in \mathbb{W}$. If μ is not quasi–coincident with, then $\mu(x) + \lambda(x)$≤1, for every $x \in \mathbb{W}$ and denoted by $\mu \tilde{q} \lambda$.

Definition 1.4.6 [70]

Let be FTS and $\mu \in T$. Then μ is said to be a quasi-coincident neighborhood of a fuzzy point x_t if $\mu^c(x) < t$. The set of all quasi-coincident neighborhoods of $x_t$is denoted by $Q(x_t)$.

μ is called a quasi-neighborhood of fuzzy point x_t in $\mathbb{W}$ if and only if there exists $\lambda \in T$such that $x_t q \lambda \leq \mu$.

Definition 1.4.7 [8]

Let $\mathcal{L}$ be a map from an FTS $(\mathbb{W}, \mathrm{T})$ to an FTS $(\mathbb{M}, \mathrm{S})$. Then:

1- $\mathcal{L}$ is said to be a fuzzy continuous (F-continuous) if and only if for each fuzzy set $\mu \in S, \mathcal{L}^{-1}(\mu) \in T$ or equivalently for each fuzzy closed set $\lambda \in F_S$, then $\mathcal{L}^{-1}(\lambda) \in F_T$.

2- $\mathcal{L}$ is a fuzzy open (closed) if and only if for each open (closed) fuzzy set λ in $(\mathbb{W}, \mathrm{T})$, $\mathcal{L}(\lambda)$ is a fuzzy open (closed) set in $(\mathbb{M}, \mathrm{S})$.

3- $\mathcal{L}$ is a fuzzy homeomorphism if it is injective, surjective, F-continuous, and $\mathcal{L}^{-1}$ is F-continuous.

Definition 1.4.8 [71]

Let $(\mathbb{W}, \mathrm{T})$ be an FTS. A family $\{\lambda_i : i \in \Gamma\}$ of fuzzy subsets of an FTS $(\mathbb{W}, \mathrm{T})$ is called a locally finite at x_t, if there exists $A \in Q(x_t)$ such that $\lambda_i q\ A$ holds except for finitely many $i \in \Gamma$.

For a fuzzy set $\mu \in I^{\mathbb{W}}$, $\mathcal{A} = \{\lambda_i : i \in \Gamma\}$ of a fuzzy subset of an FTS $(\mathbb{W}, \mathrm{T})$ is called a locally finite in A, if $\mathcal{A}$ is locally finite at every x_t, where $t \leq A(x)$ is for some $x \in \mathbb{W}$.

Definition 1.4.9 [71]

In an FTS with families $\mathbb{U}$ and $\mathbb{V}$ of fuzzy sets, $\mathbb{V}$ is a refinement of $\mathbb{U}$, written $\mathbb{V} \leq \mathbb{U}$, if and only if for each $\mu \in \mathbb{V}$ there is an $\lambda \in \mathbb{U}$ such that $\mu \leq \lambda$.

Definition 1.4.10 [72]

Let $(\mathbb{W}, \mathrm{T})$ be an FTS and $\alpha \in (0,1]$. A collection $\mathcal{B}$ of fuzzy sets is called an α-Q cover of a fuzzy set μ, if for each fuzzy point $x_\alpha$with $x_\alpha \leq \mu$, there exists $\lambda \in \mathcal{B}$ with $\lambda^c(x) < \alpha$ (That is x_α a quasi–coincident with λ). If $\mathcal{B} \subset T$, then $\mathcal{B}$ is called an open α-Q cover of μ.

Definition 1.4.11 [72]

Let $(\mathbb{W}, \mathrm{T})$ be an FTS and let $\mu \in I^{\mathbb{W}}$ and $\alpha \in [0,1]$. Then μ is said to be α-fuzzy paracompact if, for every open α-Q cover of $\mathcal{A}$ of μ, there exists an open refinement $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}$ is locally finite in μ and is an α-Q cover of μ.

μ is called a fuzzy paracompact if μ is α-fuzzy paracompact for every $\alpha \in [0,1]$. $(\mathbb{W}, \mathrm{T})$ is said to be a fuzzy paracompact if 1 fuzzy paracompact.

Definition 1.4.12 [73]

For each FTS $(\mathbb{W}, w(\tau))$, the family of crisp sets $[\tau] = \{A \subset \mathbb{W}: \chi_A \in \mathrm{T}\}$ is called the original topology of $w(\tau)$ and the crisp topological spaces $(\mathbb{W}, [\tau])$ are original topological spaces of $(\mathbb{W}, w(\tau))$.

Remark 1.4.13 [73]

A fuzzy extension of a topological property of $(\mathbb{W}, \tau)$ is said to be good when it is owned by $w(\tau)$ whenever the original property is possessed by τ.

Remark 1.4.14 [74]

$(\mathbb{W}, w(\tau))$ is a fuzzy paracompact if and only if $(\mathbb{W}, \tau)$ is paracompact, when τ is the original topology of $w(\tau)$. Thus fuzzy paracompact is a good extension (in the sense of R. Lowen) of paracompactness.

Remark 1.4.15 [72]

The concept of fuzzy metrizability (fuzzy compact) is a good extension of the crisp metrizability (compact) respectively, (in the sense of Lowen).

Definition 1.4.16 [69]

Let $\mathbb{W}$ and $\mathbb{M}$ be two FTS, then a fuzzy mapping $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is said to be fuzzy proper if:

- $\mathcal{L}$ is F-continuous mapping.

- $\mathcal{L} \times \mathrm{I}_{\mathbb{E}}: \mathbb{W} \times \mathbb{E} \rightarrow \mathbb{M} \times \mathbb{E}$ is fuzzy closed, for any fuzzy space $\mathbb{E}$.

Definition 1.4.17 [72]

A fuzzy set $\sigma \in I^{\mathbb{W}}$ is fuzzy compact iff for all family $\beta \subset T$ such that $\sup_{\mu \in \beta} \mu \geq \sigma$ and for all $\varepsilon > 0$, there exists a finite subfamily $\beta_o \subset \beta$ such that $\sup_{\mu \in \beta_o} \mu \geq \sigma - \varepsilon$. In particular, $\mathbb{W}$ is a fuzzy compact if 1 is a fuzzy compact.

Let $(\mathbb{W}, \mathrm{T})$ be an FTS. Then $\mathbb{W}$ is called a fuzzy compact space if every fuzzy open cover of $\mathbb{W}$ has a finite subcover.

Chapter

Two

Paracompact Map

2.1 Characterization of Paracompact Map

In this section, The properties of paracomact map are investigated. In addition, the relationships between a paracompact map with another important maps are discussed.

Proposition 2.1.1.

A continuous map $\mathcal{L}$ from $\mathbb{W}$ onto $\mathbb{M}$ is paracompact, if and only if the inverse image for any paracompact set in $\mathbb{M}$ is paracompact set in $\mathbb{W}$.

Proof.

Assume that $y \in \mathbb{M}$ and $\mathbb{U} = \{U_\alpha\}_{\alpha \in I}$ is an open cover of $\mathcal{L}^{-1}(\{y\})$, that is $\mathcal{L}^{-1}(\{y\}) \subseteq \cup \mathbb{U}$. Choose $\mathbb{M}$ as the open set containing y. Since $\{y\}$ is paracompact set in $\mathbb{M}$ and $\mathcal{L}$ is paracompact map, then $\mathcal{L}^{-1}(\{y\})$ is paracompact set in $\mathbb{W}$, and so the open cover $\{U_\alpha \cap \mathbb{W}\}_{\alpha \in I} = \mathbb{U}$ of $\mathcal{L}^{-1}(\{y\})$ has open refinement $\mathbb{V}$ which is locally refinement at $\mathcal{L}^{-1}(\{y\})$. Hence $\mathcal{L}$ is paracompact map.

Conversely, suppose that $\mathcal{L}$ is paracompact map and let $K \subseteq \mathbb{M}$ be paracompact. For every $y \in K$ and every open cover $\mathbb{U} = \{U_\alpha\}_{\alpha \in I}$ of $\mathcal{L}^{-1}(K)$ such that $\mathcal{L}^{-1}(\{y\}) \subseteq \cup \mathbb{U}$, then there exists a neighborhood V_y of y and $\mathcal{L}^{-1}(V_y) \subseteq \cup \mathbb{U}$ also, $\mathcal{L}^{-1} \cup V_y \subseteq \cup \mathbb{U}$. Since $\mathcal{L}^{-1}(K) \subseteq \mathcal{L}^{-1} \cup V_y$, then $\{U_\alpha \cap \mathcal{L}^{-1} \cup V_y\}_{\alpha \in I}$ is cover of $\mathcal{L}^{-1}(K)$ and it has open refinement and so $\mathbb{U}$ has open refinement, which is locally finite at $\mathcal{L}^{-1}(K)$. ■

Proposition 2.1.2.

If $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a bijective continuous map, then the following statements are equivalent:

(i) $\mathcal{L}$ is a proper map.

(ii) $\mathcal{L}$ is a paracompact map.

(iii) $\mathcal{L}$ is a closed map.

(iv) $\mathcal{L}$ is a homeomorphism of $\mathbb{W}$ on to closed subset of $\mathbb{M}$.

Proof.

(i→ii) Suppose that $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a proper map and $K \subseteq \mathbb{M}$ is a paracompact set in $\mathbb{M}$. For any open cover $\mathbb{U} = \{U_\alpha\}_{\alpha \in I}$ of $\mathcal{L}^{-1}(K)$ and $y \in K$, since $\mathcal{L}^{-1}(\{y\})$ is compact then by Theorem 1.2.46 (ii) there exists finite subcover $\mathbb{U}_y \subseteq \mathbb{U}$ such that $\mathcal{L}^{-1}(\{y\}) \subseteq \cup \mathbb{U}_y$. Now, take an open neighborhood V_y of y such that $\mathcal{L}^{-1}(V_y) \subseteq \cup \mathbb{U}_y$. Since K is paracompact, then the open cover $\{V_y : y \in K\}$ of K has locally finite open refinement $\{S_y: y \in K\}$. Thus $\{\mathcal{L}^{-1}(S_y) \cap U : y \in K, U \in \mathbb{U}_y\}$ is a locally finite open refinement of $\mathbb{U}$. Hence $\mathcal{L}^{-1}(K)$ is paracompact in $\mathbb{W}$.

(ii→iii) Let $F \subseteq \mathbb{M}$ and U be an open set in $\mathbb{W}$ such that $\mathcal{L}^{-1}(F) \subseteq U$, then $\mathcal{L}^{-1}(\{y\}) \subseteq U$ for all $y \in F$. Since $\mathcal{L}$ is a paracompact map, then there exists an open neighborhood V_y of y and {U} has open refinement $\{W_y\}$ such that $\mathcal{L}^{-1}(V_y) \subseteq W_y \subseteq U$ by Proposition 2.1.1, so $\mathcal{L}^{-1}(\bigcup V_y) \subseteq U$. But $\bigcup V_y$ is an open set and $\mathcal{L}$ is continuous, thus $\mathcal{L}^{-1}(\bigcup V_y)$ is an open set contained in U. Therefore, $\mathcal{L}$ is a closed map by Proposition 1.2.49.

(iii→iv) From (iii) $\mathcal{L}$ is a closed map and by hypothesis, $\mathcal{L}$ is continuous and bijective, so $\mathcal{L}$ is a homeomorphism map.

(iv→i) Assume that $\mathcal{L}$ is the homeomorphism map and $\mathbb{K}$ is any space, then $\mathcal{L} \times I_{\mathbb{K}}: \mathbb{W} \times \mathbb{K} \to \mathbb{M} \times \mathbb{K}$ is homeomorphism that is $\mathcal{L} \times I_{\mathbb{K}}$ is closed and $\mathcal{L}$ is continuous by hypothesis. Thus, $\mathcal{L}$ is a proper map. ■

Corollary 2.1.3.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a surjective and proper map, then the inverse image for any paracompact set in $\mathbb{M}$ is paracompact set in $\mathbb{W}$.

Proposition 2.1.4.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a surjective, open and proper, then the image for any paracompact set in $\mathbb{W}$ is paracompact set in $\mathbb{M}$.

Proof.

Since $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a proper map then $\mathcal{L}$ is a continuous. If $\mathbb{U} = \{U_\alpha\}_{\alpha \in I}$ is an open cover of $\mathcal{L}(A)$, then $\mathcal{L}^{-1}(\mathbb{U})$ is an open cover of A and so, there exists a locally finite open refinement $\mathbb{V}$ of $\mathcal{L}^{-1}(\mathbb{U})$. Since $\mathcal{L}$ is open then $\mathcal{L}(\mathbb{V})$ is an open refinement of $\mathbb{U}$. Now, to prove that $\mathcal{L}(\mathbb{V})$ is locally finite in $\mathcal{L}(A)$. Let $y \in \mathcal{L}(A)$. Since $\mathbb{V}$ is locally finite, then for each $x \in \mathcal{L}^{-1}(\{y\})$ there exists an open neighborhood D_x of x such that D_x intersects at most finitely many members of $\mathbb{V}$ and $\{D_x: x \in \mathcal{L}^{-1}(\{y\})\}$ is an open cover of $\mathcal{L}^{-1}(\{y\})$. So there exists a finite subcollection $\mathbb{D}_y$ of $\{D_x\}$ such that $\mathcal{L}^{-1}(\{y\}) \subseteq \bigcup \mathbb{D}_y$, also $\bigcup \mathbb{D}_y$ intersects at most finitely many members of $\mathbb{V}$. Since $\mathcal{L}$ is a continuous and closed map because $\mathcal{L}$ is proper then By Proposition 1.2.49, there exists an open neighborhood E_y of y in $\mathcal{L}(A)$ such that $\mathcal{L}^{-1}(E_y) \subseteq \bigcup \mathbb{D}_y$, then $\mathcal{L}^{-1}(E_y)$ intersects at most finitely many members of $\mathbb{V}$, therefore E_y intersects at most finitely many members of $\mathcal{L}(\mathbb{V})$. Thus, $\mathcal{L}(\mathbb{V})$ is locally finite in $\mathcal{L}(A)$. ■

Lemma 2.1.5.

Let $\mathbb{W}$ be a space. If $\mathbb{W} \times \{y\}$ is paracompact then $\mathbb{W}$ is paracompact, for each point y which does not belong to $\mathbb{W}$.

Proof.

Let $\mathbb{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover of $\mathbb{W}$, then $\mathbb{W} \subseteq \bigcup_{\alpha\in I} U_\alpha$ and this implies $\mathbb{W} \times \{y\} \subseteq \bigcup_{\alpha\in I} U_\alpha \times \{y\}$. Since $\mathbb{W} \times \{y\}$ is paracompact, then $\{U_\alpha\}_{\alpha\in I} \times \{y\}$ has a locally finite open refinement $\{V_\beta\}_{\beta\in J} \times \{y\}$. Therefore, $\{V_\beta\}_{\beta\in J}$ a locally finite open refinement of $\mathbb{U}$. Hence, $\mathbb{W}$ is paracompact. ■

Proposition 2.1.6.

Let $\mathcal{L}_1: \mathbb{W}_1 \to \mathbb{M}_1$ and $\mathcal{L}_2: \mathbb{W}_2 \to \mathbb{M}_2$ be bijective continuous maps, then $\mathcal{L}_1 \times \mathcal{L}_2: \mathbb{W}_1 \times \mathbb{W}_2 \to \mathbb{M}_1 \times \mathbb{M}_2$ is paracompact if and only if $\mathcal{L}_1$and $\mathcal{L}_2$are paracompact map.

Proof.

Assume that $\mathcal{L}_1$is bijective continuous. Let $A \subseteq \mathbb{M}_1$ be paracompact in $\mathbb{M}_1$ and let $y_2 \in \mathbb{M}_2$, since $\{y_2\}$ is compact then by Theorem 1.2.15, $\{y_2\}$ is paracompact. By Proposition 1.2.52, $A \times \{y_2\}$ is paracompact set in $\mathbb{M}_1 \times \mathbb{M}_2$. Since $\mathcal{L}_1 \times \mathcal{L}_2$ is paracompact, then $(\mathcal{L}_1 \times \mathcal{L}_2)^{-1}(A \times \{y_2\}) = \mathcal{L}_1^{-1}(A) \times \mathcal{L}_2^{-1}(\{y_2\})$ is paracompact, by Lemma 3.1.8, $\mathcal{L}_1^{-1}(A)$ is paracompact. Therefore, $\mathcal{L}_1$ is a paracompact map.

Similarly, to prove that $\mathcal{L}_2$ is a paracompact map.

Conversely, By Proposition 2.1.2, $\mathcal{L}_1$ and $\mathcal{L}_2$ are proper maps, thus by Proposition 1.2.47, $\mathcal{L}_1 \times \mathcal{L}_2$ is a proper map. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are surjective continuous, then $\mathcal{L}_1 \times \mathcal{L}_2$ is surjective continuous. Hence, by Proposition 2.1.2, $\mathcal{L}_1 \times \mathcal{L}_2$ is Paracompact. ■

Corollary 2.1.7. Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a paracompact map and $I_{\mathbb{E}}: \mathbb{E} \to \mathbb{E}$, then $\mathcal{L} \times I_{\mathbb{E}}: \mathbb{W} \times \mathbb{E} \to \mathbb{M} \times \mathbb{E}$ is paracompact, for any space $\mathbb{E}$.

Proposition 2.1.8.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$, $\mathcal{J}: \mathbb{M} \to \mathbb{E}$ and $\mathcal{J} o \mathcal{L}: \mathbb{W} \to \mathbb{E}$ be a map, then:

(i) If $\mathcal{J}$ and $\mathcal{L}$ are paracompact, then $\mathcal{J} o \mathcal{L}$ is paracompact map.

(ii) If $\mathcal{L}$ is paracompact and $\mathcal{J}$ is a homeomorphism, then $\mathcal{J} o \mathcal{L}$ is paracompact.

(iii) If $\mathcal{J}$ is continuous and compact and $\mathcal{L}$ is paracompact, then $\mathcal{J} o \mathcal{L}$ is a compact, where $\mathbb{W}$ is countable compact and $\mathbb{E}$ is Hausdorff space.

(iv) If $\mathcal{J}o\mathcal{L}$ is paracompact and $\mathcal{L}$ is surjective open proper, then $\mathcal{J}$ is paracompact map.

Proof.

(i) Since $\mathcal{L}$ and $\mathcal{J}$ are surjective continuous, then $\mathcal{J}o\mathcal{L}$ is surjective contin-uous.
Let K be a paracompact set in $\mathbb{E}$, then $\mathcal{J}^{-1}$(K)is paracompact set in $\mathbb{M}$, also $\mathcal{L}^{-1}(\mathcal{J}^{-1}(K)) = (\mathcal{J}o\mathcal{L})^{-1}(K)$ is paracompact in $\mathbb{W}$. Therefore, $\mathcal{J}o\mathcal{L}$ is paracompact.

(ii) Since $\mathcal{J}$ is a closed map, then $\mathcal{J}$ is homeomorphism on a closed subset of $\mathbb{E}$. Consequently, $\mathcal{J}$ is paracompact by Proposition 2.1.2, thus by (i), $\mathcal{J}o\mathcal{L}$ is paracompact.

(iii) Let K be a compact set in $\mathbb{E}$, then K a closed set. Consequently, $\mathcal{J}^{-1}$(K) is closed compact set in $\mathbb{M}$, $\mathcal{J}^{-1}$(K) is closed paracompact set by Theorem 1.2.15. Since $\mathcal{L}$ is paracompact, then $\mathcal{L}^{-1}(\mathcal{J}^{-1}(K)) = (\mathcal{J}o\mathcal{L})^{-1}(K)$ is closed paracompact in $\mathbb{W}$. But $\mathbb{W}$ is countable compact, thus $(\mathcal{J}o\mathcal{L})^{-1}(K)$ is countable compact due to Theorem 1.2.10, so $(\mathcal{J}o\mathcal{L})^{-1}(K)$ is compact by Theorem 1.2.19. Hence, $\mathcal{J}o\mathcal{L}$ is compact.

(iv) Since $\mathcal{J}o\mathcal{L}$ is surjective then $\mathcal{J}$ is surjective. Let B is an open set in $\mathbb{E}$, then $\mathcal{L}^{-1}o\mathcal{J}^{-1}(B)$ is open in $\mathbb{W}$ and so, $\mathcal{L}(\mathcal{L}^{-1}o\mathcal{J}^{-1}(B)) = \mathcal{J}^{-1}(B)$ is open in $\mathbb{M}$. Thus, $\mathcal{J}$ is continuous.Suppose that $\mathcal{J}o\mathcal{L}$ is paracompact map and K is paracompact set in $\mathbb{E}$, then $\mathcal{L}^{-1}o\mathcal{J}^{-1}(K)$ is paracompact in $\mathbb{W}$. Since $\mathcal{L}$ is open proper then, $\mathcal{L}(\mathcal{L}^{-1}o\mathcal{J}^{-1}(K)) = \mathcal{J}^{-1}(K)$ is paracompact in $\mathbb{M}$ by Corollary 2.1.3. Hence, $\mathcal{J}$ is paracompact map. ■

<u>Proposition 2.1.9.</u>

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M} = \{y\}$ be a map and $y \notin \mathbb{W}$, then $\mathcal{L}$ is paracompact if and only if $\mathbb{W}$ is paracompact set.

Proof.

Since {y} is paracompact, then $\mathcal{L}^{-1}(\{y\}) = \mathbb{W}$ is paracompact because $\mathcal{L}$ is paracompact map. Conversely, $\mathbb{M}$ is a singleton point. Consequently, $\mathcal{L}$ is a surjective continuous map and $\mathbb{W} = \mathcal{L}^{-1}(\{y\})$ is paracompact. Thus, $\mathcal{L}$ is paracompact map. ■

<u>Corollary 2.1.10.</u>

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M} = \{y\}$ be a proper map and $y \notin \mathbb{W}$, then $\mathcal{L}$ is paracompact.

Proof.
By Proposition 1.2.53, $\mathbb{W}$ is compact and so, $\mathbb{W}$ is paracompact owing to Theorem 1.2.21. Hence by Proposition 2.1.9, $\mathcal{L}$ is paracompact. ■

Proposition 2.1.11.
Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be an injective paracompact map. If H is a subset of $\mathbb{M}$, then $\mathcal{L}_H: \mathcal{L}^{-1}(H) \to H$ is paracompact.
Proof.
Since $\mathcal{L}$ is an injective paracompact map, then $\mathcal{L}$ is a proper by Proposition 2.1.2 and by Proposition 1.2.50, $\mathcal{L}_H$ is proper. Since $\mathcal{L}_H$ is a surjective and continuous map, then by Proposition 2.1.2, $\mathcal{L}_H$ is a paracompact map. ■

Proposition 2.1.12.
Let $\mathbb{W}$ be a paracompact space and $\mathbb{M}$ be any space. Then the projection map $\mathbb{Pr}_2: \mathbb{W} \times \mathbb{M} \to \mathbb{M}$ is paracompact.
Proof.
Consider the following commutative diagram:

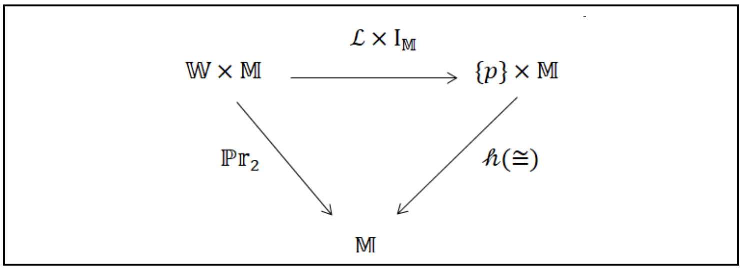

Figure 2.1 First ccommutative diagram

Notice that $\hbar: \{p\} \times \mathbb{M} \to \mathbb{M}$ is a homeomorphism onto $\mathbb{M}$, such that $p \notin \mathbb{W}$ and $\mathbb{Pr}_2: \mathbb{W} \times \mathbb{M} \to \mathbb{M}$ is the projection of $\mathbb{W} \times \mathbb{M}$ into $\mathbb{M}$. Further, since $\mathbb{W}$ is paracompact, then by Proposition 2.1.9, $\mathcal{L}: \mathbb{W} \to \{p\}$ is a paracompact map. Where $I_{\mathbb{M}}: \mathbb{M} \to \mathbb{M}$, then $\mathcal{L} \times I_{\mathbb{M}}$ is a paracompact by Corollary 2.1.7, so, by Proposition 2.1.8 (part ii, $\hbar o (\mathcal{L} \times I_{\mathbb{M}})$ is paracompact. Since $\mathbb{Pr}_2 = \hbar o (\mathcal{L} \times I_{\mathbb{M}})$, then $\mathbb{Pr}_2$ is Paracompact. ■

Proposition 2.1.13.
Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a surjective continuous map and let $\mathcal{J}: \mathbb{M} \to \mathbb{E}$ be an injective, open, and proper map such that $\mathcal{J} o \mathcal{L}: \mathbb{W} \to \mathbb{E}$ is paracompact. If $\mathbb{M}$ is Hauosdorff space, then $\mathcal{L}$ is paracompact map.

Proof.

Consider the following commutative diagram:

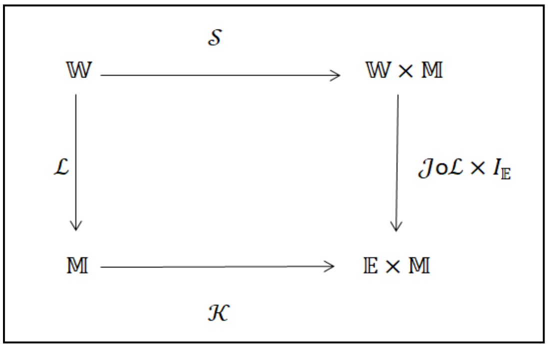

Figure 2.2 second commutative diagram

Notice that $\mathcal{S}(x) = (x, \mathcal{L}(x))$ and $\mathcal{K}(y) = (\mathcal{J}(y), y)$. By Corollary 1.2.63, $\mathcal{S}$ is a homeomorphism of $\mathbb{W}$ onto the graph $\mathcal{S}(\mathbb{W})$ of $\mathcal{L}$. Since $\mathbb{M}$ is Hauosdorff space, then by Theorem 1.2.64, the graph $\mathcal{S}(\mathbb{W})$ of $\mathcal{L}$ is closed set in $\mathbb{W}\times\mathbb{M}$. Since $\mathcal{L}$ is a surjective, then $\mathcal{S}$ is surjective and so, $\mathcal{S}$ is a homeomorphism of $\mathbb{W}$ onto $\mathbb{W}\times\mathbb{M}$, which implies to $\mathcal{S}$, is continuous. Proposition 2.1.2 infers that $\mathcal{S}$ is paracompact. Now, since $(\mathcal{J}o\mathcal{L})\times I_{\mathbb{E}}$ is a paracompact, then $((\mathcal{J}o\mathcal{L})\times I_{\mathbb{E}})o\mathcal{S}$ is paracompact. But, $((\mathcal{J}o\mathcal{L})\times I_{\mathbb{E}})o\mathcal{S} = \mathcal{K}o\,\mathcal{L}$, so $\mathcal{K}o\,\mathcal{L}$ is paracompact. Since $\mathcal{J}$ is an injective open proper, then $\mathcal{K}$ is an injective open proper. Therefore, by Proposition 1.2.51 $\mathcal{L}$ is paracompact. ■

2.2 Pa-closed Space

In this section, The concept of Pa-closed space is introduced, and its properties are investigated.

Definition 2.2.1.

A space $\mathbb{W}$ is known as Pa-closed provided each paracompact subset of $\mathbb{W}$ is closed.

Examples 2.2.2.

(1) A space $(\mathbb{Z}, \tau_D)$ is a Pa-closed.

(2) Any finite Hausdorff space is Pa-closed.

Proposition 2.2.3.

Any second countable, P-space and Hausdorff is Pa-closed space.

Proof.

Let A be a paracompact subset of a second countable space $\mathbb{W}$, then A is a second countable subspace of $\mathbb{W}$, and A is Lindelöf due to Proposition 1.2.56. Thus, by Proposition 1.2.57, A is a closed. ■

Proposition 2.2.4.

Every paracompact subset of Pa-closed and compact space is compact.

Proof.

Let $\mathbb{W}$ be a Pa-closed and compact space, and let A be a paracompact subset of $\mathbb{W}$. By Definition 2.2.1, A is a closed subset of $\mathbb{W}$. Thus by Theorem 1.2.10, A is a compact subspace of $\mathbb{W}$. ■

Proposition 2.2.5.

Let $\mathbb{W}$ be a Pa-closed space, then every compact subset of $\mathbb{W}$ is closed.

Proof.

Let $\mathbb{W}$ be a Pa-closed space and A is a compact subset of $\mathbb{W}$, then A is paracompact by Theorem 1.2.15. But $\mathbb{W}$ is Pa-closed, thus A is closed. ■

Proposition 2.2.6.

Let $\mathbb{W}$ be Pa-closed which admits a paracompact map into a space $\mathbb{M}$, then $\mathbb{M}$ is Pa-closed.

Proof.

Assume that $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a paracompact map and A is paracompact subset of $\mathbb{M}$. Then $\mathcal{L}^{-1}(A)$ is paracompact in $\mathbb{W}$. Since $\mathbb{W}$ is a Pa-closed space, then $\mathcal{L}^{-1}(A)$ is closed, thus $\mathcal{L}(\mathcal{L}^{-1}(A))$ closed by Proposition 2.1.2. But $\mathcal{L}(\mathcal{L}^{-1}(A)) = A$ due to $\mathcal{L}$ is a surjective map. Hence, $\mathbb{M}$ is Pa-closed. ■

Proposition 2.2.7.

Let $\mathbb{W}$ be paracompact space and $\mathcal{L}$ be a continuous map from $\mathbb{W}$ onto Pa-closed space $\mathbb{M}$. Then $\mathcal{L}$ is a paracompact map.

Proof.

Assume that A is paracompact subset of $\mathbb{M}$. Since $\mathbb{M}$ is Pa-closed, then A is closed, thus $\mathcal{L}^{-1}(A)$ is closed in $\mathbb{W}$ owing to $\mathcal{L}$ is continuous. Therefore $\mathcal{L}^{-1}(A)$ is paracompact in $\mathbb{W}$ by Proposition 1.2.10. Hence, $\mathcal{L}$ is a paracompact. ■

Proposition 2.2.8.

If $\mathbb{W} \times \mathbb{W}$ is Pa-closed space and $\mathbb{W}$ is locally compact, then $\mathbb{W}$ is Pa-closed space.

Proof.

Let A be a paracompact subset of $\mathbb{W}$ and $x \in \bar{A}$. We show that $x \in A$. Since $\mathbb{W}$ is locally compact then, there exist neighborhood U of x and $\bar{U}$ is compact, and so $\bar{U} \times A$ is paracompact in $\mathbb{W} \times \mathbb{W}$ by Proposition 1.2.52. Therefore, $\bar{U} \times A$ is closed in $\mathbb{W} \times \mathbb{W}$ due to $\mathbb{W} \times \mathbb{W}$ is Pa-closed space, thus $\bar{U} \times A = \overline{\bar{U} \times A} = \bar{U} \times \bar{A}$. But $(x, x) \in \bar{U} \times \bar{A}$ and $\bar{U} \times A = \bar{U} \times \bar{A}$, thus $x \in A$. This show that A is closed in $\mathbb{W}$, hence $\mathbb{W}$ is Pa-closed space. ■

Proposition 2.2.9.

Let $\mathbb{W} \times \mathbb{M}$ be Pa-closed space and $\mathbb{W}$ is paracompact, then $\mathbb{M}$ is Pa-closed space.

Proof.

Let A be a paracompact subset of $\mathbb{M}$ and $\mathbb{Pr}_2 : \mathbb{W} \times \mathbb{M} \to \mathbb{M}$ is the projection map. By Proposition 2.1.12, $\mathbb{Pr}_2$ is paracompact, thus ${\mathbb{Pr}_2}^{-1}(A)$ is paracompact in $\mathbb{W} \times \mathbb{M}$, thus ${\mathbb{Pr}_2}^{-1}(A)$ is a closed. Since $\mathbb{Pr}_2$ is closed by Proposition 2.1.2, then $\mathbb{Pr}_2({\mathbb{Pr}_2}^{-1}(A)) = A$ is closed in $\mathbb{M}$. Hence $\mathbb{M}$ is pa-closed. ■

Proposition 2.2.10.

Let $\mathbb{W}$ and $\mathbb{M}$ be compact Pa-closed spaces, then $\mathbb{W} \times \mathbb{M}$ is Pa-closed space.

Proof.

Assume that $A \times B$ is paracompact in $\mathbb{W} \times \mathbb{M}$. Since $\mathbb{W}$ is compact, then $\mathbb{P}\mathbb{r}_2 : \mathbb{W} \times \mathbb{M} \to \mathbb{M}$ is proper by Proposition 1.2.54. But $\mathbb{P}\mathbb{r}_2$ is surjective and open, thus, $\mathbb{P}\mathbb{r}_2(A \times B) = A$ is paracompact in $\mathbb{M}$ by Corollary 2.1.3. Because $\mathbb{M}$ is pa-closed, then A is closed. So ${\mathbb{P}\mathbb{r}_2}^{-1}(A) = A \times \mathbb{M}$ is closed in $\mathbb{W} \times \mathbb{M}$ due to $\mathbb{P}\mathbb{r}_2$ is continuous. By same way we show that, $\mathbb{W} \times B$ is closed in $\mathbb{W} \times \mathbb{M}$, therefore, $(A \times \mathbb{M}) \cap (\mathbb{W} \times \mathrm{B}) = \mathrm{A} \times \mathrm{B}$ is closed in $\mathbb{W} \times \mathbb{M}$. Hence, $\mathbb{W} \times \mathbb{M}$ is pa-closed space. ■

Proposition 2.2.11.

If Y is a clopen subspace of $(\mathbb{W},\tau)$. Then Y is Pa-closed if and only if $\mathbb{W}$ is Pa-closed.

Proof.

Let $(\mathbb{W},\tau)$ be a Pa-closed space and Y be an open subspace of $\mathbb{W}$. Assume that $A \subseteq Y$ is paracompact subspace of $\mathbb{W}$. First, must prove that A is paracompact in Y. Let $W = \{U \cap Y : U \in \tau\}$ be an open cover of A in Y, since Y is open then W is open cover of A in $\mathbb{W}$. Thus W has locally-finite open refinement, and so A is paracompact in Y . But Y is Pa-closed , therefore A is a closed subset of Y. We have Y is a closed subset of $\mathbb{W}$, thus A is a closed subset of W. Hence, W is a Pa-closed space. Conversely, it is clear. ■

2.3 Paracompactly Closed Map

In this section The paracompactly closed set and paracompactly closed map are given and the composition of the paracompactly closed maps is discussed.

Definition 2.3.1

Let $\mathbb{W}$ be a space, then $A \subseteq \mathbb{W}$ is said to be a paracompactly closed set if $A \cap K$ is paracompact, for every paracompact set K in $\mathbb{W}$.

Example 2.3.2

Every subset of space $\mathbb{W}$ under discrete topology is a paracompactly closed.

Theorem 2.3.3

Every closed subset of a compact and Pa-closed space $\mathbb{W}$ is a paracompactly closed.

Proof.

Let A be a closed subset of $\mathbb{W}$ and let K be a paracompact in $\mathbb{W}$. K is closed due to $\mathbb{W}$ is Pa-closed space, thus $A \cap K$ is a closed set, Theorem 1.2.10 insists that A is a compact set in $\mathbb{W}$. This implies that $A \cap K$ is a paracompact set owing to Theorem 1.2.15. Hence, A is a paracompactly closed. ■

Theorem 2.3.4.

Let $\mathbb{W}$ be a Pa-closed space. If $A \subseteq \mathbb{W}$ is paracompactly closed then it is a closed set.

Proof.

Let A be a paracompactly closed set in $\mathbb{W}$ and $x \in \bar{A}$. By Theorem 1.2.45, there exist a net $(\chi_d)_{d \in D}$ in A, such that $\chi_d \to x$. Since The set $F = \{\chi_d, x\}$ is paracompact and A is paracompactly closed, then $A \cap F$ is paracompact set in $\mathbb{W}$. We have $\mathbb{W}$ is a Pa-closed space, then $A \cap F$ is closed due to Definition 2.3.1. Because $\chi_d \to x$ and $\chi_d \in A \cap F = \overline{A \cap F}$, therefore $x \in A \cap F$ by Theorem 1.2.45, so, $x \in A$, as result $\bar{A} \subseteq A$ and $A = \bar{A}$. Hence, A is a closed set. ■

Theorem 2.3.5

Let $\mathbb{W}$ be a compact and Pa-closed space. Then $A \subseteq \mathbb{W}$ is a paracompact set if and only if it is paracompactly closed.

Proof: Let A be a paracompact subset of $\mathbb{W}$. Since $\mathbb{W}$ is a Pa-closed space, then A is a closed subset of $\mathbb{W}$. From Theorem 2.3.3, A is a paracompactly closed. Conversely, assume that A is a paracompactly closed set. Theorem 2.3.4 insists that A is a closed subset of $\mathbb{W}$. Therefore, A is a compact set due to Theorem 1.2.10. Hence, By Theorem 1.2.15, A is a paracompact subspace of $\mathbb{W}$. ■

Corollary 2.3.6

Let $\mathbb{W}$ be a compact and Pa-closed space. Then $A \subseteq \mathbb{W}$ is paracompactly closed if and only if it is compact.

Proof.

Let A be a paracompactly closed subset of $\mathbb{W}$, then by Theorem 2.3.5, A is paracompact set and so, A is compact due to Proposition 2.2.4. Conversely, Let A be a compactly subset of $\mathbb{W}$, thus A is paracompact by Theorem 1.2.15. Therefore, by Theorem 2.3.5, A is paracompactly closed. ■

Theorem 2.3.7.

If the injective map $\mathcal{L}$ of $\mathbb{W}$ onto $\mathbb{M}$ is an open and proper. Then A is a paracompactly closed set in $\mathbb{W}$ if and only if $\mathcal{L}(\mathrm{A})$ is a paracompactly closed set in $\mathbb{M}$.

Proof.

Assume that A is a paracompactly closed subset of $\mathbb{W}$ and K is a paracompact in $\mathbb{M}$. Since $\mathcal{L}$ is proper map, then $\mathcal{L}$ is paracompact due to Proposition 2.1.2. So, $\mathcal{L}^{-1}(\mathrm{K})$ is a paracompact set in $\mathbb{W}$, which implies $A \cap \mathcal{L}^{-1}(\mathrm{K})$ is a paracompact set. Corollary 2.1.3 asserts that $\mathcal{L}(A \cap \mathcal{L}^{-1}(\mathrm{K}))$=$\mathcal{L}(\mathrm{A}) \cap K$ is a paracompact set. Therefore, $\mathcal{L}(\mathrm{A})$ is a paracompactly closed subset of $\mathbb{M}$. Conversely, assume that $\mathcal{L}(\mathrm{A})$ is a paracompactly closed set. To show that A is a paracompactly closed subset of $\mathbb{W}$. Let K be a paracompact in $\mathbb{W}$, so $\mathcal{L}(\mathrm{K})$ is a paracompact set in $\mathbb{M}$ owing to Corollary 2.1.3. Then, $\mathcal{L}(\mathrm{A}) \cap \mathcal{L}(\mathrm{K})$ is paracompact because that $\mathcal{L}(\mathrm{A})$ is a paracompactly closed. So, $\mathcal{L}^{-1}(\mathcal{L}(\mathrm{A}) \cap \mathcal{L}(\mathrm{K})) = A \cap K$ is a paracompact subspace of $\mathbb{W}$ due to $\mathcal{L}$ is a paracompact map. Hence, A is a paraco-mpactly closed. ■

Definition 2.3.8.

A surjective continuous map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is called paracompactly closed providing the pre-image of any paracompactly closed set in $\mathbb{M}$ is paracompactly closed in $\mathbb{W}$.

Example 2.3.9.

Let $\mathbb{W}$ be any finite space and let $\mathbb{M}$ any space, then the surjective continuous map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is paracompactly closed.

Theorem 2.3.10.

Let $\mathbb{W}$ and $\mathbb{M}$ be a Pa-closed compact space, then the map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is paracompactly closed if and only if it is paracompact.

Proof: Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a paracompact (res. paracompactly closed) map and let K be a paracompactly closed (res. paracompact) subset of $\mathbb{M}$. Because $\mathbb{M}$ Pa-closed and compact space, K is paracompact (res. paracompactly closed) set in $\mathbb{M}$ due to Theorem 2.3.5. So, $\mathcal{L}^{-1}(K)$ is a paracompact (res. paracompactly closed) set in $\mathbb{W}$. Since $\mathbb{W}$ is Pa-closed and compact space, then $\mathcal{L}^{-1}(K)$ is a paracompactly closed (res. paracompact). Hence, $\mathcal{L}$ is a paracompactly closed (res. paracompact) map. ■

Corollary 2.3.11

Let $\mathbb{W}$ and $\mathbb{M}$ be a Pa-closed and compact space, then the map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is paracompactly closed if and only if it is compact.

Proof.

By Corollary 2.3.6. ■

Corollary 2.3.12

Let $\mathbb{W}$ and $\mathbb{M}$ be a Pa-closed and compact space. If the map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is paracompactly closed then it is a closed.

Proof.

By Theorem 2.3.3 and Theorem 2.3.4. ■

Theorem 2.3.13

Let $\mathbb{W}$ and $\mathbb{M}$ be a Pa-closed and compact space, then the continuous image of any paracompactly closed in $\mathbb{W}$ is paracompactly closed in $\mathbb{M}$.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a continuous map where $\mathbb{W}$ and $\mathbb{M}$ be a Pa-closed and compact spaces. Assume that K is a paracompactly closed in $\mathbb{W}$. By

Corollary 2.3.6 K is compact thus, $\mathcal{L}(K)$ is a compact set in $\mathbb{M}$ due to $\mathcal{L}$ is a continuous map. From Corollary 2.3.6 $\mathcal{L}(K)$ is a paracompactly closed. ■

Theorem 2.3.16

Every compact map of a Pa-closed compact space onto a Pa-closed compact space is paracompactly closed.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a compact map such that $\mathbb{W}$ and $\mathbb{M}$ are a Pa-closed compact space. Assume that K is a paracompactly closed set in $\mathbb{M}$. Consequently, K is a paracompactly closed subsets of $\mathbb{M}$ due to Corollary 2.3.6. Now, $\mathcal{L}^{-1}(K)$ is a compact set in $\mathbb{W}$ due to $\mathcal{L}$ is a compact map. Thus, $\mathcal{L}^{-1}(K)$ is a paracompactly closed set in $\mathbb{W}$ by Corollary 2.3.6. Hence, $\mathcal{L}$ is a paracompactly closed map. ■

Theorem 2.3.17

let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be closed map where $\mathbb{W}$ is a Pa-closed space and $\mathbb{M}$ is a Pa-closed compact space. Then, the image of any paracompactly closed set in $\mathbb{W}$ is parcompactly closed in $\mathbb{M}$.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a closed map where $\mathbb{W}$ is a Pa-closed space and $\mathbb{M}$ is a Pa-closed compact space. Suppose that K is a paracompactly closed set in $\mathbb{W}$. By Theorem 2.3.4 K is closed, thus, $\mathcal{L}(K)$ is a closed set in $\mathbb{M}$ due to $\mathcal{L}$ is a closed map. From Theorem 2.3.3 $\mathcal{L}(K)$ is a paracompactly closed. ■

Theorem 2.3.18

The composition of paracompactly closed maps is also a paracompactly closed map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ and $\mathcal{J}: \mathbb{M} \to \mathbb{E}$ be two paracompactly closed maps. To show that $\mathcal{J} \circ \mathcal{L}$ is also a paracompactly map. Assume that K is a paracompactly closed set in $\mathbb{E}$, For demonstrating that $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is a paracompactly closed set in $\mathbb{W}$. We have $\mathcal{J}^{-1}(\mathrm{K})$ is a paracompactly closed set in $\mathbb{M}$ since $\mathcal{J}$ is a paracompact map. Thus, $\mathcal{L}^{-1}(\mathcal{J}^{-1}(\mathrm{K}))$ is a paracompactly closed set in $\mathbb{W}$ due to, $\mathcal{L}$ is a paracompactly closed map, but $\mathcal{L}^{-1}(\mathcal{J}^{-1}(\mathrm{K})) = (\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$. So, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is a paracompactly closed set in $\mathbb{W}$. Hence, $\mathcal{J} \circ \mathcal{L}$ is paracompactly closed. ■

<u>Theorem 2.3.19</u>

Let $\mathbb{M}$ be a Pa-closed space and $\mathbb{E}$ is a Pa-closed compact space. If $\mathcal{J} \circ \mathcal{L}$: $\mathbb{W} \rightarrow \mathbb{E}$ is a paracompactly closed map and $\mathcal{J}: \mathbb{M} \rightarrow \mathbb{E}$ is a closed injective map, then $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is a paracompactly closed map.

Proof: Assume that K a paracompactly closed set in $\mathbb{M}$. Since $\mathbb{M}$ is a Pa-closed space and $\mathbb{E}$ is a Pa-closed compact space , then $\mathcal{J}(\mathrm{K})$ is paracompactly closed subspace of $\mathbb{E}$ due to Theorem 2.3.18. Thus, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathcal{J}(\mathrm{K}))$ is a paracompactly closed set in $\mathbb{W}$ because of $\mathcal{J} \circ \mathcal{L}$ is a paracompactly closed map. Therefore, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathcal{J}(\mathrm{K})) = \mathcal{L}^{-1}\left(\mathcal{J}^{-1}(\mathcal{J}(\mathrm{K}))\right) = \mathcal{L}^{-1}(\mathrm{K})$ is a paracompactly closed subspace of $\mathbb{W}$. Hence, $\mathcal{L}$ is a paracompactly closed map. ■

<u>Theorem 3.3.20</u>

Let $\mathbb{W}$ be a Pa-closed space and $\mathbb{M}$ a Pa-closed compact space. If $\mathcal{J} \circ \mathcal{L}$: $\mathbb{W} \rightarrow \mathbb{E}$ is a paracompactly closed map and $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is a closed surjective map, then $\mathcal{J}: \mathbb{M} \rightarrow \mathbb{E}$ is a paracompactly closed map.

Proof.

Suppose that K a paracompactly closed set in $\mathbb{E}$. Since $\mathcal{J} \circ \mathcal{L}$ is a paracompactly closed map then $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is paracompactly closed subspace of $\mathbb{W}$. But $\mathcal{L}$ is a surjective closed map then, $\mathcal{L}(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K}) = \mathcal{L}\left(\mathcal{L}^{-1}\left(\mathcal{J}^{-1}(\mathrm{K})\right)\right) = \mathcal{J}^{-1}(\mathrm{K})$ is paracompactly closed in $\mathbb{M}$ by Theorem 2.3.18. Hence, $\mathcal{J}$ is a paracompactly closed map. ■

Chapter Three

Certain Types of Paracompact Maps

3.1 Strong Types of Paracompact Maps

This Chapter presents a paracompact map under certain conditions. The chapter describes new weaker and strong forms of maps by using the concept of paracompactness and investigating their composition in various cases.

Definition 3.1.1.

A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is called countably paracompact providing the pre-image of any closed and countably paracompact set in $\mathbb{M}$ is countably paracompact in $\mathbb{W}$.

Theorem 3.1.2.

Every countably paracompact map of a Lindelöf space onto a Pa-closed space is paracompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a countably paracompact map such that $\mathbb{W}$ is a Lindelöf space and $\mathbb{M}$ is a Pa-closed space. Assume that K is a paracompact set in $\mathbb{M}$. So, K is a countably paracompact subset of $\mathbb{M}$ by Theorem 1.2.8. Since $\mathbb{M}$ is a Pa-closed, then K is a closed subset of $\mathbb{M}$ by Definition 2.2.1. Thus, $\mathcal{L}^{-1}(K)$ is a countably paracompact set in $\mathbb{W}$ because $\mathcal{L}$ is a countably paracompact map. In addition $\mathcal{L}^{-1}(K)$ is closed in $\mathbb{W}$ owing to $\mathcal{L}$ is continuous. Now, Theorem 1.2.9 asserts that $\mathcal{L}^{-1}(K)$ is a Lindelöf subspace of $\mathbb{W}$. Then, Theorem 1.2.26 implies that $\mathcal{L}^{-1}(K)$ is a paracompact set in $\mathbb{W}$. Hence, $\mathcal{L}$ is a paracompact map. ■

Theorem 3.1.3

Every countably compact map onto a Lindelöf and countably compact space is countably paracompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a countably compact map. Assume that K is a closed and countably paracompact set in $\mathbb{M}$. Since $\mathbb{M}$ is a Lindelöf space, then K is a Lindelöf subspace of $\mathbb{M}$ by Theorem 1.2.7. So, K is a paracompact subset of $\mathbb{M}$ owing to Theorem 1.2.20. Since $\mathbb{M}$ is a countably compact space and K is closed in $\mathbb{M}$. Then, K is a countably compact by Theorem 1.2.10. Now, Theorem 1.2.19 asserts that K is compact, and so, Theorem 1.2.17(ii) justifies that K is a countably compact subset of $\mathbb{M}$. Thus, $\mathcal{L}^{-1}(K)$ is a countably compact set in $\mathbb{W}$ because $\mathcal{L}$ is a countably compact map, and so $\mathcal{L}^{-1}(K)$ is contably paracompact in $\mathbb{W}$ due to Theorem 1.2.9. Hence, $\mathcal{L}$ is a countably paracompact map. ■

Theorem 3.1.4

Every countably paracompact map of a Pa-closed compact space onto a Pa-closed compact space is paracompactly closed map.

Proof.

Let $\mathcal{L}$: $\mathbb{W}$→$\mathbb{M}$ be a countably paracompact map such that $\mathbb{W}$ and $\mathbb{M}$ are a Pa-closed compact space. Assume that K is a paracompactly closed set in $\mathbb{M}$. So, K is a paracompact subspace of $\mathbb{M}$ due to Theorem 2.3.5 which implies K is a countably paracompact subset of $\mathbb{M}$ by Theorem 1.2.8. Since $\mathbb{M}$ is a Pa-closed, then K is a closed subset of $\mathbb{M}$. Thus, $\mathcal{L}^{-1}(K)$ is a countably paracompact set in $\mathbb{W}$ because $\mathcal{L}$ is a countably paracompact map. In addition $\mathcal{L}^{-1}(K)$ is closed in $\mathbb{W}$ owing to continuity of $\mathcal{L}$. Since $\mathbb{W}$ is a Lindelöf space, so Theorem 1.2.7 asserts that $\mathcal{L}^{-1}(K)$ is a Lindelöf subspace of $\mathbb{W}$. Then, Theorem 1.2.20 implies that $\mathcal{L}^{-1}(K)$ is a paracompact set thus, $\mathcal{L}^{-1}(K)$ is a paracompactly closed set in $\mathbb{W}$ by Theorem 2.3.5. Hence, $\mathcal{L}$ is a paracompactly closed map. ■

Theorem 3.1.5.

Every compact map onto a countably compact Pa-closed space is paracompact.

Proof.

Let $\mathcal{L}$: $\mathbb{W}$→$\mathbb{M}$ be a compact map such that $\mathbb{M}$ is a countably compact and Pa-closed space. Assume that K is a paracompact set in $\mathbb{M}$. Consequently, K is a closed subbset of $\mathbb{M}$ owing to Definition 2.2.1. Since K is closed in $\mathbb{M}$, then Theorem 1.2.10 asserts that K is a countably compact subspace of $\mathbb{M}$, which implies that K is compact by Theorem 1.2.19. So, $\mathcal{L}^{-1}(K)$ is a compact set in $\mathbb{W}$ due to $\mathcal{L}$ is a compact map. Thus, $\mathcal{L}^{-1}(K)$ is paracompact subset of $\mathbb{W}$ by Theorem 1.2.15. Hence, $\mathcal{L}$ is a paracompact mapping.

Theorem 3.1.6.

Every countably compact map onto a Lindelöf and countably compact space is countably paracompact map.

Proof.

Let $\mathcal{L}$: $\mathbb{W}$→$\mathbb{M}$ be a countably compact map. Assume that K is a closed and countably paracompact set in $\mathbb{M}$. Since $\mathbb{M}$ is a Lindelöf space, then K

is a Lindelöf subspace of $\mathbb{M}$ by Theorem 1.2.7. So, K is a paracompact subset of $\mathbb{M}$ owing to Theorem 1.2.20. Since $\mathbb{M}$ is a countably compact space and K is closed in $\mathbb{M}$. Then, K is a countably compact by Theorem 1.2.10. Now, Theorem 1.2.19 asserts that K is compact, and so, Theorem 1.2.17(ii) justifies that K is a countably compact subset of $\mathbb{M}$. Thus, $\mathcal{L}^{-1}(K)$ is a countably compact set in $\mathbb{W}$ because $\mathcal{L}$ is a countably compact map, and so $\mathcal{L}^{-1}(K)$ is contably paracompact in $\mathbb{W}$ due to Theorem 1.2.9. Hence, $\mathcal{L}$ is a countably paracompact map. ■

Definition 3.1.7.

A map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is known as a metacompact providing the pre-image of any closed and metacompact set in $\mathbb{M}$ is metacompact in $\mathbb{W}$.

Theorem 3.1.8.

Every metacompact map of a Lindelöf and normal space onto a Pa-closed space is paracompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ be a metacompact map such that $\mathbb{W}$ is a Lindelöf and normal space and $\mathbb{M}$ is a Pa-closed space. Suppose that K is a paracompact set in $\mathbb{M}$. Theorem 1.2.25 implies that K is a metacompact subset of $\mathbb{M}$. Since $\mathbb{M}$ is a Pa-closed space, then K is a closed subset of $\mathbb{M}$ by Definition 2.2.1. Thus, $\mathcal{L}^{-1}(K)$ is a metacompact set in $\mathbb{W}$ due to $\mathcal{L}$ is a metacompact map. So, $\mathcal{L}^{-1}(K)$ is a normal subspace in $\mathbb{W}$ owing to Theorem 1.2.6. As a result, $\mathcal{L}^{-1}(K)$ is a countably paracompact subspace of $\mathbb{W}$ owing to Theorem 1.2.23 and $\mathcal{L}^{-1}(K)$ Lindelöf subspace of $\mathbb{W}$ by Theorem 1.2.7. Now, Theorem 1.2.20 asserts that $\mathcal{L}^{-1}(K)$ is a paracompact set in $\mathbb{W}$. Hence, $\mathcal{L}$ is a paracompact map. ■

Theorem 3.1.9.

Every countably paracompact map of a Lindelöf space onto a normal space is a metacompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ be a countably paracompact map, where $\mathbb{W}$ is a Lindelöf space and $\mathbb{M}$ is a normal space. Assume that K is a closed and metacompact set in $\mathbb{M}$. So, K is a normal subspace of $\mathbb{M}$ by Theorem 1.2.6. Then, Theorem 1.2.23 implies that K is a countably paracompact subset of $\mathbb{M}$. Thus, $\mathcal{L}^{-1}(K)$ is a countably paracompact set in $\mathbb{W}$ due to $\mathcal{L}$ is a countably paracompact map. Hence, $\mathcal{L}^{-1}(K)$ is a countably metacompact set in $\mathbb{W}$ due to Theorem 1.2.22. Since K is a closed subset of $\mathbb{M}$, then by Theorem 1.2.7 $\mathcal{L}^{-1}(K)$ is a Lindelöf set in $\mathbb{W}$. Theorem 1.2.27 asserts that $\mathcal{L}^{-1}(K)$ is a metacompact set in $\mathbb{W}$. Hence, $\mathcal{L}$ is a metacompact map. ■

Definition 3.1.10.

A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is known as a countably metacompact providing the pre-image of any closed and countably metacompact set in $\mathbb{M}$ is countably metacompact in $\mathbb{W}$.

Theorem 3.1.11.

Every countably metacompact map of a Lindelöf space is metacompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a countably metacompact map where $\mathbb{W}$ is a Lindelöf space. Suppose that K is a closed and metacompact set in $\mathbb{M}$. So, K is a countably metacompact subset of $\mathbb{M}$ by Theorem 1.2.24. Thus, $\mathcal{L}^{-1}(K)$ is a countably metacompact set in $\mathbb{W}$ due to $\mathcal{L}$ is a countably metacompact map. Indeed, $\mathcal{L}^{-1}(K)$ is closed in $\mathbb{W}$ by the continuity of $\mathcal{L}$. Theorem 1.2.7 asserts that $\mathcal{L}^{-1}(K)$ is a Lindelöf subspace of $\mathbb{W}$ due to $\mathcal{L}^{-1}(K)$ is closed in $\mathbb{W}$. As a result, Theorem 1.2.27 implies that $\mathcal{L}^{-1}(K)$ is a metacompact set in $\mathbb{W}$. Hence, $\mathcal{L}$ is a metacompact map. ■

Corollary 3.1.12.

Every countably metacompact map of a Lindelöf space onto a Pa-closed space is paracompact map.

Proof.

From Theorem 3.1.9 and Theorem 3.1.6. ■

Definition 3.1.13.

A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is said to be a S-paracompact providing the pre-image of any closed and S-paracompact set in $\mathbb{M}$ is S-paracompact in $\mathbb{W}$.

Theorem 3.1.14.

Every S-paracompact map of a Hausdorff completely extremally disconnected space is paracompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a S-paracompact map such that $\mathbb{W}$ is a completely extremally disconnected space. Assume that K is a paracompact set in $\mathbb{M}$.

Theorem 1.2.28 implies that K is S-paracompact set in $\mathbb{M}$. Since $\mathcal{L}$ is an S-paracompact map, then $\mathcal{L}^{-1}(K)$ is a S-paracompact set in $\mathbb{W}$. Since $\mathbb{W}$ is a Hausdorff, completely extremally disconnected space, thus Theorem 1.2.2 asserts that $\mathcal{L}^{-1}(K)$ is extremally disconnected set in $\mathbb{W}$. As a result, $\mathcal{L}^{-1}(K)$ is S-paracompact and extremally disconnected set in $\mathbb{W}$, which implies that $\mathcal{L}^{-1}(K)$ is paracompact in $\mathbb{W}$ by Theorem 1.2.14. Hence, $\mathcal{L}$ is a paracompact map. ■

Definition 3.1.15.

A map $\mathcal{L}$: $\mathbb{W}\rightarrow\mathbb{M}$ is known as a β-paracompact providing the pre-image of any closed and β-paracompact set in $\mathbb{M}$ is β-paracompact in $\mathbb{W}$.

Theorem 3.1.16.

Every β-paracompact map of a Hausdorff, completely extremally disconnected and submaximal space is paracompact map.

Proof.

Let $\mathcal{L}$: $\mathbb{W}\rightarrow\mathbb{M}$ be a β-paracompact map such that $\mathbb{W}$ is a completely extremally disconnected and submaximal space. Assume that K is a paracompact set in $\mathbb{M}$. So, K is β-paracompact by Theorem 1.2.33 Then, $\mathcal{L}^{-1}(K)$ is a β-paracompact set in $\mathbb{W}$ because of $\mathcal{L}$ a β-paracompact map. Added to, $\mathcal{L}^{-1}(K)$ is extremally disconnected subspace of $\mathbb{W}$ by Theorem 1.2.2 and it is submaximal of $\mathbb{W}$ by Theorem 1.2.3. $\mathcal{L}^{-1}(K)$ is paracompact set in $\mathbb{W}$ by Theorem 1.2.34. Hence, $\mathcal{L}$ is a paracompact map. ■

Theorem 3.1.17.

Every β-paracompact map of a Hausdorff, completely extremally disconnected and submaximal space is S-paracompact map.

Proof.

Let $\mathcal{L}$: $\mathbb{W}\rightarrow\mathbb{M}$ be a β-paracompact map such that $\mathbb{W}$ is a Hausdorff completely extremally disconnected space. Assume that K is a closed S-paracompact set in $\mathbb{M}$. Thus, K is β-paracompact subset of $\mathbb{M}$ owing to Theorem 1.2.32. So, $\mathcal{L}^{-1}(K)$ is β-paracompact set in $\mathbb{W}$ because of $\mathcal{L}$ is β-paracompact. Added to, $\mathcal{L}^{-1}(K)$ is extremally disconnected subspace of $\mathbb{W}$ by Theorem 1.2.2 and it is submaximal by Theorem 1.2.3. Theorem 1.2.34. implies that K is paracmpact. Therefore, K is S-paracompact in $\mathbb{W}$ owing to Theorem 1.2.28. Hence, $\mathcal{L}$ is a S-paracompact map. ■

Definition 3.1.18.

A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is known as a fully normal providing the pre-image of any fully normal set in $\mathbb{M}$ is fully normal in $\mathbb{W}$.

Theorem 3.1.19.

Every fully normal map of a T_1-space onto a Hausdorff space is a paracompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a fully normal map such that $\mathbb{W}$ is a T_1-space and $\mathbb{M}$ is a Hausdorff space. Assume that K is a paracompact set in $\mathbb{M}$. Since $\mathbb{M}$ is a Hausdorff space, so Theorem 1.2.4 implies that K is a Hausdorff subspace of $\mathbb{M}$. Thus, K is a fully normal subset of $\mathbb{M}$ by Theorem 1.2.39. So, $\mathcal{L}^{-1}(K)$ is fully normal in $\mathbb{W}$ due to $\mathcal{L}$ is a fully normal. As a result, Theorem 1.2.38 asserts that $\mathcal{L}^{-1}(K)$ is a paracompact set in $\mathbb{W}$. Hence, $\mathcal{L}$ is a paracompact map. ■

Definition 3.1.20.

A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is said to be a fully T_4 providing the pre-image of any fully T_4 set in $\mathbb{M}$ is fully T_4 in $\mathbb{W}$.

Theorem 3.1.21.

Every fully T_4map of a T_1-space is fully normal map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a fully T_4 map such that $\mathbb{W}$ is a T_1- space. Assume that $\mathbb{K}$ is a fully normal set in $\mathbb{M}$. So, $\mathbb{K}$ is a fully T_4 set in $\mathbb{M}$ by Theorem 1.2.37. thus, $\mathcal{L}^{-1}(\mathbb{K})$ is fully T_4 in $\mathbb{W}$ due to $\mathcal{L}$ being a fully T_4 map. Since $\mathbb{W}$ is a T_1- space, then $\mathcal{L}^{-1}(\mathbb{K})$ is fully normal by Theorem 1.2.36. Hence, $\mathcal{L}$ is a fully normal map. ■

Corollary 3.1.22.

Every fully T_4 map of a T_1- space onto a Hausdorff space is paracompact map.

Proof.

From Theorem 1.2.36 and Theorem 1.2.37.

Definition 3.1.23 [26]

A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is called nearly paracompact providing the pre-image of any closed and nearly paracompact set in $\mathbb{M}$ is nearly paracompact in $\mathbb{W}$.

Theorem 3.1.24.

Every nearly paracompact map of a regular space is paracompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ be a nearly paracompact map such that $\mathbb{W}$ is a regular space. Assume that K is a paracompact set in $\mathbb{M}$. Theorem 1.2.29 implies that K is nearly paracompact set in $\mathbb{M}$. Then $\mathcal{L}^{-1}(K)$ is a nearly paracompact set in $\mathbb{W}$ owing to $\mathcal{L}$ is a nearly paracompact map. Since $\mathbb{W}$ is a regular space of $\mathbb{W}$, thus $\mathcal{L}^{-1}(K)$ is a regular subspace by Theorem 1.2.5, so, $\mathcal{L}^{-1}(K)$ is a semiregular subspace due to Theorem 1.2.31. As a result, Theorem 1.2.30 emphasizes $\mathcal{L}^{-1}(K)$ is paracompact in $\mathbb{W}$. Hence, $\mathcal{L}$ is a paracompact map. ■

Theorem 3.1.25.

Every countably compact map of a Lindelöf space onto a Lindelöf, countably compact space and Pa-closed is a paracompact map.

Proof.

From Theorem 1.2.5 and Theorem 1.2.3.

Theorem 4.1.26 [26]

Every Almost countably compact map of a Lindelöf and regular space onto a Hausdorff space is a compact map.

Note: We can replace Hausdorff space in Theorem 3.1.26 by Pa-closed space because every compact subset of Pa-closed is closed.

Corollary 3.1.27.

Every Almost countably compact map of a Lindelöf and regular space onto a countably compact and Pa-closed space is a paracompact map.

Proof.

From Theorem 3.1.26 and Theorem 3.1.5.

Theorem 3.1.28 [26]

Every weakly countably compact map of a Lindelöf and Hausdorff space onto a Hausdorff space is a compact map.

Corollary 3.1.29

Every weakly countably compact map of a Lindelöf and Hausdorff space onto a countably compact and Pa-closed space is paracompact map.

Proof.

From Theorem 3.1.28 and Theorem 3.1.5.

Theorem 3.1.30 [26]

Every sequentially compact map of a Lindelöf space onto a Hausdorff and first-countable space is a compact map.

Corollary 3.1.31

Every sequentially compact map of a Lindelöf space onto a countably compact, First-countable and Pa-closed space is paracompact map.

Proof.

From Theorem 3.1.30 and Theorem 3.1.5.

Theorem 3.1.32 [26]

Every pseudo-compact map of a Lindelöf and regular space onto a Hausdorff space is a compact map.

Corollary 3.1.33.

Every pseudo-compact map of a Lindelöf and regular space onto a countably compact and Pa-closed space is paracompact map.

Proof.

From Theorem 3.1.32 and Theorem 3.1.5.

Theorem 3.1.34. [26]

Every feebly compact map of a Lindelöf and regular space onto a Hausdorff space is a compact map.

Corollary 3.1.35.

Every feebly compact map of a Lindelöf and regular space onto a countably compact and Pa-closed space is paracompact map.

Proof.

From Theorem 3.1.34 and Theorem 3.1.5.

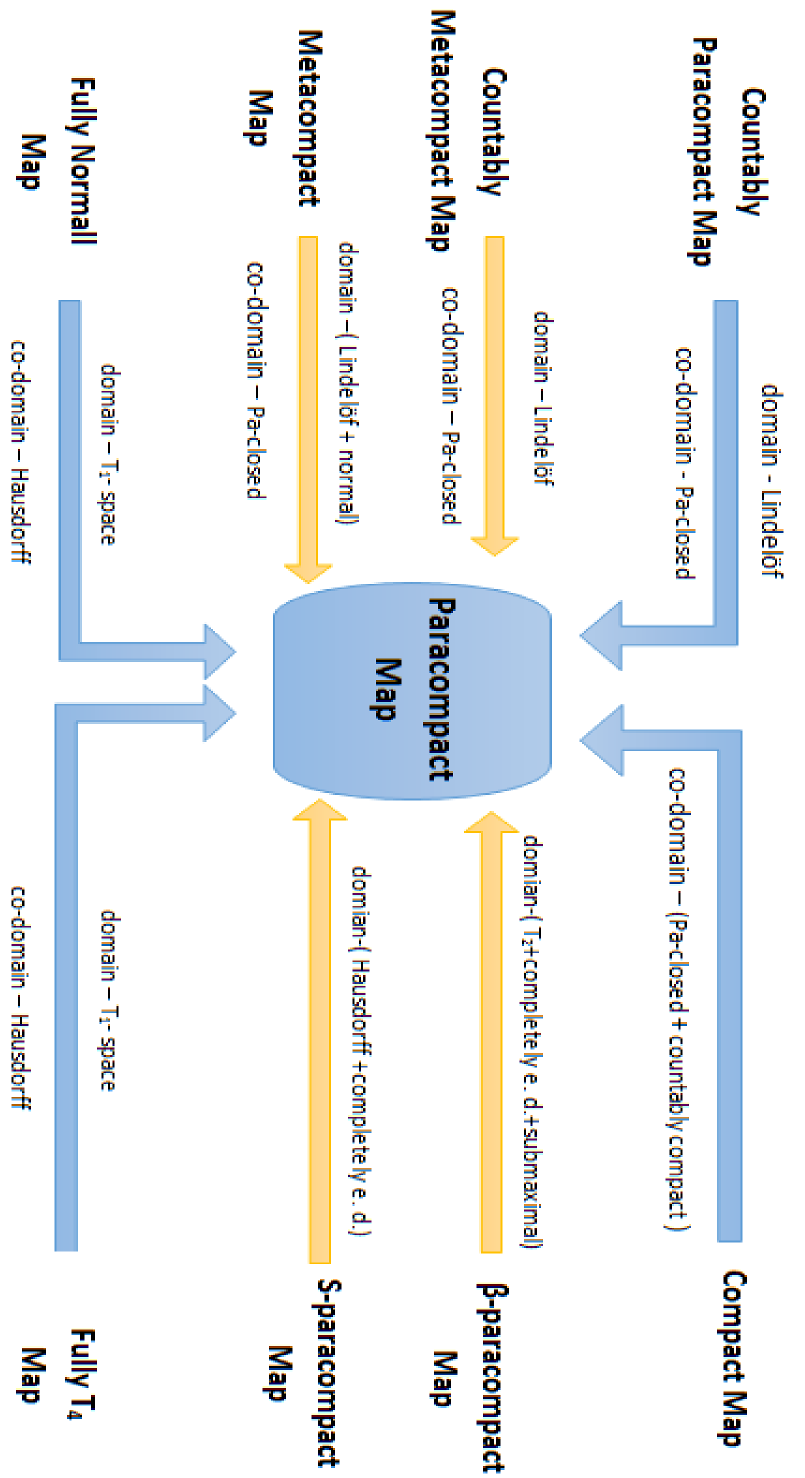

Figure 3.1 Relationships between Certain Types of Strong Paracompact Maps

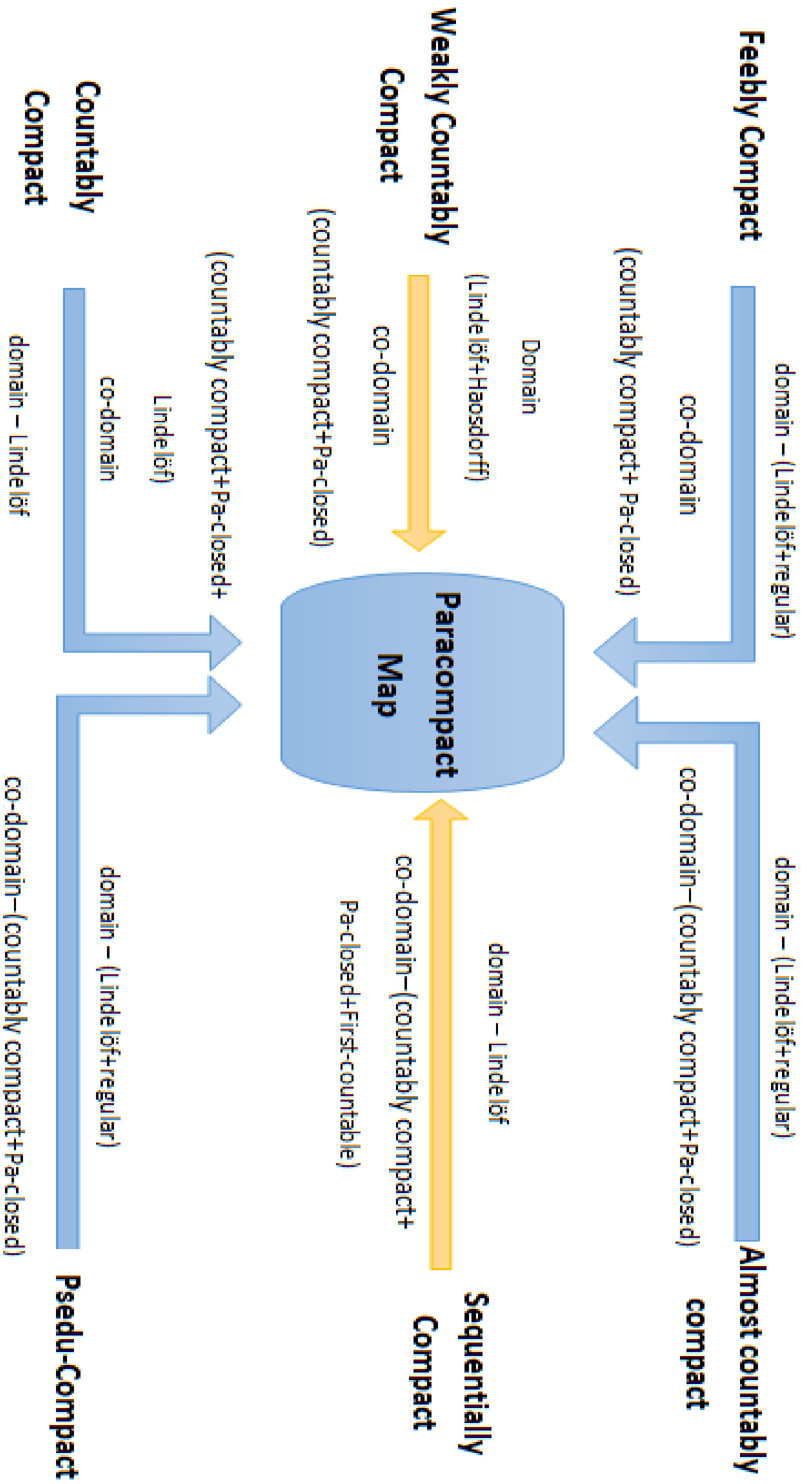

Figure 3.2 Relationships between Certain Types of Strong Compact Maps and Paracompact Map.

3.2 Weak Types of Paracompact Maps

Theorem 3.2.1.

Every paracompact map of a countably compact space onto a Haosdorff space is compact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ be a paracompact map such that $\mathbb{W}$ is a countably compact space and $\mathbb{M}$ is a Hausdorff space. Assume that K is a compact set in $\mathbb{M}$. Then K is a closed in $\mathbb{M}$ by Theorem 1.2.16. Therefore, K is also paracompact due to Theorem 1.2.15. Consequently, $\mathcal{L}^{-1}(K)$ is paraco-mpact set in $\mathbb{W}$ because $\mathcal{L}$ is paracompact map. Indeed, $\mathcal{L}^{-1}(K)$ is a closed in $\mathbb{W}$ by continuity of $\mathcal{L}$. Since $\mathbb{W}$ is a countably compact space, so $\mathcal{L}^{-1}(K)$ is countably compact subspace owing to Theorem 1.2.10. Thus, $\mathcal{L}^{-1}(K)$ is compact in $\mathbb{M}$ because of Theorem 1.2.19. Hence, $\mathcal{L}$ is compact map. ■

Theorem 3.2.2.

Every paracompact map onto a normal and Lindelöf space is a metacompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ be a paracompact map where $\mathbb{M}$ is a normal and Lindelöf space. Assume that K is a closed metacompact set in $\mathbb{M}$. Since $\mathbb{M}$ is normal, so K is a countably paracompact set due to Theorem 1.23 also, we have $\mathbb{M}$ is a Lindelöf then K is Lindelöf by Theorem 1.2.7. So, from Theorem 1.2.20 $\mathcal{L}$ is a paracompact map, then, $\mathcal{L}^{-1}(K)$ is paracompact set in $\mathbb{W}$. Now, by Theorem 1.2.25, $\mathcal{L}^{-1}(K)$ is a metacompact in $\mathbb{W}$. Hence, $\mathcal{L}$ is metacompact map. ■

Corollary 3.2.3.

Every metacompact map onto a Lindelöf space is a countably metaco-mpact map.

Proof.

From Theorem 1.2.27 and Theorem 1.2.24.

Corollary 3.2.4.

Every paracompact map onto a normal and Lindelöf space is a countably metacompact map.

Proof.

From Theorem 3.2.2 and Theorem 3.2.3.

Theorem 3.2.5.

Every paracompact map onto a Lindelöf space is a countably paracompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a paracompact map where $\mathbb{M}$ is a Lindelöf space. Assume that K is a closed contaby paracompact subset of $\mathbb{M}$. Since $\mathbb{M}$ is a Lindelöf space and K is closed,so K is a Lindelöf subspace of $\mathbb{M}$ due to Theorem 1.2.7. Then, K is a paracompact subset of $\mathbb{M}$ due to Theorem 1.2.20. Consequently, $\mathcal{L}^{-1}(K)$ is a paracompact subset of $\mathbb{W}$ due to $\mathcal{L}$ is a paracompact map. Hence, $\mathcal{L}^{-1}(K)$ is a countably paracompact as a result of Theorem 1.2.42. Hence, $\mathcal{L}$ is a countably paracompact map. ■

Theorem 3.2.6.

Every compact map of a Lindelöf space onto a countably compact, normal and a Lindelöf space is a metacompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a compact map where $\mathbb{W}$ is a Lindelöf space and $\mathbb{M}$ is a countably metacompact, normal and Lindelöf space. Assume that K is a closed metacompact subset of $\mathbb{M}$. Then K is normal due to $\mathbb{M}$ is a normal space. Theorem 1.2.23 emphasizes that K is a countably paracompact subspace of $\mathbb{M}$. Consequently, we have $\mathbb{M}$ is Lindelöf thus K is a paracompact set in $\mathbb{M}$ by Theorem 1.2.20. Since $\mathbb{M}$ is a countably compact space then K is a countably compact by Theorem 1.2.10. Then, Theorem 1.2.19 implies that K is a compact. Therefore, $\mathcal{L}^{-1}(K)$ is a compact subset of $\mathbb{W}$ due to $\mathcal{L}$ is a compact map. Theorem 1.2.17(ii) insists that $\mathcal{L}^{-1}(K)$ is a countably compact set in $\mathbb{W}$ and by Theorem 1.2.9 $\mathcal{L}^{-1}(K)$ is a countably paracompact set in $\mathbb{W}$, therefore $\mathcal{L}^{-1}(K)$ is a countably metacompact due to Theorem 1.2.22. But, we have $\mathbb{W}$ is Lindelöf therefore, $\mathcal{L}^{-1}(K)$ is a Lindelöf and countably metacompact. From Theorem 1.2.27 asserts that $\mathcal{L}^{-1}(K)$ is a metacompact. Hence, $\mathcal{L}$ is a metacompact map. ■

Corollary 3.2.7.

Every compact map of a Lindelöf space onto a countably metacompact, normal and Lindelöf space is a countably metacompact map.

Proof.

From Theorem 3.2.6 and Theorem 3.2.4. ■

Theorem 3.2.8.

Every paracompact map onto a Hausdorff and completely extremally disconnected space is an S-paracompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a paracompact map where $\mathbb{M}$ is a Hausdorff and completely extremally disconnected space. To show that $\mathcal{L}$ is an S-paracompact map. Assume that K is a closed S-paracompact subspace of $\mathbb{M}$. Theorem 1.2.2 asserts that K is an extremally disconnected subspace of $\mathbb{M}$, also K is a Hausdorff subspace of $\mathbb{M}$ thus, K is a paracompact subspace of $\mathbb{M}$ due to Theorem 1.2.14. Now, $\mathcal{L}^{-1}(K)$ is paracompact set in $\mathbb{W}$ due to $\mathcal{L}$ being a paracompact map. Theorem 1.2.28 implies that $\mathcal{L}^{-1}(K)$is an S-paracompact subspace of $\mathbb{M}$. Hence, $\mathcal{L}$ is an S-paracompact. ■

Theorem 3.2.9.

Every paracompact map onto a Hausdorff, completely extremally disconnected and submaximal space is β-paracompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a paracompact map where $\mathbb{M}$ is a completely extremally disconnected and submaximal space. Suppose that K is a closed and β-paracompact set in $\mathbb{M}$. Since K is extremally disconnected and submaximal subspace of $\mathbb{W}$ by Theorem 1.2.2 and Theorem 1.2.3, respectively, then K is paracompact set by Theorem 1.2.34. Thus, $\mathcal{L}^{-1}(K)$ is a paracompact set in $\mathbb{W}$ because of $\mathcal{L}$ is a paracompact map. Consequently, $\mathcal{L}^{-1}(K)$ is β-paracompact set in $\mathbb{W}$ for Theorem 1.2.33. Hence, $\mathcal{L}$ is a paracompact map. ■

Theorem 3.2.10.

Every paracompact map of a T_2-space onto is T_1-space is fully normal.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a paracompact map where $\mathbb{W}$ is a T_2-space and $\mathbb{M}$ is T_1-space. Suppose that K is a closed and fully normal set in $\mathbb{M}$. Then K is a paracompact set in $\mathbb{M}$ by Theorem 1.2.38. This implies $\mathcal{L}^{-1}(K)$ is paracompac in $\mathbb{W}$ owing to $\mathcal{L}$ is a paracompact map which follows

$\mathcal{L}^{-1}(K)$ is fully normal because Theorem 1.2.39. Hence, is fully normal map. ■

Theorem 3.2.11.

Every fully normal map onto a T_1-space is fully T_4.

Proof.

Let $\mathcal{L}$: $\mathbb{W}\rightarrow\mathbb{M}$ be a paracompact map where $\mathbb{W}$ is a T_1-space. Assume that K is a closed and fullyT_4 set in $\mathbb{M}$, so K a closed and fully normal set in by Theorem 1.2.36. Consequently, $\mathcal{L}^{-1}(K)$ is a fully normal set in $\mathbb{W}$ due to $\mathcal{L}$ a fully normal map. From Theorem 1.2.37 $\mathcal{L}^{-1}(K)$ is a fully T_4 set in $\mathbb{W}$. Hence, $\mathcal{L}$ is a fully T_4 map. ■

Corollary 3.2.12.

Every paracompact map of a T_2-space onto T_1-space is fully T_4.

Proof.

From Theorem 3.2.11 and Theorem 3.2.10.

Corollary 3.2.13.

Every paracompact map of a countably compact space onto a Lindelöf and Hausdorff space is a countably compact map.

Proof.

From Theorem 3.2.1 and Theorem 1.2.44(1).

Corollary 3.2.14.

Every paracompact map of a countably compact space onto a Hausdorff and Lindelöf space is a weakly countably compact map

Proof.

From Theorem 3.2.1 and Theorem 1.2.44(2).

Theorem 3.2.15.

Every paracompact map of a countably compact space onto a T_3-space and Lindelöf space is an almost countably compact map.

Proof.

From Theorem 3.2.1 and Theorem 2.2.44(3).

Corollary 3.2.16.

Every paracompact map of a countably compact space onto a T_3-space and Lindelöf space is a pseudo-compact map.

Proof.

From Theorem 3.2.1 and Theorem 1.2.44(4).

Corollary 3.2.17.

Every paracompact map of a first-countable and a countably compact space onto a T_3-space and Lindelöf space is a sequentially compact map.

Proof.

From Theorem 3.2.1 and Theorem 1.2.44(5).

Corollary 3.2.18.

Every paracompact map of a countably compact space onto a T_3-space and Lindelöf space is a feebly compact map.

Proof.

From Theorem 3.2.1 and Theorem 1.2.44(6).

Corollary 3.2.19.

Every paracompact map of a countably compact space onto Hausdorff space is a semi-compact map.

Proof.

From Theorem 3.2.1 and Theorem 1.2.44(7).

Corollary 3.2.20.

Every paracompact map of a countably compact space onto Hausdorff and a Lindelöf space is a nearly compact map.

Proof.

From Theorem 3.2.1 and Theorem 1.2.44(8).

Corollary 3.2.21.

Every paracompact map of a countably compact space onto Hausdorff and a Lindelöf space is a po-compact map.

Proof.

From Theorem 3.2.1 and Theorem 1.2.44(9).

Corollary 3.2.22.

Every paracompact map of a countably compact space onto Hausdorff and almost regular space is an almost compact map.

Proof.

From Theorem 3.2.1 and Theorem 1.2.44(10).

Theorem 3.2.23.

Every paracompact map onto a regular space is nearly paracompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a paracompact map such that $\mathbb{M}$ is a regular space. Assume that K is a closed nearly paracompact set in $\mathbb{M}$. Since $\mathbb{M}$ is a regular space, then K is a regular subspace by Theorem 1.2.5, so, $K$is a semiregular subspace of $\mathbb{M}$ due to Theorem 1.2.31. As a result, Theorem 1.2.30 emphasizes K is paracompact in $\mathbb{M}$. Then $\mathcal{L}^{-1}(K)$ is a paracompact set in $\mathbb{W}$ owing to $\mathcal{L}$ is a paracompact map. Thus $\mathcal{L}^{-1}(K)$ is a nearly paracompact by Theorem 1.2.29. Hence, $\mathcal{L}$ is a nearly paracompact map. ■

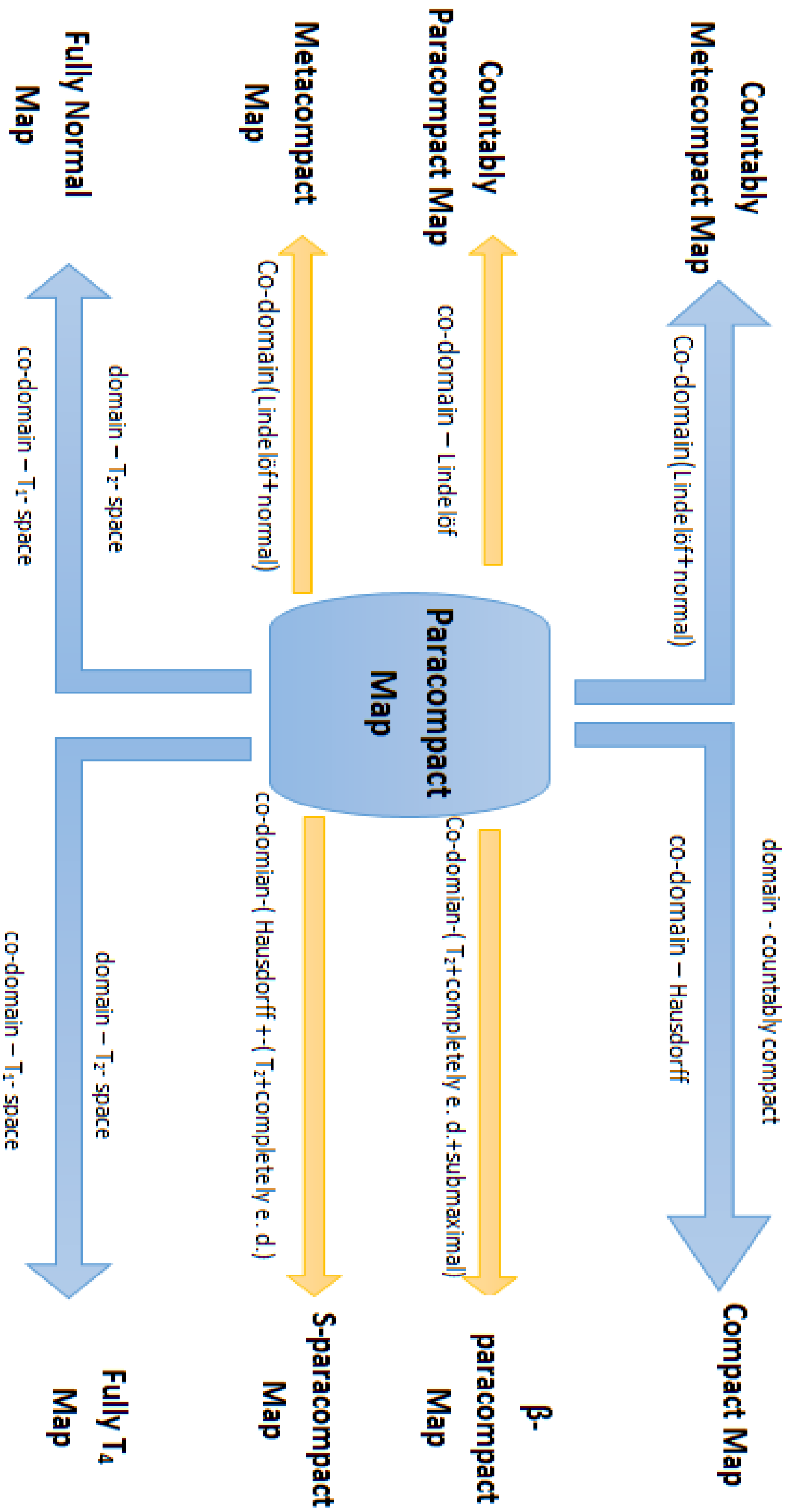

Figure 3.3 Relationships between Certain Types of weaker Paracompact Maps

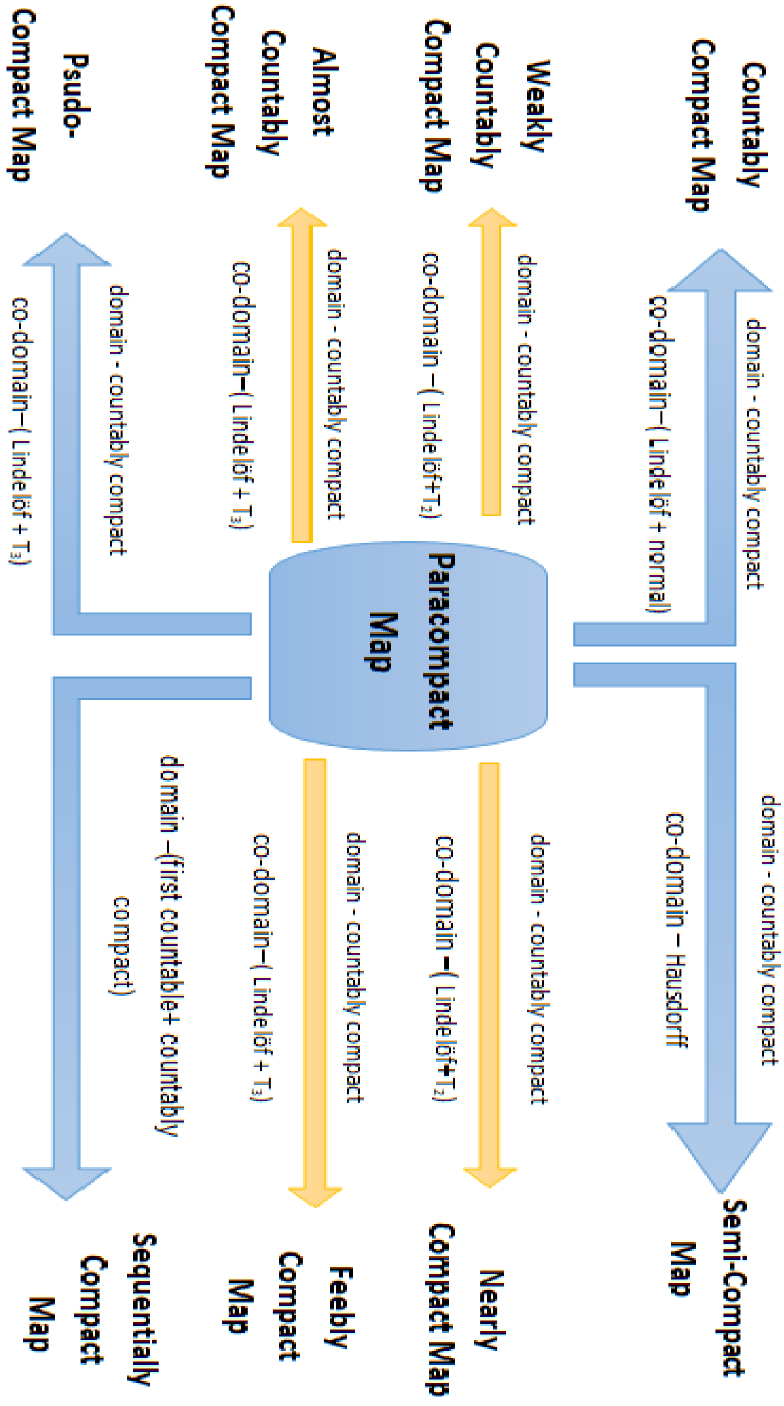

Figure 3.4 Relationships between Certain Types of weekly Compact Maps and Paracompact Map.

3.3 Composition of Certain Types of Paracompact Maps

Theorem 3.3.1.

Let $\mathbb{W}$ be a Pa-closed and compact space and let $\mathbb{M}$ is any space . Then, the continuous image of any paracompact set in $\mathbb{W}$ is parcompact in $\mathbb{M}$.

Proof.

Let $\mathcal{L}\colon \mathbb{W}\to\mathbb{M}$ be a continuous map where $\mathbb{W}$ is a Pa-closed and compact spaces. Suppose that K is a paracompact in $\mathbb{W}$. By Proposition 2.2.4, K is compact, thus, $\mathcal{L}(K)$ is a compact set in $\mathbb{M}$ due to $\mathcal{L}$ is a continuous map. From Theorem 1.2.15, $\mathcal{L}(K)$ is a paracompact. ■

Corollary 3.3.2.

Let $\mathbb{W}$ be a Pa-closed and compact space and let $\mathbb{M}$ is any space. Then, the continuous image of any closed set in $\mathbb{W}$ is paracompact in $\mathbb{M}$.

Theorem 3.3.3.

The composition of paracompact maps is also a paracompact map.

Proof.

Let $\mathcal{L}\colon \mathbb{W}\to\mathbb{M}$ and $\mathcal{J}\colon \mathbb{M}\to\mathbb{E}$ be two paracompact maps. To show that $\mathcal{J}\circ\mathcal{L}$ is also a paracompact map. Assume that K is a paracompact set in $\mathbb{E}$, For demonstrating that$(\mathcal{J}\circ\mathcal{L})^{-1}(\mathrm{K})$ is a paracompact set in $\mathbb{W}$. We have $\mathcal{J}^{-1}(\mathrm{K})$ is a paracompact set in $\mathbb{M}$ since $\mathcal{J}$ is a paracompact map. Thus, $\mathcal{L}^{-1}\left(\mathcal{J}^{-1}(\mathrm{K})\right)$ is a paracompact set in $\mathbb{W}$ due to, $\mathcal{L}$ is a paracompact map, but $\mathcal{L}^{-1}\left(\mathcal{J}^{-1}(\mathrm{K})\right)=(\mathcal{J}\circ\mathcal{L})^{-1}(\mathrm{K})$. So, $(\mathcal{J}\circ\mathcal{L})^{-1}(\mathrm{K})$ is a paracompact set in $\mathbb{W}$. Hence, $\mathcal{J}\circ\mathcal{L}$ is paracompact. ■

Theorem 3.3.4.

Let $\mathbb{M}$ be a Pa-closed compact. If $\mathcal{J}\circ\mathcal{L}:\mathbb{W}\to\mathbb{E}$ is a paracompact map and $\mathcal{J}\colon \mathbb{M}\to\mathbb{E}$ is a continuous injective map, then $\mathcal{L}\colon \mathbb{W}\to\mathbb{M}$ is a paracompact map.

Proof.

Assume that K a paracompact set in $\mathbb{M}$. Since $\mathbb{M}$ is Pa-closed and compact space, then $\mathcal{J}(\mathrm{K})$ is paracompact subspace of $\mathbb{E}$ due to Theorem 3.3.1. Thus, $(\mathcal{J}\circ\mathcal{L})^{-1}(\mathcal{J}(\mathrm{K}))$ is paracompact in $\mathbb{W}$ because $\mathcal{J}\circ\mathcal{L}$ is a paracompact map so, $(\mathcal{J}\circ\mathcal{L})^{-1}(\mathcal{J}(\mathrm{K}))$is paracompact in $\mathbb{W}$. Therefore, $(\mathcal{J}\circ\mathcal{L})^{-1}(\mathcal{J}(\mathrm{K}))=\mathcal{L}^{-1}\left(\mathcal{J}^{-1}(\mathcal{J}(\mathrm{K}))\right)=\mathcal{L}^{-1}(\mathrm{K})$ is a paracompact subspace of $\mathbb{W}$ due to $\mathcal{J}\circ\mathcal{L}$ is paracompact. Hence, $\mathcal{L}$ is a paracompact map. ■

Theorem 3.3.5.

Let $\mathbb{W}$ be a Pa-closed and compact space and. If $\mathcal{J} \circ \mathcal{L} : \mathbb{W} \to \mathbb{E}$ is a paracompact map and $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a continuous surjective map, then $\mathcal{J}: \mathbb{M} \to \mathbb{E}$ is a paracompact map.

Proof.

Suppose that K a paracompact set in $\mathbb{E}$. Since $\mathcal{J} \circ \mathcal{L}$ is a paracompact map then $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is paracompact subspace of $\mathbb{W}$. But $\mathcal{L}$ is a surjective continuous map then, $\mathcal{L}(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K}) = \mathcal{L}\left(\mathcal{L}^{-1}\left(\mathcal{J}^{-1}(\mathrm{K})\right)\right) = \mathcal{J}^{-1}(\mathrm{K})$ is paracompact in $\mathbb{M}$ by Theorem 3.3.1. Hence, $\mathcal{J}$ is a paracompact map. ■

Theorem 3.3.6.

The composition of countably paracompact maps is also a countably paracompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ and $\mathcal{J}: \mathbb{M} \to \mathbb{E}$ be two countably paracompact maps. To show that $\mathcal{J} \circ \mathcal{L}$ is also a countably paracompact map. Assume that K is a closed and countably paracompact set in $\mathbb{E}$, For demonstrating that$(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is a closed and countably paracompact set in $\mathbb{W}$. Since $\mathcal{J}^{-1}(\mathrm{K})$ is a closed countably paracompact set in $\mathbb{M}$ owing to $\mathcal{J}$ is a countably paracompact map. Thus, $\mathcal{L}^{-1}\left(\mathcal{J}^{-1}(\mathrm{K})\right)$ is a countably paracompact set in $\mathbb{W}$ because $\mathcal{L}$ is a countably paracompact map, but $\mathcal{L}^{-1}\left(\mathcal{J}^{-1}(\mathrm{K})\right) = (\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$. So, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is a countably paracompact set in $\mathbb{W}$. Hence, $\mathcal{J} \circ \mathcal{L}$ is contably paracompact. ■

Theorem 3.3.7.

Let $\mathbb{W}$ be a Lindelöf, $\mathbb{M}$ is a Pa-closed compact and $\mathbb{E}$ is Pa-closed . If $\mathcal{J} \circ \mathcal{L} : \mathbb{W} \to \mathbb{E}$ is a countably paracompact map and $\mathcal{J}: \mathbb{M} \to \mathbb{E}$ is a continuous injective map, then $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a paracompact map.

Proof.

Let $\mathcal{J} \circ \mathcal{L} : \mathbb{W} \to \mathbb{E}$ is an countably paracompact map and $\mathcal{J}: \mathbb{M} \to \mathbb{E}$ is a continuous injective map where $\mathbb{M}$ is a Pa-closed compact. Assume that K a paracompact set in $\mathbb{M}$. Then $\mathcal{J}(\mathrm{K})$ is paracompact subspace of $\mathbb{E}$ due to Theorem 3.3.1. Since $\mathbb{W}$ is a Lindelöf and $\mathbb{E}$ is Pa-closed, thus $\mathcal{J} \circ \mathcal{L}$ is a paracompact map by Theorem 3.1.2. Now, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathcal{J}(\mathrm{K})) = \mathcal{L}^{-1}\left(\mathcal{J}^{-1}(\mathcal{J}(\mathrm{K}))\right) = \mathcal{L}^{-1}(\mathrm{K})$ is a paracompact subspace of $\mathbb{W}$ due to $\mathcal{J} \circ \mathcal{L}$ is paracompact. Hence, $\mathcal{L}$ is a paracompact map. ■

Theorem 3.3.8.

Let $\mathbb{W}$ be a Pa-closed and compact space and $\mathbb{E}$ is a Lindelöf space. If $\mathcal{J} \circ \mathcal{L} : \mathbb{W} \rightarrow \mathbb{E}$ is a paracompact map and $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is a continuous surjective map, then $\mathcal{J}: \mathbb{M} \rightarrow \mathbb{E}$ is a countably paracompact map.

Proof.

Let $\mathcal{J} \circ \mathcal{L} : \mathbb{W} \rightarrow \mathbb{E}$ is a paracompact map and $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is a continuous surjective map where $\mathbb{W}$ be a Pa-closed and compact space. Suppose that K a closed countably paracompact set in $\mathbb{E}$. Since $\mathbb{E}$ is a Lindelöf space, then $\mathcal{J} \circ \mathcal{L}$ is a countably paracompact subspace of $\mathbb{E}$ owing to Theorem 3.2.5 which follows $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is a closed countably paracompact set in $\mathbb{W}$. But $\mathbb{W}$ is a compact space, thus $\mathbb{W}$ is a Lindelöf by Theorem 1.2.17(i) which implies $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is Lindelöf and so, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is paracompact. Because $\mathcal{L}$ a surjective continuous map then, $\mathcal{L}(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K}) = \mathcal{L}\left(\mathcal{L}^{-1}(\mathcal{J}^{-1}(\mathrm{K}))\right) = \mathcal{J}^{-1}(\mathrm{K})$ is paracompact in $\mathbb{M}$ by Theorem 3.3.1 therefore, $\mathcal{J}^{-1}(\mathrm{K})$ is a countably paracompact by Theorem 1.2.42. Hence, $\mathcal{L}$ is a cuontably paracompact map. ■

Theorem 3.3.9.

The composition of S-paracompact maps is also an S-paracompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ and $\mathcal{J}: \mathbb{M} \rightarrow \mathbb{E}$ be two S-paracompact maps. To show that $\mathcal{J} \circ \mathcal{L}$ is also a S-paracompact map. Assume that K is a closed and S-paracompact set in $\mathbb{E}$, For demonstrating that$(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is a closed and S-paracompact set in $\mathbb{W}$. Since $\mathcal{J}^{-1}(\mathrm{K})$ is a closed and S-paraco-mpact set in $\mathbb{M}$ owing to $\mathcal{J}$ is an S-paracompact map. Thus, $\mathcal{L}^{-1}(\mathcal{J}^{-1}(\mathrm{K}))$ is an S-paracompact set in $\mathbb{W}$ because $\mathcal{L}$ is an S-paracompact map, but $\mathcal{L}^{-1}(\mathcal{J}^{-1}(\mathrm{K})) = (\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$. So, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is a S-paracompact set in $\mathbb{W}$. Hence, $\mathcal{J} \circ \mathcal{L}$ is S-paracompact. ■

Theorem 3.3.10.

Let $\mathbb{W}$ be a Hausdorff completely extermally disconnected and $\mathbb{M}$ is a Pa-closed compact. If $\mathcal{J} \circ \mathcal{L} : \mathbb{W} \rightarrow \mathbb{E}$ is an S-paracompact map and $\mathcal{J}: \mathbb{M} \rightarrow \mathbb{E}$ is a continuous injective map, then $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is a paracompact map.

Proof.

Let $\mathcal{J} \circ \mathcal{L} : \mathbb{W} \rightarrow \mathbb{E}$ is an S-paracompact map and $\mathcal{J}: \mathbb{M} \rightarrow \mathbb{E}$ is a continuous injective map where $\mathbb{M}$ is a Pa-closed compact. Assume that K a paracompact set in $\mathbb{M}$. Then $\mathcal{J}(\mathrm{K})$ is paracompact subspace of $\mathbb{E}$ due to Theorem 3.3.1. Since $\mathbb{W}$ is a Hausdorff and completely e.d. space, thus $\mathcal{J} \circ \mathcal{L}$ is a paracompact map by Theorem 3.1.12. Now, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathcal{J}(\mathrm{K}))$ $= \mathcal{L}^{-1}(\mathcal{J}^{-1}(\mathcal{J}(\mathrm{K}))) = \mathcal{L}^{-1}(\mathrm{K})$ is a paracompact subspace of $\mathbb{W}$ due to $\mathcal{J} \circ \mathcal{L}$ is paracompact. Hence, $\mathcal{L}$ is a paracompact map. ■

Theorem 3.3.11.

Let $\mathbb{W}$ be a Pa-closed compact space, and $\mathbb{E}$ is a completely externally disconnected and Hausdorff space. If $\mathcal{J} \circ \mathcal{L}: \mathbb{W} \rightarrow \mathbb{E}$ is a paracompact map and $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is a continuous surjective map, then $\mathcal{J}: \mathbb{M} \rightarrow \mathbb{E}$ is an S-paracompact map.

Proof:

Let $\mathcal{J} \circ \mathcal{L}: \mathbb{W} \rightarrow \mathbb{E}$ is a paracompact map and $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is a continuous surjective map. Suppose that $\mathbb{K}$ is a closed S-paracompact set in $\mathbb{E}$. Since $\mathbb{E}$ is a completely e.d. and Hausdorff space, then $\mathbb{K}$ is a Hausdorff extermally disconnected subspace owing to Theorem 1.2.2 which follows $\mathbb{K}$ is a paracompact set in $\mathbb{E}$ by Theorem 1.2.34. Thus $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathbb{K})$ is a closed paracompact set in $\mathbb{W}$. But $\mathbb{W}$ is a Pa-closed compact space, thus $\mathcal{L}(\mathcal{J} \circ \mathcal{L})^{-1}(\mathbb{K}) = \mathcal{L}(\mathcal{L}^{-1}(\mathcal{J}^{-1}(\mathbb{K}))) = \mathcal{J}^{-1}(\mathbb{K})$ is a paracompact subset of $\mathbb{M}$ by Theorem 3.3.1. Therefore, $\mathcal{J}^{-1}(\mathbb{K})$ is an S-paracompact set due to Theorem 1.2.28. Hence, $\mathcal{J}$ is an S-paracompact map. ■

As a direct consequence of using similar arguments as in Theorem 3.3.11, the following results are established:

Theorem 3.3.12.

The composition of β-paracompact maps is also a β-paracompact map.

Corollary 3.3.13.

Let $\mathbb{W}$ be a Hausdorff completely e.d. and submaximal and $\mathbb{M}$ is a Pa-closed compact. If $\mathcal{J} \circ \mathcal{L}: \mathbb{W} \rightarrow \mathbb{E}$ is a β-paracompact map and $\mathcal{J}: \mathbb{M} \rightarrow \mathbb{E}$ is a continuous injective map, then $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is a paracompact map. ■

Proof.

By Theorem 3.3.1 and Theorem 3.1.14.

Chapter Four

On Paracompact Action

4.1 Paracompact Action

This Chapter focuses on introducing a paracompact action map. Some theorems relating to revised properties of paracompact action map and paracompact G-space are derived. Indeed, various types of paracompact action are presented, which are nearly paracompact, compact, fully normal and metacompact.

Definition 4.1.1

A $\mathbb{G}$-space $\mathbb{W}$ is said to be paracompact $\mathbb{G}$-space if the map $\ell: \mathbb{G} \times \mathbb{W} \to \mathbb{W} \times \mathbb{W}$ which is defined by $\ell(g,x) = (x, g.x)$ for each $(g,x) \in \mathbb{G} \times \mathbb{W}$, is a paracompact map, and the action $\varphi: \mathbb{G} \times \mathbb{W} \to \mathbb{W}$ is called paracompact**.** The triple $(\mathbb{G}, \mathbb{W}, \varphi)$ is called a (left) topological transformation group and $\mathbb{W}$ is called a paracompact $\mathbb{G}$–space on $\mathbb{G}$**.**

Example 4.1.2

Let $(\mathbb{Z}, \tau_D, +)$ be a topological group and $(\mathbb{W}, \tau_D)$ is a topological space, where $\mathbb{W}$ is a finite set and τ_D is a discrete topology, then the map $\varphi: \mathbb{Z} \times \mathbb{W} \to \mathbb{W}$ is paracompact action**.**

Proposition 4.1.3

Let $\mathbb{W}$ and $\mathbb{M}$ be a compact Pa-closed space, then $\mathbb{W} \times \mathbb{M}$ is a Pa-closed space.

Proof.

Assume that $A \times B$ is paracompact in $\mathbb{W} \times \mathbb{M}$. Since $\mathbb{W}$ is compact, then by Proposition 1.2.54, $\mathbb{Pr}_2: \mathbb{W} \times \mathbb{M} \to \mathbb{M}$ is proper. Since $\mathbb{Pr}_2$ is a surjective open map, then by Proposition 2.1.12, $\mathbb{Pr}_2(A \times B) = A$ is paracompact in $\mathbb{M}$. But $\mathbb{M}$ is Pa-closed, then A is closed. Thus, ${\mathbb{Pr}_2}^{-1}(A) = A \times \mathbb{M}$ is a closed set in $\mathbb{W} \times \mathbb{M}$. In the same way, we show that $\mathbb{W} \times B$ is closed in $\mathbb{W} \times \mathbb{M}$, therefore, $(A \times \mathbb{M}) \cap (\mathbb{W} \times \mathrm{B}) = \mathrm{A} \times \mathrm{B}$ is closed in $\mathbb{W} \times \mathbb{M}$. Hence, $\mathbb{W} \times \mathbb{M}$ is a Pa-closed space. ■

Proposition 4.1.4.

Let $\mathbb{W}$ be a compact Pa-closed and $\mathbb{G}$ is a paracompact space. Then $\mathbb{W}$ is paracompact $\mathbb{G}$-space.

Proof.

Let $\mathbb{W}$ be a compact Pa-closed and $A \times B$ is a paracompact set in $\mathbb{W} \times \mathbb{W}$. By Proposition 4.1.3, $\mathbb{W} \times \mathbb{W}$ is Pa-closed, thus $A \times B$ is a closed set. Since $\ell: \mathbb{G} \times \mathbb{W} \to \mathbb{W} \times \mathbb{W}$ is continuous, then $\ell^{-1}(A \times B)$ is closed in

$\mathbb{G} \times \mathbb{W}$. By Proposition 1.2.52 $\mathbb{G} \times \mathbb{W}$ is paracompact, therefore, $\ell^{-1}(A \times B)$ is paracompact. Hence, $\mathbb{W}$ is paracompact $\mathbb{G}$-space. ■

Proposition 4.1.5.

Let $\mathbb{W}$ be a $\mathbb{G}$-space, Then :

(i) If the action $\varphi: \mathbb{G} \times \mathbb{W} \rightarrow \mathbb{W}$ is a paracompact map and F is a closed (compact) subset of $\mathbb{W}$ and $\mathbb{G}$ is a compact topological group, then $\mathbb{G}.\mathrm{F} = \{g.x | g \in \mathbb{G}, x \in \mathrm{F}\}$ is a closed (compact) set in $\mathbb{W}$.

(ii) If the action $\varphi: \mathbb{G} \times \mathbb{W} \rightarrow \mathbb{W}$ is an injective paracompact map and F is a paracompact subset of $\mathbb{W}$ and $\mathbb{G}$ is a compact topological group, then $\mathbb{G}.\mathrm{F} = \{g.x | g \in \mathbb{G}, x \in \mathrm{F}\}$ is a paracompact set in $\mathbb{W}$.

Proof.

(i) Let F be a closed (compact) subset of $\mathbb{W}$, then $\mathbb{G} \times \mathrm{F}$ is a closed (compact) subset of $\mathbb{G} \times \mathbb{W}$. Since $\mathbb{G}.\mathrm{F} = \varphi(\mathbb{G} \times \mathrm{F})$and φ is closed and continuous, then $\mathbb{G}.\mathrm{F}$ is a closed (compact) subset of $\mathbb{W}$.

(ii) Let F be a paracompact in $\mathbb{W}$, then by Proposition 1.2.52, $\mathbb{G} \times \mathrm{F}$ is paracompact in $\mathbb{G} \times \mathbb{W}$. Since φ is an injective paracompact map, then by Proposition 2.1.2, φ is proper. But φ clear is a surjective and open, thus, by Corollary 2.1.3, $\varphi(\mathbb{G} \times \mathrm{F}) = \mathbb{G}.\mathrm{F}$ is a paracompact set in $\mathbb{W}$.

Theorem 4.1.6.

Let $\mathbb{W}$ be a paracompact $\mathbb{G}$-space and $\mathbb{G}$ is freely on $\mathbb{W}$. If H is a closed subset of $\mathbb{G}$ and B is an open subset of $\mathbb{W}$ which is invariant under A, then B is paracompact H-space.

Proof.

Assume that $\mathbb{W}$ is a paracompact $\mathbb{G}$ $-$space and $\mathbb{G}$ is freely on $\mathbb{W}$, then the map $\ell: \mathbb{G} \times \mathbb{W} \longrightarrow \mathbb{W} \times \mathbb{W}$ which is defined by $\ell(g, x) = (x, g.x)$. For each $(g, x) \in \mathbb{G} \times \mathbb{W}$, is an injective paracompact map, thus by Proposition 2.1.2, ℓ is proper We must prove the map $\mathcal{S}: H \times B \longrightarrow B \times B$ which is defined by $\mathcal{S}(h, y) = (y, h.y)$

For each $(h, y) \in H \times B$, is a paracompact map. Let $\{(h_d, y_d)\}_{d \in D}$ be a net in $H \times B$ such that $\mathcal{S}(\{(h_d, y_d)\}_{d \in D}) \propto (x, y)$ for some $(x, y) \in B \times B$. Then $\{(y_d, h_d.y_d)\} \propto (x, y)$ in $B \times B$. Let U be an open in $\mathbb{W} \times \mathbb{W}$, such that$(x, y) \in U$, then $U \cap (B \times B)$ it is open in $\mathbb{W} \times \mathbb{W}$. But $B \times B$ is an open in $\mathbb{W} \times \mathbb{W}$, then U is open in $U \cap (B \times B)$. Now, $(x, y) \in U \cap (B \times B)$ and $\{(y_d, h_d.y_d)\} \propto (x, y)$, thus, $\{(y_d, h_d.y_d)\}$ is frequently in $U \cap (B \times B)$and then $\{(y_d, h_d.y_d)\}$ is frequently in U. Thus, $\{(y_d, h_d.y_d)\} \propto (x, y)$ in $\mathbb{W} \times \mathbb{W}$. Since ℓ is a proper map, then by

Theorem 1.2.46, there exists $(h, x_1) \in \mathbb{G} \times \mathbb{W}$ such that $(h_d, y_d) \propto (h, x_1)$ and $\ell(h, x_1) = (x, y)$, thus $(x_1, h \,.\, x_1) = (x, y)$, that is $x_1 = x$. Therefore, $h_d \propto h$, since $\{h_d\}$ is a net in H, then $h \in \overline{H} = H \Longrightarrow h \in H$. Thus, by Theorem 1.2.46, $\mathcal{S}: H \times B \longrightarrow B \times B$ is proper, by Proposition 2.1.2, $\mathcal{S}$ is paracompact. ■

Since $\mathbb{G}$ is a closed subset of itself and $\mathbb{W}$ is an open subset of itself, then:

Corollary 4.1.7.

Let $\mathbb{W}$ be a paracompact $\mathbb{G}$-space and $\mathbb{G}$ is freely on $\mathbb{W}$. If B is an open subset of $\mathbb{W}$ which is invariant under $\mathbb{G}$, then B is paracompact $\mathbb{G}$ -space.

Corollary 4.1.8.

Let $\mathbb{W}$ be a paracompact $\mathbb{G}$-space and $\mathbb{G}$ is freely on $\mathbb{W}$. If H is a closed subset of $\mathbb{G}$ then $\mathbb{W}$ is paracompact H-space.

Proposition 4.1.9.

Let $(\mathbb{G}_1, \mathbb{W}, \varphi_1)$ and $(\mathbb{G}_2, \mathbb{W}, \varphi_2)$ be two topological transformation groups such that $\mathcal{J}: \mathbb{G}_1 \to \mathbb{G}_2$ is a bijective open proper map onto group homomorphism and $\mathcal{L}: \mathbb{W} \longrightarrow \mathbb{M}$ is an equivariant map. If $\mathcal{L}$ is a surjective open proper map, and $\mathbb{W}$ is paracompact $\mathbb{G}_1$-space, then $\mathbb{M}$ is a paracompact $\mathbb{G}_2$-space.

Proof.

Let $\ell_1: \mathbb{G}_1 \times \mathbb{W} \longrightarrow \mathbb{W} \times \mathbb{W}$ and $\ell_2: \mathbb{G}_2 \times \mathbb{M} \longrightarrow \mathbb{M} \times \mathbb{M}$ be the maps that are defined by $\ell_1(g_1, x) = (x, g_1.x)$ and $\ell_2(g_2, x) = (x, g_2.x)$ for each $(g_1, x) \in \mathbb{G} \times \mathbb{W}$ and $(g_2, x) \in \mathbb{G} \times \mathbb{M}$. Consequently, the following diagram can be realized based on the maps $\mathcal{L}$, $\mathcal{J}$, ℓ_1, and ℓ_2.

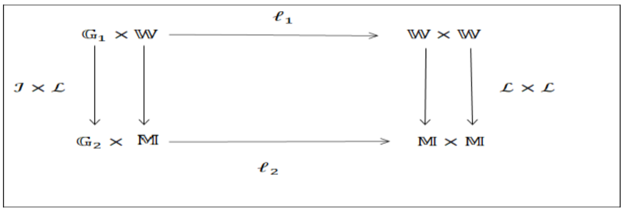

Figure 4.1 First diagram on maps

Let $(g_1, x) \in \mathbb{G} \times \mathbb{W}$, then according to the above diagram:

$[(\mathcal{L} \times \mathcal{L}) o \ell_1]\ (g_1, x) = (\mathcal{L} \times \mathcal{L})\ \big(x, \varphi_1(g_1, x)\big)$

$=(\mathcal{L}(\mathrm{x}), \mathcal{L}(\varphi_1(g_1, x))) = (\mathcal{L}(x), \varphi_2(\mathcal{J}(g_1), \mathcal{L}(x)))$
$=[\ell_2 o(\mathcal{J} \times \mathcal{L})](g_1, x)$.
So the diagram is commutative. Since $\mathcal{L}: \mathbb{W} \longrightarrow \mathbb{M}$ is a surjective proper, then $\mathcal{L} \times \mathcal{L}$ is a bijective proper map. By Proposition 2.1.2, $\mathcal{L} \times \mathcal{L}$ is a paracompact map. Since $\ell_1$is paracompact, then by Proposition 2.1.8(i), $(\mathcal{L} \times \mathcal{L})o\ell_1$ is paracompact. Thus, $\ell_2 o(\mathcal{J} \times \mathcal{L})$ is a paracompact map. But, by Proposition 2.1.8(iv), ℓ_2 is paracompact map. Hence, $\mathbb{M}$ is a paracompact $\mathbb{G}_1$-space. ■

Proposition 4.1.10.

Let $\mathbb{W}$ be a paracompact $\mathbb{G}$-space, $x \in \mathbb{W}$ such that $\{x\}$ is open and $H = \{x\} \times \mathbb{W}$, then $\ell_H: \ell^{-1}(H) \longrightarrow H$ is a paracompact map, where $\ell: \mathbb{G} \times \mathbb{W} \rightarrow \mathbb{W} \times \mathbb{W}$ and $\ell(g, x) = (x, g.x)$, for each $(g, x) \in \mathbb{G} \times \mathbb{W}$.

Proof.

Let $(x, y) \in \{x\} \times \mathbb{W}$ and $\mathbb{U} = \{U_\alpha \times V_\beta | \alpha \in I, \beta \in \Delta\}$ is an open cover of $\ell^{-1}(\{(x, y)\})$. Since ℓ is the Paracompact map than by Proposition 2.1.1, there exists a neighborhood $D_{(x,y)}$ of (x, y) such that $\{(U_\propto \times V_\beta) \cap \ell^{-1}(D_{(x,y)})\}_{\alpha \in I, \beta \in \Delta}$ has an open refinement $\mathbb{V}$ such that $\mathbb{V}$ is locally finite at $\ell^{-1}(\{(x, y)\})$. Since $\{U_\alpha \times \{x\} | \alpha \in I\}$ is an open cover of $\ell^{-1}(\{(x, y)\})$ in $\mathbb{G} \times \{x\}$, then it has a locally finite open refinement at $\ell^{-1}(\{(x, y)\})$. Since $\ell_H: \ell^{-1}(H) = \mathbb{G} \times \{x\} \rightarrow H$ is continuous and surjective map, then ℓ_H is paracompact. ■

Lemma 4.1.11.

The quotient map $P: \mathbb{W} \longrightarrow \mathbb{W}/\mathbb{G}$ is open.

Proof.

Let A be an open set in $\mathbb{W}$. Then $P^{-1}(P(A)) = \cup_{g \in \mathbb{G}} A_g$, which is a union of open sets. Thus, $P(A)$ is open. ■

Proposition 4.1.12.

Let $\mathbb{W}$ be a paracompact $\mathbb{G}$-space with the action $\varphi: \mathbb{G} \times \mathbb{W} \rightarrow \mathbb{W}$, $\varphi(g, x) = g.x$, $\forall (g, x) \in \mathbb{G} \times \mathbb{W}$. Then for each $x \in \mathbb{W}$, such that $\{x\}$ is open.

- **(i)** The map $\varphi_x: \mathbb{G} \longrightarrow \mathbb{W}$ which is defined by $\varphi_x(g) = \varphi(g, x)$, is paracompact map.
- **(ii)** The stabilizer subgroup $\mathbb{G}_x$ of $\mathbb{G}$ at x, is paracompact.
- **(iii)** The map $\mathcal{L}: \mathbb{G}/\mathbb{G}_x \longrightarrow \mathbb{G}.x$ is paracompact, where $\mathbb{G}$ is compact Pa-closed space.

(iv) The orbit at x, $\mathbb{G}.x$ is closed.

Proof.

(i) Let $H = \{x\} \times \mathbb{W} \subseteq \mathbb{W} \times \mathbb{W}$, then by Proposition 4.1.10, $\ell_H : \ell^{-1}(H) \longrightarrow H$ is a paracompact map, where $\ell : \mathbb{G} \times \mathbb{W} \longrightarrow \mathbb{W} \times \mathbb{W}$ and $\ell(g, x) = (x, g.x)$, for each $(g, x) \in \mathbb{G} \times \mathbb{W}$. Now, we can write $\varphi_x : \mathbb{G} \overset{f}{\cong} \mathbb{G} \times \{x\} \xrightarrow{\ell_H} \{x\} \times \mathbb{W} \overset{h}{\cong} \mathbb{W}$ such that f and h are homeomorphisms. Let U be a paracompact set in $\mathbb{W}$, then $h^{-1}(U) = \{x\} \times U$, $x \in \mathbb{W}$. Since ℓ_H is a paracompact map, then, ${\ell_H}^{-1}\left(h^{-1}(U)\right) = {\ell_H}^{-1}(\{x\} \times U)$ is a paracompact set in $\mathbb{G} \times \{x\}$, but f is a homeomorphism, then $f^{-1}({\ell_H}^{-1}\left(h^{-1}(U)\right))$ is paracompact in $\mathbb{G}$. Since $f^{-1}({\ell_H}^{-1}\left(h^{-1}(U)\right)) = {\varphi_x}^{-1}(U)$, then a map $\varphi_x : \mathbb{G} \longrightarrow \mathbb{W}$ is paracompact.

(ii) By (i), $\varphi_x : \mathbb{G} \longrightarrow \mathbb{W}$ is a paracompact map, since $\{x\}$ is a paracompact set for each $x \in \mathbb{W}$. Then, ${\varphi_x}^{-1}(\{x\})$ is paracompact. But ${\varphi_x}^{-1}(\{x\}) = \mathbb{G}_x$, hence $\mathbb{G}_x$ is a paracompact set in $\mathbb{G}$.

(iii) Let $\psi : \mathbb{G} \longrightarrow \mathbb{G}/\mathbb{G}_x$ be a quotient map, then ψ is well defined continuous and surjective. We will define the map $\mathcal{L} : \mathbb{G}/\mathbb{G}_x \longrightarrow \mathbb{G}.x$, by $\mathcal{L}(g\,\mathbb{G}_x) = g.x$, $\forall g \in \mathbb{G}$ .To prove that $\mathcal{L}$ is an injective map. Let $g_1\mathbb{G}_x, g_2\mathbb{G}_x \in \mathbb{G}/\mathbb{G}_x$ such that $\mathcal{L}(g_1\mathbb{G}_x) = \mathcal{L}(g_2\mathbb{G}_x)$. Then, $g_1.x = g_2.x$, this implies that ${g_2}^{-1}(g_1.x) = x$. Hence, $({g_2}^{-1}g_1).x = x$. So that, $({g_2}^{-1}g_1).x = x$ leads to ${g_2}^{-1}g_1 \in \mathbb{G}_x$. Thereby, $g_1\mathbb{G}_x = g_2\mathbb{G}_x$. Therefore, $\mathcal{L}$ is an injective map.

Now, $$\underset{\varphi_x}{\overline{\mathbb{G} \xrightarrow{\psi} \mathbb{G}/\mathbb{G}_x \xrightarrow{\mathcal{L}} \mathbb{G}.x}}$$

$\mathcal{L}o\psi \equiv (\varphi_x)_{\mathbb{G}.x}$, where $(\varphi_x)_{\mathbb{G}.x} : {\varphi_x}^{-1}(\mathbb{G}.x) \longrightarrow \mathbb{G}.x$. Since φ_x is injective, and $\mathbb{G}.x$ is open then by Proposition 4.1.10, $(\varphi_x)_{\mathbb{G}.x}$ is paracompact. Since $\mathbb{G}$ is a Pa-closed and $\mathbb{G}_x$ is paracompact, then $\mathbb{G}_x$ is closed, therefore by Lemma 4.1.12, $\mathbb{G}/\mathbb{G}_x$ is Hausdorff. By Lemma 4.1.11 and Proposition 1.2.52, ψ is an open proper map. Thus, by Proposition 2.1.13, $\mathcal{L}$ is paracompact.

(iv) Since $\varphi_x : \mathbb{G} \rightarrow \mathbb{W}$ is a paracompact, then by Proposition 2.1.2, φ_x is a closed map. Clear, $\mathbb{G}$ is closed set in $\mathbb{G}$, therefore $\varphi_x(\mathbb{G})$ is a closed set in $\mathbb{W}$. But, $\varphi_x(\mathbb{G}) = \mathbb{G}.x$, hence $\mathbb{G}.x$ is a closed set in $\mathbb{W}$. ■

Proposition 4.1.13.

Let $\mathbb{W}$ be a $\mathbb{G}$-space, such that $\mathbb{G}$ acts freely on $\mathbb{W}$. Then $\mathbb{W}$ is paracompact $\mathbb{G}$-space if and only if the graph $\mathbb{G}R(R)$ of the equivalence relation R which is defined by $\mathbb{G}$ on $\mathbb{W}$ is a closed set in $\mathbb{W} \times \mathbb{W}$ and the map $\Phi: \mathbb{G}R(R) \longrightarrow \mathbb{G}$ which is defined by $\Phi(\mathrm{x}, \mathrm{y}) = g$, such that $g.x = y$, is a continuous map.

Proof.

Let $\mathbb{W}$ be a paracompact $\mathbb{G}$-space. Then the map $\ell: \mathbb{G} \times \mathbb{W} \rightarrow \mathbb{W} \times \mathbb{W}$ is a paracompact map, by Proposition 2.1.2, ℓ is closed. Thus, $\ell(\mathbb{G} \times \mathbb{W})$ is a closed set in $\mathbb{W} \times \mathbb{W}$. But, $\ell(\mathbb{G} \times \mathbb{W}) = \mathbb{G}R(R)$, then $\mathbb{G}R(R)$ is closed set in $\mathbb{W} \times \mathbb{W}$. Since $\mathbb{G}$ acts freely on $\mathbb{W}$, then by Remarks 1.3.8 (ii), the map ℓ is injective, thus by Proposition 2.1.2, ℓ is a homeomorphism from $(\mathbb{G} \times \mathbb{W})$ onto a closed set in $\mathbb{W} \times \mathbb{W}$. Let $\hat{\ell}: (\mathbb{G} \times \mathbb{W}) \longrightarrow \mathbb{G}R(R)$, which is defined by $\hat{\ell}(g, x) = (g, x)$, for each $(g, x) \in \mathbb{G} \times \mathbb{W}$. Therefore, $\hat{\ell}$ is a homeomorphism map and so, $\hat{\ell}$ is bijective. Thus, $\hat{\ell}$ has inverse map $\hat{\ell}^{-1}: \mathbb{G}R(R) \longrightarrow (\mathbb{G} \times \mathbb{W})$, which is defined by: $\hat{\ell}^{-1}(g, x) = (\Phi(\mathrm{x}, \mathrm{y}), \mathrm{x})$, for each $(x, y) \in \mathbb{G}R(R)$, that is, $\hat{\ell}^{-1} \equiv \Phi\Delta\mathbb{P}\mathbb{r}_1$, such that $\mathbb{P}\mathbb{r}_1: \mathbb{G}R(R) \longrightarrow \mathbb{W}$ is a projection map, $\mathbb{P}\mathbb{r}_1(x, y) = x$, for each $(x, y) \in \mathbb{G}R(R)$. Since $\hat{\ell}$ is a homeomorphism, then $\hat{\ell}$ is an open map, i.e., $\hat{\ell}^{-1}$ is continuous, thus, $\Phi\Delta\mathbb{P}\mathbb{r}_1$ is a continuous map. Now, we must prove that Φ is continuous. Let U be an open set in $\mathbb{G}$, then $U \times \mathbb{W}$ is an open in $\mathbb{G} \times \mathbb{W}$, thus, $(\Phi\Delta\mathbb{P}\mathbb{r}_1)^{-1}(U \times \mathbb{W}) = \Phi^{-1}(U) \cap [\mathbb{P}\mathbb{r}_1{}^{-1}(\mathbb{W}) \cap \mathbb{G}R(R) = \Phi^{-1}(U) \cap [(\mathbb{W} \times \mathbb{W}) \cap \mathbb{G}R(R)] = \Phi^{-1}(U) \cap \mathbb{G}R(R) = \Phi^{-1}(U)$. Therefore, $\Phi^{-1}(U)$ is an open in $\mathbb{G}R(R)$. Hence $\Phi: \mathbb{G}R(R) \longrightarrow \mathbb{G}$ is a continuous map.

Conversely, Since $\ell: \mathbb{G} \times \mathbb{W} \rightarrow \mathbb{W} \times \mathbb{W}$ is a continuous and injective map, then $\hat{\ell}: (\mathbb{G} \times \mathbb{W}) \longrightarrow \mathbb{G}R(R)$ is a continuous and bijective map. Now, to prove that $\hat{\ell}$ is an open map. Let $U \times V$ be an open subset of $\mathbb{G} \times \mathbb{W}$, then $\hat{\ell}^{-1}(U \times V) = (\Phi\Delta\mathbb{P}\mathbb{r}_1)^{-1}(U \times \mathbb{W}) = \Phi^{-1}(U) \cap \mathbb{P}\mathbb{r}_1{}^{-1}(\mathbb{W}) \cap \mathbb{G}R(R)$, but the map $\mathbb{P}\mathbb{r}_1$ is continuous, therefore, $\mathbb{P}\mathbb{r}_1{}^{-1}(\mathbb{W}) \cap \mathbb{G}R(R)$ is open in $\mathbb{G}R(R)$, and since $\Phi^{-1}(U)$ is an open set in $\mathbb{G}R(R)$, then $\Phi^{-1}(U) \cap \mathbb{P}\mathbb{r}_1{}^{-1}(\mathbb{W}) \cap \mathbb{G}R(R)$ is open in $\mathbb{G}R(R)$. Hence, $\hat{\ell}^{-1}$ is a continuous map, that is $\hat{\ell}$ is an open map, thus ℓ a homeomorphism of $\mathbb{G} \times \mathbb{W}$ onto closed subset $\mathbb{W} \times \mathbb{W}$. Thus, by proposition 3.1.4 the map $\ell: \mathbb{G} \times \mathbb{W} \rightarrow \mathbb{W} \times \mathbb{W}$ is paracompact map. Hence, $\mathbb{W}$ is a paracompact $\mathbb{G}$-space. ■

Proposition 4.1.14.

Let $\mathbb{W}$ and $\mathbb{G}$ be a compact Pa-closed. If the action map $\ell: \mathbb{G} \times \mathbb{W} \longrightarrow \mathbb{W} \times \mathbb{W}$ is a paracompact then, for every compact subset B of $\mathbb{W}$, the transporter set $\mathbb{G}_B = \{g \in \mathbb{G} | (g.B) \cap B \neq \emptyset\}$ is paracompact.

Proof.

Assume that ℓ is a paracompact map. By definition of ℓ, $\mathbb{G}_B = \mathbb{P}\mathrm{r}_1(\ell^{-1}(B \times B))$ and $\mathbb{P}\mathrm{r}_1$ is the projection $\mathbb{G} \times \mathbb{W} \rightarrow \mathbb{G}$. Since $B \times B$ is a paracompact set, so $\ell^{-1}(B \times B)$ is paracompact in $\mathbb{G} \times \mathbb{W}$. Thus, by Proposition 2.1.12, $\mathbb{G}_B$ is paracompact set. ■

4.2 Certain Types of Paracompact Actions

Definition 4.2.1.

A $\mathbb{G}$-space $\mathbb{W}$ is said to be nearly paracompact $\mathbb{G}$-space if the map $\ell: \mathbb{G} \times \mathbb{W} \to \mathbb{W} \times \mathbb{W}$ which is defined by $\ell(g, x) = (x, g.x)$ for each $(g, x) \in \mathbb{G} \times \mathbb{W}$, is a nearly paracompact map.

Theorem 4.2.2.

Let $\mathbb{W}$ be a compact Pa-closed and Huasdorff space, and $\mathbb{G}$ be a paracompact space. Then $\mathbb{W}$ is a nearly paracompact $\mathbb{G}$-space.

Proof.

To show the map $\ell: \mathbb{G} \times \mathbb{W} \to \mathbb{W} \times \mathbb{W}$ is nearly paracompac. Assume that K is a nearly paracompact set in $\mathbb{W} \times \mathbb{W}$. Since $\mathbb{W} \times \mathbb{W}$ is a compact Hausdorff space, then $\mathbb{W} \times \mathbb{W}$ is a regular space, so $\mathbb{W} \times \mathbb{W}$ semi-regular by Theorem 1.2.31. Thus, K is a semi-regular by hereditary of semi-regular property, which implies that K is a paracompact subspace of $\mathbb{W} \times \mathbb{W}$ due to Theorem 1.2.30. Now, ℓ is paracompact map owing to Theorem 4.1.4, so $\ell^{-1}(K)$ is a paracompact set in $\mathbb{G} \times \mathbb{W}$. Therefore, $\ell^{-1}(K)$ is nearly paracompact in $\mathbb{G} \times \mathbb{W}$ by Theorem 1.2.29. Hence, $\mathbb{W}$ is a nearly paracompact $\mathbb{G}$-space.

Theorem 4.2.3.

Let $\mathbb{W}$ and $\mathbb{G}$ be regular spaces and $\mathbb{W}$ is nearly paracompact $\mathbb{G}$-space such that $\mathbb{G}$ is freely on $\mathbb{W}$. If H is a closed subset of $\mathbb{G}$ and B is an open subset of $\mathbb{W}$ which is invariant under H, then B is a nearly paracompact H-space.

Proof.

Suppose that $\mathbb{W}$ and $\mathbb{G}$ are regular spaces, thus $\mathbb{G} \times \mathbb{W}$ is regular space. We have $\ell: \mathbb{G} \times \mathbb{W} \to \mathbb{W} \times \mathbb{W}$ is nearly paracompact map, so ℓ is a paracompact map due to Theorem 3.1.24. Thus, B is a paracompact H-space by Theorem 4.1.6, that is $\mathcal{S}: H \times B \longrightarrow B \times B$ is paracompact map. But $B \times B$ is a regular because of B is a regular subspace of $\mathbb{W}$ and so, $\mathcal{S}$ is a nearly paracompact map by Theorem 3.2.25. Hence, B is a nearly paracompact H-space.

Definition 4.2.4.

A $\mathbb{G}$-space $\mathbb{W}$ is said to be fully normal $\mathbb{G}$-space if the map $\ell: \mathbb{G} \times \mathbb{W} \to \mathbb{W} \times \mathbb{W}$ which is defined by $\ell(g, x) = (x, g.x)$ for each $(g, x) \in \mathbb{G} \times \mathbb{W}$, is a fully normal map.

Theorem 4.2.5.

Let $\mathbb{W}$ be a compact Pa-closed and Huasdorff space, and $\mathbb{G}$ is paracompact and Huasdorff space. Then $\mathbb{W}$ is a fully normal $\mathbb{G}$-space.

Proof.

The map $\ell: \mathbb{G} \times \mathbb{W} \rightarrow \mathbb{W} \times \mathbb{W}$ is paracompact due to Theorem 4.1.4. Assume that K is a fully normal set in $\mathbb{W} \times \mathbb{W}$. Since $\mathbb{W} \times \mathbb{W}$ is Huasdorff, then K is a Huasdorff subspace of $\mathbb{W} \times \mathbb{W}$ and so, K is T_1-space. From Theorem 1.2.38, K is a paracompact subspace of $\mathbb{W} \times \mathbb{W}$. so $\ell^{-1}(K)$ is a paracompact set in $\mathbb{G} \times \mathbb{W}$. Therefore, $\ell^{-1}(K)$ is fully normal in $\mathbb{G} \times \mathbb{W}$ by Theorem 1.2.39. Hence, $\mathbb{W}$ is a fully paracompact $\mathbb{G}$-space.

Theorem 4.2.6.

Let $\mathbb{W}$ and $\mathbb{G}$ be a Huasdorff space, and $\mathbb{W}$ is a fully normal $\mathbb{G}$-space such that $\mathbb{G}$ is freely on $\mathbb{W}$. If H is a closed subset of $\mathbb{G}$ and B is an open subset of $\mathbb{W}$ which is invariant under H, then B is a fully normal H-space.

Proof.

Suppose that $\mathbb{W}$ and $\mathbb{G}$ are a Huasdorff space, then $\mathbb{G} \times \mathbb{W}$ is Huasdorff space and so, $\mathbb{G} \times \mathbb{W}$ is T_1-space and $\mathbb{W} \times \mathbb{W}$ is a Huasdorff space. Since $\ell: \mathbb{G} \times \mathbb{W} \rightarrow \mathbb{W} \times \mathbb{W}$ is fully normal map, then ℓ is a paracompact map due to Theorem 3.1.17. Thus, B is a paracompact H-space, that is $\mathcal{S}: H \times B \longrightarrow B \times B$ is a paracompact map. But $B \times B$ is Hausdorff because of B is a Hausdorff subspace of $\mathbb{W}$, thus $B \times B$ is T_1-space and $H \times B$ is a Hausdorff subspace so, $\mathcal{S}$ is a fully normal map by Theorem 3.2.10. Hence, B is a fully normal H-space.

Definition 4.2.7.

A $\mathbb{G}$-space $\mathbb{W}$ is said to be compact $\mathbb{G}$-space if the map $\ell: \mathbb{G} \times \mathbb{W} \rightarrow \mathbb{W} \times \mathbb{W}$ which is defined by $\ell(g, x) = (x, g.x)$ for each $(g, x) \in \mathbb{G} \times \mathbb{W}$, is a compact map.

Theorem 4.2.8.

Let $\mathbb{W}$ be a compact Pa-closed space, and $\mathbb{G}$ is compact space. Then $\mathbb{W}$ is a compact $\mathbb{G}$-space.

Proof.

Assume that K is a compact set in $\mathbb{W} \times \mathbb{W}$. Since $\mathbb{W} \times \mathbb{W}$ is Pa-closed, then K is a closed subset of $\mathbb{W} \times \mathbb{W}$ by Definition 2.2.1. So, $\ell^{-1}(K)$ is a closd subset of $\mathbb{G} \times \mathbb{W}$ by continuity of ℓ. we have $\mathbb{G} \times \mathbb{W}$ is compact space, thus $\ell^{-1}(K)$ is compact. Therefore, ℓ is a compact map. Hence, $\mathbb{W}$ is a compact $\mathbb{G}$-space.

Theorem 4.2.9.

Every paracompact map of countably compact space onto a Pa-closed space is compact.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a paracompact map. Assume that K is a compact subset of $\mathbb{M}$. Then K is a paracompact set by Theorem 1.2.15, and K is closed because of $\mathbb{M}$ is a Pa-closed space. Thus, $\mathcal{L}^{-1}(K)$ is a paracompact subspace of $\mathbb{W}$ due to $\mathcal{L}$ is paracompact. Also, $\mathcal{L}^{-1}(K)$ is closed owing to $\mathcal{L}$ is continuous map. Since $\mathbb{W}$ is a countably compact space then $\mathcal{L}^{-1}(K)$ is a countably compact subpace, so $\mathcal{L}^{-1}(K)$ is a compact subspace of $\mathbb{W}$ by Theorem 1.2.19. Hence $\mathcal{L}$ is compact map.

Theorem 4.2.10.

Let $\mathbb{W}$ is a compact $\mathbb{G}$-space where $\mathbb{W}$ is a compact Pa-closed space, and $\mathbb{G}$ is a compact freely on $\mathbb{W}$. If H is a closed subset of $\mathbb{G}$ and B is an clopen subset of $\mathbb{W}$ which is invariant under H, then B is a compact H-space.

Proof.

Suppose that $\ell: \mathbb{G} \times \mathbb{W} \to \mathbb{W} \times \mathbb{W}$ is compact map where $\mathbb{W}$ is a compact Pa-closed space and $\mathbb{G}$ is a compact space, then $\mathbb{W} \times \mathbb{W}$ is a compact Pa-closed space by Theorem 2.2.10, and so, $\mathbb{W} \times \mathbb{W}$ is a countably compact Pa-closed space. Thus, ℓ is a paracompact map because of Theorem 4.1.4, so B is a paracompact H-space due to Theorem, that is $\mathcal{S}: H \times B \longrightarrow B \times B$ is a paracompact map. But $B \times B$ is clopen which implies $B \times B$ Pa-closed subspace of $\mathbb{W} \times \mathbb{W}$ because of Theorem 2.2.11. We have $\mathbb{G} \times \mathbb{W}$ is compact space, so $\mathbb{G} \times \mathbb{W}$ is countably compact by Theorem 1.2.17(ii). Since $H \times B$ is a closed subset of $\times \mathbb{W}$, then $H \times B$ is countably compact. therefore $\mathcal{S}$ is compact by Theorem 4.2.9. Hence, B is a compact H-space.

Definition 4.2.11.

A $\mathbb{G}$-space $\mathbb{W}$ is said to be metacompact $\mathbb{G}$-space if the map $\ell: \mathbb{G} \times \mathbb{W} \to \mathbb{W} \times \mathbb{W}$ which is defined by $\ell(g, x) = (x, g.x)$ for each $(g, x) \in \mathbb{G} \times \mathbb{W}$, is a metacompact map.

<u>Theorem 4.2.12.</u>

Let $\mathbb{W}$ be a compact Pa-closed and normal space, and $\mathbb{G}$ is compact and normal space. Then, $\mathbb{W}$ is a metacompact $\mathbb{G}$-space.

Proof: Since $\mathbb{G}$ is compact, then $\mathbb{G}$ is a paracompact space. Thus, $\mathbb{W}$ is a paracompact $\mathbb{G}$-space by Theorem 4.1.4, so $\ell: \mathbb{G} \times \mathbb{W} \to \mathbb{W} \times \mathbb{W}$ is a paracompact map. Assume that K is a closed metacompact set in $\mathbb{W} \times \mathbb{W}$. Since $\mathbb{W} \times \mathbb{W}$ is normal space then K is countably paracompact by Theorem 1.2.23. We have $\mathbb{W} \times \mathbb{W}$ is Lindelöf, then K is Lindelöf, so K is a paracompact subspace of $\mathbb{W} \times \mathbb{W}$ by Theorem 1.2.20. Therefore, $\ell^{-1}(K)$ is paracompact in $\mathbb{G} \times \mathbb{W}$, so $\ell^{-1}(K)$ is paracompact owing to Theorem 1.2.23. Thus, ℓ is a metacompact map. Hence, $\mathbb{W}$ is a metacompact $\mathbb{G}$-space.

<u>Lemma 4.2.13.</u>

Let $\mathbb{W}$ be counatably compact, normal and Lindelöf, then any closed metacompact subspace of $\mathbb{W}$ is compact.

Proof.

From Theorem 1.2.23, Theorem 1.2.20 and Theorem 1.2.15.

<u>Theorem 4.2.14.</u>

Let $\mathbb{W}$ is a metacompact $\mathbb{G}$-space where $\mathbb{W}$ is a compact Pa-closed space, and $\mathbb{G}$ is a compact freely on $\mathbb{W}$. If H is a closed subset of $\mathbb{G}$ and B is an clopen subset of $\mathbb{W}$ which is invariant under H, then B is a metacompact H-space.

Proof.

Assume that $\ell: \mathbb{G} \times \mathbb{W} \to \mathbb{W} \times \mathbb{W}$ is metacompact map where $\mathbb{W}$ is a compact Pa-closed and normal space, and $\mathbb{G}$ is a compact space, then $\mathbb{W} \times \mathbb{W}$ is a compact Pa-closed and normal space, also $\mathbb{G} \times \mathbb{W}$ is a compact. Thus, ℓ is a compact map because of Theorem 4.2.8, so B is a compact H-space due to Theorem, that is $\mathcal{S}: H \times B \longrightarrow B \times B$ is a compact map. But $B \times B$ is closed which implies $B \times B$ is compact and norma subspace of $\mathbb{W} \times \mathbb{W}$, so $B \times B$ is normal, countably compact and Lindelöf, therefore from Lemma 4.2.13 , $\mathcal{S}$ is compact map. Hence, B is a compact H-space.

<u>Theorem 4.2.15.</u>

The composition of metacompact maps is a metacompact map.

Proof.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ and $\mathcal{J}: \mathbb{M} \to \mathbb{E}$ be two metacompact maps. Suppose that K is a closed and metacompact set in $\mathbb{E}$, then $\mathcal{J}^{-1}(\mathrm{K})$ is a closed metacompact set in $\mathbb{M}$ owing to $\mathcal{J}$ is a metacompact map. Thus, $\mathcal{L}^{-1}\big(\mathcal{J}^{-1}(\mathrm{K})\big)$ is a metacompact set in $\mathbb{W}$ because $\mathcal{L}$ is a metacompact map, but $\mathcal{L}^{-1}(\mathcal{J}^{-1}(\mathrm{K})) = (\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$. So, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is a metacompact set in $\mathbb{W}$. Hence, $\mathcal{J} \circ \mathcal{L}$ is metacompact.

Theorem 4.2.16.

Let $\mathbb{W}$ be a countably compact, Lindelöf and normal space. If $\mathcal{J} \circ \mathcal{L}: \mathbb{W} \to \mathbb{E}$ is a metacompact map and $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a continuous surjective map, then $\mathcal{J}: \mathbb{M} \to \mathbb{E}$ is a metacompact map.

Proof.

Suppose that K a closed metacompact set in $\mathbb{E}$. Since $\mathcal{J} \circ \mathcal{L}$ is a metac-ompact map then $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is metacompact set in $\mathbb{W}$, also, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is closed due to continuity of $\mathcal{J} \circ \mathcal{L}$. Thus, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is compact by Lemma 4.2.13. But $\mathcal{L}$ is a continuous surjective map then, $\mathcal{L}(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K}) = \mathcal{L}\left(\mathcal{L}^{-1}\big(\mathcal{J}^{-1}(\mathrm{K})\big)\right) = \mathcal{J}^{-1}(\mathrm{K})$ is compact, so $\mathcal{J}^{-1}(\mathrm{K})$ is a metacompact set in $\mathbb{M}$. Hence, $\mathcal{J}$ is a metacompact map.

Theorem 4.2.17.

Let $\mathbb{G}_1$, $\mathbb{W}$ and $\mathbb{M}$ be compact and normal spaces, and $(\mathbb{G}_1, \mathbb{W}, \varphi_1)$ and $(\mathbb{G}_2, \mathbb{M}, \varphi_2)$ be two a group actions such that a bijective map $\mathcal{J}: \mathbb{G}_1 \to \mathbb{G}_2$ is open and proper onto group homomorphism and $\mathcal{L}: \mathbb{W} \longrightarrow \mathbb{M}$ is an equivariant map. If $\mathcal{L}$ is a surjective open proper map, and $\mathbb{W}$ is metacompact $\mathbb{G}_1$-space, then $\mathbb{M}$ is a metacompact $\mathbb{G}_2$-space.

Proof.

Let $\ell_1: \mathbb{G}_1 \times \mathbb{W} \longrightarrow \mathbb{W} \times \mathbb{W}$ and $\ell_2: \mathbb{G}_2 \times \mathbb{M} \longrightarrow \mathbb{M} \times \mathbb{M}$ be maps are defined by $\ell_1(g_1, x)$=$(x, \varphi_1(g_1, x))$ and $\ell_2(g_2, x)$=$(x, \varphi_2(g_2, x))$. Let $(g_1, x) \in \mathbb{G} \times \mathbb{W}$, then according to diagram below:

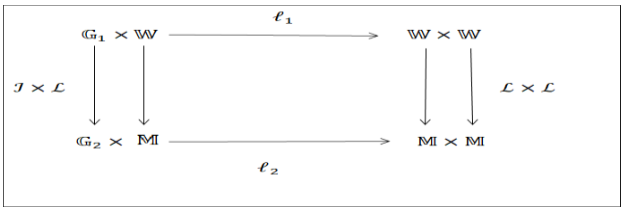

Figure 4.2 Second diagram on maps

$[(\mathcal{L} \times \mathcal{L}) \circ \ell_1]\,(g_1, x)$

$=(\mathcal{L}\times\mathcal{L})\left(x,\varphi_1(g_1,x)\right)$

$=(\mathcal{L}(\mathrm{x}),\mathcal{L}(\varphi_1(g_1,x)))=\left(\mathcal{L}(x),\varphi_2(\mathcal{J}(g_1),\mathcal{L}(x))\right)$

$=[\ell_2\circ(\mathcal{J}\times\mathcal{L})](g_1,x)$.

So the diagram is commutative. We have $\mathcal{L}:\mathbb{W}\longrightarrow\mathbb{M}$ is a surjective proper, then $\mathcal{L}\times\mathcal{L}$ is a bijective proper map. By Theorem 2.1.2, $\mathcal{L}\times\mathcal{L}$ is a paracompact map, so $\mathcal{L}\times\mathcal{L}$ is metacompact owing to Theorem 3.2.2. But $\ell_1$is metacompact, then $(\mathcal{L}\times\mathcal{L})\circ\ell_1$ is paracompact by Theorem 4.2.15. Thus, $\ell_2\circ(\mathcal{J}\times\mathcal{L})$ is a metacompact map. Thus, ℓ_2 is a metacompact map by Theorem 4.2.16. Hence, $\mathbb{M}$ is a metacompact $\mathbb{G}_1$-space.

Chapter Five

Paracompact Map on the Components of Fuzzy Topographic Topological Mapping (FTTM)

5.1 On the Fuzzy Paracompact Map in FTTM

This Chapter is mainly devoted to introducing a fuzzy paracompact map and applying it to a mathematical model of fuzzy topographic topological mapping ($FTTM$).

Definition 5.1.1.

A surjective F-continuous map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is called a fuzzy paracompact if the inverse image for any fuzzy paracompact set in $I^{\mathbb{M}}$ is fuzzy paracompact set in $I^{\mathbb{W}}$.

Remark 5.1.2.

Every subspace of a metrizable space is metrizable.

Theorem 5.1.3

A surjective F-continuous map $\mathcal{L}$ from a fuzzy metrizable space $(\mathbb{W}, w(\tau))$ onto a fuzzy topological space $(\mathbb{M}, \mathrm{S})$ is a fuzzy paracompact map, where $w(\tau)$ has the original topology τ such that $(\mathbb{W}, \tau)$ is a metrizable space.

Proof.

Assume that $\mu \in I^{\mathbb{M}}$ is a fuzzy paracompact subspace of $(\mathbb{M}, \mathrm{S})$. To prove that $\mathcal{L}^{-1}(\mu) \in I^{\mathbb{W}}$ is a fuzzy paracompact. A subspace $(\mathcal{L}^{-1}(\mu), w(\tau_{\mathcal{L}^{-1}(\mu)}))$ of $(\mathbb{W}, \mathrm{T})$ has original topological space $(\mathcal{L}^{-1}(\mu), \tau_{\mathcal{L}^{-1}(\mu)})$ which is a subspace of a topological space $(\mathbb{W}, \tau)$. Since $(\mathbb{W}, \tau)$ is a metrizable space, then $(\mathcal{L}^{-1}(A), \tau_{\mathcal{L}^{-1}(A)})$ is a paracompact subspace of $\mathbb{W}$ by Remark 5.1.2. Therefore $(\mathcal{L}^{-1}(A), w(\tau_{\mathcal{L}^{-1}(A)}))$ is a fuzzy paracompact set due to Remark 1.4.14. Hence, $\mathcal{L}$ is a fuzzy paracompact map. ■

Theorem 5.1.4.

A F-continuous map $\mathcal{L}$ of an FTS $(\mathbb{W}, \mathrm{T})$ onto an FTS $(\mathbb{M}, \mathrm{S})$ is a fuzzy closed if and only if for every fuzzy set $\lambda \in I^{\mathbb{M}}$ and every fuzzy open set $\mu \in I^{\mathbb{W}}$ which satisfies $\mathcal{L}^{-1}(\lambda) \leq \mu$, there exists a fuzzy open set $\sigma \in I^{\mathbb{M}}$ containing λ such that $\mathcal{L}^{-1}(\sigma) \leq \mu$.

Proof.

Assume that $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a fuzzy closed map. Let λ be a fuzzy set in $I^{\mathbb{M}}$ and μ is a fuzzy open set in $I^{\mathbb{W}}$ which holding $\mathcal{L}^{-1}(\lambda)$. The fuzzy set $\sigma = 1 - \mathcal{L}(1-\mu))$ is open in $I^{\mathbb{M}}$ and contains λ. Thus, $\mathcal{L}^{-1}(\sigma) = \mathcal{L}^{-1}(1 - \mathcal{L}(1-\mu)) \leq 1 - (1-\mu) = \mu$. Conversely, suppose that $\mathcal{L}$

satisfies the condition in the Theorem, and let us take a fuzzy closed $\varphi \in I^{\mathbb{W}}$. For a fuzzy set $\mu = 1 - \varphi$ is open, and $\lambda = 1 - \mathcal{L}(\varphi)$ we have $\mathcal{L}^{-1}(\lambda) = \mathcal{L}^{-1}(1 - \mathcal{L}(\varphi)) \leq 1 - \varphi = \mu$. Thus there exists fuzzy open $\sigma \in I^{\mathbb{M}}$ such that $1 - \mathcal{L}(\varphi) \leq \sigma$ and $\mathcal{L}^{-1}(\sigma) \leq \mu$ i.e. $\mathcal{L}^{-1}(\sigma) \wedge \varphi = 0$ which implies that $\sigma \wedge \mathcal{L}(\varphi) = 0$ i.e. $\sigma \leq 1 - \mathcal{L}(\varphi)$. Thus $\mathcal{L}(\varphi) = 1 - \sigma$, and this show that the fuzzy set $\mathcal{L}(\varphi)$ is closed. Hence, $\mathcal{L}$ is a fuzzy closed map. ■

Theorem 5.1.5.

Let $\mathcal{L}: (\mathbb{W}, w(\tau)) \to (\mathbb{M}, w(\dot{\tau}))$ be a fuzzy paracompact map, where FTS $(\mathbb{M}, w(\dot{\tau})$ has the original topological τ such $(\mathbb{M}, \dot{\tau})$. Then for every singleton $y_\mu \in I^{\mathbb{M}}$ and every family $\beta \subseteq I^{\mathbb{W}}$; $\beta = \{\lambda_i : i \in \Gamma_i\}$ satisfying $\mathcal{L}^{-1}(y_t) \leq \bigvee_{i \in \Gamma_i} \beta$, there exists a quasi-neighborhood A_{y_t} of y_t and partial open refinement $\beta_o = \{\acute{\lambda}_j : j \in \Gamma_j\}$ of β such that $\mathcal{L}^{-1}(A) \leq \bigvee_{j \in \Gamma_j} \acute{\lambda}_j$ and β_o is a locally finite at $\mathcal{L}^{-1}(y_\mu)$. ■

Proof.

Assume that $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a fuzzy paracompact map and $y_\mu \in I^{\mathbb{M}}$. Since $\{y\}$ is a paracompact in $(\mathbb{M}, \dot{\tau})$ when τ is the original topology of $w(\dot{\tau})$, thus $\{y_\mu\}$ is a fuzzy paracompact in $I^{\mathbb{M}}$ by Remark 1.4.14. Thus $\mathcal{L}^{-1}(y_\mu)$ is a fuzzy paracompact in $I^{\mathbb{W}}$ due to Definition 5.1.1, then for each family $\beta \subseteq I^{\mathbb{W}}$; $\beta = \{\lambda_i : i \in \Gamma_i\}$ is open α-Q cover of μ, there is an open refinement $\{\rho_k : k \in \Gamma_k\}$. Let A be an open quasi neighborhood of y_μ. Take an open quasi neighborhood G_{a_ω} of a_ω for each $a_\omega \in A$ such that $\mathcal{L}^{-1}(a_\omega) \in \bigvee_{k \in \Gamma_k} \rho_k$. Then the fuzzy cover $\mathbb{G} = \{G_{a_\omega} : a_\omega \in A\}$ of A is an open cover of y_μ. But $\{y_t\}$ is a fuzzy compact by Remark 2.4.15, thus $\mathbb{G}$ has a finite subcover $\{D_{a_\omega} : a_\omega \in A\}$. So, for all ε with $0 < \varepsilon \leq \alpha$, there exists a locally finite refinement $\beta_o = \{\mathcal{L}^{-1}(D_{a_\omega}) \wedge \rho_k : a_\omega \in A, k \in \Gamma_k\}$ of β such that $\bigvee_{a_\omega \in A, k \in \Gamma_k} \mathcal{L}^{-1}(D_{a_\omega}) \wedge \rho_k \geq \alpha - \varepsilon$ and $\mathcal{L}^{-1}(A) \leq (\bigvee \mathcal{L}^{-1}(D_{a_\omega})) \wedge (\bigvee W_k)$. ■

Theorem 5.1.6.

A fuzzy paracompact map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is closed.

Proof.

Let λ be a fuzzy set in $I^{\mathbb{M}}$ and σ is a fuzzy open set in $I^{\mathbb{W}}$ such that $\mathcal{L}^{-1}(\lambda) \leq \sigma$, then $\mathcal{L}^{-1}(y_\mu) \leq \sigma$ for all y_μ contain in λ. Since $\mathcal{L}$ is a fuzzy paracompact map, then there exists a fuzzy open quasi neighborhood V_{y_μ} of y_μ and $\{\sigma\}$ has open refinement $\{W_{y_\mu}\}$ such that $\mathcal{L}^{-1}(V_{y_\mu}) \leq W_{y_\mu} \leq \sigma$ due to Theorem 5.1.5, so $\mathcal{L}^{-1}(\vee V_{y_\mu}) \leq \sigma$. But $\vee V_{y_\mu}$ is a fuzzy open set contained in σ. Hence, $\mathcal{L}$ is a fuzzy closed map by Theorem 5.1.4. ■

Lemma 5.1.7.

If $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is A bijective fuzzy closed, then $\mathcal{L}^{-1}: \mathbb{M} \to \mathbb{W}$ is F-continuous.

Proof.

Let $\lambda \in I^{\mathbb{W}}$ be a closed set, then $\mathcal{L}(\lambda)$ is a fuzzy closed set in $\mathbb{M}$. But $(\mathcal{L}^{-1})^{-1}(\lambda) = \mathcal{L}(\lambda)$ due to $\mathcal{L}$ being bijective, hence $\mathcal{L}^{-1}$ is F-continuous. ■

Theorem 5.1.8.

An injective fuzzy paracompact map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a fuzzy homeomorphism.

Proof.

From Theorem 5.1.7 and Definition 5.1.1. ■

Theorem 5.1.9.

An injective fuzzy paracompact map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a fuzzy proper.

Proof.

Since $\mathcal{L}$ is an injective fuzzy paracompact map, then $\mathcal{L}$ is a fuzzy homeomorphism by Theorem 5.1.9. Let $\mathbb{E}$ be any fuzzy space, then $\mathcal{L} \times \mathrm{I}_{\mathbb{E}}: \mathbb{W} \times \mathbb{E} \to \mathbb{M} \times \mathbb{E}$ is a fuzzy homeomorphism, so $\mathcal{L} \times \mathrm{I}_{\mathbb{E}}$ is a fuzzy closed. But $\mathcal{L}$ is F-continuous by hypothesis. Hence, $\mathcal{L}$ is a fuzzy homeomorphism. ■

Theorem 5.1.10.

Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$, $\mathcal{J}: \mathbb{M} \to \mathbb{E}$ and $\mathcal{J}o\mathcal{L}: \mathbb{W} \to \mathbb{E}$ be fuzzy maps. If $\mathcal{L}$ and $\mathcal{J}$ are fuzzy paracompact, then $\mathcal{J}o\mathcal{L}$ is a fuzzy paracompact map.

Proof.

Since $\mathcal{L}$ and $\mathcal{J}$ are surjective F-continuous maps, then $\mathcal{J}o\mathcal{L}$ is surjective F-continuous. Let K be a fuzzy paracompact set in $\mathbb{E}$, then $\mathcal{L}^{-1}(\mathcal{J}^{-1}(K))$ is paracompact in $\mathbb{W}$ due to $\mathcal{L}$ and $\mathcal{J}$ are fuzzy paracompact. Since $\mathcal{J}o\mathcal{L}^{-1}(K) = \mathcal{L}^{-1}(\mathcal{J}^{-1}(K))$, then $\mathcal{J}o\mathcal{L}$ is a fuzzy paracompact. ■

For boundedness of FM_i, $i=1,2,\ldots,n$ such that $n \in \mathbb{Z}^+$, it is included in

$\mathbb{R}^3$. The metric function d: $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^+$ caused the metric function d_{FM} : $FM \times FM \to \mathbb{R}^+$ on FM. A metric function d_{FM} is induced by the topology τ_{FM} on FM that is defined by: $\tau_{FM} = \{\theta_k \cap FM$: θ_k is an open subset of $\mathbb{R}^3$, $k \in \Omega$, Ω is an index set$\}$

That is, FM_i is a fuzzy metrizable subspace of a metrizable space $\mathbb{R}^3$.

The next theorem gives certain characteristics to the map f_i: $FM_i \to FM_{i+1}$ between the component FM_i in $FTTM_i$ and the component FM_{i+1} in $FTTM_{i+1}$.

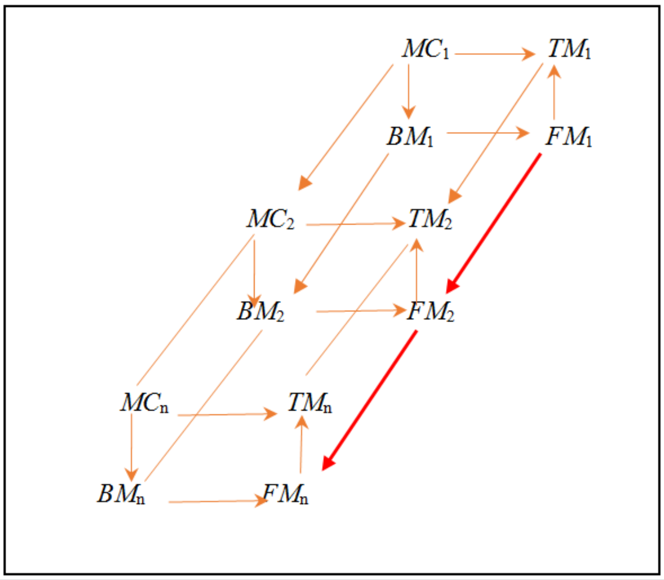

Figure 5.1 Sequence of n versions of FTTM

<u>Corollary 5.1.11.</u>

In FTTM, for each $i = 1, 2, 3, \ldots, n$ such that $i + 1 \leq n$ and $n \in \mathbb{Z}^+$.

1) The fuzzy map f_i: $FM_i \to FM_{i+1}$ is paracompact.
2) The fuzzy map f_i: $FM_i \to FM_{i+1}$ is closed.
3) The fuzzy map f_i: $FM_i \to FM_{i+1}$ is a homeomorphism.
4) The fuzzy map f_i: $FM_i \to FM_{i+1}$ is proper.
5) The composition of f_i: $FM_i \to FM_{i+1}$ and The map f_{i+1}: $FM_{i+1} \to FM_{i+2}$ is a paracompact fuzzy map.

Proof.

1) From Theorem 6.1.3.
2) From Theorem 6.1.6.
3) From Theorem 6.1.8.
4) From Theorem 6.1.9.
1) From Theorem 6.1.10. ■

5.2 On the Paracompact Map in FTTM

Theorem 5.2.1.

Each continuous map of a metrizable space $(\mathbb{W}, \tau_1)$ onto a topological space $(\mathbb{M}, \tau_2)$ is a paracompact.

Proof.

Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ be a surjective continuous map such that $\mathbb{W}$ is a metrizable space. Assume that K is a paracompact set in $\mathbb{M}$. Consequently, $\mathcal{L}^{-1}(K)$ is a metrizable subspace of $\mathbb{W}$ due to Remark 5.1.2. Thus, by Theorem 1.2.59, $\mathcal{L}^{-1}(K)$ is paracompact in $\mathbb{W}$. Hence $\mathcal{L}$ is paracompact map.

Corollary 5.2.2.

The maps between the components of $FTTM$ are paracompact.

Proof.

(i) bm: $MC \rightarrow BM$ such that :

$$bm((x,y)_0, B_{Z\,(x,y)}) = ((x,y)_{-h}, B_{Z\,(x,y)}),$$
$$\forall ((x,y)_0, B_{Z\,(x,y)}) \in MC$$

The component MC is a metrizable space by Theorem 1.2.61 and bm is a surjective continuous map due to components of $FTTM$ are homeomorphic. Then because Theorem 5.2.1 bm is a paracompact map

(ii) fm: $BM \rightarrow FM$ such that:

$$fm((x,y)_{-h}, B_{Z\,(x,y)}) = ((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}),$$
$$\forall ((x,y)_{-h}, B_{Z\,(x,y)}) \in BM$$

From Corollary 1.2.62 and Theorem 5.2.1.

(iii) tm: $FM \rightarrow TM$ such that :

$$tm((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}) = (x,y,z),\ \forall ((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}) \in FM$$

with $-h<0$ is a constant, and $B_z \in B \subseteq \mathbb{R}$.

From Theore\m 1.2.62 and Theorem 5.2.1.

Theorem 5.2.3.

In $FTTM$, For each $i = 1,2,3,..,n$ such that $i+1 \leq n$ and $n \in \mathbb{Z}^+$.

1) The map f_i: $MC_i \rightarrow MC_{i+1}$ is paracompact.
2) The map f_i: $BM_i \rightarrow BM_{i+1}$ is paracompact.
3) The map f_i: $FM_i \rightarrow FM_{i+1}$ is paracompact.
4) The map f_i: $TM_i \rightarrow TM_{i+1}$ is paracompact.

Proof.

From Theorem 1.2.62 and Theorem 5.2.1.

Corollary 5.2.4.

The maps between the components of FTTM are countably paracompact.

Proof.

(i) bm: $MC \rightarrow BM$ such that :

$$bm((x,y)_0, B_{Z\,(x,y)}) = ((x,y)_{\text{-}h}, B_{Z\,(x,y)}),$$
$$\forall((x,y)_0, B_{Z\,(x,y)}) \in MC$$

The component BM is a Lindelöf space by Theorem 1.2.61. Since bm is a paracompact map due to Theorem 5.2.2. Then from Theorem 3.2.5 bm is a countably paracompact map.

(ii) fm: $BM \rightarrow FM$ such that:

$$fm((x,y)_{-h}, B_{Z\,(x,y)}) = ((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}),$$
$$\forall((x,y)_{\text{-}h}, B_{Z\,(x,y)}) \in BM$$

From Theorem 1.2.62, Theorem 5.2.2. and Theorem 3.2.5

(iii) tm: $FM \rightarrow TM$ such that :

$$tm((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}) = (x,y,z), \forall((x,y)_{\text{-}h}, \mu_{B_{Z\,(x,y)}}) \in FM$$

with $-h<0$ is a constant, and $B_z \in B \subseteq \mathbb{R}$.

From Theorem 1.2.62, Theorem 5.2.2. and Theorem 3.2.5

Corollary 5.2.5.

In $FTTM$, f\or each $i = 1,2,3,..,n$ such that $i+1 \leq n$ and $n \in \mathbb{Z}^+$.

1) The map f_i: $MC_i \rightarrow MC_{i+1}$ is countably paracompact.
2) The map f_i: $BM_i \rightarrow BM_{i+1}$ is countably paracompact.
3) The map f_i: $FM_i \rightarrow FM_{i+1}$ is countably paracompact.
4) The map f_i: $TM_i \rightarrow TM_{i+1}$ is countably paracompact.

Proof.

From Theorem 1.2.62, Theorem 5.2.2. and Theorem 3.2.5

Corollary 5.2.6.

The maps between the components of FTTM are compact.

Proof.

(i) bm: $MC \rightarrow BM$ such that :

$$bm((x,y)_0, B_{Z\,(x,y)}) = ((x,y)_{\text{-}h}, B_{Z\,(x,y)}),$$
$$\forall((x,y)_0, B_{Z\,(x,y)}) \in MC$$

The component MC is a countably compact space Theorem 1.2.61 and The component BM is Hausdorff space owing to Corollary 1.2.62. Since bm is a paracompact map due to Theorem 5.2.2. Then from Theorem 3.2.1 bm is a compact map.

(ii) fm: $BM \rightarrow FM$ such that:

$$fm((x,y)_{-h}, B_{Z\,(x,y)}) = ((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}),$$
$$\forall((x,y)_{\text{-}h}, B_{Z\,(x,y)}) \in BM$$

By the same way.

(iii) tm: $FM \rightarrow TM$ such that :

$$tm((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}) = (x,y,z),\ \forall((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}) \in FM$$

with $-h<0$ is a constant, and $B_z \in B \subseteq \mathbb{R}$.

By the same way.

<u>Theorem 5.2.7.</u>

In $FTTM$, For each $i = 1,2,3,..,n$ such that $i+1 \leq n$ and $n \in \mathbb{Z}^+$.

Proof.

1) The map f_i: $MC_i \rightarrow MC_{i+1}$ is compact.

By Theorem 1.2.62 the component MC_i is a countably compact space and the component MC_{i+1} is Hausdorff space. Since bm is a paracompact map due to Theorem 5.2.3. Then from Theorem 3.2.1 bm is a compact map.

2) The map f_i: $BM_i \rightarrow BM_{i+1}$ is compact.

By the same way.

3) The map f_i: $FM_i \rightarrow FM_{i+1}$ is compact.

By the same way.

4) The map f_i: $TM_i \rightarrow TM_{i+1}$ is compact.

By the same way.

<u>Theorem 5.2.8.</u>

The maps between the components of $FTTM$ are countably compact.

Proof.

(i) bm: $MC \rightarrow BM$ such that :

$$bm((x,y)_0, B_{Z\,(x,y)}) = ((x,y)_{-h}, B_{Z\,(x,y)}),$$
$$\forall((x,y)_0, B_{Z\,(x,y)}) \in MC$$

The component MC is countably compact space by Theorem 1.2.61 and BM is a Lindelöf and Hausdorff space by Theorem 1.2.62. Since bm is a paracompact map due to Theorem 5.2.2. Then from Theorem 3.2.13, bm is a countably compact map.

(ii) fm: $BM \rightarrow FM$ such that:

$$fm((x,y)_{-h}, B_{Z\,(x,y)}) = ((x,y)_{-h}, \mu_{B_{Z\,(x,y)}},$$
$$\forall((x,y)_{-h}, B_{Z\,(x,y)}) \in BM$$

By the same way.

(iii) tm: $FM \rightarrow TM$ such that :

$$tm((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}) = (x,y,z),\ \forall((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}) \in FM \quad (3.13)$$

with $-h<0$ is a constant, and $B_z \in B \subseteq \mathbb{R}$.

By the same way.

Theorem 5.2.9.

In $FTTM$, for each $i = 1, 2, 3, .., n$ such that $i + 1 \leq n$ and $n \in \mathbb{Z}^+$.

1) The map f_i: $MC_i \rightarrow MC_{i+1}$ is countably compact.
2) The map f_i: $BM_i \rightarrow BM_{i+1}$ is countably compact.
3) The map f_i: $FM_i \rightarrow FM_{i+1}$ is countably compact
4) The map f_i: $TM_i \rightarrow TM_{i+1}$ is countably compact.

Proof.

From Theorem 1.2.62, Theorem 5.2.3 and Theorem 3.2.13.

Theorem 5.2.10.

The maps between the components of FTTM are metacompact (res. countably metacompact).

Proof.

(i) *bm*: $MC \rightarrow BM$ such that :

$$bm((x,y)_0, B_{Z\,(x,y)}) = ((x,y)_{-h}, B_{Z\,(x,y)}),$$
$$\forall ((x,y)_0, B_{Z\,(x,y)}) \in MC$$

The component BM is a normal and Lindelöf space by Theorem 1.2.62. Since bm is a paracompact map due to Theorem 5.2.2. Then from Theorem 3.2.2 (res. Theorem 3.2.4) bm is a metacompact (res. countably metacompact) map.

(ii) *fm*: $BM \rightarrow FM$ such that:

$$fm((x,y)_{-h}, B_{Z\,(x,y)}) = ((x,y)_{-h}, \mu_{B_{Z\,(x,y)}},$$
$$\forall ((x,y)_{-h}, B_{Z\,(x,y)}) \in BM$$

By the same way.

(iii) *tm*: $FM \rightarrow TM$ such that :

$$tm((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}) = (x,y,z), \forall ((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}) \in FM \quad (3.13)$$

with $-h<0$ is a constant, and $B_z \in B \subseteq \mathbb{R}$.

By the same way.

Theorem 5.2.11.

In $FTTM$, for each $i = 1, 2, 3, .., n$ such that $i + 1 \leq n$ and $n \in \mathbb{Z}^+$.

1) The map f_i: $MC_i \rightarrow MC_{i+1}$ is metacompact (res. countably metacompact).
2) The map f_i: $BM_i \rightarrow BM_{i+1}$ is metacompact (res. countably metacompact).
3) The map f_i: $FM_i \rightarrow FM_{i+1}$ is metacompact (res. countably metacompact).
4) The map f_i: $TM_i \rightarrow TM_{i+1}$ is metacompact (res. countably metacompact).

Proof.

From Theorem 1.2.62, Theorem 5.2.3 and Theorem 3.2.2 (res. Theorem 3.2.4).

Theorem 5.2.12.

The maps between the components of FTTM are nearly compact.

Proof.

(i) bm: MC → BM such that :

$$\mathrm{bm}((\mathrm{x},\mathrm{y})_0, \mathrm{B}_{\mathrm{Z}\,(\mathrm{x},\mathrm{y})}) = ((\mathrm{x},\mathrm{y})_{\text{-h}}, \mathrm{B}_{\mathrm{Z}\,(\mathrm{x},\mathrm{y})}),$$

$$\forall((\mathrm{x},\mathrm{y})_0,\mathrm{B}_{\mathrm{Z}\,(\mathrm{x},\mathrm{y})})\in \mathrm{MC}$$

The component *BM* is a normal and Lindelöf space by Theorem and Theorem 1.2.62. Since *bm* is a paracompact map due to Theorem 5.2.2. Then from Theorem 3.2.20, *bm* is a nearly compact map.

(ii) *fm*: *BM*→ *FM* such that:

$$fm((x,y)_{-h}, B_{Z\,(x,y)}) = ((x,y)_{-h}, \mu_{B_{Z\,(x,y)}},$$

$$\forall((x,y)_{-h},B_{Z\,(x,y)})\in BM$$

By the same way.

(iii) *tm*: *FM*→ *TM* such that :

$$tm\left((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}\right) = (x,y,z),\ \forall((x,y)_{-h},\mu_{B_{Z\,(x,y)}})\in FM$$

with $-h<0$ is a constant, and $B_z\in B\subseteq \mathbb{R}$.

By the same way.

Theorem 5.2.13.

In $FTTM$, for each $i = 1,2,3,..,n$ such that $i+1 \le n$ and $n \in \mathbb{Z}^+$.

1) The map f_i: MC_i→ MC_{i+1} is nearly compact.
2) The map f_i: BM_i→ BM_{i+1} is nearly compact.
3) The map f_i: FM_i→ FM_{i+1} is nearly compact.
4) The map f_i: TM_i→ TM_{i+1} is nearly compact.

Proof.

From Theorem 1.2.62, Theorem 5.2.3 and Theorem 3.2.20.

Theorem 5.2.14.

The maps between the components of *FTTM* are fully normal (res. fully T_4).

Proof.

(i) *bm: MC → BM such that :*

$$bm((x,y)_0, B_{Z\,(x,y)}) = ((x,y)_{-h}, B_{Z\,(x,y)}),$$

$$\forall((x,y)_0,B_{Z\,(x,y)})\in MC \qquad (3.11)$$

The component MC is a T_2-space by Theorem 1.2.61 and the component BM is a T_1-space owing to Theorem 1.2.62 .Since *bm* is a paracompact map due to Theorem 5.2.2. Then from Theorem 3.2.10 (res. 3.2.12), *bm* is a fully T_4 (res. fully normal) map.

(ii) *fm: BM→FM such that:*

$$fm((x,y)_{-h}, B_{Z\,(x,y)}) = ((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}),$$
$$\forall((x,y)_{-h}, B_{Z\,(x,y)}) \in BM$$

By the same way.

(iii) *tm: FM→TM such that :*

$$tm((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}) = (x,y,z),\ \forall((x,y)_{-h}, \mu_{B_{Z\,(x,y)}}) \in FM$$

with $-h<0$ *is a constant, and* $B_z \in B \subseteq \mathbb{R}$.

By the same way.

Corollary 5.2.15.

In $FTTM$, for each $i = 1,2,3,..,n$ such that $i+1 \leq n$ and $n \in \mathbb{Z}^+$.

1) The map f_i: $MC_i \to MC_{i+1}$ is fully normal (res. fully T_4).

2) The map f_i: $BM_i \to BM_{i+1}$ is fully normal (res. fully T_4).

3) The map f_i: $FM_i \to FM_{i+1}$ is fully normal (res. fully T_4).

4) The map f_i: $TM_i \to TM_{i+1}$ is fully normal (res. fully T_4).

Proof.

From Theorem 1.2.62, Theorem 5.2.2 and Theorem 3.2.10 (res. Theorem 3.2.12).

Conclusion and Future Work

Conclusions

The study has introduced a new type of map called the paracompact map, which utilizes paracompact spaces to address ambiguous real-life problems. The relationships between the paracompact map and other important maps have been studied, and two classes of maps have been identified: the strong form and the weaker form of the paracompact map. The composition operations of paracompact maps have been examined, and a motivational utilization of G-space has led to defining a paracompact G-space. The study has also presented different types of paracompact actions, including nearly paracompact, compact, fully normal, and metacompact, and introduced the concept of the fuzzy paracompact map.

In conclusion, this study has provided valuable insights into the paracompact map and its relationships with other important maps. The identification of two classes of maps and the examination of composition operations have practical implications for addressing ambiguous real-life problems. The introduction of the fuzzy paracompact map and its generalization to fuzzy topographic topological mapping (FTTM) have further expanded the scope of the study.

Future Works

Future work can focus on exploring the practical applications of the paracompact map and its extensions in various fields, including engineering, computer science, and data analysis. Additionally, the study can be extended to examine the relationships between the fuzzy paracompact map and other important maps in more depth.

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المستخلص

تدور هذه الدراسة حول الدالة المضغوطة (Paracompact map) من خلال إعتمادنا في العمل على الفضاء المضغوطة (paracompact space) ومحاولة معالجة بعض مشاكل الحياة الواقعية. يساعد إستخدام العلاقات بين الدالة المضغوطة وبعض الدوال المهمة الأخرى كدالة بورباكي الفعلية أو الدالة المغلقة على تحديد بعض الخصائص الاساسية لهذه الدالة. كما تم تعميم مفهوم الدالة المضغوطة الى أنواع جديدة أخرى والربط بينها وبين هذه الدوال الجديدة حيث تم تصنيف هذه الدوال الى فئتين الأولى تسمى بالشكل القوي والتي تؤدي الى الدالة المضغوطة ضمن شروط معينة, أما الفئة الثانية فتدعى بالشكل الأضعف والتي تؤدي إليها الدالة المضغوطة في ظل شروط معينة أيضا, بعدها تمت دراسة التركيب لهذه الدوال الجديدة.كما تم تعريف دالة الفعل المضغوطة (paracompact action) ودراسة أهم خصائصها وإثبات أن الدالة الفعلية هي دالة الفعل المضغوطة تحت شروط خاصة, وبالاعتماد على دالة الفعل المضغوطة تم تعريف فضاء جديد يدعى (الفضاء-G المضغوط) كذلك تم تعريف أنواع جديدة لهذا الفضاء كالفضاء-G شبه المضغوط , الطبيعي و ما وراء الضغط وتمت دراسة بعض الخصائص لها. إضافة الى ذلك قدم مفهوم الدالة المضغوطة الضبابية وتم دراسة خصائصها من خلال بعض النظريات بعد ذلك قمنا بتطبيق هذه الدالة على راسم الدوال التبولوجية الضبابية (FTTM).

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رسالة مقدمة الى

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من قبل

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وبإشراف

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