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Identification of the most suitable probability distribution models

A Dissertation

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By

Zahra Mahmood Fadhil

Supervised by

Prof.Dr. Kareema Abad AL-Kadim

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Contents

List of Figures		X
List of Symbols		Xiii
Chapter one	General Introduction	1-9
1.1	Introduction	2
1.2	Literature Review	3
1.3	The Problem Statement	7
1.4	Aims of the Study	8
1.5	The study Outline	8
Chapter Two	Preliminaries	10-17
2.1	Definitions and some Basic Concepts.	11
	The Hazard Function	11
	Order Statistic	12
	Maximum Likelihood Method	13
2.2	Useful Distributions	14
2.3	Number of Statistical Measurements.	16
Chapter Three	Transmuted Survival of Lindley	18-37
3.1	The Transmuted Survival Lindley Distribution	19
3.1.2	The Statistical Characteristics	26
3.2	Mode of TSLD.	34
3.3	Order Statistic of TSLD	35
3.4	Maximum Likelihood Estimators Of TSLD	36
Chapter Four	General Proposed nth Degree Hazard Rate Distribution	38-69

4.1	General Proposed nth Degree Hazard Rate model	39
4.1.1	Shape of GPHRD	41
4.1.2	The Statistical Characteristics of GPHRD	45
4.1.3	Mode of GPHRD	53
4.1.4	Order statistic of GPHRD	54
4.1.5	Maximum Likelihood Estimators of GPHRD	55
4.2	The Three degree Hazard Rate Distribution THRD.	56
4.2.1	General Proposed Hazard Rate Distribution where $n = 3$	56
4.2.3	The Statistical Characteristics of THRD	61
4.2.4	Mode of THRD	66
4.2.4	Order statistic of THRD	67
4.2.5	Maximum Likelihood Estimators of THRD	68
Chapter Five	Transmuted Hazard Model	70-100
5.1	The Transmuted Hazard Formula (TH)	71
5.2	Transmuted Hazard Models	79
5.2.1	The Transmuted Constant Hazard Rate Model	79
5.2.1	The Transmuted Power Hazard Rate Model	87
5.2.1	The Transmuted Linear Hazard Rate Model	95
Chapter Six	Transmuted Hazard General Proposed nth Degree Hazard Rate Distribution	101-153
6.1	The Transmuted Hazard General Proposed nth Degree Hazard Rate Formal	102
6.1.1	Shape of THGPHRF	105
6.1.2	Statistical Properties	109
6.1.3	MODE	116
6.1.4	Moments Generating Function of THGPHRD	117

6.1.5	Characteristic Function of THGPHRD	117
6.1.6	The Order statistic of THGPHRD	117
6.1.7	Maximum Likelihood Estimators of THGPHRD	118
6.2	The Transmuted Hazard Three Degree Hazard Rate Distribution	120
6.2.1	Shape of THTHRD	122
6.2.2	Statistical Properties	127
6.2.3	Mode of THTHRD	131
6.2.4	The Order Statistic of THTHRD	133
6.2.5	Maximum Likelihood Estimators of THTHRD	134
6.3	Transmuted Hazard Linear Exponential Distribution	138
6.3.1	Moment of THLED	142
6.3.2	Mode of THTHRD	148
6.3.3	Moments Generating Function of THTHRD	149
6.3.4	Characteristic Function of THTHRD	149
6.3.5	The Order statistic of THTHRD	150
6.3.6	Maximum Likelihood Estimators of THTHRD	151
Chapter Seven	Application	154-
7.1	Applications of TSLD	155
7.2	Application of GPHRD	157
7.3	Application of THRD	162
7.4	Application of the THGPHRD where $n=2$	166
7.5	Application of THTHRD	170
7.6	Application of THLED	175

Dedication

To my God

To the ladies of the people of Paradise, My Lady Fatimah Al-Zahra, My Lady Maryam Bint Imran, My Lady Khadija Bint Khuwaylid, My Lady Asia

To the light of my way and the sparkle of my life My Mother

To the candle of my life Muhammad

To my princess Rawan

To my presence Ali

Researcher

Zahra M.

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Researcher

Zahra M.

Abstract

In our life there are many phenomena that need to modeling according to a new appropriate distribution to be more accurate and flexible in dealing with these data .

It is aimed to find such new distributions by using the hazard rate function. This function is important in the mathematical statistics because it has wide application in many fields of life such as modern medicine and engineering etc. It is found that a new general formula for the hazard function.

The general formula for the hazard function is found that new life distributions for n of the parameters. In particular, this formula is discussed in the case of $n = 3$ and $n = 4$.

A new transmuted hazard formula is created based on the hazard Function , In addition, a set is extracted of distributions and proposed Some of their statistical properties as well as combined the two Formulas and extracted from them a new transformation formula in a General case and we proposed three special cases of them in the case of $n = 2, n = 3$ and $n = 4$.

Also, the parameters of some distributions are estimated by using the Method of Maximum likelihood and comparison between the proposed

distributions and the previous distributions, the preference of this distributions is reached by using real data using the MATLAB program.

List of Figures		
Figure	Description	page
3.1	The Density Function of TSLD	22
3.2	The Cumulative Function TSLD	23
3.3	The Survival Function of TSLD	23
3.4	The Hazard Function of TSLD	27
4.1	The Density Function of GPHRD in case $n=2$	43
4.2	The Cumulative Function of GPHRD in case $n=2$	43
4.3	The Survival Function of GPHRD in case $n=2$	44
4.4	The Hazard Function of GPHRD in case $n=2$	44
5.1	The Hazard Function of THCHRM	72
5.2	The Density Function of THCHRM	73
5.3	The Cumulative Function of THCHRM	75
5.4	The Survival Function of THCHRM	76
5.5	The Hazard Function of THPHRM	80
5.6	The PDF of THPHRM	81
5.7	The PDF of THPHRM	81
5.8	The CDF of THPHRM	83
5.9	The Survival Function of THPHRM	84
5.10	The Hazard Function of THLHRM	88
5.11	The PDF function of THLHRM	89
5.12	The PDF function of THLHRM	90

5.13	The Survival Function of THLHRM	92
6.1	The Density Function of THGPHRD in case $n=2$	98
6.2	The CDF of THGPHRD in case $n=2$	99
6.3	The Survival Function of THGPHRD in case $n=2$	99
6.4	The Hazard Function of THGPHRD in case $n=2$	101
6.5	The Density Function of THTHRD	116
6.6	The Cumulative Function of THTHRD	116
6.7	The Survival Function of THTHRD	117
6.8	The Hazard Function of THTHRD	117
6.9	The Density Function of THLED	131
6.10	The Cumulative Function of THLED	131
6.11	The Survival Function of THLED	132
6.12	The Hazard Function of THLED	133
7.1	The PDF of GPHRD in case $n=2$, when the sample size 50	160
7.2	The Survival Function of GPHRD in case $n=2$, when the sample size 50.	161
7.3	The Hazard Function of GPHRD in case $n=2$, when the sample size 50.	161
7.4	The Density Function of THRD, when the sample size 50	163
7.5	The CDF of THRD, when the sample size 50.	164
7.6	The Survival Function of THRD, when the sample size 50.	164

7.7	The Hazard Function of THRD , when the sample size 50.	165
7.8	The Density Function of THGPHRD where $n=2$, when the sample size 50 .	168
7.9	The survival function of THGPHRD where $n=2$, when the sample size 50 .	169
7.10	The CDF of THGPHRD where $n=2$, when the sample size 50	159
7.11	The hazard function of THGPHRD where $n=2$, when the sample size 50 .	170
7.12	The density function of THTHRD , when the sample size 50 .	173
7.13	The Survival Function of THTHRD, when the sample size 50 .	174
7.14	The CDF of THTHRD , when the sample size 50 .	174
7.15	The hazard function of THTHRD , when the sample size 50	165
7.16	The Density Function of LED , when the sample size 50 .	178
7.17	The Survival Function of LED, when the sample size 50	178
7.18	The hazard Function of LED , when the sample size 50	179
7.19	The Density Function of THLED , when the sample size 50 .	180
7.20	The Survival Function of THLED, when the sample size 50	180
7.21	The CDF of THLED , when the sample size 50 .	181
7.22	The hazard function of THLED , when the sample size 50 .	182

List of Abbreviation

Symbol	Description
TF	Transmuted Formula
TS	Transmuted Survival
LD	Lindley distribution
HR	Hazard Rate
ELED	exponentiated linear exponential distribution
LED	Linear exponential distribution
TH	Transmuted Hazard
BIC	The Bayesian Information Criterion
AIC	Akaike Information Criterion
TSLD	The Transmuted Survival Lindley Distribution
RD	Rayleigh distribution
WD	Weibull distribution
GED	Generalized exponential distribution
GRD	Generalized Rayleigh distribution
NGLED	A new generalized linear exponential distribution
EWD	Exponential Weibull distribution
TRD	Transmuted Rayleigh distribution
TED	Transmuted exponential distribution
MWD	Modified Weibull distribution
THRD	Three degree Hazard Rate Distribution
GPHRD	General Proposed nth Degree Hazard Rate

	Distribution
TCHRM	Transmuted Constant Hazard Rate Model
TSLD	The Transmuted Survival Lindley Distribution
GPHRD	General Proposed nth Degree Hazard Rate Distribution
THCHRM	The Transmuted Hazard Constant Hazard Rate Model
THPHRM	Transmuted Hazard Power Hazard Rate Model
THLHRM	The Transmuted Hazard Linear Hazard Rate Model
LHRM	linear hazard rate model
THGPHRF	Transmuted Hazard General Proposed nth Degree Hazard Rate Formal
THTHRD	Transmuted Hazard Three degree Hazard Rate Distribution.
THLED	Transmuted Hazard Linear Exponential Distribution

List of Symbols	
Symbol	Description
PDF	The density function
$E(T)$	The mean
CV	The Coefficient of Variation
σ	the Standard deviation
MLE	Maximum Likelihood Estimators
THF	Transmuted Hazard Formula
T	be a random variable
$Var_2(t)$	the variance of transmuted
$E_2(T)$	the mean of transmuted
ψ	Coefficient of kurtosis
$F_{Y_1}(t)$	The Cumulative Distribution Function of the minimum order statistic
$f_{Y_1}(t)$	The Density Function of the minimum order statistic
$F_{Y_n}(t)$	The Cumulative Function of the maximum order statistic
$f_{Y_n}(t)$	The Density Function of the maximum order statistic
γ	Coefficient of Skewness
Mse	Mean square error

الخلاصة

يوجد في حياتنا العديد من الظواهر التي تحتاج إلى نمذجة وفقاً لتوزيع جديد مناسب لتكون أكثر دقة ومرونة في التعامل مع هذه البيانات ، ولهذه الحاجة هدفنا إيجاد توزيعات جديدة باستخدام دالة معدل الخطر وهذه الدالة مهمة في الإحصاء الرياضي نظراً لتطبيقاتها الواسعة في العديد من مجالات الحياة ، الطب الحديث والهندسة وما إلى ذلك ، لذا قمنا بإيجاد صيغة عامة جديدة لدالة الخطورة ، ومنها اوجدنا توزيعات حياتية جديدة لـ n من المعلمات. على وجه الخصوص ، ناقشنا دراسة هذه الصيغة العامة في حالة $n = 3$ ، $n = 4$. كما أنشأنا صيغة تحويل جديدة تعتمد على دالة الخطورة وبالإضافة إلى ذلك استخرجت مجموعة من التوزيعات منها ودرسنا بعض خصائصها الإحصائية ومن ثم جمعنا الصيغتين العامتين واستخلصنا منهما صيغة تحويل جديدة بشكل عام. ودرست ثلاث حالات خاصة منها في حالة $n = 4$ ، $n = 3$ ، $n = 2$. كما قدرت معاملات بعض التوزيعات التي تم إيجادها باستخدام طريقة الامكان الاعظم والمقارنة بين التوزيعات المقترحة والتوزيعات السابقة ، ووصلنا إلى افضلية توزيعاتنا في مطابقة البيانات الحقيقية باستخدام برنامج MATLAB لإيجاد النتائج والأشكال العددية.



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زهراء محمود فاضل ابراهيم

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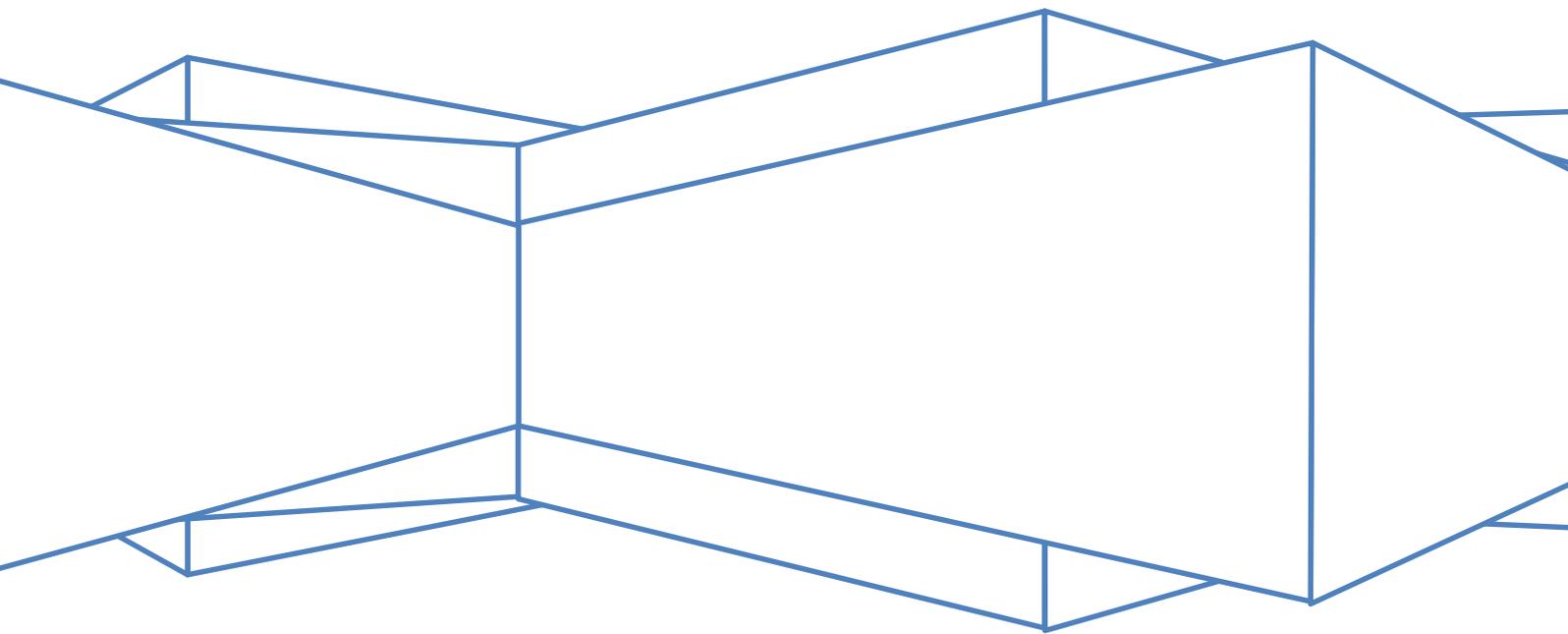
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Chapter One

General Introduction



1.1 Introduction

If you take a close look at living things such as humans and other animals, machines such as phones and computers, social and economic groups like families and churches, and political groups like governments, you'll see that they all have clearly defined states at any given point in their existence. The prognosis for a patient receiving medical care might range from alive and well to fatal. Workers can be injured and out of work or in operation, and equipment can be inoperable or operational. All of these organizations are subject to the same level of hazard.

Biostatisticians devote a lot of attention to the hazard rate function, which plays a critical role in estimating levels of hazard.

Over the years, researchers have been estimating the hazard rate function using the Kaplan-Meier and Nelson-Aalen. Hazard rate function is an important concept that can be used to postulate life distributions in the presence of several competing risk factors, it is perhaps the most popular of the techniques used in modeling and analyzing survival data. The most common use of the function is to model an entity's chance of death as a function of their age. It can also be used to model any other time-dependent event of interest.

The function models the occurrence of only one event, namely the first event, whereas the intensity function models the occurrence of a sequence of events over time. This is due to its intuitive interpretation as the amount of risk associated with a unit at age t . Another reason for its popularity is that it is a special case of the intensity function for a non-homogeneous Poisson process. Intuitively, if we have data with discrete times in weeks, months, or years, we could get an intuitive idea of the hazard rate. The hazard rate is the unobserved rate at which events occur.

the hazard rate function is more informative about the underlying mechanism of failure than the other techniques used in analyzing lifetime distributions. The hazard function assumes other aliases in other fields: Force of mortality or force of decrement in demography and actuarial sciences; intensity function in stochastic processes; in vital statistics and in the life sciences, it is known as age-specific failure rate; inverse of the Mill's ratio in economics; in point process and extreme value theory it is known as the rate function or intensity function; in the engineering sciences,

The hazard function is known as the failure rate and in reliability analysis, It is known as conditional failure rate. The failure density (PDF) measures the overall speed of failures while the hazard rate measures the dynamic speed of failures. The hazard rate for the occurrence of events may be increasing, decreasing, constant, bathtub-shaped or hump-shaped.

The hazard rate when it is converted increases and then decreases. An example of this is death during surgery. The hazard function is positive.

1.2 Literature Review.

The following are some of the related studies .

Combining the Poisson and Lindley distributions (Sankaran ,1970)[1] came up with the discrete Poisson-Lindley distribution.

The research conducted by(Greenwich,1992) [2] culminated in the use of a unimodal hazard rate function in modeling failure rate that had a relatively high rate of failure in the middle of expected life time.

Boland et al. [3] has adjudicated that hazard rate ordering was an ordering for random variables which compared lifetimes with their hazard rate functions. They maintained that the hazard rate ordering was particularly

useful in reliability theory and survival analysis, owing to the importance of the hazard rate function in those areas.

Blackstone [4] has noted that the hazard function was the instantaneous rate of occurrence of a time-related event, such as death and indicated that there were methods, we could use to determine the hazard function from clinical outcome data, to identifying risk factors for higher hazard, and to generating patient-specific predictions.

(Warren Gilchrist, 2000) [5] paid great mind to the transmutation map, especially the quadratic form.

The Lindley distribution, first proposed by Lindley is under consideration here(Lee ,2003) [6].

In survival analysis, some researchers(Clark et al.,2003) [7] who study the timing and occurrence of event, often analyze the probability distributions of the time preceding the occurrence of the event. They focus mostly on the end result of the process, rather than the processes that generated the end results, but in real life, apart from chance events, most events do not just hap pen, there may be some underlying developments preceding the events. When researchers consider the underlying processes leading to the end result, it might improve the understanding of the concept generating the end result, some studies have revealed that the hazard function plays a crucial role in characterizing the aging process.

They found by (M. Zenga ,2007) in [8] the exponentiated linear exponential distribution ELED and proved that the hazard rate function can be increasing, decreasing and bathtub shaped.

Obtaining a zero-truncated and a size-biased variant of the Poisson-Lindley distribution (Ghitany et al.,2008) [9].

(Ghitany et al., 2008)[11] went on to analyze the benefits and drawbacks of both variants. Many estimating strategies for the discrete Poisson-Lindley distribution were described by Ghitany and AlMutairi .

It can be noted that the LED where it is only increasing Hazard Rate(HR) is cognition for modeling lifetime data in reliability analysis. Several authors set forth a generalization LED to create statistical model with decreasing in(HR) unimodal and bathtub HRF The transmuted (LFRD) with increasing, decreasing, unimodal and bathtub HR shapes were introduced by (Sarhan , 2009) [12].

In the study [13] (J. Zhang , 2009) the researcher was able to prove that the hazard function can be widely applied in the survival analysis.

Several authors have considered the generalization Linear exponential distribution for the GLED (Mahmoud ,2010) [14].

The researcher was able in (S.K. Upadhyay, (2010)) [15] to apply the unilateral hazard function to know the measure of the failure rate found in survival modeling, as he found that there is a relatively high failure rate in half life expectancy in the middle of life expectancy.

When looking at stress-strength reliability models, the Lindley distribution is crucial. Furthermore, several academics have suggested new classes of distributions based on adaptations of the Lindley distribution, complete with features. Every single crucial thought is steered by incorporating earlier ,when looking at stress-strength reliability models, the Lindley distribution is crucial. Furthermore, several academics have suggested new classes of distributions based on adaptations of the Lindley distribution, complete with features. Every

single crucial thought is steered by incorporating earlier structures from fixed distributions to dynamic ones (Deniz and Ojeda ,2011) [16] proposed an insurance-related use for a discrete variant of this distribution.

Lifetime data with conflicting hazards was the topic of discussion between (Mazucheli and Achcar,2011) [17].

After obtaining an enlarged Lindley distribution(Bakouch et al., 2012) [18] examined its features and applications.

The researchers found in the study [19] the interpretation of the hazard and the procedures for its occurrence are important to reduce the disaster.

(Asi and Rama Mishra,2013) [20] looked at a distribution that was quasi Lindley In order to better analyses survival data . Another generalization of LED was known as the generalized Linear exponential distribution (GLED) .

The researchers explained in (L.K. Read and R.M. Vogel, (2016))[22] that the classification of hazard is a classification and arrangement of random variables by comparing age with the function of hazard that is specific to it.

Recently, (C.H.Lee ,2017)[23] provided some notes on GLED proposed a new transformation called inverted GLED. The fundamental point of this paper is to study how the various estimators of the unknown parameters of a GLED behave for various sample sizes and different parameter values.

We note in the research (M.A.W. Mahmoud, M.G.M. Ghazal, (2017)) [24] that the researcher touched on the idea that the hazard function can be used and benefited from in survival.

In the research (D. Kvrniasari, R. Widyarini,(2019)). [25] the researchers found The study of the hazard function is a key point in the survival analysis.

(Mohamaad,S.F. and AL-Kadim , K.A. 2021) [26] Researchers have found a new formulation (TS) called Transmuted Survival. Modeling and analyzing lifespan data are essential in many practical fields, including medical, engineering, and finance. Such data has been modeled using a number of different lifespan distributions. Such distributions include the generalizations of the exponential, Weibull, gamma, and Rayleigh families . The failure rate function might be monotonically declining, monotonically growing, or constant; it can also be nonmonotone, bathtub-shaped, or even unimodal, all of which give rise to distinctive distributions.

in the paper (S.Y. Park, 2021) [27], the researchers dealt with the concept of life survival at the time of the event and the time of the occurrence of the event.

In the research(UCLA Statistical Consulting Group ,2021) [28] the researchers noted that the constant hazard function is The gravity function of the exponential distribution is an exponential distribution.

We take advantage of reliability survival analysis, In our research, we benefited from two main topics, the hazard function and transmuted formal. We found a set of mathematical formulas through which we can find a set of new distributions for n of parameters.

1.3 The Problem Statement

The urgent need to obtain new distributions for modeling many phenomena of life is the main motivation in finding new formulas for distributions that are more accurate and flexible in dealing with that data.

1.4 Aims of the Study

Where our goal is to build two formulas, The first formula enables us to find new distributions based on the hazard function, and the second formula can produce a new distribution by taking the risk function of the traditional distribution as well as finding the mathematical and statistical properties of the new distributions. And comparing the new distributions with the original distributions in terms of validity and performance, using information standards and quantitative statistics such as Akai Standard and Quality of Fitting Quantities.

1.5 The study Outline

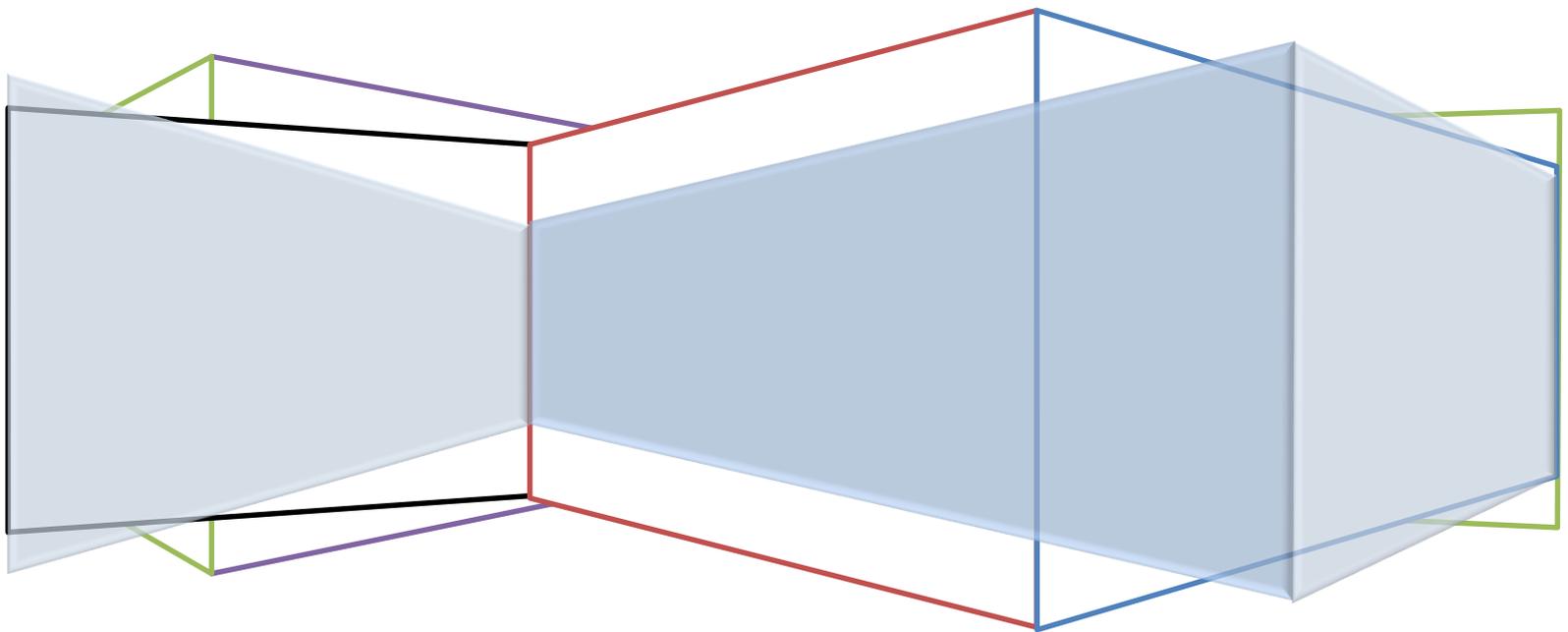
- **Chapter two** : In this chapter includes some basic concepts and definitions that used in this dissertation.
- **Chapter three**: In this chapter , a new distribution of life is found by taking advantage of TS and the LD , denoted TSLD .it is more accurate and flexible in dealing with data . LD is openly used for survival study where applied in areas of life as medical field .
- **Chapter four**: In this chapter, we studied General Proposed Hazard Rate Distribution of n^{th} , Degree based on the formula of the hazard function . The first section deals with finding the distribution in the case of n – parameters , taking a special case when $n = 2$, and the second section with $n = 3$, And in each section some statistical properties are studied , survival function, cumulative function, the r^{th} moment was introduced about the mean and about the origin, statistical rank .
- **Chapter five** : In this chapter, a new transmuted hazard formula **TH** is introduced , where the new formula is constructed the transmuted survival formula and hazard function. It is a necessary

and important in the field of biostatistics and analysis of survival data. from this new formula **TH** to get new distributions whose results in analyzing life data are better than the distributions of the original distribution of life, we studied some its statistical properties ,and using real data to test its performance .

- **Chapter six** : The formula (4.1) from chapter four and the formula (5.5) from chapter five have been merged in this chapter to generate a new formula, which we will refer to as transmuted hazard general proposed nth degree hazard rate formal. Special distribution of this new formula were also investigated. There are three major topics that have been covered in this chapter.
- **Chapter seven** : The focus of this chapter was on comparing and contrasting two sets of actual data. The first team collected information from people who had been diagnosed with bladder cancer. Or the second category is a collection of tools. Where we have applied the obtained distributions to the data, compared them to prior distributions, and concluded that our distributions are always superior.

Chapter Two

Preliminaries



This chapter includes some basic concepts and definitions that used in this dissertation.

2.1 Definitions and Some Basic Concepts

The section includes some important definitions and concepts , To find two new formulas, we have added three basic concepts: the hazard function ,transmuted formula(TF) and transmuted survival formula(TS).

2.1.1 The Hazard Function[21]

Let T be a positive random variable of a distribution have the parameter set $\Theta = \{\theta_1, \dots, \theta_k\}$ and the density function $f(t, \Theta)$ where the survival time has parameter Θ , The cumulative function is $F(t, \Theta)$, survival function $S(t, \Theta)$, The conditional probability of failure rate *Pro*, short period of time Δt , interval $(\Delta + t)$ and hazard function $h(t, \Theta)$

$$h(t, \Theta) = \frac{f(t, \Theta)}{S(t, \Theta)} \quad (1.1)$$

From the equation (1.1)

$$-h(t, \Theta) = -\frac{f(t, \Theta)}{S(t, \Theta)} = \frac{d}{dt} \ln(S(t, \Theta))$$

$$\text{Then } S(t, \Theta) = e^{-\int_0^t h(y) dy} \quad (1.2)$$

2.1.2 The Conditions of the Hazard Function .[22]

Here are three conditions which if any function satisfies, them the can be considered a hazard function.

- 1) $h(t, \Theta) \geq 0, \forall t \in \mathcal{R}^+$.
- 2) $\int_0^{\infty} h(t, \Theta) dt = \infty$.
- 3) $h(t, \Theta)$ is increasing and then decreasing or on the contrary ,over time

2.1.3 The Types of Hazard Function [2]

1- Monotonous

The function is of this type if it decreases in the case of improving the product or increases in the case of consuming the product.

2- Non monotonous

The function is of this type if it decreases in the case of improving the product, followed by a state of stability, and increases in the case of consuming the product, or vice versa.

2.1.4 Transmuted Formula TF [25]

The Transmuted Formula is

$$F_2(t, \theta) = (1 + \lambda)F(t, \theta) - \lambda F^2(t, \theta) \quad (1.3)$$

2.1.5 Transmuted Survival TS [26]

The Transmuted Survival Formula is

$$S_2(t, \theta) = (1 + \lambda)S^2(t, \theta) - \lambda S(t, \theta) \quad (1.4)$$

2.1.6 Order Statistic [6]

The order statistic is used in statistical theory, where they play an important role in statistics, Let $(T_{(1)}, T_{(2)}, \dots, T_{(n)})$ be random simple continuous iid, $\forall T_i \exists f(t)$ and $F(t)$. then $T_{(1)} \leq T_{(2)} \leq \dots \leq T_{(n)}$ be the order statistics, so that the $\forall T_{(i)}$, is the i^{th} smallest order statistics, especially.

$$T_{(1)} = Y_1 = \text{Min}(T_1, T_2, \dots, T_n).$$

$$Y_2 = \text{the 2nd order of } T_1, T_2, \dots, T_n.$$

$$T_{(n)} = Y_n = \text{Max}(T_1, T_2, \dots, T_n).$$

The cumulative distribution function of the minimum order statistic is .

$$\begin{aligned} F_{Y_1}(t) &= P(Y_1 \leq t) = \\ &= 1 - \prod_{i=1}^n P(T_i > t) = 1 - (1 - F_T(t))^n, Y_1 \in \mathcal{R}. \end{aligned} \quad \dots (1.5)$$

The density function of the minimum order statistic is .

$$f_{Y_1}(t) = n(1 - F_T(t))^{n-1} f_T(t) \quad \text{Where } f_{Y_1}(t) = F_{Y_1}'(t) \quad \dots (1.6)$$

The cumulative function of the maximum order statistic is .

$$F_{Y_n}(t) = (F_T(t))^n, Y_n \in \mathbb{R} \quad \dots (1.7)$$

Where $P(Y_n \leq t) = P(Y_1 \leq t, \dots, Y_n \leq t) = \prod_{i=1}^n P(Y_i \leq t)$

The density function of the maximum order statistic is .

$$f_{Y_n}(t) = n(F_T(t))^{n-1} f(t). \quad (1.8)$$

2.1.7 Maximum Likelihood Method (ML)[10]

It is considered one of the important methods in estimating the parameters, as the researcher Fisher found this method in the year 1922. As the method (ML) has the potential to be applied in building statistical estimators, these estimators are characterized by having the characteristics of a good estimator. This method finds the largest value of the log of likelihood function and it is used with large data and the installation of parametric statistical models and life data.

Let (T_1, T_2, \dots, T_n) are random simple of continuous iid

$\forall T_i \exists f(t, \theta)$, then the likelihood function

$$L(\theta) = \prod_{i=1}^m f(t_i, \theta) \quad \dots (1.9)$$

The second step is to take \ln to both sides of the equation (1.9)

$$\ln L(\Theta) = \ln \prod_{i=1}^m f(t_i, \Theta) \quad (1.10)$$

The third step is to derive the function (1.10) with respect to each parameter and confirm it in terms of the parameters and equate the derivatives to zero. Thus, we get the estimators. This method may need simple mathematical steps to find the estimators of the distribution, other times it is only solved by numerical methods.

2.2 Useful Distributions.

In the section, we will present some distributions that used in this study .

Lindley Distribution LD [20]

The probability density function of the lindley distribution is.

$$f(t) = \frac{\theta^2(1+t)e^{-\theta t}}{1+\theta} \quad \theta, t > 0$$

The survival function of the Lindley distribution

$$S(t) = \frac{(1+\theta t+\theta)e^{-\theta t}}{1+\theta} \quad \theta, t > 0 \dots (1.11)$$

Exponential distribution (ED) [23]

The probability density function of the exponential distribution is.

$$f(t; \beta) = \beta e^{-\beta t}, \beta > 0, t > 0 \quad \dots (1.12)$$

Rayleigh distribution (RD) [12]

The probability density function of the rayleigh distribution is.

$$f(t; \theta) = \theta t e^{-\frac{\theta t^2}{2}}, \theta > 0. \quad \dots (1.13)$$

Linear Exponential Distribution (LED)[23]

The Probability Density Function of the Linear exponential distribution is.

$$f(t; \beta, \theta) = (\beta + \theta t)e^{-(\beta t + \theta \frac{t^2}{2})}, \beta > 0, \theta > 0 \quad (1.14)$$

Weibull distribution (WD) [23]

The Probability Density Function of the WD is.

$$f(t; \theta, \gamma) = \theta \gamma t^{\gamma-1} e^{-\theta t^\gamma}, \theta > 0, \gamma > 0. \quad \dots (1.15)$$

Generalized exponential distribution (GED) [9]

The Probability Density Function of the Generalized exponential distribution GED is.

$$f(t; \alpha, \beta) = \alpha \beta e^{-\beta t} (1 - e^{-\beta t})^{\alpha-1}, \alpha > 0, \beta > 0. \quad \dots (1.16)$$

Generalized Rayleigh distribution (GRD) [15]

The Probability Density Function of the GRD is.

$$f(t; \alpha, \theta) = \alpha \theta t e^{\frac{-\theta t^2}{2}} \left(1 - e^{\frac{-\theta t^2}{2}}\right)^{\alpha-1}, \alpha > 0, \theta > 0 \dots (1.17)$$

Generalized linear exponential distribution (GLED) [21]

The Probability Density Function of the GLED is.

$$f(t; \alpha, \beta, \theta) = \alpha(\beta + \theta t)e^{-t(\beta + \frac{\theta t^2}{2})} \left(1 - e^{-t(\beta + \frac{\theta t^2}{2})}\right)^{\alpha-1} \quad \forall \alpha, \beta, \theta > 0. \quad (1.18) \text{ A generalized linear}$$

exponential distribution (NGLED)[20]

The Probability Density Function of the NGLED is.

$$f(t; \alpha, \beta, \theta, \gamma) = \alpha(\beta + \theta \gamma t^{\gamma-1})e^{-(\beta x + \theta t^\gamma)}(1 - e^{-(\beta x + \theta t^\gamma)})^{\alpha-1} \quad \forall \alpha, \beta, \theta, \gamma > 0 (1.19)$$

Modified Weibull distribution (MWD)[15]

The Probability Density Function of the MWD is.

$$f(t; \beta, \theta, \gamma) = (\beta + \theta \gamma t^{\gamma-1})e^{-(\beta x + \theta t^\gamma)}, \quad \forall \beta, \theta, \gamma > 0. \quad (1.20)$$

Exponential Weibull distribution (EWD)[15]

The Probability Density Function of the EWD is.

$$f(t; \alpha, \theta, \gamma) = \alpha \theta \gamma t^{\gamma-1} e^{-\theta t^\gamma} (1 - e^{-\theta t^\gamma})^{\alpha-1}, \forall \alpha, \theta, \gamma > 0. \quad (1.21)$$

Transmuted exponential distribution (TED)[15]

The Probability Density Function of the TED is.

$$f(t; \beta, \lambda) = \beta e^{-\beta t} (1 - \lambda + 2\lambda e^{-\beta t}), \forall \beta > 0, |\lambda| \leq 1. \quad (1.22)$$

Transmuted Rayleigh distribution (TRD)[15]

The Probability Density Function of the TRD is.

$$f(t; \theta, \lambda) = \theta t e^{-\frac{\theta t^2}{2}} (1 - \lambda + 2\lambda e^{-\frac{\theta t^2}{2}}), \forall \theta > 0, |\lambda| \leq 1. \quad (1.23)$$

2.3 Number of Statistical Measurements

To know the quality and accuracy of the new distribution, the following measures are used. The idea of these measures is for the researcher to make sure that the initial values do not differ from the expected ones and are very close to them.

2.3.1 Information Criterion[12]

Akaike Information Criterion (AIC)

was studied in 1973 by Hirotugu Akaike, as it is used in statistical analysis on real-life data to see the relative quality of that data. Its equation is as follows

$$AIC = -2 \ln L + 2q \quad (1.24)$$

If we have an AIC that has a large negative value, then Hurvich and Tsai take the corrected Akaike information criterion, because the

CAIC of this criterion can be estimated with a less slope than the first criterion and its following formula

$$CAIC = -2 \ln L + \frac{2qm}{m-q-1} \quad (1.25)$$

The Bayesian Information Criterion (BIC)

Schwarz (1978) studied this standard and used it to compare between statistical models

And his formula

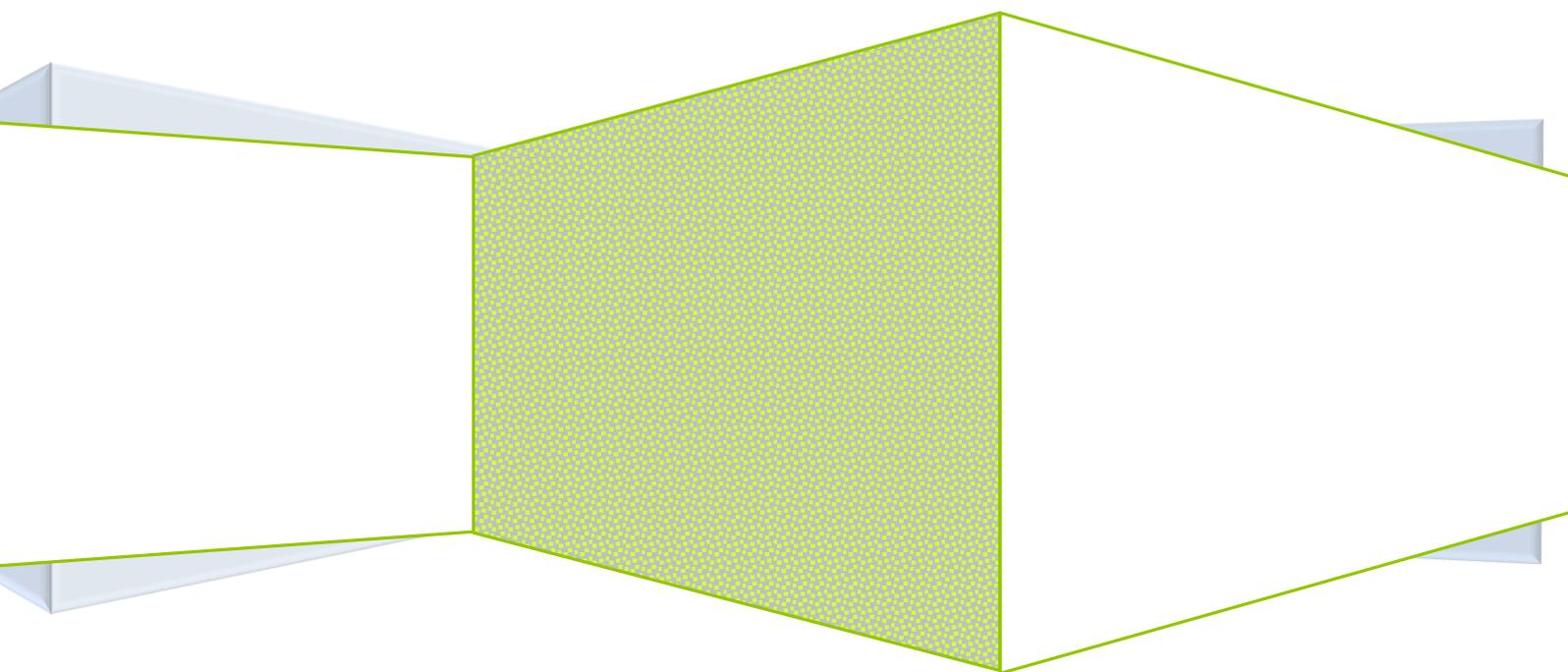
$$BIC = -2 \ln L + q \ln (m), \quad (1.26)$$

Note : Where m represents the sample size of the real data that we have studied .And q represents the magnitude of the parameters of the distribution we're working with.

/

Chapter Three

Transmuted Survival of Lindley Distribution



In this chapter, a new distribution was presented as a survival model by utilizing the survival function of the quadratic-transmuted distribution, where the quadratic-transmuted Lindley di

tribution was used to derive the transmuted survival (TSL) Lindley distribution, which is essential because it is more flexible and accurate in data applications. Since there are occasionally data points that do not meet the standard distribution, the new distribution provides more accurate results when applied to the data, and the probability density function and cumulative probability function are extracted. New deployment properties with reliable performance were derived from a statistical and mathematical perspective. We also estimated a dataset using traditional methods.

3.1 The Transmuted Survival Lindley Distribution (TSLD)

Let T be a random variable with a Lindley distribution. Substituting the survival function of Lindley distribution LD (1.11) into Transmuted survival formal TS (1.3), the survival function of the new distribution Transmuted survival Lindley distribution (TSLD) is obtained.

$$S_2(t) = \frac{(1 + \lambda)(1 + \theta t + \theta)^2 e^{-2\theta t}}{(1 + \theta)^2} - \frac{\lambda(1 + \theta t + \theta)e^{-\theta t}}{1 + \theta} \quad \dots (3.1)$$

the Cumulative Function of (TSLD)

$$F_2(t) = 1 - \frac{(1 + \lambda)(\theta t + \theta + 1)^2 e^{-2(\theta t)}}{(1 + \theta)^2} + \frac{\lambda(\theta t + \theta + 1)e^{-(\theta t)}}{(1 + \theta)}. \quad (3.2)$$

The Density Function of (TSLD)

The PDF of TSLD is derivative of the cumulative function TSLD

$$f_2(t) = -\frac{(2\theta + 2\theta\lambda)[-(\theta t + \theta + 1)^2 e^{-2\theta t} + (\theta t + \theta + 1)e^{-\theta t}]}{(1 + \theta)^2} + \frac{\lambda(\theta)[-(\theta t + \theta + 1)e^{-\theta t} + e^{-\theta t}]}{1 + \theta} \quad \text{where } \theta > 0, -1 \leq \lambda \leq 1$$

$$f_2(t) = \frac{(1 + \lambda)}{(1 + \theta)^2} (2\theta^2)(\theta t + \theta + 1)e^{-2\theta t}[(1 + t)] - \frac{\lambda\theta^2}{1 + \theta} e^{-\theta t}[(1 + t)]$$

$$f_2(t) = \frac{(\theta^2)(1+t)e^{-\theta t}}{(1+\theta)} \left[\frac{2(1+\lambda)}{(1+\theta)} (\theta t + \theta + 1)e^{-\theta t} - \lambda \right] \dots\dots(3.3)$$

The function $f_2(t)$ should satisfy the conditions of the density function

The first condition $f_2(t) \geq 0 \quad \forall \quad t > 0$ and at

$$[2(1 + \lambda)(1 + \theta t + \theta)e^{-\theta t} - \lambda(1 + \theta)] \geq 0$$

$$\frac{2(\theta t + \theta + 1)e^{-\theta t}}{(\theta + 1) - 2(\theta t + \theta + 1)e^{-\theta t}} \geq \lambda$$

The second condition $\int_0^\infty f_2(t)dt = 1$

$$\int_0^\infty f_2(t)dt = \int_0^\infty \left[\frac{(1+\lambda)}{(1+\theta)^2} (2\theta^2)(\theta t + \theta + 1)e^{-2\theta t}(1 + t) - \frac{\lambda\theta^2}{1+\theta} e^{-\theta t}(1 + t) \right] dt$$

$$= \int_0^\infty \frac{(1+\lambda)(2\theta^2)(\theta t + \theta + 1)(1+t)e^{-2\theta t}}{(1+\theta)^2} dt - \int_0^\infty \frac{\lambda(\theta^2)e^{-\theta t}(1+t)}{1+\theta} dt$$

Let

$$L1 = \frac{(1+\lambda)(2\theta^2)}{(1+\theta)^2} \int_0^\infty (\theta t + \theta + 1)(1+t)e^{-2\theta t} dt$$

And

$$L2 = \frac{\lambda (\theta^2)}{1 + \theta} \int_0^{\infty} e^{-\theta t} (1 + t) dt$$

$$L1 = (1 + \lambda) \frac{(2\theta^2)}{(1 + \theta)^2} \int_0^{\infty} ((1 + \theta) + (2\theta + 1)t + \theta t^2) e^{-2\theta t} dt$$

$$L1 = (1 + \lambda) \frac{(2\theta^2)}{(1 + \theta)^2} \left((1 + \theta) \int_0^{\infty} e^{-2\theta t} dt + (2\theta + 1) \int_0^{\infty} t e^{-2\theta t} dt + \theta \int_0^{\infty} t^2 e^{-2\theta t} dt \right)$$

$$L1 = (1 + \lambda) \frac{(2\theta^2)}{(1 + \theta)^2} \left(\frac{(1 + \theta)}{2\theta} + \frac{(2\theta + 1)}{4\theta^2} + \frac{1}{4\theta^2} \right)$$

$$L1 = (1 + \lambda) \frac{(2\theta^2)}{(1 + \theta)^2} \left(\frac{(2\theta + 4\theta^2)}{4\theta^2} + \frac{(2\theta + 2)}{4\theta^2} + \frac{1}{4\theta^2} \right)$$

$$L1 = \frac{(1 + \lambda)(2\theta^2)}{(1 + \theta)^2} \left(\frac{(\theta + 1)^2}{2\theta^2} \right)$$

$$L1 = (1 + \lambda)$$

$$L2 = \int_0^{\infty} \left[\frac{\lambda}{1 + \theta} (\theta^2) e^{-\theta t} (1 + t) \right] dt$$

$$L2 = \frac{\lambda}{1 + \theta} (\theta^2) \int_0^{\infty} [(1 + t) e^{-\theta t}] dt$$

$$L2 = \frac{\lambda}{1 + \theta} (\theta^2) \int_0^{\infty} [(1 + t) e^{-\theta t}] dt$$

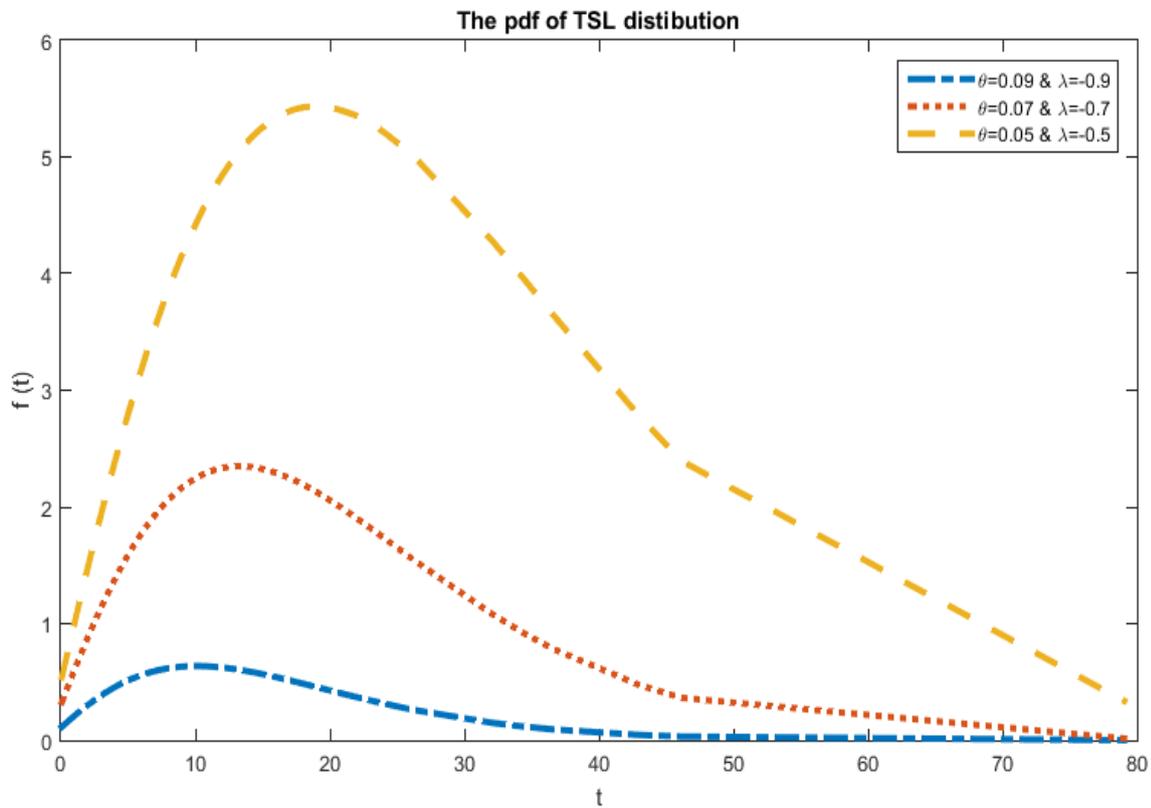
$$L2 = \lambda$$

Then

$$\int_0^{\infty} f_2(t) dt = 1$$

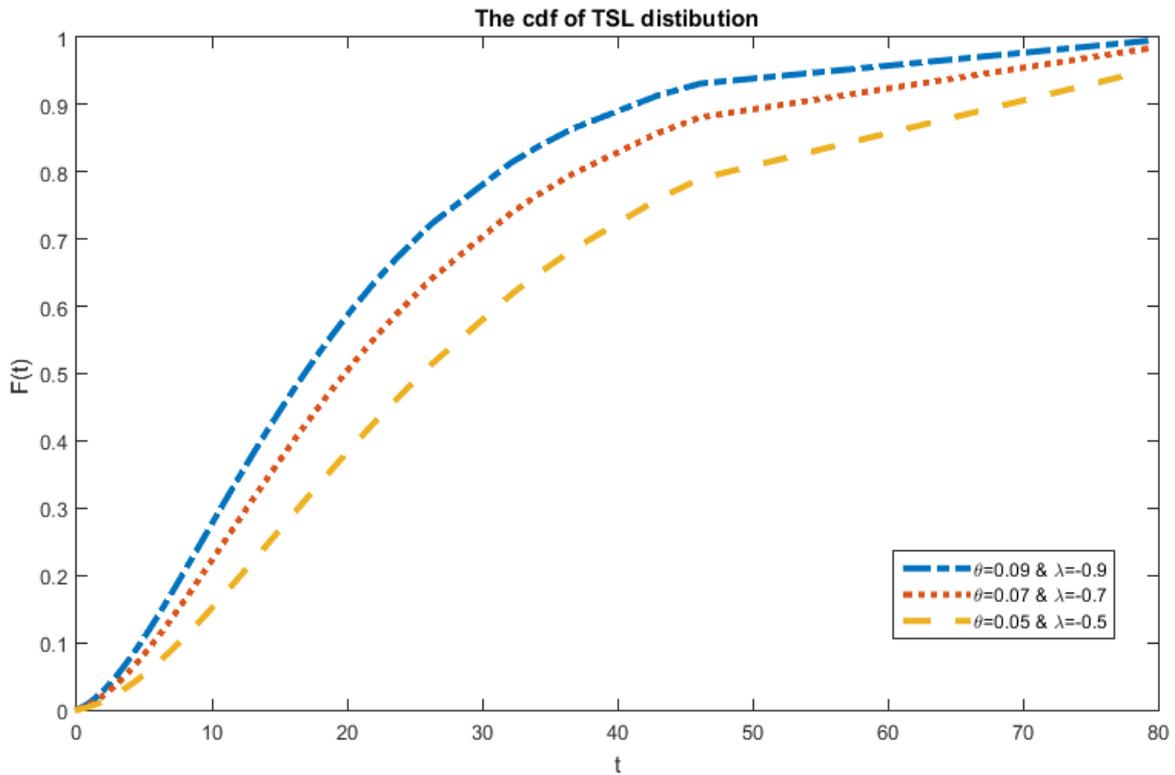
$f_2(t)$ density function is obtained .

The following figures some shapes for The Density Function of the TSLD, The Cumulative Function of the TSLD, The Survival Function of the TSLD, The Hazard Function of the TSLD, for appointed values of the parameters θ, λ



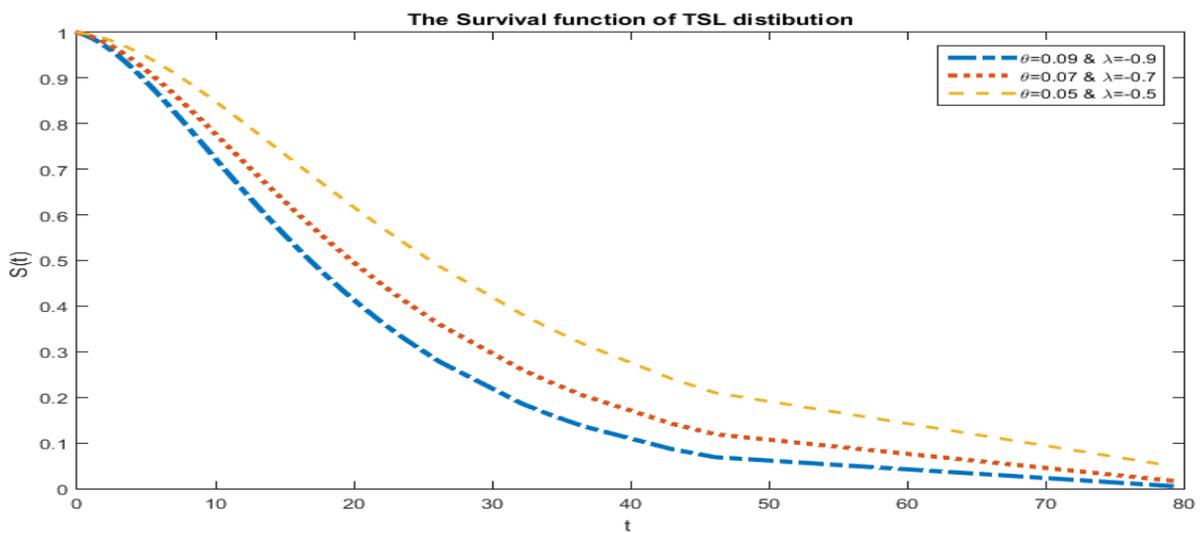
Figure(3.1): The Density Function of TSLD with different values of θ and λ ,

It is observed the function is increasing and then decreasing with increasing time, it is twisted to the right and has one value.



Figure(3.2) : The Cumulative Function TSLD with different values of θ and λ .

the Distribution function is monotonously increasing with increasing time.



Figure(3.3): The Survival Function of TSLD with different values of θ and λ . the Survival Function monotonously decreasing with increasing time, it is twisted to the right .

3.1.1 Shape of TSLD

3.1.1.2 The Limit of PDF of TSLD

$$\lim_{t \rightarrow \infty} f_2(t) = \frac{(\theta^2)}{1 + \theta} \lim_{t \rightarrow \infty} \left[\frac{(2)(1 + \lambda)}{(1 + \theta)} (1 + \theta t + \theta) e^{-2\theta t} (1 + t) - \lambda e^{-\theta t} (1 + t) \right]$$

$$\lim_{t \rightarrow \infty} f_2(t) = \frac{(\theta^2)}{1 + \theta} \left[\frac{(2)(1 + \lambda)}{-2\theta(1 + \theta)} (\theta) 0 - \frac{\lambda 0(1)}{-\theta} \right]$$

$$\lim_{t \rightarrow \infty} f_2(t) = 0.$$

NOW,

$$\lim_{t \rightarrow 0} f_2(t) = \frac{(\theta^2)}{1 + \theta} \lim_{t \rightarrow 0} \left[\frac{(2)(1 + \lambda)}{(1 + \theta)} (1 + \theta t + \theta) e^{-2\theta t} (1 + t) - \lambda e^{-\theta t} (1 + t) \right]$$

$$\lim_{t \rightarrow 0} f_2(t) = \frac{(\theta^2)}{1 + \theta} \left[\frac{(2)(1 + \lambda)}{(1 + \theta)} (1 + \theta 0 + \theta) e^{-2\theta 0} (1 + 0) - \lambda e^{-\theta 0} (1 + 0) \right]$$

$$\lim_{t \rightarrow 0} f_2(t) = \frac{(\theta^2)}{1 + \theta} \left[\frac{(2)(1 + \lambda)}{(1 + \theta)} (1 + \theta) - \lambda \right]$$

$$\lim_{t \rightarrow 0} f_2(t) = \frac{(\theta^2)}{1 + \theta} [(2 + 2\lambda) - \lambda]$$

$$\lim_{t \rightarrow 0} f_2(t) = \frac{(\theta^2)(2 + \lambda)}{(1 + \theta)}.$$

3.1.1.2. The Limit of CDF and Survival Function of TSLD

$$\lim_{t \rightarrow \infty} F_2(t) = \lim_{t \rightarrow \infty} \left(1 - (1 + \lambda) \frac{1}{(1 + \theta)^2} (1 + \theta t + \theta)^2 e^{-2\theta t} + \lambda \frac{1}{1 + \theta} (1 + \theta t + \theta) e^{-\theta t} \right)$$

$$\lim_{t \rightarrow \infty} F_2(t) = \left(1 - \frac{(1 + \lambda)}{(1 + \theta)^2} \lim_{t \rightarrow \infty} (1 + \theta t + \theta)^2 e^{-2\theta t} + \frac{\lambda}{1 + \theta} \lim_{t \rightarrow \infty} (1 + \theta t + \theta) e^{-\theta t} \right)$$

$$\lim_{t \rightarrow \infty} F_2(t) = \left(1 - \frac{(1 + \lambda)}{(1 + \theta)^2} 0 + \frac{\lambda}{1 + \theta} 0 \right)$$

$$\lim_{t \rightarrow \infty} F_2(t) = 1$$

This mean If t convergent to infinity then the limit of the cdf is equal to one is obtained.

Now

$$\lim_{t \rightarrow 0} F_2(t) = \lim_{t \rightarrow 0} \left(1 - (1 + \lambda) \frac{1}{(1 + \theta)^2} (1 + \theta(1 + t) + \theta(1 + t) + \theta^2(1 + t)^2) e^{-2\theta t} + \lambda \frac{1}{1 + \theta} (1 + \theta t + \theta) e^{-\theta t} \right)$$

$$\lim_{t \rightarrow 0} F_2(t) = \lim_{t \rightarrow 0} \left(1 - (1 + \lambda) \frac{1}{(1 + \theta)^2} (1 + \theta(1 + 0) + \theta(1 + 0) + \theta^2(1 + 0)^2) e^{-2\theta 0} + \lambda \frac{1}{1 + \theta} (1 + \theta 0 + \theta) e^{-\theta 0} \right)$$

$$\lim_{t \rightarrow 0} F_2(t) = \lim_{t \rightarrow 0} \left(1 - (1 + \lambda) \frac{1}{(1 + \theta)^2} (1 + 2\theta + \theta^2) + \lambda \frac{1}{1 + \theta} (1 + \theta) \right)$$

$$\lim_{t \rightarrow 0} F_2(t) = 0$$

If t convergent to zero than the limit of the cdf is equal to zero is obtained .

It is concluded that the range of the CDF function is closed interval and its domain is the positive real numbers and that it is a positive which satisfies the conditions of cumulative Distribution function.

and vice versa with the survival function .

$$\lim_{t \rightarrow 0} S_2(t) = 1$$

$$\lim_{t \rightarrow \infty} S_2(t) = 0$$

3.1.2 The Statistical Characteristics

These characteristics present formulas of The Hazard Function , Moments, Moment Generating Function, Order Statistics of TSLD and Maximum Likelihood Estimators Of TSLD.

3.1.2.1 The Hazard Function: [8]

The Hazard Function is defined by

$$h_2(t) = \frac{f_2(t)}{S_2(t)}$$

$$h_2(t) = \frac{f(t)(2(1 + \lambda)S(t) - \lambda)}{S(t)((1 + \lambda)S(t) - \lambda)}$$

$$h_2(t) = h(t) \frac{(2(1 + \lambda)S(t) - \lambda)}{((1 + \lambda)S(t) - \lambda)}$$

$$h_2(t) = h(t) \left[\frac{\lambda}{((1 + \lambda)S(t) - \lambda)} + 2 \right] \dots\dots\dots(3.4)$$

Then the Hazard Function of TSLD .

$$h_2(t) = \frac{\theta(1+t)}{1 + \theta(1+t)} \left[\frac{\lambda}{\left((1+\lambda) \frac{1}{1+\theta} (1 + \theta t + \theta) e^{-\theta t - \lambda} \right)} + 2 \right] \dots\dots\dots(3.5)$$

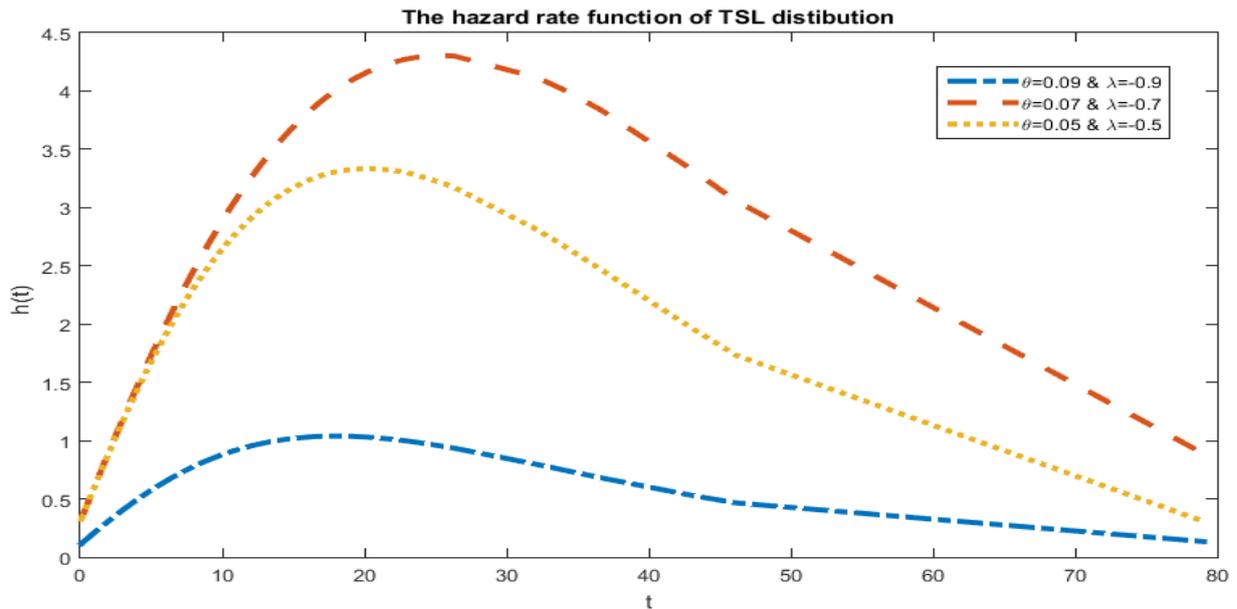


Figure (3.4): The Hazard Function of TSLD with Altered Values of θ and λ ,

Now we can find the limit of hazard function when T goes to zero and infinite .

$$\lim_{t \rightarrow 0} h_2(t, \theta, \lambda) = \lim_{t \rightarrow 0} \frac{\theta(1+t)}{1 + \theta(1+t)} \left[\frac{\lambda}{\left((1+\lambda)^{\frac{1}{1+\theta}} (1 + \theta t + \theta) e^{-\theta t - \lambda} \right)} + 2 \right]$$

$$\lim_{t \rightarrow 0} h_2(t, \theta, \lambda) = \frac{\theta(\lambda+2)}{1 + \theta}$$

And

$$\lim_{t \rightarrow \infty} h_2(t, \theta, \lambda) = \lim_{t \rightarrow \infty} \frac{\theta(1+t)}{1 + \theta(1+t)} \left[\frac{\lambda}{\left((1+\lambda)^{\frac{1}{1+\theta}} (1 + \theta t + \theta) e^{-\theta t - \lambda} \right)} + 2 \right]$$

$$\lim_{t \rightarrow \infty} h_2(t, \theta, \lambda) = \infty$$

Cumulative Hazard Function of TSLD

$$H_2(t) = \log S_2(x)$$

$$H_2(t) = \log S(t) + \log((1 + \lambda)S(t) - \lambda)$$

$$H_2(t) = \log\left(\frac{1}{1+\theta} (1 + \theta t + \theta)e^{-\theta t}\right) + \log\left[\left(\frac{(1+\lambda)}{1+\theta} (1 + \theta t + \theta)e^{-\theta t} - \lambda\right)\right]$$

$$H_2(t) = (\log(1 + \theta t + \theta) - \log(1 + \theta) - \theta t) + \log\left[\left(\frac{(1+\lambda)(1 + \theta t + \theta)e^{-\theta t} - \lambda(1 + \theta)}{(1 + \theta)}\right)\right]$$

$$H_2(t) = (\log(1 + \theta t + \theta) - 2\log(1 + \theta) - \theta t) + \log\left((1 + \lambda)(1 + \theta t + \theta)e^{-\theta t} - \lambda(1 + \theta)\right) \dots\dots\dots(3.6)$$

3.1.2.2 Moment of TSLD:[3][11]

This section presents the *r*th moment and moment generating function of TSLD.

Theorem 3.1

If $T \sim \text{TSLD}(\lambda, \theta)$, the *r*th central moment about the origin ,and the *r*th central moment and about the mean μ as follows :

$$E_2(T^r) = \frac{(1+\lambda)r!}{(2\theta)^r} \left(1 + \frac{r^2+5r+4\theta r}{4(1+\theta)^2}\right) - \frac{\lambda r!}{1+\theta} \left(\frac{\theta + r + 1}{(\theta)^r}\right). \dots(3.7)$$

$$E_2(T - \mu)^r = \sum_{k=0}^r C_k^r \left[\frac{k!}{(2\theta)^k} \left[(1 + \lambda) \left(1 + \frac{k(k+5+4\theta)}{4(1+\theta)^2}\right) - \frac{\lambda k!}{1+\theta} \left(\frac{\theta + k + 1}{(\theta)^r}\right) \right] \right] \left(- \left(\frac{1}{(2\theta)} \left[(1 + \lambda) \left(1 + \frac{(3+2\theta)}{2(1+\theta)^2}\right) - \frac{2\lambda(\theta + 2)}{1+\theta} \right] \right) \right)^{r-k} \dots(3.8)$$

Proof

The r th central moment about the origin is given

$$E_2(T^r) = \int_0^{\infty} t^r f_2(t) dt = \int_0^{\infty} t^r \left[\frac{(1+\lambda)}{(1+\theta)^2} (2\theta^2)(1+\theta t + \theta) e^{-2\theta t} (1+t) \right] dt - \int_0^{\infty} t^r \left[\frac{\lambda}{1+\theta} (\theta^2) e^{-\theta t} (1+t) \right] dt$$

Suppose that

$$L1 = \int_0^{\infty} t^r \left[\frac{(1+\lambda)}{(1+\theta)^2} (2\theta^2)(1+\theta t + \theta)(1+t) e^{-2\theta t} \right] dt$$

And

$$L2 = \int_0^{\infty} t^r \left[\frac{\lambda}{1+\theta} (\theta^2) e^{-\theta t} (1+t) \right] dt$$

Now

$$L1 = \frac{(1+\lambda)(2\theta^2)}{(1+\theta)^2} \int_0^{\infty} t^r [(1+\theta t + \theta) e^{-2\theta t} (1+t)] dt$$

$$L1 = \frac{(1+\lambda)(2\theta^2)}{(1+\theta)^2} \int_0^{\infty} t^r ((1+\theta) + t(2\theta+1) + t^2\theta) e^{-2\theta t} dt$$

$$L1 = \frac{(1+\lambda)(2\theta^2)}{(1+\theta)^2} \int_0^{\infty} ((1+\theta)t^r + (2\theta+1)t^{r+1} + \theta t^{r+2}) e^{-2\theta t} dt$$

$$L1 = \frac{(1+\lambda)(2\theta^2)}{(1+\theta)^2} \left((1+\theta) \int_0^{\infty} t^r e^{-2\theta t} dt + (2\theta+1) \int_0^{\infty} t^{r+1} e^{-2\theta t} dt + \theta \int_0^{\infty} t^{r+2} e^{-2\theta t} dt \right)$$

$$L1 = \frac{(1+\lambda)(2\theta^2)}{(1+\theta)^2} \left(\frac{(1+\theta) r!}{(2\theta)^{r+1}} + \frac{(1+2\theta) (r+1)!}{(2\theta)^{r+2}} + \frac{(\theta) (r+2)!}{(2\theta)^{r+3}} \right)$$

$$L2 = \frac{\lambda}{1+\theta} (\theta^2) \int_0^{\infty} t^r [e^{-\theta t} (1+t)] dt$$

$$L2 = \frac{\lambda}{1+\theta} (\theta^2) \left(\int_0^{\infty} t^r e^{-\theta t} dt + \int_0^{\infty} t^r t e^{-\theta t} dt \right)$$

$$L_2 = \frac{\lambda}{1+\theta} (\theta^2) \left(\frac{r!}{(\theta)^{r+1}} + \frac{(r+1)!}{(\theta)^{r+2}} \right)$$

$$L_2 = \frac{\lambda r!}{1+\theta} \frac{(\theta + (r+1))}{(\theta)^r}$$

$$\int_0^\infty t^r f_2(t) dt = \frac{(1+\lambda)(2\theta^2)}{(1+\theta)^2} \left(\frac{(1+\theta)r!}{(2\theta)^{r+1}} + \frac{(1+2\theta)(r+1)!}{(2\theta)^{r+2}} + \frac{(\theta)(r+2)!}{(2\theta)^{r+3}} \right) - \frac{\lambda}{1+\theta} \left(\frac{(\theta+r+1)r!}{(\theta)^r} \right)$$

$$E_2(T^r) = \frac{(1+\lambda)(2\theta^2)}{(1+\theta)^2} \left(\frac{(1+\theta)r!}{(2\theta)^{r+1}} + \frac{(1+2\theta)(r+1)!}{(2\theta)^{r+2}} + \frac{(\theta)(r+2)!}{(2\theta)^{r+3}} \right) - \frac{\lambda}{1+\theta} \left(\frac{(\theta+r+1)r!}{(\theta)^r} \right)$$

$$E_2(T^r) = \frac{(1+\lambda)r!(2\theta^2)}{(2\theta)^r(1+\theta)^2} \left(\frac{(1+\theta)}{(2\theta)} + \frac{(1+2\theta)(r+1)}{(2\theta)^2} + \frac{(\theta)(r+1)(r+2)}{(2\theta)^3} \right) - \frac{\lambda}{1+\theta} \left(\frac{(\theta+r+1)r!}{(\theta)^r} \right)$$

$$E_2(T^r) = \frac{(1+\lambda)r!(2\theta^2)}{(2\theta)^r(1+\theta)^2} \left(\frac{4\theta(1+\theta)+2(r+1+2\theta+2\theta r)+(r^2+3r+2)}{8(\theta)^2} \right) - \frac{(2)^r \lambda r!}{1+\theta} \left(\frac{(\theta+r+1)}{(2\theta)^r} \right)$$

$$E_2(T^r) = \frac{(1+\lambda)r!}{(2\theta)^r(1+\theta)^2} \left(\frac{8\theta+4\theta^2+5r+4+r^2+4\theta r}{4} \right) - \frac{(2)^r \lambda r!}{1+\theta} \left(\frac{(\theta+r+1)}{(2\theta)^r} \right)$$

$$E_2(T^r) = \frac{(1+\lambda)r!}{(2\theta)^r} \left(1 + \frac{r^2+5r+4r\theta}{4(1+\theta)^2} \right) - \frac{\lambda r!}{(\theta)^r} \left(\frac{(\theta+r+1)}{1+\theta} \right).$$

It is prove second part from theorem based on Binomial theorem

$$E_2(T - \mu)^r = \sum_{k=0}^r C_k^r \left[\frac{(1+\lambda)r!}{(2\theta)^r} \left(1 + \frac{k^2+5k+4k\theta}{4(1+\theta)^2} \right) - \frac{\lambda k!}{(\theta)^k} \left(\frac{(\theta+k+1)}{1+\theta} \right) \right] \left(-\frac{(1+\lambda)}{2\theta} \left(1 + \frac{3+2\theta}{2(1+\theta)^2} \right) + \frac{\lambda}{\theta} \left(\frac{2+\theta}{1+\theta} \right) \right)^{r-k}$$

Depending on the first part of the proof , it is find the expected value and variance

The Expected Value of TSLD .

$$E_2(T) = \frac{(1 + \lambda)}{2\theta} \left(1 + \frac{3 + 2\theta}{2(1 + \theta)^2} \right) - \frac{\lambda}{\theta} \left(\frac{2 + \theta}{1 + \theta} \right)$$

$$E_2(T^2) = \frac{2(1+\lambda)}{(4)\theta^2} \left(1 + \frac{7+4\theta}{2(1+\theta)^2} \right) - \frac{2\lambda}{\theta^2} \left(\frac{3 + \theta}{1 + \theta} \right) \dots (3.9) .$$

The Variance of TSLD .

$$Var_2(t) = \frac{2(1+\lambda)}{(4)\theta^2} \left(1 + \frac{7+4\theta}{2(1+\theta)^2} \right) - \frac{2\lambda}{\theta^2} \left(\frac{3 + \theta}{1 + \theta} \right) - \left(\frac{(1+\lambda)}{2\theta} \left(1 + \frac{3 + 2\theta}{2(1+\theta)^2} \right) - \frac{\lambda}{\theta} \left(\frac{2+\theta}{1+\theta} \right) \right)^2 \dots(3.10)$$

Now

Note based on part two from theorem (4.1.1), we can get the *SK* and *KU* is obtained

Let

$$SK = E(T - \mu)^3 =$$

$$SK = \sum_{k=0}^3 C_k^3 \left[\frac{3(1+\lambda)}{4(\theta)^3} \left(1 + \frac{k^2+5k+4k\theta}{4(1+\theta)^2} \right) - \frac{\lambda k!}{(\theta)^k} \left(\frac{\theta + k + 1}{1 + \theta} \right) \right] \left(-\frac{(1+\lambda)}{2\theta} \left(1 + \frac{3 + 2\theta}{2(1+\theta)^2} \right) + \frac{\lambda}{\theta} \left(\frac{2+\theta}{1+\theta} \right) \right)^{3-k} \dots(3.11)$$

And

$$KU = E(T - \mu)^4 =$$

$$KU = \sum_{k=0}^4 C_k^4 \left[\frac{3(1+\lambda)}{2(\theta)^4} \left(1 + \frac{k^2+5k+4k\theta}{4(1+\theta)^2} \right) - \frac{\lambda k!(\theta + k + 1)}{(\theta)^k(1+\theta)} \right] \left(\frac{-(1+\lambda)}{2\theta} \left(1 + \frac{3 + 2\theta}{2(1+\theta)^2} \right) + \frac{\lambda(2+\theta)}{\theta(1+\theta)} \right)^{4-k} \dots\dots\dots(3.12)$$

The Coefficient of Variation of TSLD

$$CV = \left(\frac{2(1+\lambda)}{(4)\theta^2} \left(1 + \frac{7+4\theta}{2(1+\theta)^2} \right) - \frac{2\lambda}{\theta^2} \left(\frac{3+\theta}{1+\theta} \right) - \left(\frac{(1+\lambda)}{2\theta} \left(1 + \frac{3+2\theta}{2(1+\theta)^2} \right) - \frac{\lambda}{\theta} \left(\frac{2+\theta}{1+\theta} \right) \right)^2 \right)^{1/2} \left(\frac{(1+\lambda)}{2\theta} \left(1 + \frac{3+2\theta}{2(1+\theta)^2} \right) - \frac{\lambda}{\theta} \left(\frac{2+\theta}{1+\theta} \right) \right)^{-1} \dots (3.13)$$

Standard deviation of TSLD .

$$\sigma = (V_2(t))^{1/2}$$

$$\sigma = \left(\frac{2(1+\lambda)}{(4)\theta^2} \left(1 + \frac{7+4\theta}{2(1+\theta)^2} \right) - \frac{2\lambda}{\theta^2} \left(\frac{3+\theta}{1+\theta} \right) - \left(\frac{(1+\lambda)}{2\theta} \left(1 + \frac{3+2\theta}{2(1+\theta)^2} \right) - \frac{\lambda}{\theta} \left(\frac{2+\theta}{1+\theta} \right) \right)^2 \right)^{\frac{1}{2}} . (3.14)$$

Coefficient of Skewness of TSLD .

$$\gamma = SK (V_2(t))^{-3/2}$$

$$\gamma = \sum_{k=0}^3 C_k^3 \left[\frac{3(1+\lambda)}{4(\theta)^3} \left(1 + \frac{k^2+5k+4k\theta}{4(1+\theta)^2} \right) - \frac{\lambda k!}{(\theta)^k} \left(\frac{\theta+k+1}{1+\theta} \right) \right] \left(-\frac{(1+\lambda)}{2\theta} \left(1 + \frac{3+2\theta}{2(1+\theta)^2} \right) + \frac{\lambda}{\theta} \left(\frac{2+\theta}{1+\theta} \right) \right)^{3-k} \left(\frac{2(1+\lambda)}{(4)\theta^2} \left(1 + \frac{7+4\theta}{2(1+\theta)^2} \right) - \frac{2\lambda}{\theta^2} \left(\frac{3+\theta}{1+\theta} \right) - \left(\frac{(1+\lambda)}{2\theta} \left(1 + \frac{3+2\theta}{2(1+\theta)^2} \right) - \frac{\lambda}{\theta} \left(\frac{2+\theta}{1+\theta} \right) \right)^2 \right)^{-3/2} \dots (3.15)$$

Coefficient of kurtosis of TSLD

$$\psi = KU (V_2(t))^{-2}$$

$$\psi = \left[\sum_{k=0}^4 C_k^4 \left[\frac{3(1+\lambda)}{2(\theta)^4} \left(1 + \frac{k^2+5k+4k\theta}{4(1+\theta)^2} \right) - \frac{\lambda k!(\theta+k+1)}{(\theta)^k(1+\theta)} \right] \left(\frac{-(1+\lambda)}{2\theta} \left(1 + \frac{3+2\theta}{2(1+\theta)^2} \right) + \frac{\lambda(2+\theta)}{\theta(1+\theta)} \right)^{4-k} \right] \left(\frac{2(1+\lambda)}{(4)\theta^2} \left(1 + \frac{7+4\theta}{2(1+\theta)^2} \right) - \frac{2\lambda}{\theta^2} \left(\frac{3+\theta}{1+\theta} \right) - \left(\frac{(1+\lambda)}{2\theta} \left(1 + \frac{3+2\theta}{2(1+\theta)^2} \right) - \frac{\lambda}{\theta} \left(\frac{2+\theta}{1+\theta} \right) \right)^2 \right)^{-2} \dots\dots\dots(3.16)$$

3.1.2 Moments Generating Function of TSLD

$M_T(z) = E_2(e^{zT}) = \int_0^\infty e^{zT} f_2(t) dt$ where $-h < z < h$ and $h > 0, h \in R$

$M_T(z) = E_2(e^{zT}) = \sum_{r=1}^\infty \frac{z^r}{r!} E_2(t^r)$

$M_T(z) = \sum_{r=1}^\infty \frac{(z)^r}{r!} \frac{r!}{(2\theta)^r} \left[(1+\lambda) \left(1 + \frac{r(r+5+4\theta)}{4(1+\theta)^2} \right) - (2)^r \lambda \left(\frac{(\theta+r+1)}{1+\theta} \right) \right] \dots\dots\dots(3.17)$

3.1.3. Characteristic Function of TSLD

$M_t(ix) =$

$\sum_{r=1}^\infty \frac{(ix)^r}{r!} \frac{r!}{(2\theta)^r} \left[(1+\lambda) \left(1 + \frac{r(r+5+4\theta)}{4(1+\theta)^2} \right) - (2)^r \lambda \left(\frac{(\theta+r+1)}{1+\theta} \right) \right] \dots\dots\dots(3.18)$

3.1.4 Quantile function of TSLD

The function is inverse of cdf . such that $t = Q(p) = F^{-1}(p)$ where

$0 < p < 1$

$p = 1 - \frac{(1+\lambda)(1+\theta t+\theta)^2 e^{-2\theta t}}{(1+\theta)^2} + \frac{\lambda(1+\theta t+\theta)e^{-\theta t}}{1+\theta}$

$1 - p = \frac{(1+\theta t+\theta)e^{-\theta t}}{1+\theta} \left(\frac{(1+\lambda)(1+\theta t+\theta)e^{-\theta t}}{(1+\theta)} + \lambda \right)$

$\ln(1 - p) = \ln \left[\frac{(1+\theta t+\theta)e^{-\theta t}}{1+\theta} \left(\frac{(1+\lambda)(1+\theta t+\theta)e^{-\theta t}}{(1+\theta)} + \lambda \right) \right] \dots\dots\dots(3.19)$

The equation that the researchers got(3.19) can only be solved for t using numerical methods .

3.2 Mode of TSLD

$$\frac{\partial f_2(t)}{\partial t} = 0$$

$$\frac{\partial f_2(t)}{\partial t} = \frac{\left(\frac{(1+\lambda)(2\theta^2)((1+\theta) + (1+2\theta)t + t^2\theta)e^{-2\theta t}}{(1+\theta)^2} - \frac{\lambda(\theta^2)e^{-\theta t}[(1+t)]}{1+\theta} \right)}{\partial t}$$

$$\frac{\partial f_2(t)}{\partial t} =$$

$$\frac{(1+\lambda)(2\theta^2)e^{-2\theta t}}{(1+\theta)^2} (-2\theta^2 t^2 - 4\theta^2 t - 2\theta^2 + 1) - \frac{\lambda(\theta^2)e^{-\theta t}}{1+\theta} ((-\theta) + (-\theta)t + 1)$$

$$\frac{\partial f_2(t)}{\partial t} = 0$$

$$\frac{(1+\lambda)(2\theta^2)e^{-2\theta t}}{(1+\theta)^2} ((-2\theta^2)(1+t)^2 + 1) - \frac{\lambda(\theta^2)e^{-\theta t}}{1+\theta} ((-\theta)(1+t) + 1) = 0$$

$$(1+\lambda)(2\theta^2)e^{-2\theta t} ((-2\theta^2)(1+t)^2 + 1) - \lambda((-\theta)(1+t) + 1)(1+\theta) = 0$$

$$(1+\lambda)(2\theta^2)e^{-2\theta t} ((-2\theta^2)(1+t)^2 + 1) = \lambda((-\theta t - \theta^2 t) + 1 - \theta^2)$$

$$(1 + \lambda)(2)e^{-\theta t} ((-2\theta^2)(1+t)^2 + 1) + \lambda(\theta t + \theta^2 t) = \lambda(1 - \theta^2) \dots (3.20)$$

The researchers can solve equation (3.20) numerically .

3.3 Order Statistic of TSLD .

The Cumulative Distribution Function of the minimum order statistic is

$$F_{Y_1}(t) = 1 - \left((1 + \lambda) \frac{1}{(1 + \theta)^2} (1 + \theta t + \theta)^2 e^{-2\theta t} - \lambda \frac{(1 + \theta t + \theta)e^{-\theta t}}{1 + \theta} \right)^n \dots (3.21)$$

The Density Function of the minimum order statistic is .

$$f_{Y_1}(t) =$$

$$\frac{n \left(1 - (1 + \lambda) \frac{1}{(1 + \theta)^2} (1 + \theta t + \theta)^2 e^{-2\theta t} + \lambda \frac{1}{1 + \theta} (1 + \theta t + \theta) e^{-\theta t} \right)^{n-1} (\theta^2) e^{-\theta t} (1 + t) \left[\frac{(2)(1 + \lambda)}{(1 + \theta)} (1 + \theta t + \theta) e^{-\theta t} - \lambda \right]}{1 + \theta}$$

... (3.22)

The Cumulative Function of the maximum order statistic is .

$$F_{Y_n}(t) =$$

$$\left(1 - (1 + \lambda) \frac{1}{(1 + \theta)^2} (1 + \theta t + \theta)^2 e^{-2\theta t} + \lambda \frac{1}{1 + \theta} (1 + \theta t + \theta) e^{-\theta t} \right)^n \dots (3.23)$$

The Density Function of the maximum order statistic is .

$$f_{Y_n}(t) =$$

$$\frac{n \left(1 - (1 + \lambda) \frac{1}{(1 + \theta)^2} (1 + \theta t + \theta)^2 e^{-2\theta t} - \lambda \frac{1}{1 + \theta} (1 + \theta t + \theta) e^{-\theta t} \right)^{n-1} (\theta^2) e^{-\theta t} (1 + t) \left[\frac{(2)(1 + \lambda)}{(1 + \theta)} (1 + \theta t + \theta) e^{-\theta t} - \lambda \right]}{1 + \theta}$$

... (3.24)

3.4 Maximum Likelihood Estimators Of TSLD

In this section , the unknown parameters of the TSLD are estimate, by using maximum likelihood estimation method .

The Likelihood function of the PDF is :

$$L(\lambda, \theta; t_1, t_2, \dots, t_n) = \prod_{I=1}^n f_2(\lambda, \theta; t_i)$$

$$= \prod_{I=1}^n \frac{(\theta^2) e^{-\theta t} (1+t) [(2)(1+\lambda)(1+\theta t + \theta) e^{-\theta t} - \lambda(1+\theta)]}{(1+\theta)^2}$$

$$L(\lambda, \theta; t_1, t_2, \dots, t_n) = L$$

$$= \frac{(\theta^2)^n e^{-\sum_{I=1}^n \theta t_I} \prod_{I=1}^n (1+t_I) \prod_{I=1}^n [(2)(1+\lambda)(1+\theta t + \theta) e^{-\theta t} - \lambda(1+\theta)]}{(1+\theta)^{2n}}$$

$$\ln L = 2n \ln(\theta) - \theta \sum_{I=1}^n t_I + \sum_{I=1}^n \ln(1+t_I) - 2n \ln(1+\theta)$$

$$+ \sum_{I=1}^n \ln \left((2)(1+\lambda)(1+\theta t + \theta) e^{-\theta t} - \lambda(1+\theta) \right)$$

$$\frac{d \ln L}{d \theta} =$$

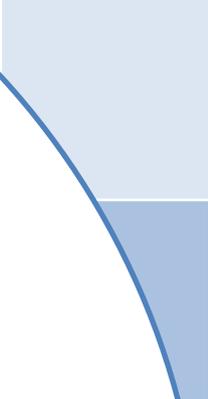
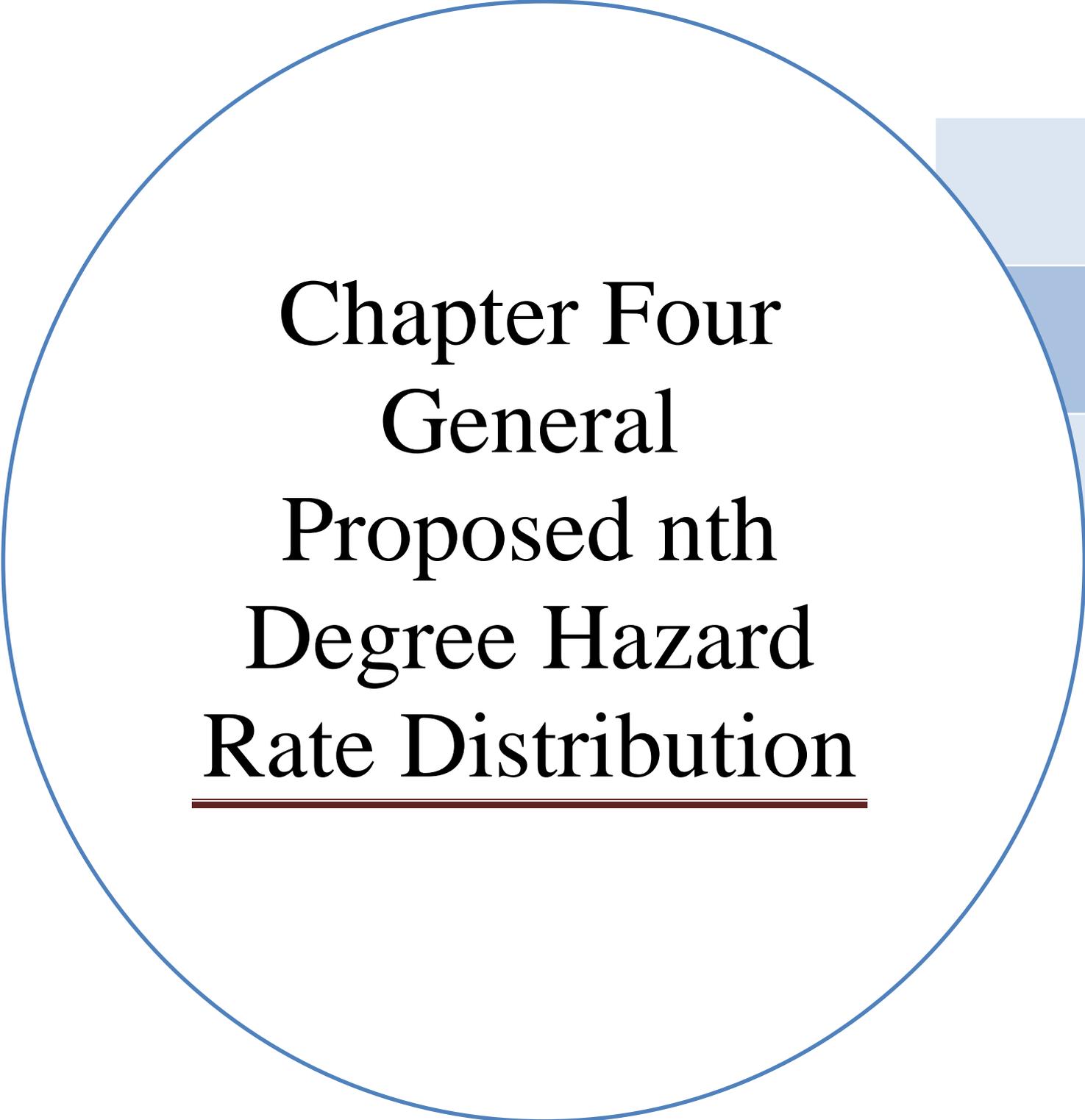
$$\frac{2n}{\theta} - \sum_{I=1}^n t_I - \frac{2n}{1+\theta} + \sum_{I=1}^n \frac{(2)(1+\lambda)((1+t) e^{-\theta t} - \theta(1+\theta t + \theta) e^{-\theta t}) - \lambda}{(2)(1+\lambda)(1+\theta t + \theta) e^{-\theta t} - \lambda(1+\theta)} = 0 \dots (3.25)$$

$$\frac{\partial L}{\partial \lambda} = \sum_{I=1}^n \frac{(2)(1+\theta t + \theta) e^{-\theta t} - (1+\theta)}{(2)(1+\lambda)(1+\theta t + \theta) e^{-\theta t} - \lambda(1+\theta)} = 0 \dots (3.26)$$

$$\frac{2n + \theta 2n - \theta 2n}{\theta(1+\theta)} + \sum_{I=1}^n \frac{(2)(1+\lambda)((1+t) e^{-\theta t} - \theta(1+\theta t + \theta) e^{-\theta t}) - \lambda}{(2)(1+\lambda)(1+\theta t + \theta) e^{-\theta t} - \lambda(1+\theta)} = \sum_{I=1}^n t_I$$

$$\frac{2n}{\theta(1+\theta)} + \sum_{l=1}^n \frac{(2)(1+\lambda)((1+t)e^{-\theta t} - \theta(1+\theta t + \theta)e^{-\theta t})^{-\lambda}}{(2)(1+\lambda)(1+\theta t + \theta)e^{-\theta t} - \lambda(1+\theta)} = \sum_{l=1}^n t_l$$

The numerical method is used to solve the above equations (3.25) and (3.26)



Chapter Four
General
Proposed n th
Degree Hazard
Rate Distribution



In this chapter, a new distribution GPHRD is found based on the formula of the hazard function. we find the PDF, the CDF and some of its survival function. General Proposed Hazard Rate Distribution of $n = 2$ is discussed with its statistical prospered estimation its parameters by maximum likelihood method. Finally, using real life data to test its general proposed second degree hazard rate distribution.

4.1 General Proposed n^{th} Degree Hazard Rate Distribution GPHRD

Let T be a positive random variable with function

$$h(t) = \sum_{i=0}^n a_i t^i \quad t > 0 \quad \text{where } a_i \geq 0 \quad \forall i = 0, 1, \dots, n \text{ only } a_{n-1} \in R \quad (4.1)$$

Whereas, this function $h(t)$ satisfies the conditions of the hazard function, that is, its means

1. $h(t) > 0$ for all $t > 0 \quad \forall i = 0, 1, \dots, n$

Where $a_0, a_1, a_2, \dots, a_n \in \mathbb{R}^+$ except this $a_{n-1} \in \mathbb{R}$

That is

$$\sum_{i=0}^n a_i t^i \geq 0$$

$$a_0 + a_1 t^2 + a_2 t^2 + \dots + a_{n-1} t^{n-1} + a_n t^n \geq 0$$

$$a_0 + a_1 t^2 + a_2 t^2 + \dots + a_n t^n \geq -a_{n-1} t^{n-1}$$

2. $\int_0^{\infty} h(t) dt = \sum_{i=0}^n a_i \int_0^{\infty} t^i dt = \infty$

The Cumulative Hazard Rate Function

$$H(t) = \sum_{i=0}^n a_i \frac{t^{i+1}}{i+1} \quad (4.2)$$

Then the PDF of the GPHRD

it is product of $h(t)$ and $e^{-H(t)}$

$$f(t) = \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \quad (4.3)$$

The general proposed second degree GPHRD .

In the case $n = 2$, the density function GPHRD is

$$f(t) = \sum_{i=0}^2 a_i t^i e^{-\sum_{i=0}^2 a_i \frac{t^{i+1}}{i+1}} \quad (4.4)$$

A function $f(\cdot)$ that defined as $f: R \rightarrow [0, \infty]$ is a probability density function since

$$1) f(t) \geq 0 \text{ for all } t > 0$$

$$\text{And } \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \geq 0$$

$$\sum_{i=1}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \geq -a_{n-1} t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}$$

$$\sum_{i=1}^n a_i t^i \geq -a_{n-1} t^i$$

$$2) \int_0^{\infty} f(t) dt = 1$$

The second property is shown below:

$$\int_0^{\infty} f(t) dt = \int_0^{\infty} \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} dt$$

$$\text{let } u = \sum_{i=0}^n a_i \frac{t^{i+1}}{i+1} \text{ then } du = \left(\sum_{i=0}^n a_i t^i \right) dt$$

$$\int_0^{\infty} f(t) dt = \int_0^{\infty} e^{-u} du = -e^{-u} \Big|_0^{\infty} = 1$$

The CDF of GPHRD is

$$F(t) = 1 - e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \quad \dots(4.5)$$

The Cumulative Function where $n = 2$

$$F(t) = 1 - e^{-\sum_{i=0}^2 a_i \frac{t^{i+1}}{i+1}} \quad \dots(4.6)$$

The Survival Function of GPHRD :

$$S(t) = e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \quad \dots(4.7)$$

Remark(4.1): Special Cases

1-If $a_i = 0 \forall i > 0$ then Equation (4.3) is the PDF of the exponential distribution .

2-If $a_i = 0 \forall i > 1$ then Equation (4.3) is the PDF of the linear exponential distribution .

3-If $a_i = 0 \forall i > 1$ and $a_0 = 0$ then Equation (4.3) is the PDF of the Rayleigh distribution .

4-If $a_i = 0 \forall i \geq 0$ only $a_n \neq 0$ then Equation (4.3) is the PDF of the Weibull distribution .

4.1.1 Shape of GPHRD

4.1.1.1 The Limit of Density Function of GPHRD

the limit of PDF

$$\begin{aligned} \lim_{t \rightarrow 0} f(t) &= \lim_{t \rightarrow 0} \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \\ &= \lim_{t \rightarrow 0} a_0 = a_0 \end{aligned}$$

And

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} = 0$$

4.1.1.2 The Limit of CDF of GPHRD

$$\lim_{t \rightarrow 0} F(t) = \lim_{t \rightarrow 0} \left(1 - e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right) = 1 - \lim_{t \rightarrow 0} \left(e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right) = 0$$

And

$$\lim_{t \rightarrow \infty} F(t) = \lim_{t \rightarrow \infty} \left(1 - e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right) = 1 - \lim_{t \rightarrow \infty} \left(e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right) = 1$$

4.1.1.3 The Limit of Survival function of GPHRD

$$\lim_{t \rightarrow 0} S(t) = \lim_{t \rightarrow 0} \left(e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right) = 1$$

And

$$\lim_{t \rightarrow \infty} S(t) = \lim_{t \rightarrow \infty} \left(e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right) = 0$$

4.1.1.4 The Limit of Hazard Function of GPHRD

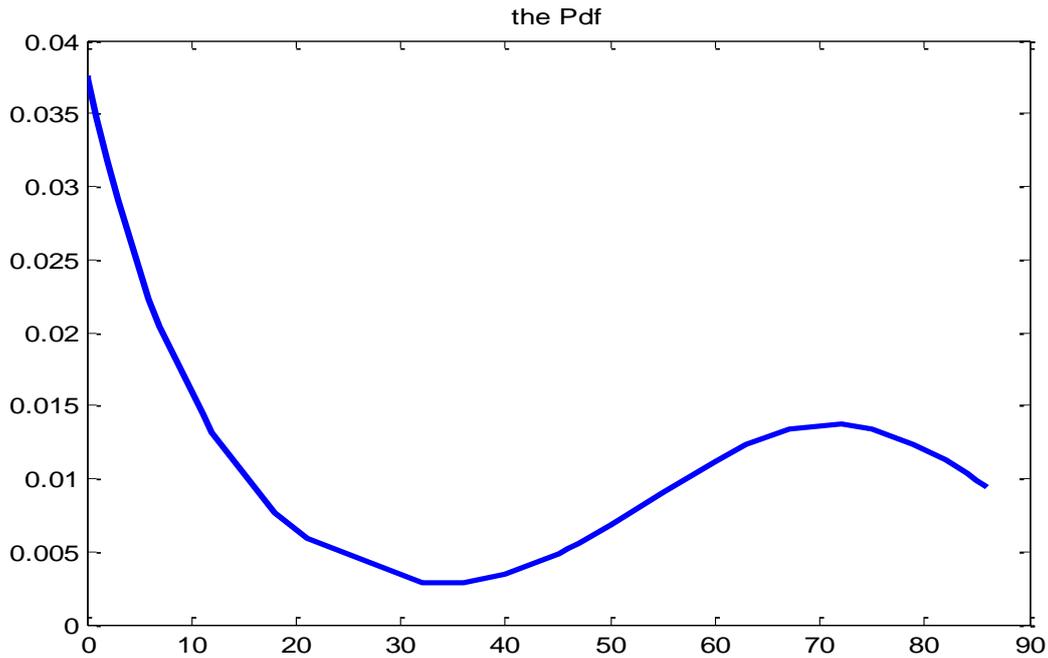
$$\lim_{t \rightarrow 0} h(t) = \lim_{t \rightarrow 0} \sum_{i=0}^n a_i t^i = a_0$$

And

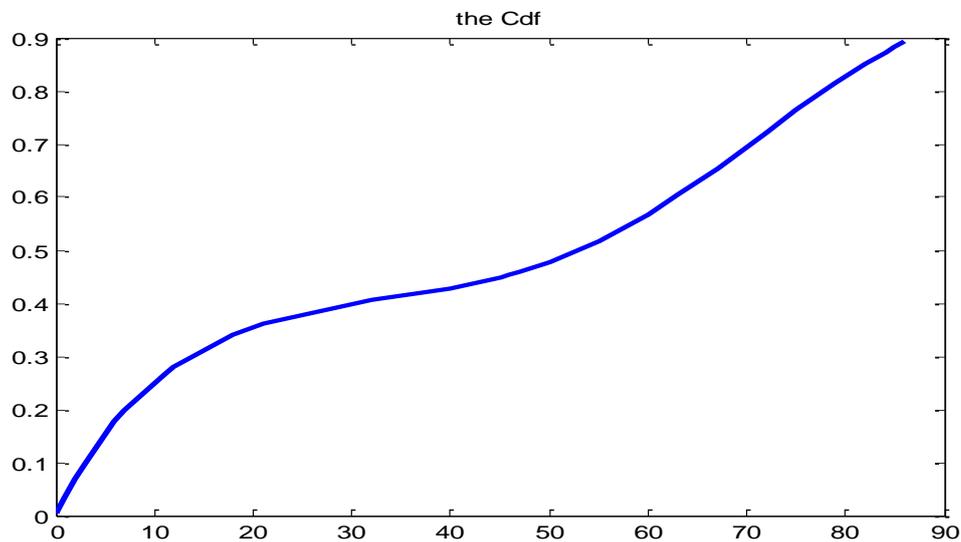
$$\lim_{t \rightarrow \infty} h(t) = \lim_{t \rightarrow \infty} \sum_{i=0}^n a_i t^i = \infty$$

The following figures show the shape of each of PDF, CDF, The Survival Function and hazard function of the second degree GPHRD.

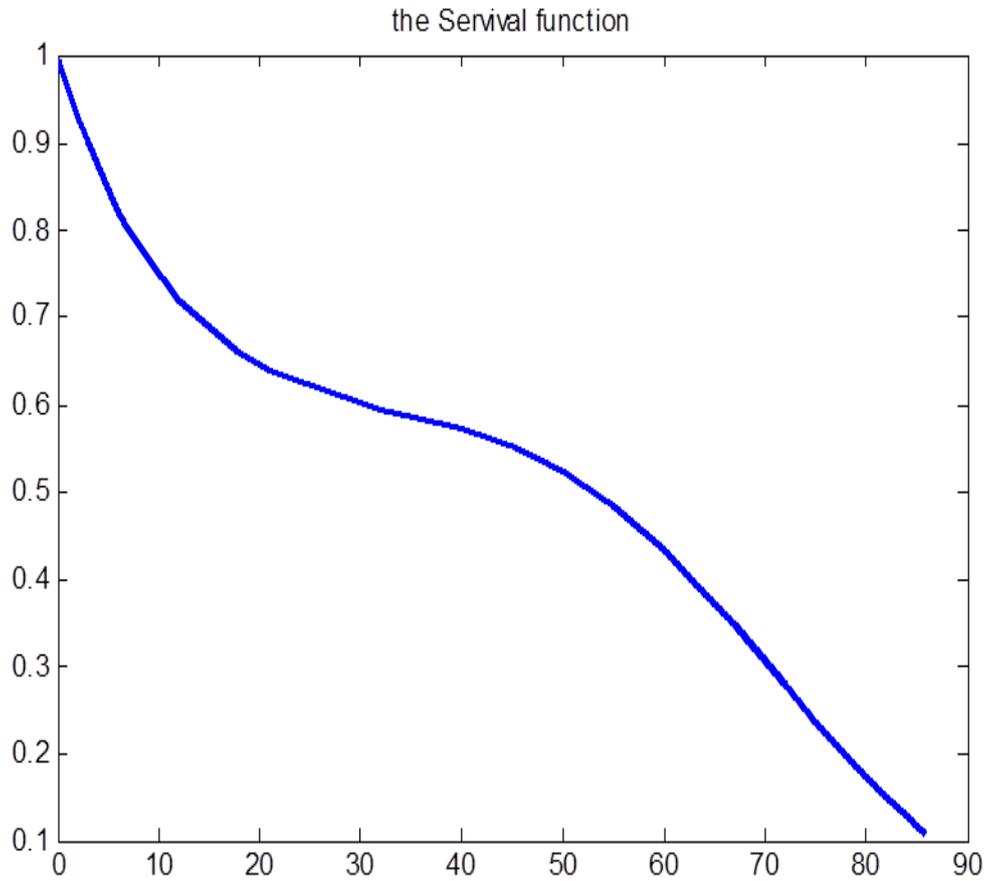
We noted that each figure conformity this reality limit.



Figure(4.1): The Density Function of GPHRD in case $n=2$, where $a_0 = .038$; $a_1 = -.002$; $a_2 = 0.00003$.it is noted the PDF starts from the value $a_0 = .038$ and then decreases until it reaches zero as long as increasing T.

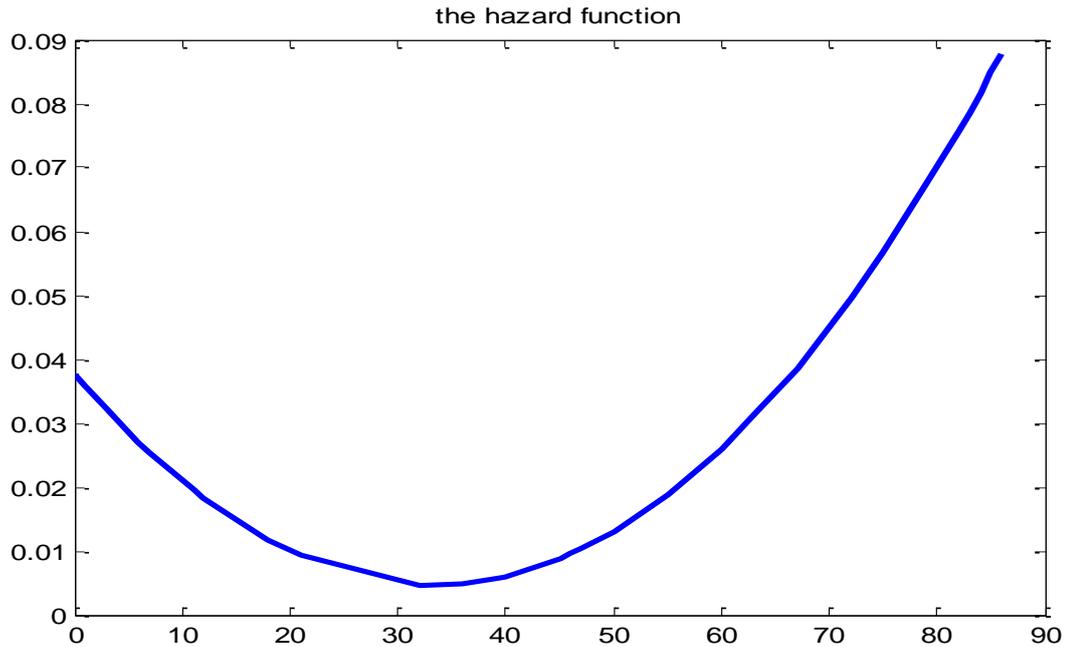


Figure(4.2): The Cumulative Function of GPHRD in case $n=2$, where $a_0 = .038$; $a_1 = -.002$; $a_2 = 0.00003$. it is noted that the cdf starts from the value 0 and then increasing until it reaches 1 as long as increasing T.



Figure(4.3): The Survival Function of GPHRD in case $n=2$,

where $a_0 = .038$; $a_1 = -.002$; $a_2 = 0.00003$. we notice that this function starts from the value 1 and then non increasing until it reaches 0 when it approaches t to ∞ .



Figure(4.4): The Hazard Function of GPHRD in case $n=2$, where $a_0 = .038$; $a_1 = -.002$; $a_2 = 0.00003$. it is noted that the hazard function starts from the value $a_0 = .038$ and it is increases to that $h(t) \rightarrow \infty$ as $t \rightarrow \infty$.and it takes the shape of a swimming pool .

4.1.2 Statistical Properties

This section we study some of its statistical properties of GPHRD . specifically the r^{th} moment and moment generating function

4.1.2.1 Moment of GPHRD

Moments are necessary and important in any statistical analysis especially in applications . It can be used to study the most features and characteristics of a distribution (skewness and kurtosis) .

Theorem4.2

If $T \sim \text{GPHRD} (a_i)$ then the r^{th} moment about the origin, and about the mean μ are given by

$$1 - E(T)^r =$$

$$\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i + \sum_{i=0}^n (i+1)j_{(i+1)} + 2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_i)}} \quad (4.8)$$

$$2 - E_2(\mu - T)^r =$$

$$\sum_{k=0}^r C_k^r \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+k+i+\sum_{i=0}^n (i+1)j_i)}} \right]^* \\ \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_i)}} \right)^{r-k} \quad (4.9)$$

Proof

Starting with proof the first part

$$\text{Take: } E(T^r) = \int_0^{\infty} t^r f(t) dt$$

$$E(T^r) = \int_0^{\infty} t^r \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} dt$$

$$\text{Then } E(T^r) = \sum_{i=0}^n a_i \int_0^{\infty} t^{r+i} e^{-a_0 t} e^{-\sum_{i=1}^n a_i \frac{t^{i+1}}{i+1}} dt \quad (4.10)$$

Again

$$E(T^r) = \sum_{i=0}^n a_i \int_0^{\infty} t^{r+i} e^{-a_0 t} e^{-a_2 \frac{t^2}{2}} e^{-a_1 \frac{t^3}{3}} \dots \dots e^{-a_n \frac{t^{n+1}}{n+1}} dt \quad (4.11)$$

But expansion of e^{-t} are given by

$$e^{-a_i \frac{t^{i+1}}{i+1}} = \sum_{j=0}^{\infty} \frac{\left(-\frac{a_i}{i+1}\right)^j}{j!} (t^{i+1})^j \quad (4.12)$$

Substituting (4.12) into (4.11) in order to get .

$$E(T^r) =$$

$$\sum_{i=0}^n a_i \int_0^{\infty} t^{r+i} e^{-a_0 t} \sum_{j_1=0}^{\infty} \frac{(-a_1 \frac{t^2}{2})^{j_1}}{j_1!} \sum_{j_2=0}^{\infty} \frac{(-a_2 \frac{t^3}{3})^{j_2}}{j_2!} \dots \dots \sum_{j_n=0}^{\infty} \frac{(-a_n \frac{t^{n+1}}{n+1})^{j_n}}{j_n!} dt$$

$$E(T^r) =$$

$$\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \dots \left(\frac{-a_n}{n+1}\right)^{j_n} \int_0^{\infty} t^{r+i} (t^{2j_1})(t^{3j_2}) \dots (t^{(n+1)j_n}) e^{-a_0 t} dt.$$

$$E(T^r) =$$

$$\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \dots \left(\frac{-a_n}{n+1}\right)^{j_n}}{j_1! j_2! \dots j_n!} \int_0^{\infty} t^{r+i+2j_1+3j_2 \dots +(n+1)j_n} e^{-a_0 t} dt$$

$$E(T^r) =$$

$$\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \dots \left(\frac{-a_n}{n+1}\right)^{j_n}}{j_1! j_2! \dots j_n!} \frac{\Gamma(r+i+\sum_{i=0}^n (i+1)j_{(i+1)} + 1)}{a_0^{(r+i+\sum_{i=0}^n (i+1)j_{(i+1)} + 1)}}$$

Proof 2 Since $(T - \mu)^r = \sum_{k=0}^r C_k^r (T)^k (-\mu)^{r-k}$, then

$$E(T - \mu)^r =$$

$$\sum_{k=0}^r C_k^r \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \dots \left(\frac{-a_n}{n+1}\right)^{j_n} \Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)} + 1)}{a_0^{(k+i+\sum_{i=0}^n (i+1)j_{(i+1)} + 1)}} \right] * \\ \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \dots \left(\frac{-a_n}{n+1}\right)^{j_n} \Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)} + 2)}{a_0^{(i+\sum_{i=0}^n (i+1)j_{(i+1)} + 2)}} \right)^{r-k} .$$

Corollary 4.3.

If $T \sim \sim GPHRD (a_i)$ then the mean , variance , coefficients of variation ,skewness and kurtosis of general proposed nth degree hazard rate distribution are respectively as:

$$E(T) =$$

$$\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)} + 2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \quad (4.13)$$

$$\text{var}(T) =$$

$$\sum_{k=0}^r C_k^2 \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)} + 1)}{a_0^{(k+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right]^* \\ \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)} + 2)}{a_0^{(i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right)^{2-k} \quad (4.14)$$

$$\gamma =$$

$$\sum_{k=0}^3 C_k^3 \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)} + 1)}{a_0^{(1+k+i+\sum_{i=0}^n (i+1)j_i)}} * \\ \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)} + 2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_i)}} \right)^{3-k} \\ \left(\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)} + 3)}{a_0^{(3+i+\sum_{i=0}^n (i+1)j_i)}} - \right. \\ \left. \left(\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)} + 2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_i)}} \right)^2 \right)^{-\frac{3}{2}} \quad (4.15)$$

$$\psi =$$

$$\begin{aligned}
& \sum_{k=0}^4 C_k^4 \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+k+i+\sum_{i=0}^n (i+1)j_i)}} \right] * \\
& \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_i)}} \right)^{4-k} * \\
& \left(\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+3)}{a_0^{(3+i+\sum_{i=0}^n (i+1)j_i)}} - \right. \\
& \left. \left(\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_i)}} \right)^2 \right)^{\frac{1}{2}-2} \quad (4.16)
\end{aligned}$$

CV =

$$\begin{aligned}
& = \left(\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+3)}{a_0^{(3+i+\sum_{i=0}^n (i+1)j_i)}} - \right. \\
& \left. \left(\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_i)}} \right)^2 \right)^{1/2} * \\
& \left(\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_i)}} \right)^{-1} \quad (4.17)
\end{aligned}$$

proof

Based on (4.10), then mean is defined when $r = 1$

$$E(T) = \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_i)}} .$$

Since (4.9), then

$$\text{var}(t) = E(T - \mu)^2 =$$

$$\sum_{k=0}^2 C_k^2 \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+k+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right] * \\ \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right)^{2-k} .$$

The Coefficient of Variation of (GPHRD) is defined as

$$CV = (\text{var}(T))^{1/2} (E(T))^{-1}$$

$$CV =$$

=

$$\left(\sum_{k=0}^2 C_k^2 \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+k+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right] * \right. \\ \left. \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right)^{2-k} \right)^{1/2} * \\ \left(\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right)^{-1} .$$

the Standard deviation of GPHRD is given by

$$\sigma = (\text{Var}(t))^{\frac{1}{2}}$$

$\sigma =$

$$\left(\sum_{k=0}^2 C_k^2 \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+k+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right]^* \right. \\ \left. \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right)^{2-k} \right)^{\frac{1}{2}} \quad (4.18)$$

Note Based on part two from theorem (4.2), it is get the *SK* and *KU*.

Take $SK = E(T - \mu)^3 =$

$$\sum_{k=0}^3 C_k^3 \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+k+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right]^* \\ \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right)^{3-k} \quad (4.19)$$

Suppose that $KU = E(T - \mu)^4 =$

$=$

$$\sum_{k=0}^4 C_k^4 \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+k+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right]^* \\ \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right)^{4-k} \quad (4.20)$$

So Coefficient of Skewness of (GPHRD) is defined as the

$$\gamma = SK\sigma^{-3}$$

$\gamma =$

$$\sum_{k=0}^3 C_k^3 \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+k+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right]^* \\ \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{i+1}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right)^{3-k} \\ \left(\sum_{k=0}^2 C_k^2 \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+k+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right]^* \right. \\ \left. \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{i+1}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right)^{2-k} \right)^{\frac{-3}{2}}$$

And defined the Coefficient of kurtosis of (GPHRD) is

$$\Psi = KU \sigma^{-4}$$

$$\Psi =$$

$$\sum_{k=0}^4 C_k^4 \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+k+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right]^* \\ \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{(i+1)}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right)^{4-k} * \\ \left(\sum_{k=0}^2 C_k^2 \left[\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(k+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+k+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right]^* \right. \\ \left. \left(- \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_i \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(-\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{\Gamma(i+\sum_{i=0}^n (i+1)j_{i+1}+2)}{a_0^{(2+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \right)^{2-k} \right)^{-2}$$

Moments Generating Function of GPHRD .

$$\text{Since } M_T(z) = E(e^{zT}) = \int_0^{\infty} e^{zT} f(t) dt$$

where $-h < z < h$ and $h > 0, h \in IR$

$$e^{zt} = \sum_{r=1}^{\infty} \frac{(zt)^r}{r!}$$

$$M_T(z) = E(e^{zT}) = \int_0^{\infty} \sum_{r=1}^{\infty} \frac{(zT)^r}{r!} f(t) dt$$

$$M_T(z) = E(e^{zT}) = \sum_{r=1}^{\infty} \frac{z^r}{r!} E(T^r)$$

$$M_T(z) =$$

$$\sum_{r=1}^{\infty} \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} \frac{z^r}{r!} \frac{a_i \left(-\frac{a_1}{2}\right)^{j_1} \left(-\frac{a_2}{3}\right)^{j_2} \dots \left(-\frac{a_n}{n+1}\right)^{j_n}}{j_1! j_2! \dots j_n!} \frac{\Gamma(r+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+r+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \quad (4.21)$$

Characteristic Function of GPHRD

$$M_t(iz) =$$

$$\sum_{r=1}^{\infty} \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} \frac{(iz)^r}{r!} \frac{a_i \left(-\frac{a_1}{2}\right)^{j_1} \left(-\frac{a_2}{3}\right)^{j_2} \dots \left(-\frac{a_n}{n+1}\right)^{j_n}}{j_1! j_2! \dots j_n!} \frac{\Gamma(r+i+\sum_{i=0}^n (i+1)j_{(i+1)}+1)}{a_0^{(1+r+i+\sum_{i=0}^n (i+1)j_{(i+1)})}} \quad (4.22)$$

4.1.3 Mode of GPHRD .

$$\frac{\partial f(t)}{\partial t} = 0$$

$$\frac{\partial \left[\sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right]}{\partial t}$$

$$= - \sum_{i=0}^n a_i t^i \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} + e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \sum_{i=1}^n (i a_i) t^{i-1}$$

$$\frac{\partial \left[\sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right]}{\partial t} = e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left[- \sum_{i=0}^n a_i t^i \sum_{i=0}^n a_i t^i + \sum_{i=1}^n (i a_i) t^{i-1} \right] = 0$$

$$\left[- \sum_{i=0}^n a_i t^i \sum_{i=0}^n a_i t^i + \sum_{i=1}^n (i a_i) t^{i-1} \right] = 0$$

$$\left[\sum_{i=1}^n (i a_i) t^{i-1} \right] = \left[\sum_{i=0}^n a_i t^i \right]^2$$

Which can be solved numerically .

4.1.4 Order statistics of GPHRD

Let be T_1, T_2, \dots, T_n a random sample of size n from new distribution, if $T_1 \leq \dots \leq T_n$ denotes the order statistics of the random sample from a continuous population with CDF and PDF then the PDF of distribution is given by

$$Y_1 = \text{Min}(T_1, T_2, \dots, T_n)$$

$$Y_2 = \text{the 2nd smallest of } T_1, T_2, \dots, T_n.$$

$$Y_n = \text{Max}(T_1, T_2, \dots, T_n)$$

Then the CDF of the min order statistic

$$F_{Y_1}(t) = P(Y_1 \leq t) = 1 - \prod_{i=1}^n P(T_i > t) = 1 - (1 - F_T(t))^n$$

$$F_{Y_1}(t) = 1 - \left(e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right)^n \quad \dots(4.23)$$

Then the PDF of Y_1 .

$$\frac{dF_{Y_1}(t)}{dt} = f_{Y_1}(t)$$

$$f_{Y_1}(t) = n \left(e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right)^{n-1} \left[\sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right] \quad \dots(4.24)$$

Then CDF of the max order statistic

$$F_{Y_n}(t) = \left(1 - e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}\right)^n \quad \dots(4.25)$$

Then the PDF of Y_n .

$$f_{Y_n}(t) = n \left(1 - e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}\right)^{n-1} \left[\sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}\right] \quad \dots(4.26)$$

4.1.5 Maximum Likelihood Estimators of GPHRD

If (T_1, \dots, T_n) denoted a random sample have a GPHRD, then the likelihood function is given by

$$\begin{aligned} L = L(a_0, a_1, a_2, \dots, a_n; t_1, t_2, \dots, t_m) &= \prod_{j=1}^m f_2(a_0, a_1, a_2, \dots, a_n; t_j) \\ &= \prod_{j=1}^m \left[\sum_{i=0}^n a_i t_j^i e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} \right] \\ &= e^{-\sum_{j=0}^m \sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} \prod_{j=1}^m \left[\sum_{i=0}^n a_i t_j^i \right] \end{aligned}$$

Take ln of likelihood function .

$$\ln L = \left[-\sum_{j=0}^m \sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1} \right] + \left[\sum_{j=0}^m \ln \sum_{i=0}^n a_i t_j^i \right]$$

Then, by taking the partial derivatives of lnL with respect to unidentified parameters a_i and the result equal zero .

$$\frac{\partial \ln L}{\partial a_0} = \left[-\sum_{j=0}^m t_j \right] + \sum_{j=0}^m \frac{1}{(a_0 + \sum_{i=1}^n a_i t_j^i)} \quad \dots(4.27)$$

$$\frac{\partial \ln L}{\partial a_1} = \left[-\sum_{j=0}^m \frac{t_j^2}{2} \right] + \sum_{j=0}^m \frac{t_j}{(a_0 + \sum_{i=1}^n a_i t_j^i)} \quad \dots(4.28)$$

$$\frac{\partial \ln L}{\partial a_2} = \left[-\sum_{j=0}^m \frac{t_j^3}{3} \right] + \sum_{j=0}^m \frac{t_j^2}{(a_0 + \sum_{i=1}^n a_i t_j^i)} \quad \dots(4.29)$$

$$\frac{\partial \ln L}{\partial a_i} = \left[-\sum_{j=0}^m \frac{t_j^{i+1}}{i+1} \right] + \sum_{j=0}^m \frac{t_j^i}{(a_0 + \sum_{i=1}^n a_i t_j^i)} \quad \dots(4.30)$$

When $\frac{\partial \ln L}{\partial a_i} = 0$, there is no closed solution of (4.28), (4.29), (4.30) and (4.27)

therefor, numerical technique (Newton- Raphson method) should be apply to solve.

4.2 The Three Degree Hazard Rate Distribution

In this section, we dealt with a special case of the General Proposed Hazard Rate Distribution when $n=3$ and it called the three-degree hazard rate distribution.

Finding it in the PDF, the CDF, the survival function, and some of the statistical properties It also estimated the parameters using the maximum likelihood method.

4.2.1 General Proposed Hazard Rate Distribution where $n = 3$.

In the following, we assume that T is a positive random variable with a function.

$$h_3(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \quad a_\varphi, t > 0 \quad \forall \varphi = 0, 1, 2, 3 \quad \dots (4.31)$$

In contrast, this function $h(t)$ satisfies the hazard function's conditions, that is, its means.

1. $h(t) > 0$ for all $a_\varphi, t > 0 \quad \forall i = 0, 1, 2, 3$

2. $\int_0^\infty h_3(t) dt = \int_0^\infty a_0 + a_1 t + a_2 t^2 + a_3 t^3 dt = a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \frac{a_3 t^4}{4} \Big|_0^\infty = \infty$

The Cumulative Hazard Rate Function

$$H(t) = a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \frac{a_3 t^4}{4} \quad \dots (4.32)$$

The PDF of the THR Distribution

It is the result of multiplying $h(t)$ by $\exp(-H(t))$

$$f(t) = (a_0 + a_1t + a_2t^2 + a_3t^3) e^{-\left(a_0t + \frac{a_1t^2}{2} + \frac{a_2t^3}{3} + \frac{a_3t^4}{4}\right)} \quad \dots (4.33)$$

A probability density function $f(\cdot)$ defined by $f: R \rightarrow [0, \infty]$ is a density function if and only if

$$1) f(t) \geq 0 \text{ for all } t > 0 \text{ and } (a_0 + a_1t + a_3t^3) \geq -a_2t^2$$

$$2) \int_0^{\infty} f(t)dt = 1,$$

The second property is shown below:

$$\int_0^{\infty} f(t)dt = \int_0^{\infty} (a_0 + a_1t + a_2t^2 + a_3t^3) e^{-\left(a_0t + \frac{a_1t^2}{2} + \frac{a_2t^3}{3} + \frac{a_3t^4}{4}\right)} dt$$

$$\text{let } u = \left(a_0t + \frac{a_1t^2}{2} + \frac{a_2t^3}{3} + \frac{a_3t^4}{4}\right) \text{ then } du = (a_0 + a_1t + a_2t^2 + a_3t^3)dt$$

$$\int_0^{\infty} f(t)dt = \int_0^{\infty} e^{-u} du = -e^{-u} \Big|_0^{\infty} = 1$$

The CDF of THR distribution is:

$$F(t) = 1 - e^{-\left(a_0t + \frac{a_1t^2}{2} + \frac{a_2t^3}{3} + \frac{a_3t^4}{4}\right)} \quad \dots(4.34)$$

The Survival Function of THR distribution :

$$S(t) = e^{-\left(a_0t + \frac{a_1t^2}{2} + \frac{a_2t^3}{3} + \frac{a_3t^4}{4}\right)} \quad \dots(4.35)$$

Remark(4.4)

- If $a_\varphi = 0 \forall \varphi > 0$ then Equation (4.33) is the PDF of the exponential distribution .

- If $a_\varphi = 0 \forall \varphi > 1$ then we get from Equation (4.34) is the PDF of the linear exponential distribution .

4.2.2 Shape of THRD

4.2.2.1 The Limit of PDF of THR Distribution

$$\lim_{t \rightarrow 0} f(t) = \lim_{t \rightarrow 0} (a_0 + a_1 t + a_2 t^2 + a_3 t^3) e^{-\left(a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \frac{a_3 t^4}{4}\right)} = a_0$$

And

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} (a_0 + a_1 t + a_2 t^2 + a_3 t^3) e^{-\left(a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \frac{a_3 t^4}{4}\right)} = 0$$

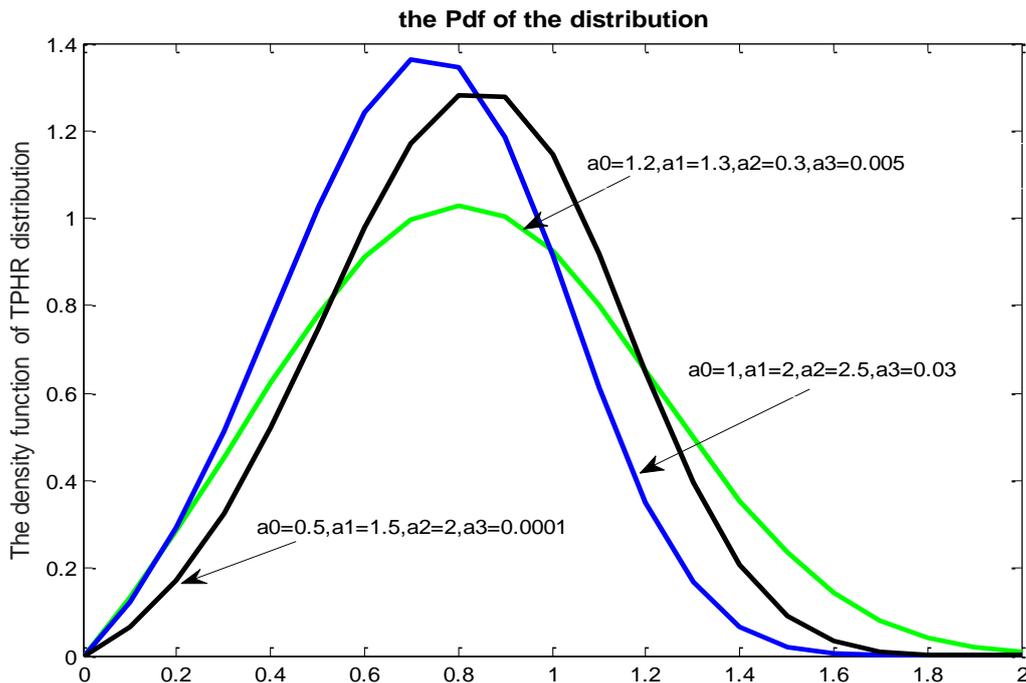


Figure 4.5. The density function of THR Distribution with different values of a_0, a_1, a_2, a_3 , When we look closely, we can see that the function has a convex shape. At this point $f(t) = 0$, the function starts to moment increasing , and once it achieves its maximum value, it starts to decrease, forming a bell curve.

4.2.2.2 The limit of CDF of THR Distribution

$$\lim_{t \rightarrow 0} F(t) = 1 - \lim_{t \rightarrow 0} e^{-\left(a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \frac{a_3 t^4}{4}\right)} = 1 - 1 = 0.$$

$$\lim_{t \rightarrow \infty} F(t) = 1 - \lim_{t \rightarrow \infty} e^{-\left(a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \frac{a_3 t^4}{4}\right)} = 1 - 0 = 1.$$

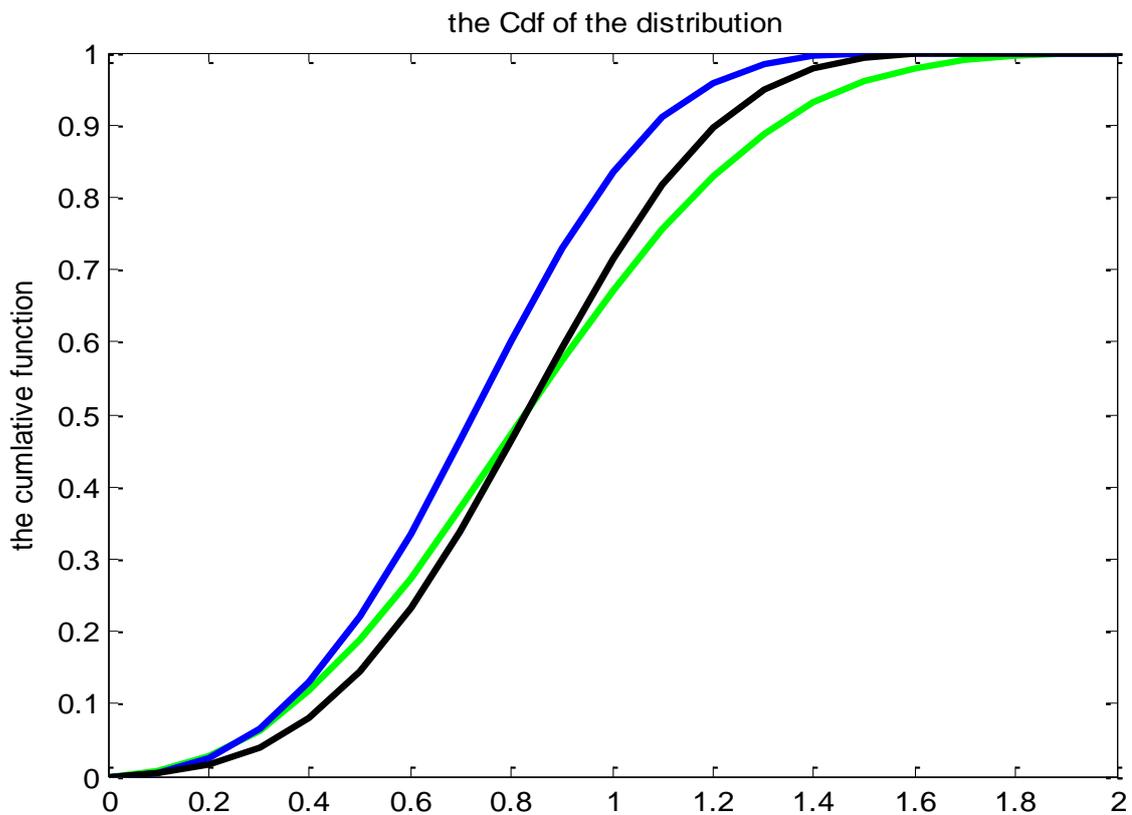


Figure4.6. The Cumulative Function of THR Distribution with different values of a_0, a_1, a_2, a_3 , A close examination reveals that the function .It starts at zero and increases over time as we observe $t \rightarrow \infty$ then $F(t) \rightarrow 1$.

4.2.2.3 The Limit of Survival Function of THR Distribution

$$\lim_{t \rightarrow 0} S(t) = \lim_{t \rightarrow 0} e^{-\left(a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \frac{a_3 t^4}{4}\right)} = 1. \text{ And}$$

$$\lim_{t \rightarrow \infty} S(t) = \lim_{t \rightarrow \infty} e^{-\left(a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \frac{a_3 t^4}{4}\right)} = 0$$

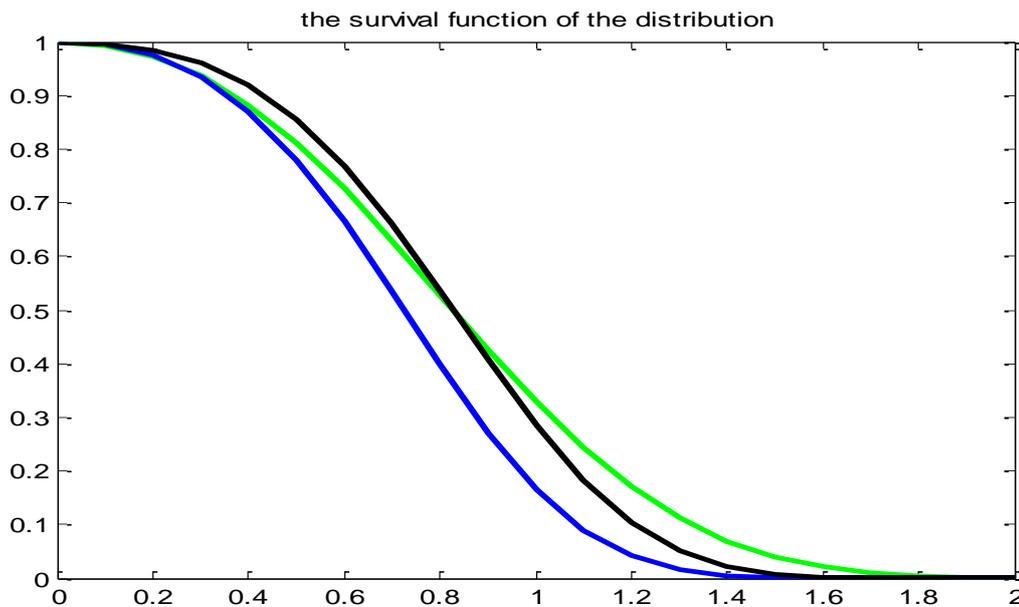


Figure 4.7. The Survival Function of THR Distribution with different values of a_0, a_1, a_2, a_3 . After examining the function in detail, as we can see, it begins at one and increases over time as we observe $t \rightarrow \infty$ then $S(t) \rightarrow 0$.

4.2.2.4 The Limit of Hazard Function of THR Distribution

$$\lim_{t \rightarrow 0} h_3(t) = \lim_{t \rightarrow 0} (a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_0$$

$$\text{And } \lim_{t \rightarrow \infty} h_3(t) = \lim_{t \rightarrow \infty} (a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_0 + \infty = \infty$$

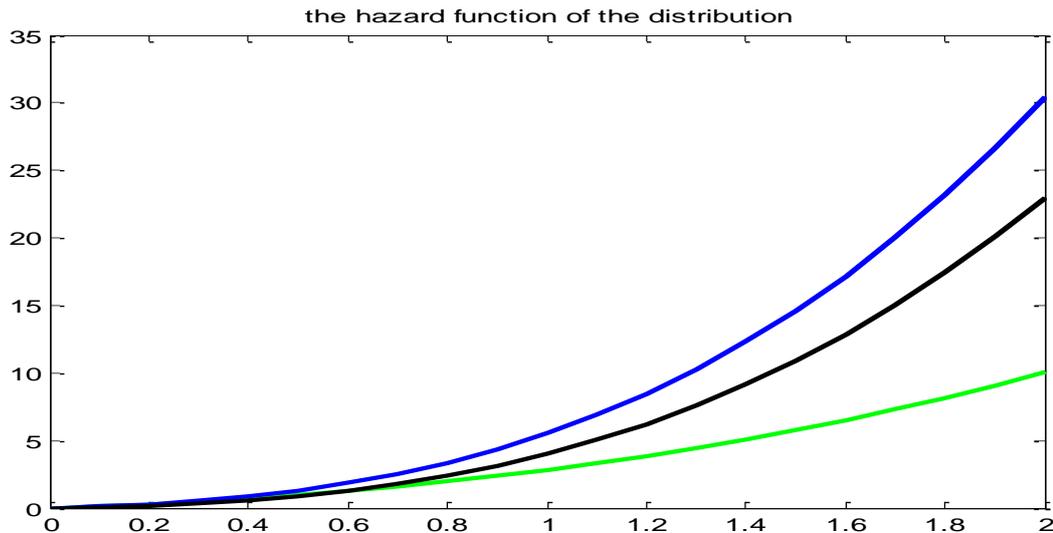


Figure 4.8. The hazard function of THR distribution with different values of a_0, a_1, a_2, a_3 . As a result, we can observe that the function is increasing.

4.2.3 Statistical Properties

In this section, it analyses the THR distribution statistically. in particular, the moment and moment generating functions .

4.2.3.1 Moment

Any statistical study, and particularly those with practical implications, requires the use of moments. It's a great tool for analysing distributional parameters like skewness and kurtosis.

Theorem 4.5

If $T \sim \text{THRD}(a_\varphi)$ then the r^{th} moments about the origin , and about the mean μ are given as

$$1 - E(T)^r =$$

$$\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_\varphi \frac{\left(\frac{-a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{-a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(\frac{-a_3}{4}\right)^{j_3}}{j_3!} \frac{\Gamma(\varphi + \sum_{\varphi=0}^3 (i+1)j_{(\varphi+1)} + 2)}{a_0^{(2+\varphi + \sum_{\varphi=0}^3 (i+1)j_\varphi)}}. \quad (4.36)$$

$$\begin{aligned}
& .2 - E(\mu - T)^r = \\
& \sum_{k=0}^r C_k^r \left[\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_{\varphi} \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(-\frac{a_3}{4}\right)^{j_n}}{j_n!} \frac{\Gamma(k+\varphi+\sum_{\varphi=0}^3(i+1)j_{(\varphi+1)}+1)}{a_0^{(1+k+\varphi+\sum_{\varphi=0}^3(i+1)j_{(\varphi+1)})}} \right] * \\
& \left(-\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_n=0}^{\infty} a_{\varphi} \frac{\left(-\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(-\frac{a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(-\frac{a_3}{4}\right)^{j_n}}{j_n!} \frac{\Gamma(\varphi+\sum_{\varphi=0}^3(i+1)j_{(\varphi+1)}+2)}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(i+1)j_{\varphi})}} \right)^{r-k} \dots (4.37)
\end{aligned}$$

Proof

Depending on the equation (4. 8)

$$E(T^r) = \int_0^{\infty} t^r (a_0 + a_1 t + a_2 t^2 + a_3 t^3) e^{-\left(a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \frac{a_3 t^4}{4}\right)} dt$$

Then

$$E(T^r) = \int_0^{\infty} t^r e^{-(a_0 t)} (a_0 + a_1 t + a_2 t^2 + a_3 t^3) e^{-a_1 \frac{t^2}{2}} e^{-a_2 \frac{t^3}{3}} e^{-a_3 \frac{t^4}{4}} dt$$

Again

$$\begin{aligned}
E(T^r) &= a_0 \int_0^{\infty} t^r e^{-(a_0 t)} e^{-a_1 \frac{t^2}{2}} e^{-a_2 \frac{t^3}{3}} e^{-a_3 \frac{t^4}{4}} dt + \\
& a_1 \int_0^{\infty} t^{r+1} e^{-(a_0 t)} e^{-a_1 \frac{t^2}{2}} e^{-a_2 \frac{t^3}{3}} e^{-a_3 \frac{t^4}{4}} dt + a_2 \int_0^{\infty} t^{r+2} e^{-(a_0 t)} e^{-a_1 \frac{t^2}{2}} e^{-a_2 \frac{t^3}{3}} e^{-a_3 \frac{t^4}{4}} dt + \\
& a_3 \int_0^{\infty} t^{r+3} e^{-(a_0 t)} e^{-a_1 \frac{t^2}{2}} e^{-a_2 \frac{t^3}{3}} e^{-a_3 \frac{t^4}{4}} dt \dots (4.38)
\end{aligned}$$

But expansion of $e^{-a_1 \frac{t^2}{2}}$, $e^{-a_2 \frac{t^3}{3}}$ and $e^{-a_3 \frac{t^4}{4}}$ are given by

$$e^{-a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} = \sum_{j=0}^{\infty} \frac{\left(-\frac{a_{\varphi}}{\varphi+1}\right)^j}{j!} (t^{\varphi+1})^j \quad \text{where } \varphi = 1, 2, 3 \dots (4.39)$$

Substituting (4.39) into (4.38) Given this, it's clear that .

$$E(T^r) = \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{-a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(\frac{-a_3}{4}\right)^{j_3}}{j_3!} \int_0^{\infty} t^{r+\varphi+2j_1+3j_2+4j_3} e^{-a_0 t} dt$$

$$E(T^r) = \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{-a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(\frac{-a_3}{4}\right)^{j_3}}{j_3!} \frac{\Gamma(r+\varphi+2j_1+3j_2+4j_3+1)}{a_0^{(r+\varphi+2j_1+3j_2+4j_3+1)}}$$

Now it will prove second part from theorem based on Binomial theorem

$$E(T - \mu)^r =$$

$$\sum_{k=0}^r C_k^r \left[\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(k+\varphi+2j_1+3j_2+4j_3+1)}{a_0^{-(k+\varphi+2j_1+3j_2+4j_3+1)}} \right]^* \\ \left(- \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+2)}{a_0^{(\varphi+2j_1+3j_2+4j_3+2)}} \right)^{r-k}$$

Corollary 4.6

If $T \sim \text{THRD}(a_{\varphi})$ then the mean, variance, Coefficient of Variation, Skewness and Kurtosis of General Proposed nth Degree Hazard Rate Distribution are respectively.

$$E(T) =$$

$$\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{-a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(\frac{-a_3}{4}\right)^{j_3}}{j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+2)}{a_0^{(\varphi+2j_1+3j_2+4j_3+2)}} \dots (4.40)$$

$$\text{Var}(T) = \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{-a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(\frac{-a_3}{4}\right)^{j_3}}{j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+3)}{a_0^{(\varphi+2j_1+3j_2+4j_3+3)}} - \\ \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{-a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(\frac{-a_3}{4}\right)^{j_3}}{j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+2)}{a_0^{(\varphi+2j_1+3j_2+4j_3+2)}} \right)^2 \dots (4.41)$$

Note Based on part two from theorem (4.5), it is get the SK and KU .

$$SK = E(T - \mu)^3 =$$

$$\sum_{k=0}^3 C_k^3 \left[\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(k+\varphi+2j_1+3j_2+4j_3+1)}{a_0^{(k+\varphi+2j_1+3j_2+4j_3+1)}} \right] * \\ \left(- \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+2)}{a_0^{(\varphi+2j_1+3j_2+4j_3+2)}} \right)^{3-k} \dots (4.42)$$

$$\text{And } KU = E(T - \mu)^4 =$$

$$\sum_{k=0}^4 C_k^4 \left[\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(k+\varphi+2j_1+3j_2+4j_3+1)}{a_0^{-(k+\varphi+2j_1+3j_2+4j_3+1)}} \right] * \\ \left(- \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+2)}{a_0^{(\varphi+2j_1+3j_2+4j_3+2)}} \right)^{4-k} \dots (4.43)$$

$$CV = \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+3)}{a_0^{(\varphi+2j_1+3j_2+4j_3+3)}} \right) - \\ \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+2)}{a_0^{(\varphi+2j_1+3j_2+4j_3+2)}} \right)^2 \right)^{1/2} * \\ \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+2)}{a_0^{(\varphi+2j_1+3j_2+4j_3+2)}} \right)^{-1} \dots (4.44)$$

$$\sigma = \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{-a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(\frac{-a_3}{4}\right)^{j_3}}{j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+3)}{a_0^{(\varphi+2j_1+3j_2+4j_3+3)}} - \right. \\ \left. \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{-a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(\frac{-a_3}{4}\right)^{j_3}}{j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+2)}{a_0^{(\varphi+2j_1+3j_2+4j_3+2)}} \right)^2 \right)^{\frac{1}{2}} \dots (4.45)$$

$\gamma =$

$$\sum_{k=0}^3 C_k^3 \left[\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(k+\varphi+2j_1+3j_2+4j_3+1)}{a_0^{(k+\varphi+2j_1+3j_2+4j_3+1)}} \right] * \\ \left(- \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+2)}{a_0^{(\varphi+2j_1+3j_2+4j_3+2)}} \right)^{3-k} * \\ \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+3)}{a_0^{(\varphi+2j_1+3j_2+4j_3+3)}} - \right. \\ \left. \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+2)}{a_0^{(\varphi+2j_1+3j_2+4j_3+2)}} \right)^2 \right)^{-3/2} \dots (4.46)$$

$$\Psi = \sum_{k=0}^4 C_k^4 \left[\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \Gamma(k + \varphi + 2j_1 + \right. \\ \left. 3j_2 + 4j_3 + 1) a_0^{-(k+\varphi+2j_1+3j_2+4j_3+1)} \right]^{4-k} * \\ \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1} \left(\frac{-a_2}{3}\right)^{j_2} \left(\frac{-a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+3)}{a_0^{(\varphi+2j_1+3j_2+4j_3+3)}} - \right.$$

$$\left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{-a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(\frac{-a_3}{4}\right)^{j_3}}{j_3!} \frac{\Gamma(\varphi+2j_1+3j_2+4j_3+2)}{a_0^{(\varphi+2j_1+3j_2+4j_3+2)}} \right)^2 \Big)^{-2} \dots (4.47)$$

proof

The proof can be obtained based on the corollary(4.3).

4.2.3.2 Moments Generating Function of THR Distribution

$$M_T(z) =$$

$$\sum_{r=1}^{\infty} \frac{z^r}{r!} \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{-a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(\frac{-a_3}{4}\right)^{j_3}}{j_3!} \frac{\Gamma(r+\varphi+2j_1+3j_2+4j_3+1)}{a_0^{(r+\varphi+2j_1+3j_2+4j_3+1)}} (4.48)$$

4.2.3.3 Characteristic Function of THR Distribution

$$M_t(ix) =$$

$$\sum_{r=1}^{\infty} \frac{(ix)^r}{r!} \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{-a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{-a_2}{3}\right)^{j_2}}{j_2!} \frac{\left(\frac{-a_3}{4}\right)^{j_3}}{j_3!} \frac{\Gamma(r+\varphi+2j_1+3j_2+4j_3+1)}{a_0^{(r+\varphi+2j_1+3j_2+4j_3+1)}} \dots (4.49)$$

4.2.4 Mode of THRD

$$\frac{\partial f(t)}{\partial t} = 0$$

$$\frac{\partial \left[(a_0 + a_1 t^1 + a_2 t^2 + a_3 t^3) e^{-\left(a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \frac{a_3 t^4}{4}\right)} \right]}{\partial t}$$

$$= - \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} + e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \sum_{\varphi=1}^3 (a_{\varphi} \varphi) t^{\varphi-1}$$

$$= e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left[- \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} + \sum_{\varphi=1}^3 (a_{\varphi} \varphi) t^{\varphi-1} \right] = 0$$

$$\left[- \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} + \sum_{\varphi=1}^3 (a_{\varphi} \varphi) t^{\varphi-1} \right] = 0$$

$$\left[\sum_{\varphi=1}^3 (a_{\varphi} \varphi) t^{\varphi-1} \right] = \left[\sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \right]^2$$

Which can be solved numerically .

4.2.5 Order Statistic of THR Distribution:[6]

Let be T_1, T_2, \dots, T_n a random sample of size n from new , if $T_1 \leq \dots \leq T_n$ denotes the order statistics of the random sample from a continuous population with CDF and PDF then the PDF of distribution is given by

The CDF of the min order statistic

$$F_{Y_1}(t) = 1 - \left(e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right)^n \dots\dots\dots(4.50)$$

Then the PDF of Y_1

$$f_{Y_1}(t) = n \left[\sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right] \left(e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right)^{n-1} \dots\dots\dots(4.51)$$

Then CDF of the max order statistic

$$F_{Y_n}(t) = \left(1 - e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right)^n \dots\dots\dots(4.52)$$

Then the max PDF of Y_n .

$$f_{Y_n}(t) = n \left(1 - e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right)^{n-1} \left[\sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right] \dots\dots\dots (4.53)$$

4.2.6 Maximum Likelihood Estimators of THR Distribution

It is apply this method in order to estimate the parameters of the THR distribution

,Let $\Upsilon = (a_0, a_1, a_2, a_3)^T$ be the factor vector of that we want to guess and

(t_1, \dots, t_n) be a random variables

$$\begin{aligned}
 L = L(a_0, a_1, a_2, a_3; t_1, t_2, \dots, t_m) &= \prod_{j=1}^m f_2(a_0, a_1, a_2, a_3; t_j) \\
 &= \prod_{j=1}^m \left[\sum_{\varphi=0}^3 a_{\varphi} t_j^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \right] \\
 &= e^{-\sum_{j=0}^m \sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \prod_{j=1}^m \left[\sum_{\varphi=0}^3 a_{\varphi} t_j^{\varphi} \right]
 \end{aligned}$$

It can be presented the log likelihood function for the vector of parameters $(a_{\varphi})^T$ as

$$\ln L = \left[-\sum_{j=0}^m \sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1} \right] + \left[\sum_{j=0}^m \ln \sum_{\varphi=0}^3 a_{\varphi} t_j^{\varphi} \right]$$

Then, by taking the partial derivatives of $\ln L$ with respect to unidentified ,

$t_0 = 1$ parameters (a_{φ}) as

$$\frac{\partial \ln L}{\partial a_0} = \left[-\sum_{j=0}^m t_j \right] + \frac{1}{\sum_{j=0}^m \ln(a_0 + \sum_{\varphi=1}^3 a_{\varphi} t_j^{\varphi})} \dots \dots \dots (4.54)$$

$$\frac{\partial \ln L}{\partial a_1} = \left[-\sum_{j=0}^m \frac{t_j^2}{2} \right] + \sum_{j=0}^m \frac{t_j}{(a_0 + \sum_{\varphi=1}^3 a_{\varphi} t_j^{\varphi})} \dots \dots \dots (4.55)$$

$$\frac{\partial \ln L}{\partial a_2} = \left[-\sum_{j=0}^m \frac{t_j^3}{3} \right] + \sum_{j=0}^m \frac{t_j^2}{(a_0 + \sum_{\varphi=1}^3 a_{\varphi} t_j^{\varphi})} \dots \dots \dots (4.56)$$

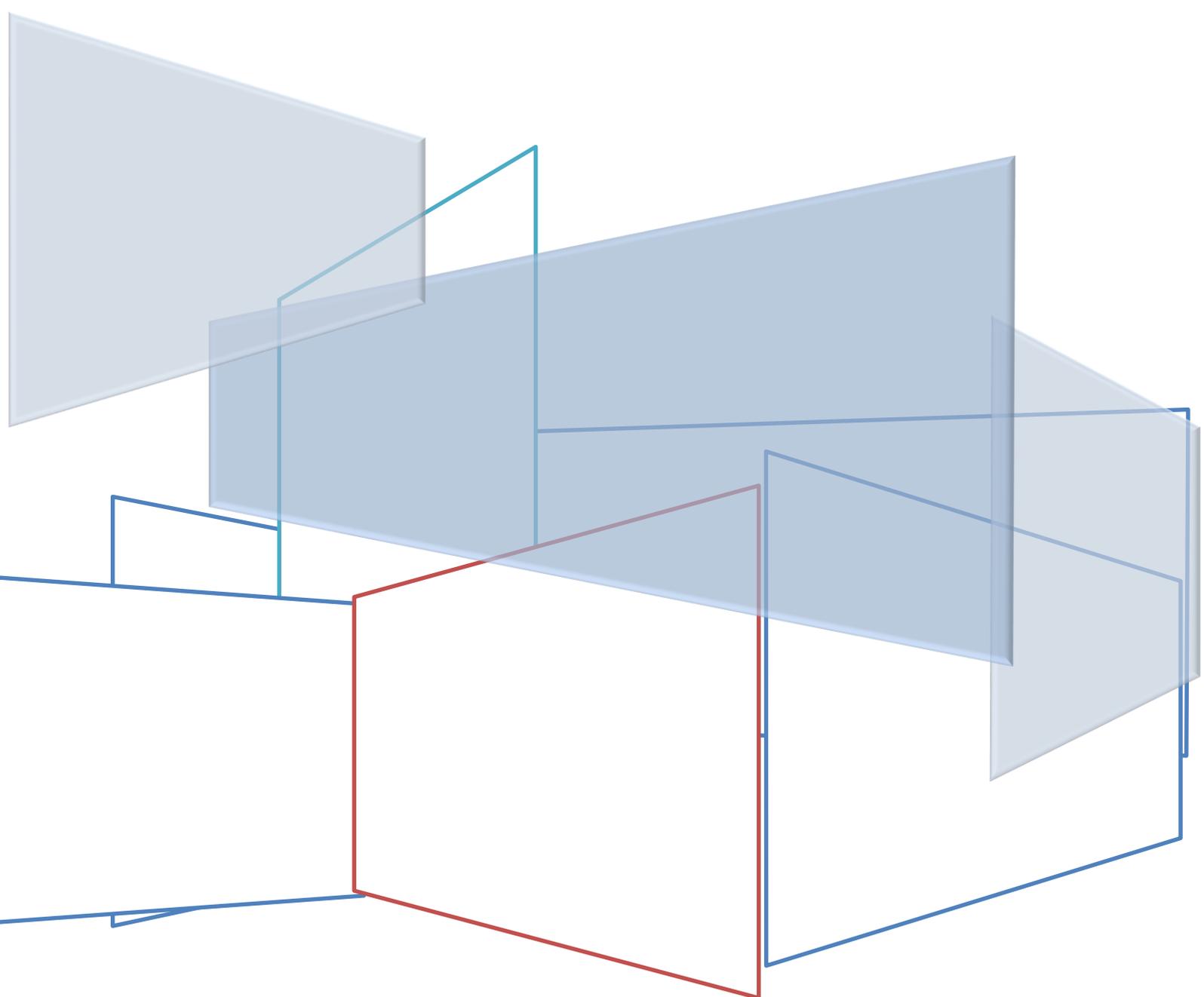
$$\frac{\partial \ln L}{\partial a_3} = \left[-\sum_{j=0}^m \frac{t_j^4}{4} \right] + \sum_{j=0}^m \frac{t_j^3}{(a_0 + \sum_{\varphi=1}^3 a_{\varphi} t_j^{\varphi})} \dots \dots \dots (4.57)$$

When $\frac{\partial \ln L}{\partial a_\varphi} = 0$, there is no closed solution of (4.54), (4.55), (4.56) and

(4.57) therefore, numerical technique (Newton-Raphson method) should be applied to solve.

Chapter Five

Transmuted Hazard Model



In this chapter, a new transmuted hazard formula (THF) is proposed by combining the transmuted survival formula with the hazard function. Because a hazard function is considered important and necessary in the field of biostatistics and analyzing survival data, this new formula (THF) yields new distributions whose results in analyzing life data are superior to those of the original life distribution, from which we derived the linear exponential distribution. We applied the formula to it and demonstrated, using actual data and maximum likelihood estimators, that the new distribution is superior to the original distribution.

5.1 The Transmuted Hazard Formula (TH)

Equation (1.2) will be put into formula (1.4) to get the new transmuted formula hazard **TH**.

$$e^{-\int_0^t h_2(y)dy} = e^{-2\int_0^t h(y)dy} + \lambda e^{-2\int_0^t h(y)dy} - \lambda e^{-\int_0^t h(y)dy} \quad \dots(5.1)$$

Again

$$e^{-\int_0^t h_2(y)dy} = e^{-\int_0^t h(y)dy} \left[\left(e^{-\int_0^t h(y)dy} + \lambda e^{-\int_0^t h(x)dx} \right) - \lambda \right] \quad \dots(5.2)$$

Taking the logarithm to both sides .

$$\ln e^{-\int_0^t h_2(y)dy} = \ln \left(e^{-\int_0^t h(y)dy} \left[\left(e^{-\int_0^t h(y)dy} + \lambda e^{-\int_0^t h(y)dy} \right) - \lambda \right] \right)$$

Then

$$-\int_0^t h_2(y)dy = -\int_0^t h(y)dy + \ln \left[e^{-\int_0^t h(y)dy} + \lambda \left(e^{-\int_0^t h(y)dy} - 1 \right) \right]$$

By deriving both sides with respect to t

$$-h_2(t) = -h(t) - \frac{(1 + \lambda)h(t) e^{-\int_0^t h(x)dx}}{\left[(1 + \lambda) e^{-\int_0^t h(x)dx} - \lambda \right]}$$

Some illustrative steps.

$$h_2(t) = h(t) \left[1 + \frac{(1 + \lambda) e^{-\int_0^t h(x) dx}}{\left[(1 + \lambda) e^{-\int_0^t h(x) dx} - \lambda \right]} \right]$$

By adding subtracting λ .

$$h_2(t) = h(t) \left[1 + \frac{(1 + \lambda) e^{-\int_0^t h(x) dx} - \lambda + \lambda}{\left[(1 + \lambda) e^{-\int_0^t h(x) dx} - \lambda \right]} \right]$$

$$h_2(t) = h(t) \left[\frac{\lambda}{\left[(1 + \lambda) e^{-2 \int_0^t h(x) dx} - \lambda \right]} + 2 \right]$$

$$h_2(t) = h(t) \left[\frac{(1+\lambda) e^{-\int_0^t h(y) dy}}{\left[(1+\lambda) e^{-\int_0^t h(y) dy} - \lambda \right]} + 1 \right] \quad \dots (5.3)$$

Therefore, this is the new formula, known as the transmuted hazard formula TH , that is discussed further down:

$$h_2(t) = h(t) \left[\frac{(1+\lambda) e^{-\int_0^t h(y) dy} - \lambda + \lambda}{\left[(1+\lambda) e^{-\int_0^t h(y) dy} - \lambda \right]} + 1 \right] \quad \dots (5.4)$$

$$h_2(t) = h(t) \left[\frac{\lambda}{\left[(1+\lambda) e^{-2 \int_0^t h(y) dy} - \lambda \right]} + 2 \right] \quad \dots (5.5)$$

The function $h_2(t)$ is a hazard function if it satisfies the following conditions .

the first property is satisfied $\forall t \in (0, \infty)$ as well

For (5.4) and (5.6)

$$h_2(t) \geq 0 \text{ since } h(t) \geq 0 \text{ and } when(1 + \lambda)e^{-\int_0^t h(y)dy} > \lambda$$

Consequently, given the above.

$$e^{-\int_0^t h(y)dy} = e^{\ln R} = S(t) \rightarrow 0 \text{ when } t \rightarrow \infty, \text{ then } h_2(t) = h(y)$$

$$\text{and } S(t) \rightarrow 1 \text{ when } t \rightarrow 0, \text{ then } h_2(t) = (2 + \lambda)h(t) \geq 0$$

The second property is next

$$\int_0^\infty h_2(t) dt = \left[\int_0^\infty h(t)dt + \left[\ln \left((1 + \lambda) e^{-\int_0^t h(y)dy} - \lambda \right) \right]_0^\infty \right]$$

$$\int_0^\infty h_2(t) dt =$$

$$= \left[\int_0^\infty h(t)dt + \ln((1 + \lambda)S(\infty) - \lambda) - \ln((1 + \lambda)S(0) - \lambda) \right]$$

$$\int_0^\infty h_2(t) dt = [\infty + \ln(-\lambda) - \ln(1)] = \infty$$

The Cumulative Hazard Function of TH

The purpose of cumulative hazards will be determined by its definition.

$$H_2(t) = \int_0^t h_2(t)dy = \left[\int_0^t \frac{\lambda h(t)}{(1+\lambda)e^{-H(t)}-\lambda} dt + 2H(t) \right] \dots\dots\dots(5.6)$$

The Density Function of TH

It is know that

$$f_2(t) = h_2(t) e^{-\int_0^t h_2(y)dy}$$

$$e^{-H_2(t)} = e^{-\int_0^t \frac{\lambda h(y)}{\left(e^{-\int_0^t h(y)dy} + \lambda \left(e^{-\int_0^t h(y)dy} - 1\right)\right)} dy} e^{-2 \int_0^t h(y)dy}$$

The density function law will be used.

$$e^{-H_2(t)} = e^{-\int_0^t \frac{\lambda h(y)}{\left(e^{-\int_0^t h(y)dy} + \lambda \left(e^{-\int_0^t h(y)dy} - 1\right)\right)} dy} e^{-2 \int_0^t h(y)dy}$$

$$f_2(t) =$$

$$\left[\frac{\lambda h(t)}{\left((1 + \lambda)e^{-\int_0^t h(x)dx} - \lambda\right)} + 2h(t) \right] e^{-\int_0^t \frac{\lambda h(t)}{\left((1 + \lambda)e^{-\int_0^t h(x)dx} - \lambda\right)} dx} e^{-\left(2 \int_0^t h(t)dx\right)}$$

Thus

$$f_2(t) = \left[\frac{\lambda h(t)e^{-\left(2 \int_0^t h(t)dx\right)}}{\left((1 + \lambda)e^{-\int_0^t h(x)dx} - \lambda\right)} + 2h(t)e^{-\left(2 \int_0^t h(t)dx\right)} \right] e^{-\int_0^t \frac{\lambda h(t)}{\left((1 + \lambda)e^{-\int_0^t h(x)dx} - \lambda\right)} dx}$$

$$f_2(t) = \left[\frac{\lambda h(t) e^{-\left(\int_0^t h(x) dx\right)} e^{-\left(\int_0^t h(x) dx\right)}}{\left((1 + \lambda) e^{-\int_0^t h(x) dx} - \lambda \right)} + 2h(t) e^{-\left(\int_0^t h(t) dx\right)} e^{-\left(\int_0^t h(x) dx\right)} \right] e^{-\int_0^t \frac{\lambda h(t)}{\left((1 + \lambda) e^{-\int_0^t h(x) dx} - \lambda \right)} dx}$$

$$f_2(t) = \frac{\lambda f(t) e^{-\left(\int_0^t h(x) dx\right)} e^{-\int_0^t \frac{\lambda h(t)}{\left((1 + \lambda) e^{-\int_0^t h(x) dx} - \lambda \right)} dx}}{\left((1 + \lambda) e^{-\int_0^t h(x) dx} - \lambda \right)}$$

$$+ 2f(t) e^{-\left(\int_0^t h(x) dx\right)} e^{-\int_0^t \frac{\lambda h(t)}{\left((1 + \lambda) e^{-\int_0^t h(x) dx} - \lambda \right)} dx}$$

$$f_2(t) = \left[\frac{\lambda f(t)}{\left((1 + \lambda) e^{-\int_0^t h(x) dx} - \lambda \right)} + 2f(t) \right] e^{-\int_0^t h(x) + \frac{\lambda h(x)}{\left((1 + \lambda) e^{-\int_0^t h(x) dx} - \lambda \right)} dx}$$

$$f_2(t) = \left[\frac{\lambda f(t)}{\left((1 + \lambda) e^{-\int_0^t h(x) dx} - \lambda \right)} + 2f(t) \right] e^{-\int_0^t h(x) + \frac{\lambda h(x)}{\left((1 + \lambda) e^{-\int_0^t h(x) dx} - \lambda \right)} dx}$$

$$f_2(t) = \left[\frac{\lambda f(t)}{\left((1 + \lambda)e^{-\int_0^t h(x)dx} - \lambda \right)} + 2f(t) \right] e^{-\int_0^t \frac{h(x)\left((1+\lambda)e^{-\int_0^t h(x)dx} - \lambda \right) + \lambda h(x)}{(1+\lambda)e^{-\int_0^t h(x)dx} - \lambda} dx}$$

$$f_2(t) = f(t)e^{-\int_0^t \frac{h(y)\left((1+\lambda)e^{-\int_0^t h(y)dy} \right)}{(1+\lambda)e^{-\int_0^t h(y)dy} - \lambda} dy} \left[\lambda \frac{1}{\left((1 + \lambda)e^{-\int_0^t h(y)dy} - \lambda \right)} + 2 \right]$$

$$f_2(t) = f(t) \left[\lambda \frac{e^s}{\left((1+\lambda)e^{-\int_0^t h(y)dy} + \lambda(e^{-\int_0^t h(y)dy} - 1) \right)} + 2e^s \right] \dots\dots\dots(5.8)$$

Using the following hypothesis

$$S = - \int_0^t \frac{h(y) \left(\lambda e^{-\int_0^t h(y)dy} + e^{-\int_0^t h(y)dy} \right)}{\left(e^{-\int_0^t h(y)dy} + \lambda \left(e^{-\int_0^t h(y)dy} - 1 \right) \right)} dy$$

let $v = e^{-\int_0^t h(y)dy} + \lambda \left(e^{-\int_0^t h(y)dy} - 1 \right),$

$dv = -h(y) \left(e^{-\int_0^t h(y)dy} + \lambda e^{-\int_0^t h(y)dy} \right) dy$

$$S = \int_0^t \frac{dv}{v}$$

$$S = \ln \left[\left(e^{-\int_0^t h(y)dy} + \lambda \left(e^{-\int_0^t h(y)dy} - 1 \right) \right) \right] \Big|_0^t$$

It is get $S = \ln \left(e^{-\int_0^t h(y)dy} + \lambda \left(e^{-\int_0^t h(y)dy} - 1 \right) \right) \dots \dots \dots (5.9)$

And therefore , the function $f_2(t)$ then

$$f_2(t) = f(t) \left[2e^{-\int_0^t h(y)dy} + \lambda \left(2e^{-\int_0^t h(y)dy} - 1 \right) \right] \dots \dots \dots (5.10)$$

It can be shown that the density function $f_2(t)$ must satisfy the following two conditional.

[1] The first condition that the function is bigger or equal to zero is satisfied for $\forall t \in (0, \infty)$, in addition

$$f(t) \left[2e^{-\int_0^t h(y)dy} - \lambda \left(2e^{-\int_0^t h(y)dy} - 1 \right) \right] \geq 0$$

$$[-\lambda] \left(1 - 2e^{-\int_0^t h(y)dy} \right) \geq -2e^{-\int_0^t h(y)dy}$$

$$[-\lambda] \geq \frac{-2e^{-\int_0^t h(y)dy}}{\left(1 - 2e^{-\int_0^t h(y)dy} \right)} \dots \dots \dots (5.11)$$

we product both sides in equation(5.11) by the value (-1) ,we get

$$[\lambda] \leq \frac{2e^{-\int_0^t h(y)dy}}{\left(1 - 2e^{-\int_0^t h(y)dy} \right)}$$

[2] The second condition

$$\int_0^\infty f_2(t) dt = \int_0^\infty f(t) \left[2(1 + \lambda)e^{-\int_0^t h(y)dy} - \lambda \right] dt$$

$$\int_0^{\infty} f_2(t) dt = \int_0^{\infty} 2(1 + \lambda)f(t)e^{-\int_0^t h(y)dy} dt - \int_0^{\infty} [\lambda f(t)] dt$$

$$\int_0^{\infty} f_2(t) dt = (2 + 2\lambda) \int_0^{\infty} f(t)e^{-\int_0^t h(y)dy} dt - \lambda$$

Since $e^{-\int_0^t h(y)dy} = s(t)$ then let $F(t) = v$

$$\int_0^{\infty} f_2(t) dt = (2 + 2\lambda) \int_0^1 (1 - v)dv - \lambda$$

$$\int_0^{\infty} f_2(t) dt = (2 + 2\lambda) \left(1 - \frac{1}{2}\right) - \lambda$$

$$\int_0^{\infty} f_2(t) dt = 1$$

Remark(5.1)

After finding the probability mass function, we will discuss proving the third condition of the hazard function.

The third property

$$h_2(t) \geq 0 \text{ and } \lim_{t \rightarrow 0} f_2(t) = \lim_{t \rightarrow 0} h_2(t); f_2(t) \geq h_2(t) \forall t \in \mathcal{R}^+$$

There must be a period of time, that increases the $h_2(t)$ function or decreases.

The contrary, over time.

The Cumulative Function of THF .

$$F_2(t) = 1 - e^{-\int_0^t h_2(t)dy} =$$

$$F_2(t) = 1 - e^{-\int_0^t \frac{-\lambda h(y)}{(1+\lambda)e^{-\int_0^t h(y)dy - \lambda}} dy} e^{-2 \int_0^t h(y)dy} \dots\dots\dots(5.12)$$

The Survival Function of THF .

$$S_2(t) = e^{-\int_0^t h_2(t)dy} = e^{-\int_0^t \frac{-\lambda h(y)}{(1+\lambda)e^{-\int_0^t h(y)dy - \lambda}} dy} e^{-2 \int_0^t h(y)dy} \dots\dots\dots(5.13)$$

5.2 Transmuted Hazard Models .

discussed the basics of three types of Transmuted Hazard models.

mngcas

5.2.1 The Transmuted Hazard Constant Hazard Rate Model

THCHRM

IF The hazard function $h(t)$ is constant

$$h(t) = a \quad \forall t > 0, a > 0 \dots\dots\dots(5.14)$$

Using the formula (5.5) to find the new hazard function of THCHRM is

$$h_2(t) = a \left[\frac{\lambda}{[(1+\lambda) e^{-\int_0^t a dy - \lambda}] + 2} \right]$$

The hazard function of THCHRM

$$h_2(t) = a \left[\frac{\lambda}{[(1+\lambda) e^{-at - \lambda}] + 2} \right] \dots\dots\dots(5.15)$$

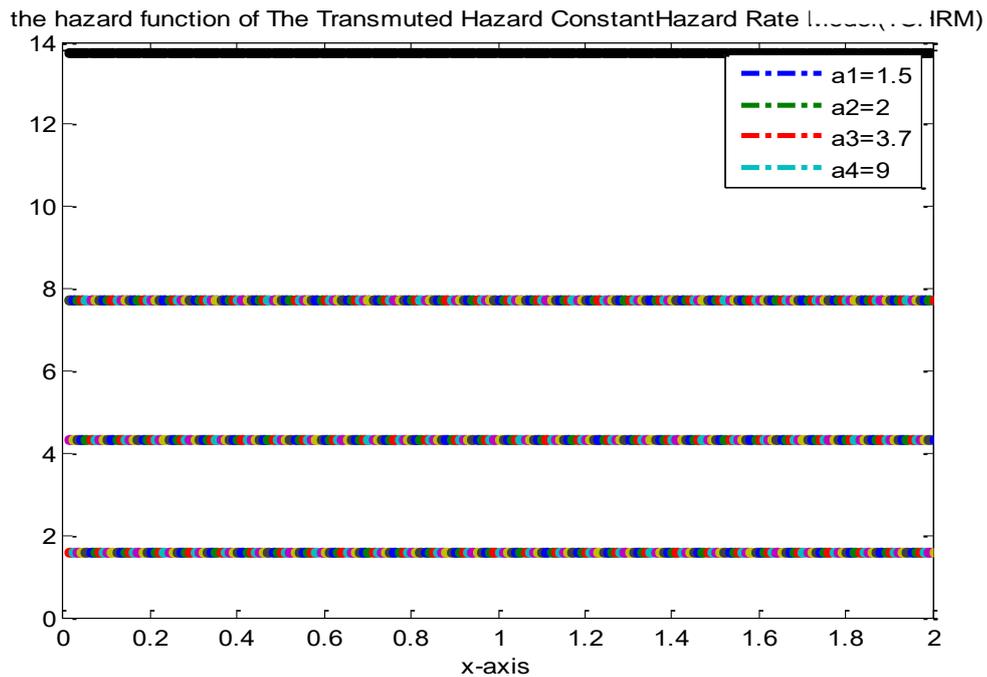
Since the function $h_2(t)$ satisfy the conditions of the hazard function as follows .

1. $h_2(t) \geq 0 \quad \forall t \geq 0$ and for all $\lambda \geq 2(1 - 2 e^{-at})^{-1} e^{-at}$.

$$2. \int_0^\infty h_2(t) dt = \left[\int_0^\infty \frac{a(1+\lambda) e^{-at}}{[(1+\lambda) e^{-at} - \lambda]} dt + \int_0^\infty a dt \right]$$

$$\int_0^\infty h_2(t) dt$$

$$= [\ln(-\lambda) - \ln(1) + \infty] = \infty$$



Figure(5.1): The hazard function of THCHRM, where the parameter a take different values .

It's been brought to our attention that The hazard function of THCHRM is monotone in the sense that the risk of harm grows with the unit's age, but reduces when its condition improves.

The Density Function of THCHRM .

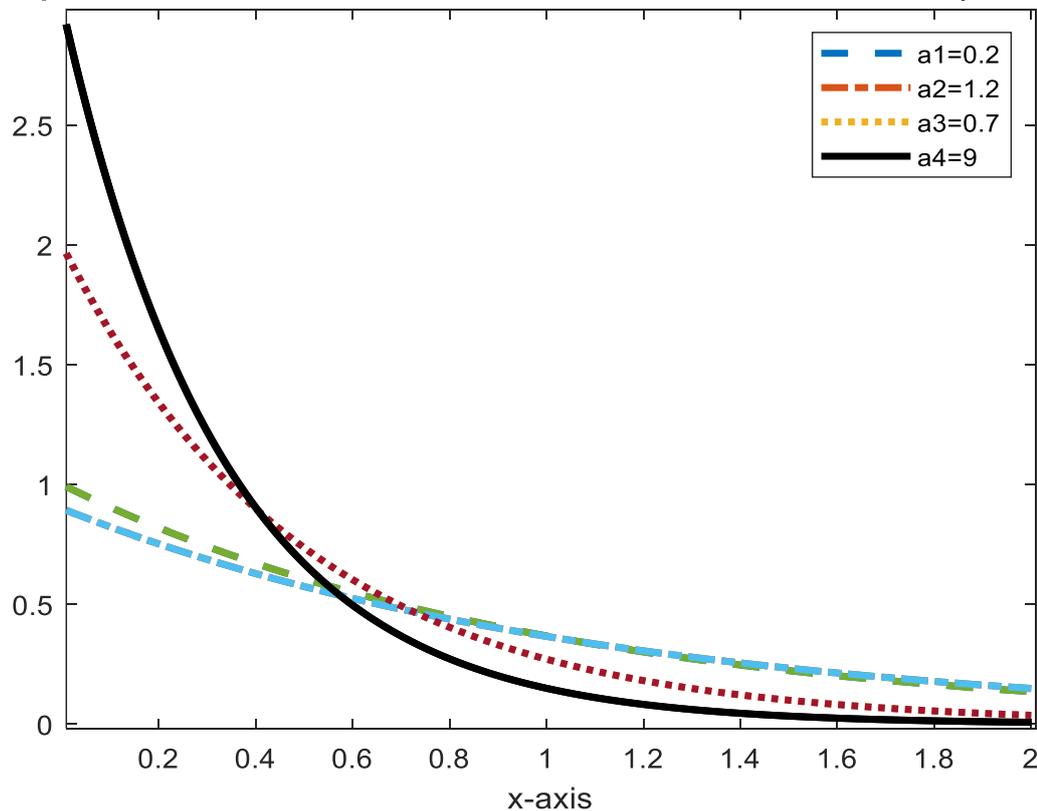
$$f_2(t) = a e^{-at} [2(1 + \lambda) e^{-at} - \lambda] \dots \dots \dots (5.16)$$

Check the density function conditions

$$1 - f_2(t) = a e^{-at} [2(1 + \lambda) e^{-at} - \lambda] \geq 0$$

$$2 - \int_0^{\infty} f_2(t) dt = \int_0^{\infty} a e^{-at} [2(1 + \lambda) e^{-at} - \lambda] dt = 1$$

the pdf of The Transmuted Hazard Constant Hazard Rate Model (THCHRM)



Figure(5.2): The Density Function of THCHRM.

Where the parameter a take different values . Notice that the function is increasing and then decreasing as you get closer $t \rightarrow \infty$.

Remark(5.2)

Another technique to show that $h_2(t)$ meets the requirements of the hazard function, using the density function and survival function as evidence.

$$1) f_2(t) = a e^{-at} [2(1 + \lambda) e^{-at} - \lambda] \geq 0 ;$$

$$\text{And } S_2(t) = (1 + \lambda)S^2(t) - \lambda S(t) > 0 \text{ thus}$$

$$h_2(t) = \frac{f_2(t)}{s_2(t)} \geq 0.$$

$$2) \int_0^{\infty} h_2(t) dt = - \int_0^{\infty} d \ln s_2(t) = - \ln s_2(t) \Big|_0^{\infty} = \ln s_2(0) - \ln s_2(\infty) = \ln 1 - \ln 0 = \infty$$

$$3) h_2(t) \geq 0 \text{ and } \lim_{t \rightarrow 0} f_2(t) = \lim_{t \rightarrow 0} h_2(t); f_2(t) \geq h_2(t) \forall t > 0$$

To prove this

$$\lim_{t \rightarrow 0} f_2(t) = a \lim_{t \rightarrow 0} e^{-at} [2(1 + \lambda) e^{-at} - \lambda] = a(2 + \lambda).$$

$$\lim_{t \rightarrow 0} h_2(t) = \lim_{t \rightarrow 0} \left[\frac{\lambda}{[(1 + \lambda) e^{-at} - \lambda]} + 2 \right] = a(2 + \lambda).$$

And

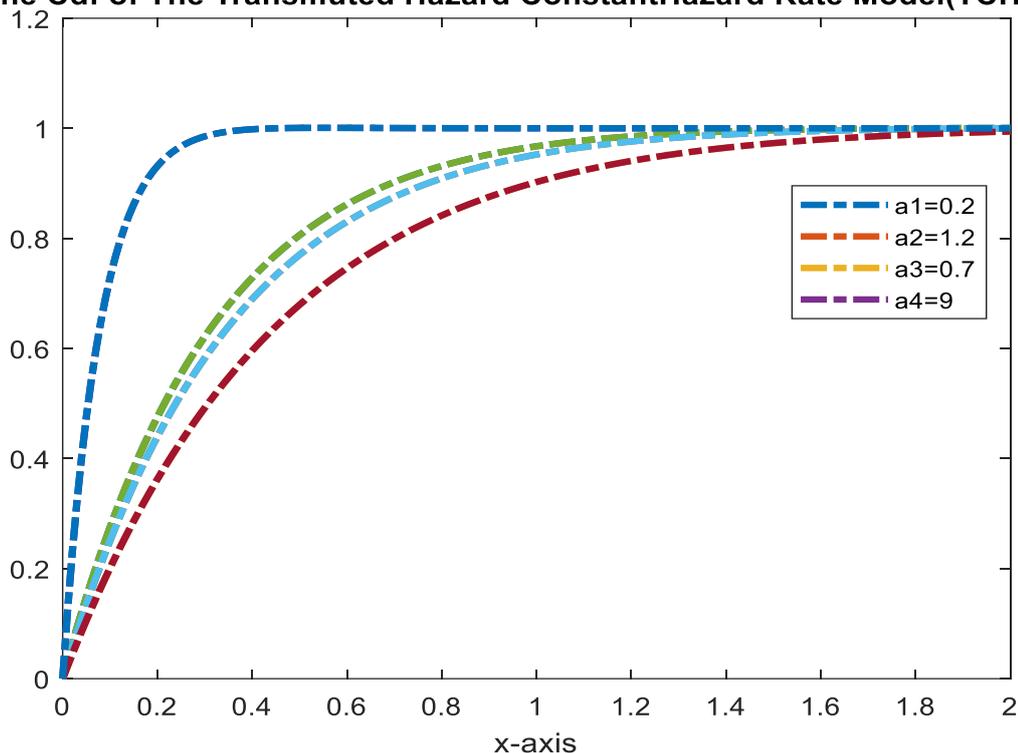
$$\text{since it's } f_2(t) = h_2(t) e^{-\int_0^t h_2(y) dy} \text{ and } e^{-\int_0^t h_2(y) dy} \geq 0$$

$$\text{then } f_2(t) \geq h_2(t).$$

The Cumulative Function of THCHRM .

$$F_2(t) = 1 - e^{-\int_0^t h_2(y) dy} = 1 - ((1 + \lambda)e^{-at} - \lambda) e^{-at} \dots\dots\dots(5.17)$$

the Cdf of The Transmuted Hazard Constant Hazard Rate Model(TCHRM)

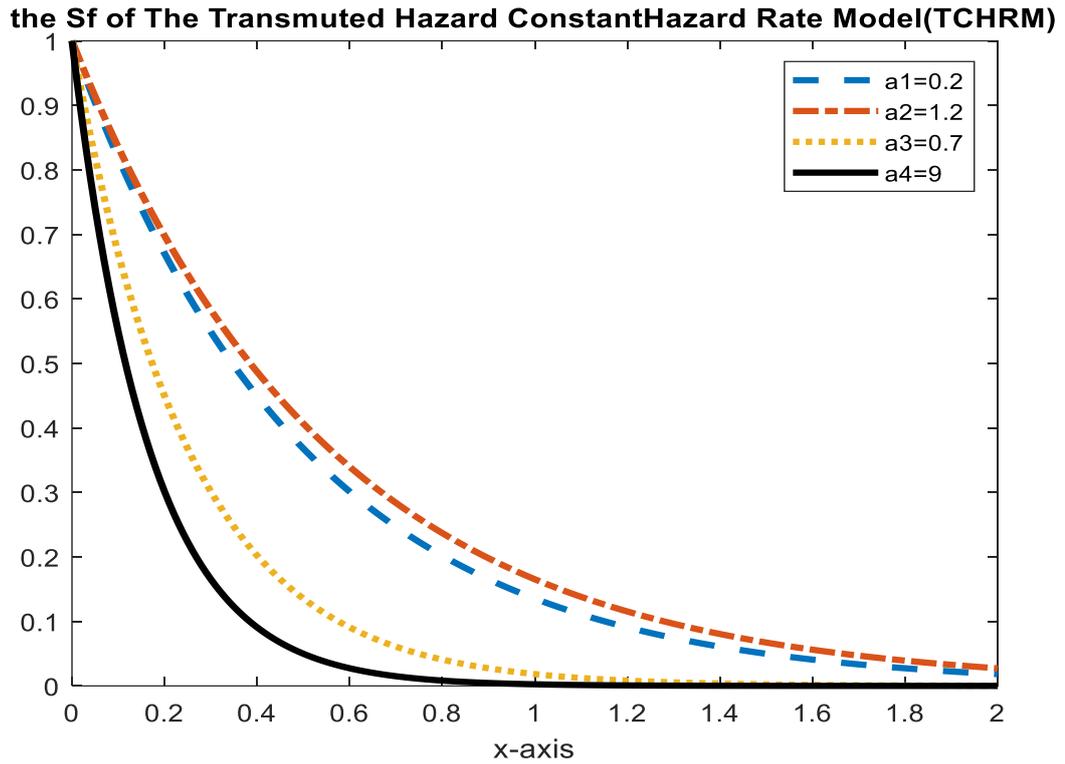


Figure(5.3): The Cumulative Function of THCHRM.

Where the parameter λ takes number 1 . This function begins at zero and steadily increases as it moves closer and closer to the value of one when $t \rightarrow \infty$.

The Survival Function of THCHRM .

$$S_2(t) = e^{-\int_0^t h_2(t)dy} = e^{-\lambda a \int_0^t \frac{1}{((1+\lambda)e^{-at}-\lambda)} dy} e^{-2at} \dots\dots\dots(5.18)$$



Figure(5.4): The Survival Function of THCHRM, where the parameter a take different values and $\lambda = -1$. This function starts at one and continues decreasing until it approaches zero where $t \rightarrow \infty$.

5.2.1.1 Shape of THCHRM .

5.2.1.1.1 The Limit of Cumulative Function of THCHRM

$$\lim_{t \rightarrow 0} F_2(t) = \lim_{t \rightarrow 0} 1 - ((1 + \lambda)e^{-at} - \lambda) e^{-at}$$

$$\lim_{t \rightarrow 0} F_2(t) = 1 - \lim_{t \rightarrow 0} ((1 + \lambda)e^{-at} - \lambda) \lim_{t \rightarrow 0} e^{-at} = 1 - 1 = 0$$

And

$$\lim_{t \rightarrow \infty} F_2(t) = 1 - \lim_{t \rightarrow \infty} ((1 + \lambda)e^{-at} - \lambda) e^{-at}$$

$$\lim_{t \rightarrow \infty} F_2(t) = 1 - \lim_{t \rightarrow \infty} ((1 + \lambda)e^{-at} - \lambda) \lim_{t \rightarrow \infty} e^{-at} = 1$$

5.2.1.1.2 The Limit of survival function of THCHRM .

$$\lim_{t \rightarrow 0} S_2(t) = \lim_{t \rightarrow 0} ((1 + \lambda)e^{-at} - \lambda) \quad e^{-at} = 1$$

$$\text{And } \lim_{t \rightarrow \infty} S_2(t) = 0$$

5.2.1.2 Moment THCHRM

Theorem(5.3)

If $T \sim \text{THCHRM}(a)$ then the r_{th} moment about the origin, and about the mean μ one help hey as of T , say μ'_r , is given as

$$1. E_2(T^r) = \left[\frac{2(1+\lambda)a}{(2a+1)^{r+1}} r! - \frac{\lambda}{(a)^{r+1}} r! \right]. \quad \dots(5.19)$$

$$2. E_2(T - \mu)^r =$$

$$3. \sum_{k=0}^r C_k^r \left[\frac{2(1+\lambda)a}{(2a+1)^{k+1}} k! - \frac{\lambda}{(a)^{k+1}} k! \right] \left[\frac{2(1+\lambda)a}{(2a+1)^2} - \frac{\lambda}{a^2} \right]^{r-k} \quad \dots (5.20)$$

Proof

Proof part one from theorem

$$\textbf{Take } E_2(T^r) = \int_0^{\infty} t^r \theta e^{-\theta t} [2(1 + \lambda) e^{-\theta t} - \lambda] dt$$

$$E_2(T^r) = \left[2(1 + \lambda)\theta \int_0^{\infty} t^r e^{-(2\theta+1)t} dt - \lambda\theta \int_0^{\infty} t^r e^{-\theta t} dt \right]$$

Suppose that $L_1 = \int_0^{\infty} t^r e^{-\theta t} dt$ and $L_2 = \int_0^{\infty} t^r e^{-(2\theta+1)t} dt$ and

Let $y = \theta t$ than $dy = \theta dt$

$$L_1 = \int_0^{\infty} \left(\frac{y}{\theta}\right)^r e^{-y} \frac{dy}{\theta}$$

$$L_1 = \left(\frac{1}{\theta}\right)^{r+1} \Gamma(r + 1)$$

Same that $L_2 = \int_0^{\infty} t^r e^{-(2\theta+1)t} dt$

Let $z = (2\theta + 1)t$, $dz = (2\theta + 1)dt$

$$L_2 = \left(\frac{1}{2\theta + 1}\right)^{r+1} \int_0^{\infty} (z)^r e^{-z} dz$$

$$L_2 = \left(\frac{1}{2\theta + 1}\right)^{r+1}$$

As a result,

$$E_2(T^r) = \left[\frac{2(1+\lambda)a}{(2a+1)^{r+1}} r! - \frac{\lambda}{(a)^{r+1}} r! \right]$$

Proof two part

Depending on the equations (5.19) and Binomial theorem.

When $E_2(T) = \frac{2(1+\lambda)a}{(2a+1)^2} - \frac{\lambda}{a^2}$.

Then

$E_2(T - \mu)^r =$

$$= \sum_{k=0}^r C_k^r \left[\frac{2(1+\lambda)a}{(2a+1)^{k+1}} k! - \frac{\lambda}{(a)^k} k! \right] \left[-\frac{2(1+\lambda)a}{(2a+1)^2} + \frac{\lambda}{a} \right]^{r-k}$$

Remark(5.4)

If a is hazard function of exponential distribution then substitute it into the Transmuted Hazard Formula (TH). We will get Transmuted Survival exponential distribution.

5.2.2 Transmuted Hazard Power Hazard Rate Model THPHRM

The hazard function $h(t)$ of The Power Hazard Rate Model

$$h(t) = ct^{c-1} \quad \forall t \geq 0, c > 0 \quad \dots (5.21)$$

The new hazard function of THPHRM using (5.5).

$$h_2(t) = \left[\frac{\lambda ct^{c-1}}{\left((1+\lambda)e^{-\int_0^t ct^{c-1} dy} - \lambda \right)} + 2ct^{c-1} \right]$$

The hazard function of THPHRM .

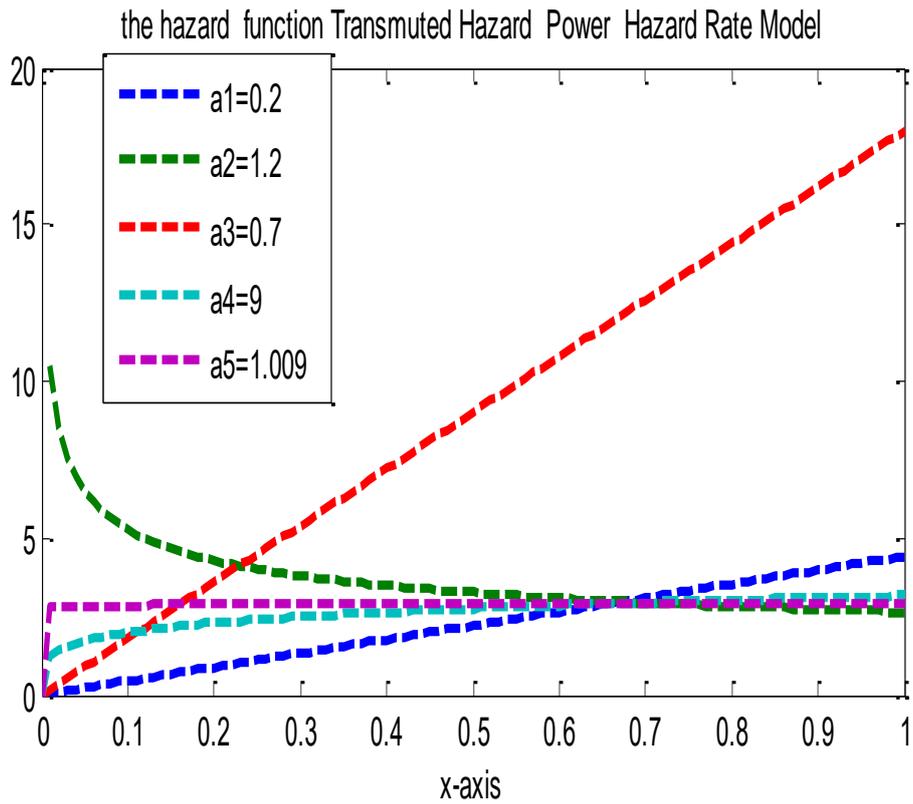
$$h_2(t) = ct^{c-1} \left[\frac{\lambda}{\left((1+\lambda)e^{-t^c} - \lambda \right)} + 2 \right] \quad \dots(5.22)$$

Since the function $h_2(t)$ satisfy the conditions of the hazard function as follows .

$$1. h_2(t) = ct^{c-1} \left[\frac{\lambda}{\left((1+\lambda)e^{-t^c} - \lambda \right)} + 2 \right] \geq 0 \quad \forall t \geq 0 \quad \text{and for all } \lambda \leq 2e^{-t^c} (1 - 2e^{-t^c})^{-1} .$$

$$2. \int_0^{\infty} h_2(t) dt = \left[\int_0^{\infty} \frac{(1+\lambda)ct^{c-1}e^{-t^c}}{\left[(1+\lambda)e^{-t^c} - \lambda \right]} dt + \int_0^{\infty} ct^{c-1} dt \right]$$

$$\int_0^{\infty} h_2(t) dt = [\ln(-\lambda) - \ln(1) + \infty] = \infty$$



Figure(5.5): The Hazard Function of THPHRD.

Where the parameters c takes different values and the parameter λ take one value . Based on the preceding diagram, we can see that the function has adopted a semi-linear shape.

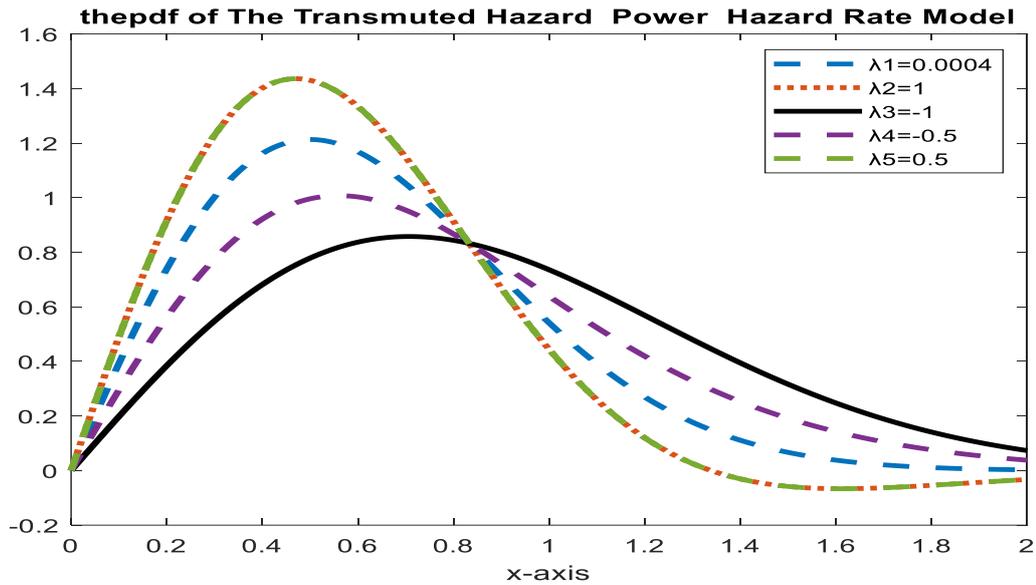
5.3.2.2 The Density Function of THPHRM

$$f_2(t) = ct^{c-1}e^{-t^c} [2(1 + \lambda)e^{-t^c} - \lambda] . \quad \dots(5.23)$$

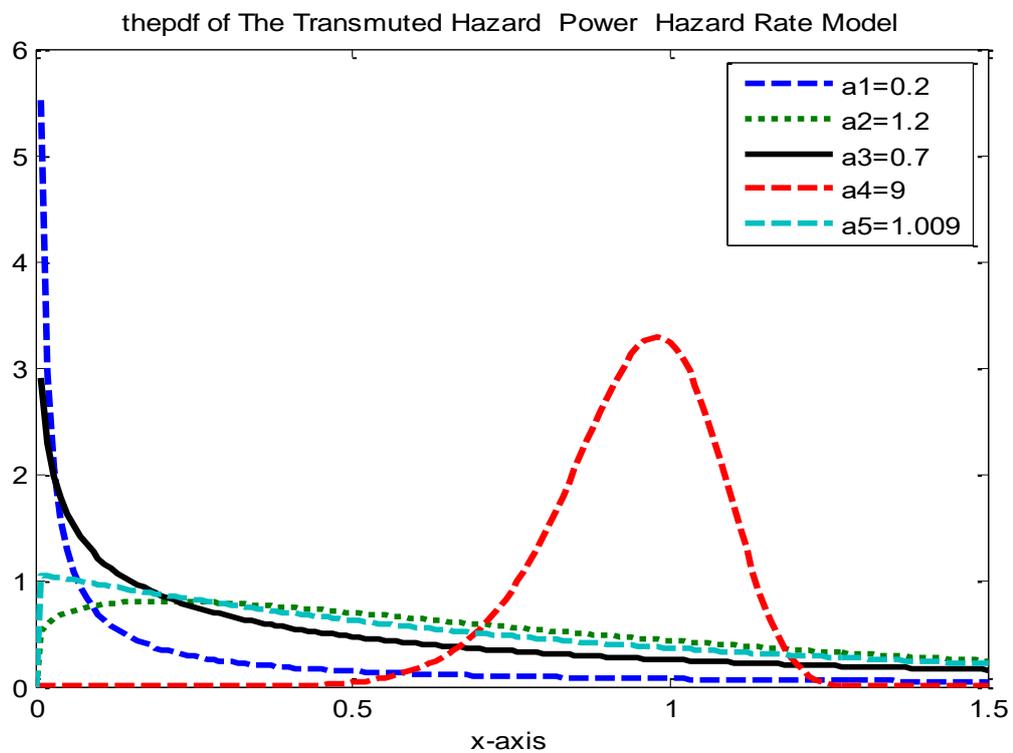
Conditions for the density function should be checked .

$$1 - f_2(t) = ct^{c-1}e^{-t^c} [2(1 + \lambda)e^{-t^c} - \lambda] \geq 0$$

$$2 - \int_0^\infty f_2(t)dt = \int_0^\infty ct^{c-1}e^{-t^c} [2(1 + \lambda)e^{-t^c} - \lambda] dt = 1$$



Figure(5.6): The PDF of THPHRM , where the parameters λ takes different values and the parameter $c = 2$ take one value . We note that the function, depending on the above form, started from zero, then increased, and then decreased, taking the form of convexity.



Figure(5.7): The PDF of THPHRM, where the parameters c takes different values and the parameter λ take one value .

It's important to notice that the shape of the function depends on the values of the parameters, as seen in the aforementioned diagram.

It has become monotone at times, and at other times it is non-monotone.

Remark(5.5)

Evidence from the density function and survival function may also be used to prove that $h_2(t)$ satisfies the hazard function's assumptions

$$\int_0^{\infty} h_2(t) dt = - \int_0^{\infty} d \ln s_2(t) = - \ln s_2(t) \Big|_0^{\infty} = \ln s_2(0) - \ln s_2(\infty) = \ln 1 - \ln 0 = \infty$$

$$h_2(t) \geq 0 \text{ and } \lim_{t \rightarrow 0} f_2(t) = \lim_{t \rightarrow 0} h_2(t); f_2(t) \geq h_2(t) \forall t > 0$$

To prove this

$$\lim_{t \rightarrow 0} ct^{c-1} e^{-t^c} [2(1 + \lambda)e^{-t^c} - \lambda] = 0.$$

$$\lim_{t \rightarrow 0} h_2(t) = \lim_{t \rightarrow 0} ct^{c-1} \left[\frac{\lambda}{((1 + \lambda)e^{-t^c} - \lambda)} + 2 \right] = 0.$$

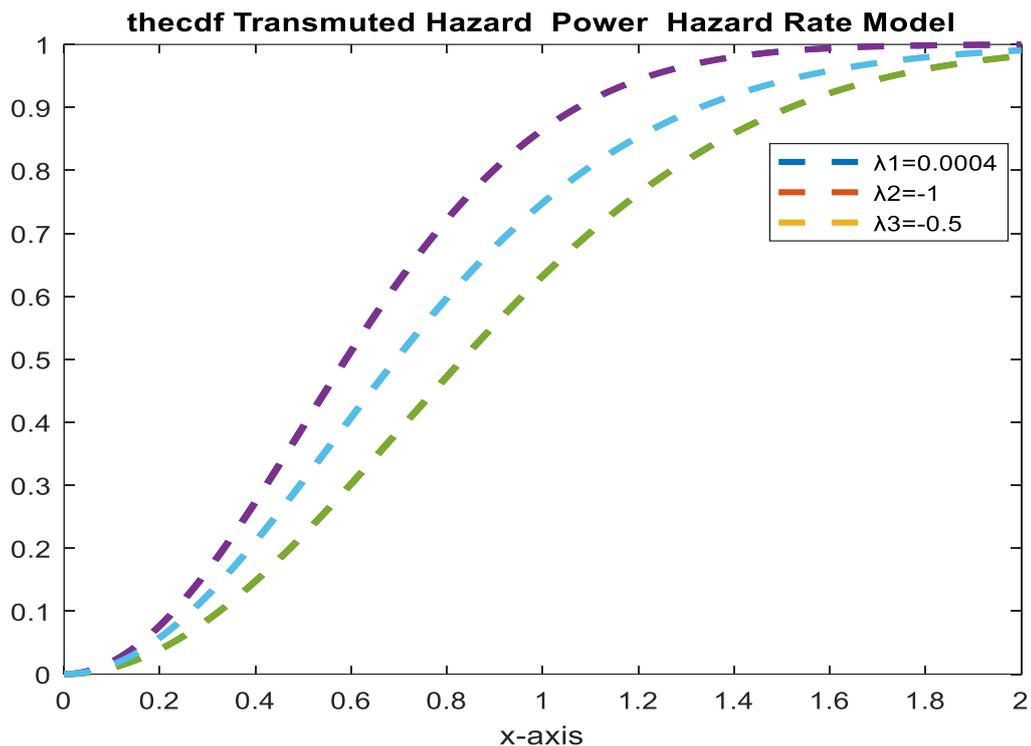
And

since it's $f_2(t) = h_2(t) e^{-\int_0^t h_2(y) dy}$ and $e^{-\int_0^t h_2(y) dy} \geq 0$

then $f_2(t) \geq h_2(t)$.

5.3.2.3 The Cumulative Function of (THPHRM)

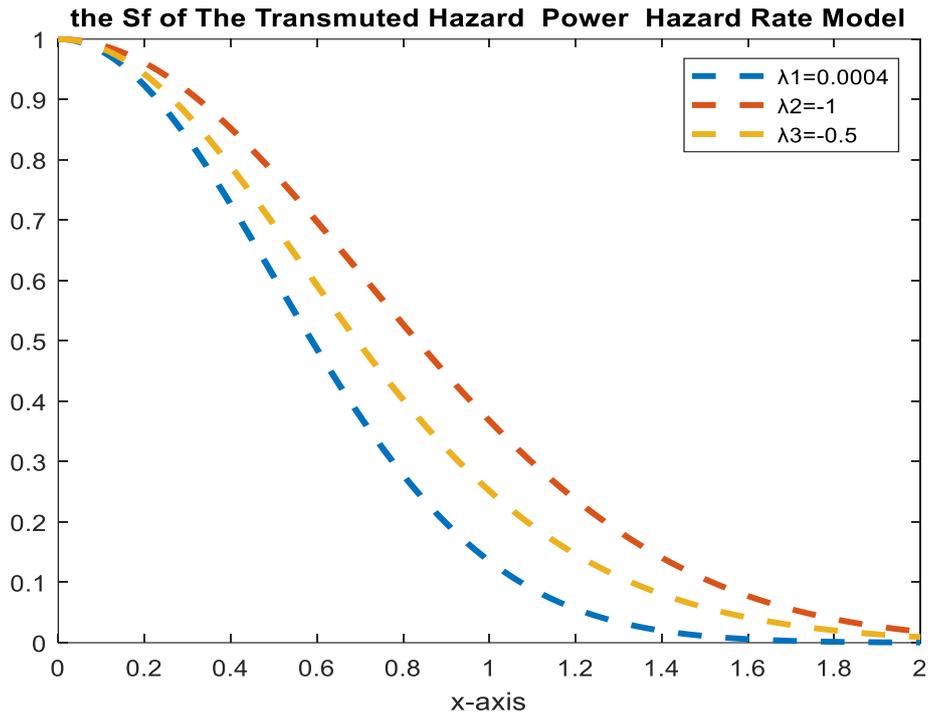
$$F_2(t) = 1 - \left[e^{-t^c} \left((1 + \lambda)e^{-t^c} - \lambda \right) \right] \dots\dots\dots(5.23)$$



Figure(5.8): The CDF of THPHRM, where the parameters λ takes different values and the parameter take c one value . It is important to note that the form of the function starts at zero, increases, and now settles at one.

The Survival Function of THPHRM .

$$S_2(t) = e^{-t^c} \left((1 + \lambda)e^{-t^c} - \lambda \right) \dots\dots\dots(5.24)$$



Figure(5.9): The Survival Function of THPHRM, where the parameters λ takes different values and the parameter c take one value. It is important to note that the form of the function starts at 1 and decreases, now settling at 0.

5.2.2.1 Shape of THPHRM

5.2.2.1.1 The Limit of Cumulative Function of THPHRM

$$\lim_{t \rightarrow 0} F_2(t) = \lim_{t \rightarrow 0} 1 - \left[e^{-t^c} \left((1 + \lambda)e^{-t^c} - \lambda \right) \right] = 1 - 1 = 0$$

And

$$\lim_{t \rightarrow \infty} F_2(t) = \lim_{t \rightarrow \infty} 1 - \left[e^{-t^c} \left((1 + \lambda)e^{-t^c} - \lambda \right) \right] = 1$$

5.2.2.1.2 The Limit of Survival Function of THPHRM

$$\lim_{t \rightarrow 0} S_2(t) = \lim_{t \rightarrow 0} 1 - \left[e^{-t^c} \left((1 + \lambda)e^{-t^c} - \lambda \right) \right] = 1$$

And $\lim_{t \rightarrow \infty} S_2(t) = 0$

5.2.2 Moment THPHRM

Theorem(5.3)

If $T \sim$ THPHRM (a) then the r_{th} moment about the origin, and about the mean μ one help hey as of T , say μ'_r , is given as

$$E_2(T^r) = \left[(1 + \lambda) \left(\frac{1}{2}\right)^{\frac{r}{c}} \Gamma\left(\frac{r}{c} + 1\right) - \lambda \Gamma\left(\frac{r}{c} + 1\right) \right] \dots \dots \dots (5.25)$$

$$E_2(T^r) = \sum_{k=0}^r C_k^r \left[(1 + \lambda) \left(\frac{1}{2}\right)^{\frac{k}{c}} \Gamma\left(\frac{k}{c} + 1\right) - \lambda \Gamma\left(\frac{k}{c} + 1\right) \right] * \left[(1 + \lambda) \left(\frac{1}{2}\right)^{\frac{1}{c}} \Gamma\left(\frac{1}{c} + 1\right) - \lambda \Gamma\left(\frac{1}{c} + 1\right) \right]^{r-k} \quad (5.26)$$

Proof

Proof one part from theorem

Take $E_2(T^r) = \int_0^{\infty} t^r ct^{c-1} e^{-t^c} [2(1 + \lambda)e^{-t^c} - \lambda] dt$

$$E_2(T^r) = \int_0^{\infty} [2(1 + \lambda) ct^{c-1} t^r e^{-2t^c} - \lambda ct^{c-1} t^r e^{-t^c}] dt$$

Suppose that $L_1 = \int_0^{\infty} 2(1 + \lambda) ct^{c-1} t^r e^{-2t^c} dt$

and $L_2 = \int_0^{\infty} [\lambda ct^{c-1} t^r e^{-t^c}] dt$ and

Let $y = 2t^c$ than $dy = 2ct^{c-1} dt$

$$\sqrt[c]{\frac{y}{2}} = t$$

$$L_1 = (1 + \lambda) \int_0^{\infty} \left(\frac{y}{2}\right)^{\frac{r}{c}} e^{-2y} dy$$

$$L_1 = (1 + \lambda) \left(\frac{1}{2}\right)^{\frac{r}{c}} \Gamma\left(\frac{r}{c} + 1\right)$$

Same that $L_2 = \int_0^{\infty} [\lambda ct^{c-1} t^r e^{-t^c}] dt$

Let $z = t^c$, $dz = ct^{c-1} dt$

$$L_2 = \int_0^{\infty} (z)^{\frac{r}{c}} e^{-z} dz$$

$$L_2 = \lambda \Gamma\left(\frac{r}{c} + 1\right)$$

As a result,

$$E_2(T^r) = \left[(1 + \lambda) \left(\frac{1}{2}\right)^{\frac{r}{c}} \Gamma\left(\frac{r}{c} + 1\right) - \lambda \Gamma\left(\frac{r}{c} + 1\right) \right]$$

Proof two part

Depending on the equations (5.25) and Binomial theorem.

When $E_2(T) = (1 + \lambda) \left(\frac{1}{2}\right)^{\frac{1}{c}} \Gamma\left(\frac{1}{c} + 1\right) - \lambda \Gamma\left(\frac{1}{c} + 1\right)$.

Then

$$E_2(T^r) = \sum_{k=0}^r C_k^r \left[(1 + \lambda) \left(\frac{1}{2}\right)^{\frac{k}{c}} \Gamma\left(\frac{k}{c} + 1\right) - \lambda \Gamma\left(\frac{k}{c} + 1\right) \right] * \left[(1 + \lambda) \left(\frac{1}{2}\right)^{\frac{1}{c}} \Gamma\left(\frac{1}{c} + 1\right) - \lambda \Gamma\left(\frac{1}{c} + 1\right) \right]^{r-k}$$

5.2.3 The Transmuted Hazard Linear Hazard Rate Model

THLHRM

The hazard function $h(t)$ of The linear hazard rate model (LHRM) is

$$h(t) = a + bt \quad \forall a, b \text{ and } t \in (0, \infty) \dots\dots\dots(5.27)$$

the new hazard function of (THLHRM) using (5.5).

$$h_2(t) = \left[\frac{\lambda(a+bt)}{\left((1+\lambda)e^{-\int_0^t (a+bt)dy} - \lambda \right)} + 2(a + bt) \right]$$

The hazard function of (THLHRM) .

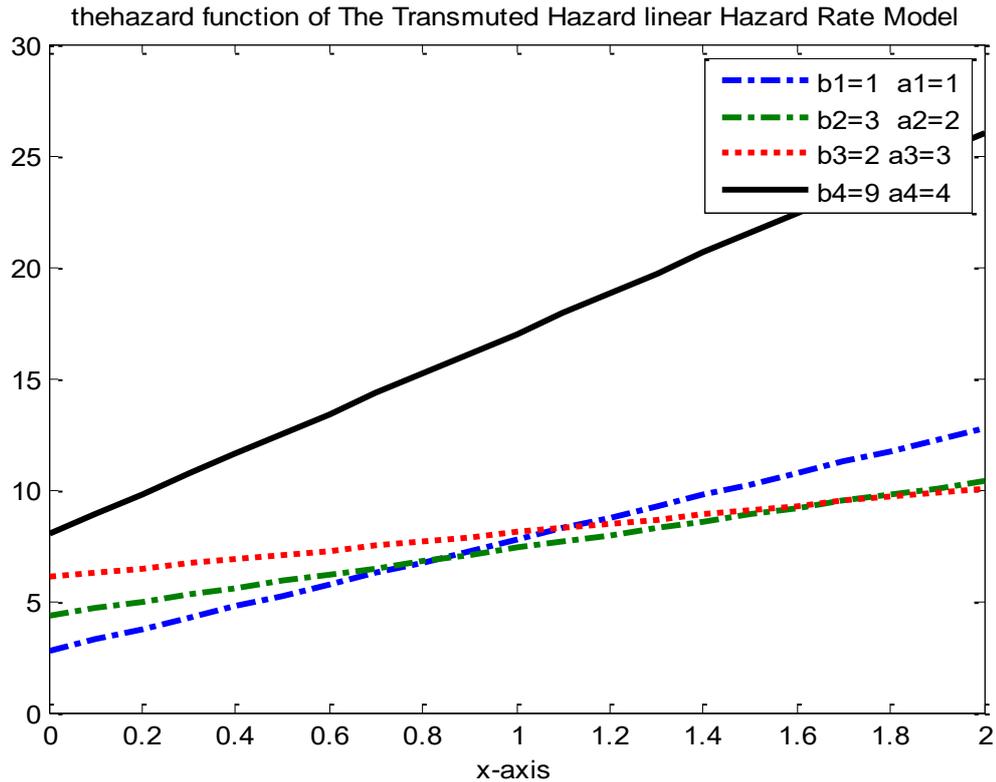
$$h_2(t) = \left[\frac{(1+\lambda)(a+bt)e^{-at-\frac{bt^2}{2}}}{\left((1+\lambda)e^{-at-\frac{bt^2}{2}} - \lambda \right)} + (a + bt) \right] \dots\dots\dots(5.28)$$

Since the function $h_2(t)$ satisfy the conditions of the hazard function

$$1. h_2(t) \geq 0 \quad \forall t \geq 0 \quad \text{and for all} \left[\lambda \leq \frac{2e^{-a-\frac{bt^2}{2}}}{\left(1-2e^{-a-\frac{bt^2}{2}} \right)} \right] .$$

$$2. \int_0^\infty h_2(t) dt = \left[\int_0^\infty \frac{(a+bt)\left((1+\lambda)e^{-a-\frac{bt^2}{2}} \right)}{\left((1+\lambda)e^{-a-\frac{bt^2}{2}} - \lambda \right)} dt + \int_0^\infty (a + bt)dt \right]$$

$$\int_0^\infty h_2(t) dt = [\ln(-\lambda) - \ln(1) + \infty] = \infty$$



Figure(5.10): The hazard function of THLHRM, where the parameters a and b take different values . As shown in the graphic above, the function is linear but skewed and increasing.

5.3.3.2 The Density Function of THLHRM .

The density function $f(t)$ of The LHRM is

$$f(t) = (a + bt)e^{-\int_0^t (a+bt)dy} \tag{5.29}$$

A new density function of (THLHRM) can be found as follows .

$$f(t) = (a + bt)e^{-a-\frac{bt^2}{2}} .$$

$$f_2(t) = f(t) \left[2(1 + \lambda)e^{-\int_0^t h(y)dy} - \lambda \right] .$$

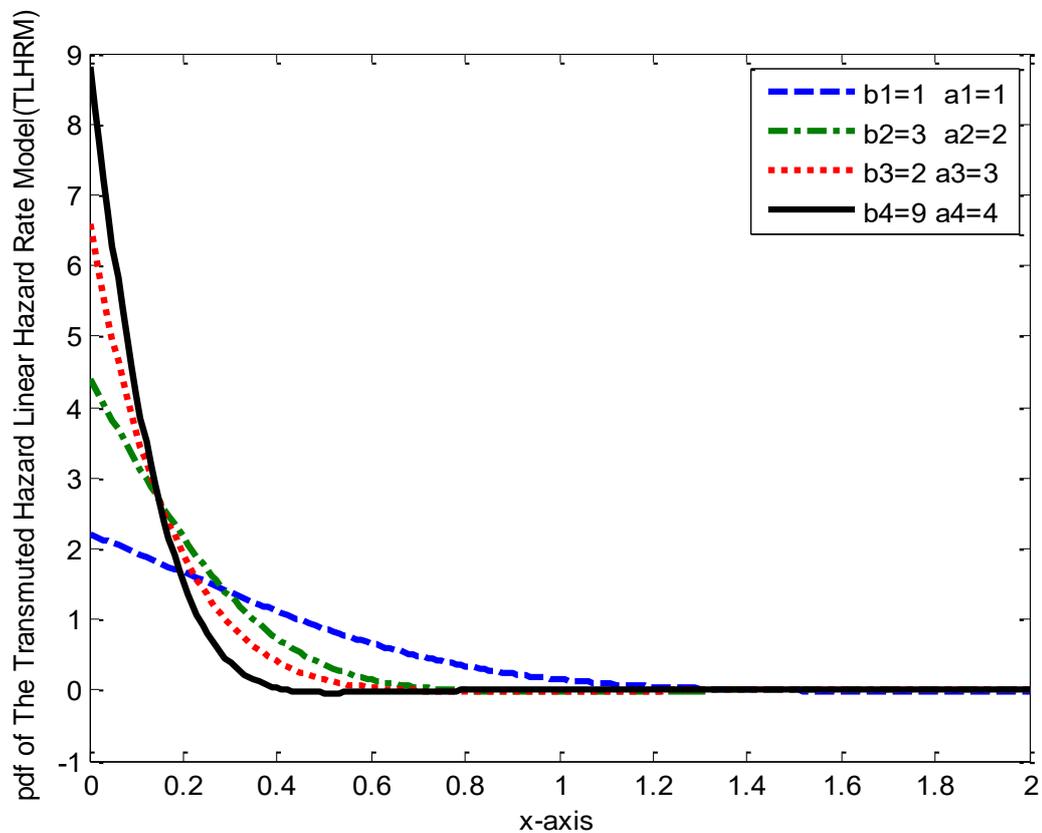
$$f_2(t) = (a + bt)e^{-at-\frac{bt^2}{2}} \left[2(1 + \lambda)e^{-at-\frac{bt^2}{2}} - \lambda \right] \tag{5.30}$$

As this function $f_2(t)$ is to density function since it satisfies .

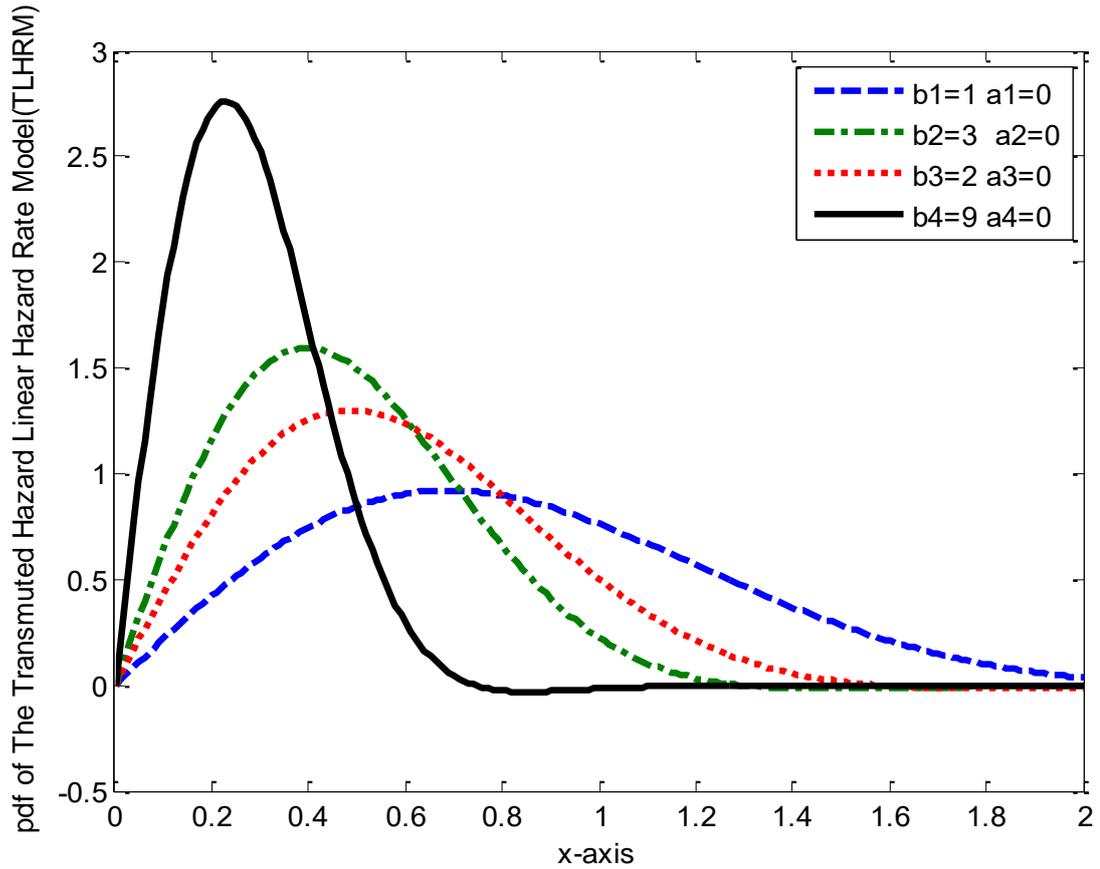
- 1) The first condition $f_2(t) \geq 0$ satisfied for all $t > 0$, and at

$$\lambda \leq \frac{2e^{-a-\frac{bt^2}{2}}}{\left(1 - 2e^{-a-\frac{bt^2}{2}}\right)}$$

$$2) \int_0^\infty f_2(t) = \left[(1 + \lambda) \int_0^\infty 2(a + bt)e^{-2\left(a+\frac{bt^2}{2}\right)} dt - \lambda \right] = 1$$



Figure(5.11): The PDF function of THLHRM, where the parameters a and b take different values . It is noticed that the function is decreasing continuously until it applies to the x-axis when $T \rightarrow \infty$.



Figure(5.12): The PDF function of THLHRM, where the parameter b take different values and $a= 0$. It is noted that the function starts from zero, then continues to increase, then decreases, taking the form of convexity, and then becomes semi-linear.

The Cumulative Function of THLHRM .

$$F_2(t) = 1 - \left(e^{-at-\frac{bt^2}{2}} + \lambda \left(e^{-at-\frac{bt^2}{2}} - 1 \right) \right) e^{-t\left(a+\frac{bt}{2}\right)} \dots\dots\dots(31)$$

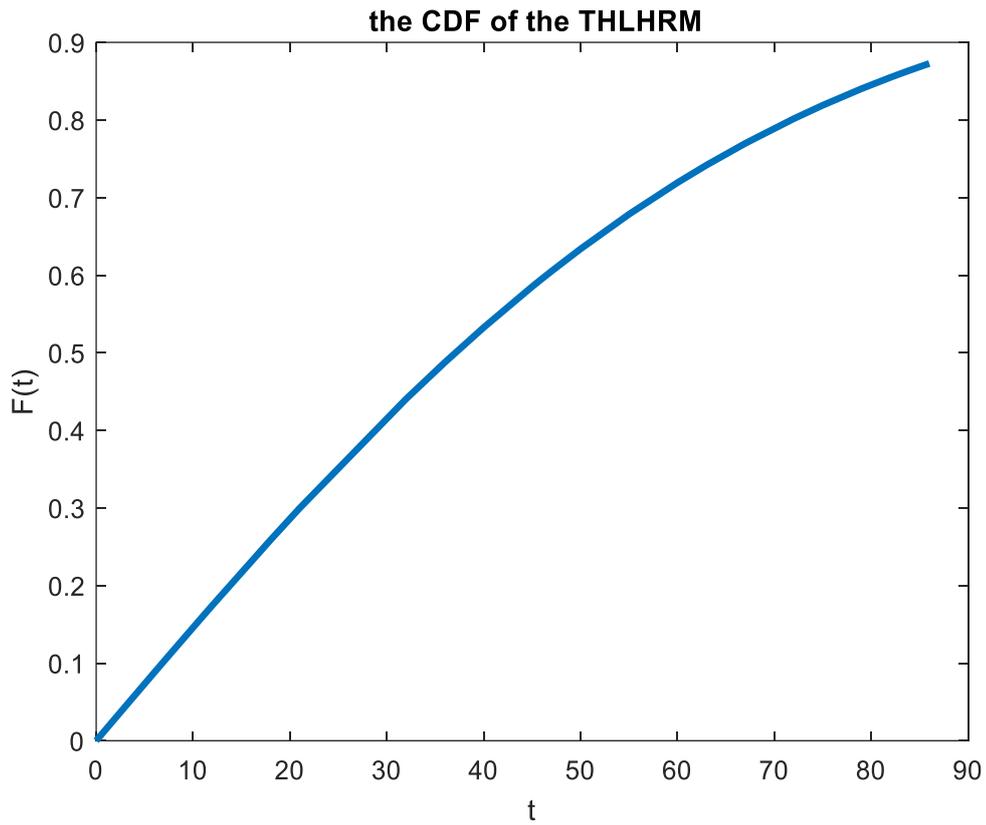
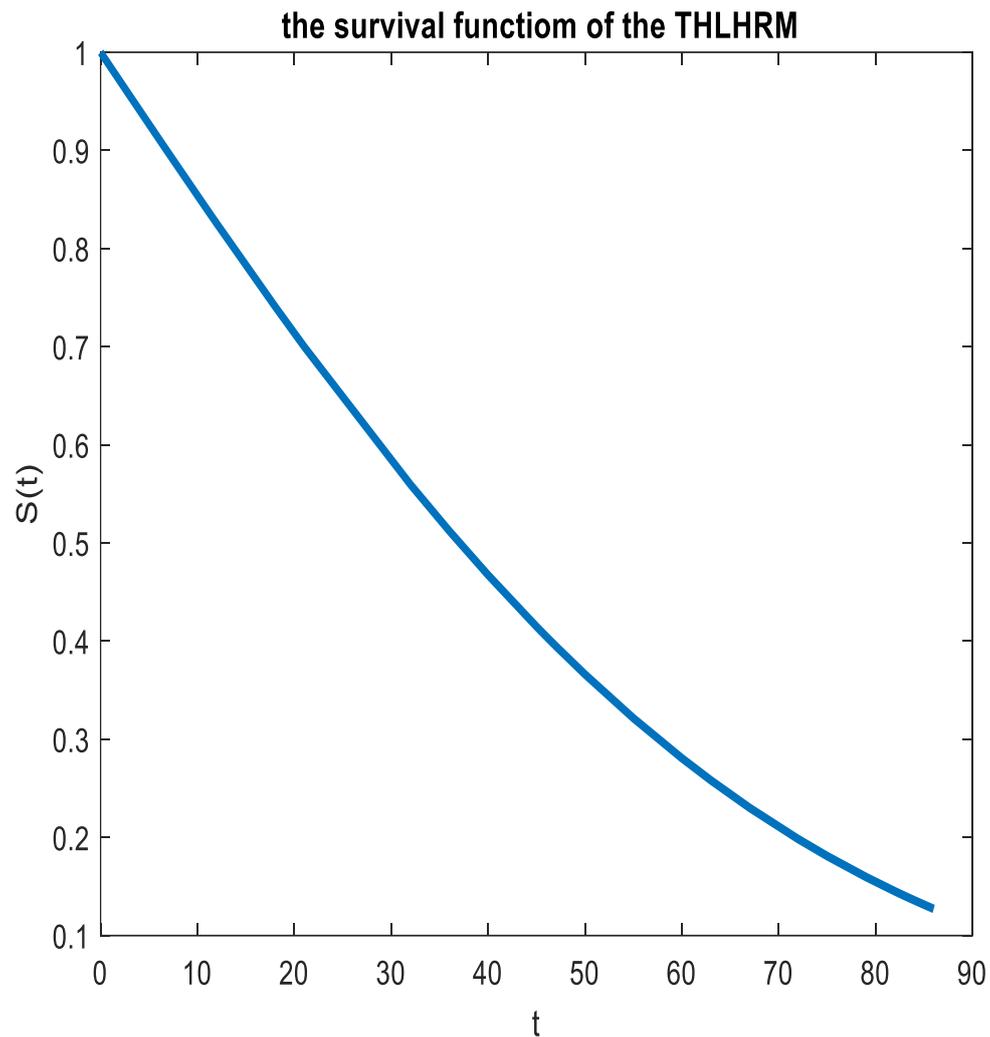


Figure (5.12): The CDF of THLHRM, where the parameters $a = 1$ and $b = 6$. The function is shown to begin at zero and increase towards one.

The Survival function of THLHRM

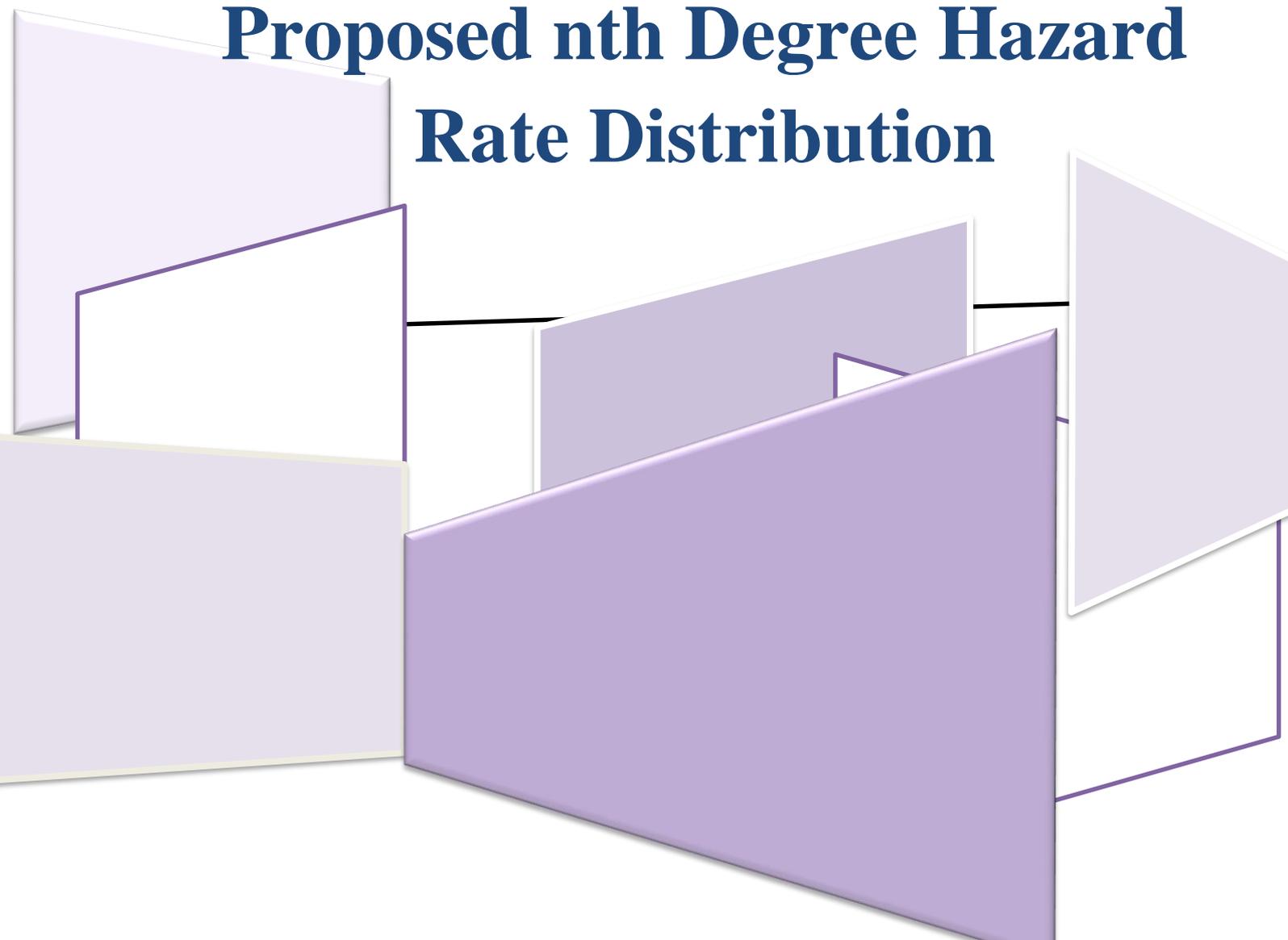
$$S_2(t) = \left((1 + \lambda)e^{-at - \frac{bt^2}{2}} - \lambda \right) e^{-\left(a + \frac{bt}{2}\right)} \dots\dots\dots(5.32)$$



Figure(5.13): The Survival Function of THLHRM, where the parameters $a = 1$ and $b = 6$. It is noted that the function starts from the number one and then decreases to approach the number one.

Chapter Six

Transmuted Hazard General Proposed n th Degree Hazard Rate Distribution



The formula (4.1) from chapter four and the formula (5.5) from chapter five have been merged in this chapter to generate a new formula, which we will refer to as transmuted hazard general proposed nth degree hazard rate formal. Special distribution of this new formula were also investigated. There are three major topics that have been covered in this chapter. In the first section, we dealt with the new formula THGPHRF, studied the density function, the CDF, the survival function, and some statistical properties, as well as the estimation method. In the second section, we dealt with a special case of the new THGPHR formula in the case of $n = 3$ as well as found the density function, cumulative function survival function and studied the statistical properties and the method maximal likelihood estimation. In the third section, it is dealt with the linear exponential distribution, which is a special case of the new formula in the case of $n = 3$ and as well as found the density function, cumulative function survival function and the statistical properties and the method maximal likelihood estimation.

6.1 The Transmuted Hazard General Proposed nth Degree Hazard Rate Formal THGPHRF

A random variable T is said to have THGPHRF with parameters a_i .

Substitution the hazard function (4.1) into the formula (5.5), we will get a new function is.

$$h_2(t) = \left[\frac{(1+\lambda) \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}}{\left((1+\lambda) e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right)} + \sum_{i=0}^n a_i t^i \right] \quad (6.1)$$

On the other hand, this function $h_2(t)$ meets the conditions of the hazard function.

the first property is satisfied for all $t > 0$, and at

$$h_2(t) = \left[\frac{(1+\lambda) \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}}{\left((1+\lambda) e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right)} + \sum_{i=0}^n a_i t^i \right] \geq 0$$

$$\frac{2e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}}{\left(1 - 2e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}\right)} \geq \lambda$$

We will get that the hazard function $h_2(t)$ is greater than zero

The second property is next:

$$\int_0^\infty h_2(t)dt = \left[\int_0^\infty \frac{(1 + \lambda) \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}}{\left((1 + \lambda)e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda\right)} dt + \sum_{i=0}^n a_i \int_0^\infty t^i dt \right]$$

Simplistic

$$\begin{aligned} &= \left[-\ln\left((1 + \lambda)e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda\right) + \sum_{i=0}^n a_i \frac{t^{i+1}}{i+1} \right]_0^\infty \\ &= [-\ln((1 + \lambda)e^{-\infty} - \lambda) + \infty] - [-\ln((1 + \lambda)e^0 - \lambda) + 0] \\ &= [-\ln(-\lambda) + \infty] - [-\ln(1) + 0] = \infty \end{aligned}$$

The Cumulative Hazard Rate Function .

$$H_2(t) = \sum_{i=0}^n a_i \frac{t^{i+1}}{i+1} - \ln\left((1 + \lambda)e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda\right) \dots\dots\dots(6.2)$$

The Density Function of THGPHRF .

the PDF of the new distribution is defined according to(1.3),that is

$$f_2(t) = h_2(t)e^{-H_2(t)}.$$

$$f_2(t) = \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left(2(1 + \lambda)e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda\right) \dots\dots\dots(6.3)$$

As this function $f_2(t)$ is to be checked, here are two conditions for the density function:

the first property $f_2(t) \geq 0$ is satisfied for all $t > 0$, and at

$$\frac{\left(2e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}\right)}{\left(1 - 2e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}\right)} \geq \lambda$$

The second property is $\int_0^{\infty} f_2(t)dt = 2(1 + \lambda) \sum_{i=0}^n a_i \int_0^{\infty} t^i e^{-2 \sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} dt - \lambda \int_0^{\infty} \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} dt$

Suppose that

$$l_1 = \int_0^{\infty} \sum_{i=0}^n a_i t^i e^{-2 \sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} dt \quad , \quad l_2 = \int_0^{\infty} \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} dt$$

Where as

$$l_1 = \int_0^{\infty} \sum_{i=0}^n a_i t^i e^{-2 \sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} dt$$

$$\text{Let } g = 2 \sum_{i=0}^n a_i \frac{t^{i+1}}{i+1} \text{ then } dg = 2 \left(\sum_{i=0}^n a_i t^i \right) dt$$

Again

$$l_1 = \int_0^{\infty} e^{-g} \frac{dg}{2} = \frac{1}{2} (-e^{-\infty} + e^{-0}) = \frac{1}{2}$$

As well as for

$$l_2 = \int_0^{\infty} \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} dt$$

$$\text{let } u = \sum_{i=0}^n a_i \frac{t^{i+1}}{i+1} \text{ then } du = \left(\sum_{i=0}^n a_i t^i \right) dt$$

Again

$$\int_0^{\infty} l_2(t)dt = \int_0^{\infty} e^{-u} du = -e^{-u} \Big|_0^{\infty} = 1$$

It is get

$$\int_0^{\infty} f_2(t)dt = \frac{2(1 + \lambda)}{2} - \lambda = 1$$

Depending on the property (1) and property (2) this $f_2(t)$ is density function .

The Cumulative Function of THGPHRF .

$$F(t) = 1 - e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left((1 + \lambda)e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right) \dots\dots\dots(6.4)$$

The Survival Function of THGPHRF

The survival function of this distribution is:

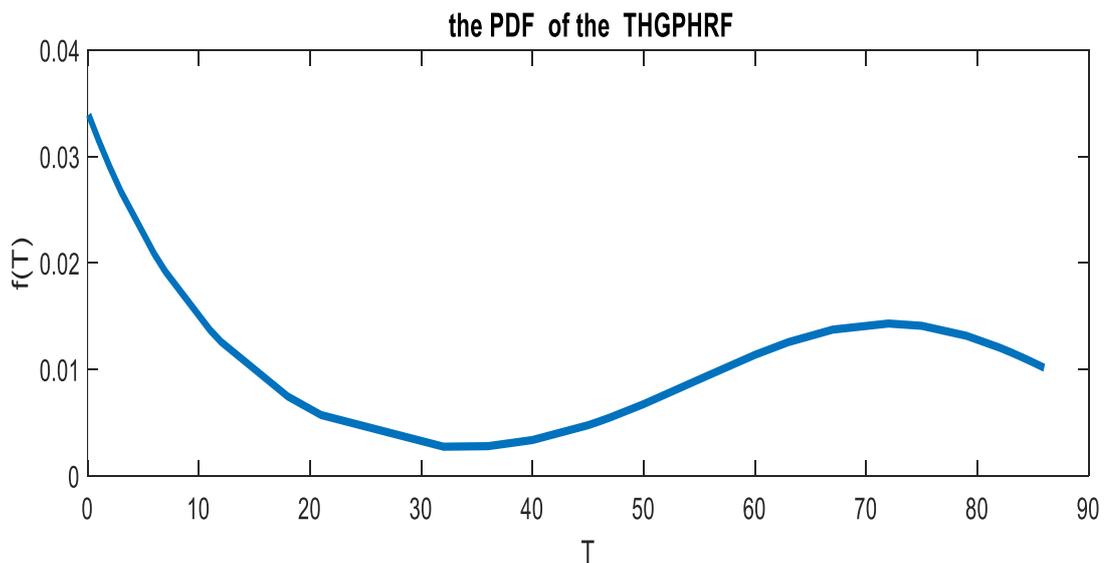
$$S(t) = e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left((1 + \lambda)e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right) \dots\dots\dots(6.5)$$

6.1.1 Shape of THGPHRF

6.1.1. 1 The Limit of PDF of THGPHRF

$$\begin{aligned} \lim_{t \rightarrow 0} f_2(t) &= \lim_{t \rightarrow 0} \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left(2(1 + \lambda)e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right) \\ &= a_0(2(1 + \lambda) - \lambda) = a_0(2 + \lambda) \dots\dots\dots(6.6) \end{aligned}$$

$$\begin{aligned} \lim_{t \rightarrow \infty} f_2(t) &= \lim_{t \rightarrow \infty} \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left(2(1 + \lambda)e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right) \\ &= \lim_{t \rightarrow \infty} (a_0 + \infty)(0)(2(1 + \lambda)(0) - \lambda) \\ &= (a_0 + \infty)(0)(-\lambda) \end{aligned}$$



Figure(6.1): The Density Function of THGPHRD in case n=2 , where a₀ =

.038; $a_1 = -.002$; $a_2 = 0.00003$.it is noted that the PDF starts from the value $a_0 = .038$ and then decreases until it reaches zero as long as increasing T .

6.1.1.2 The Limit of CDF of THGPHRF

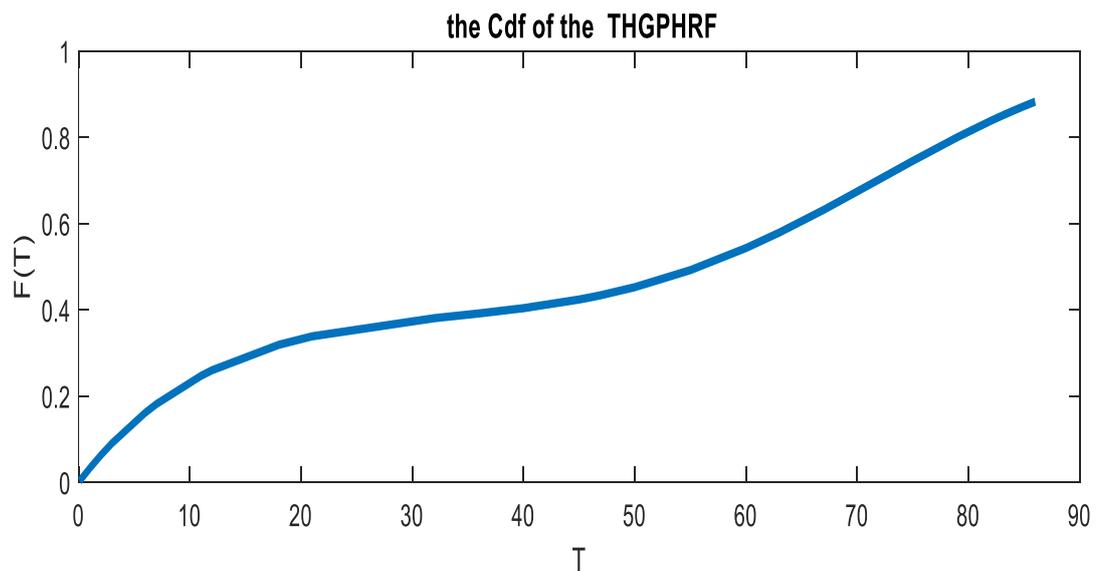
$$\lim_{t \rightarrow 0} F(t) = 1 - \lim_{t \rightarrow 0} e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left((1 + \lambda) e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right)$$

$$= 1 - ((1 + \lambda) - \lambda) = 0$$

$$\lim_{t \rightarrow \infty} F(t) = 1 - \lim_{t \rightarrow \infty} e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left((1 + \lambda) e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right)$$

$$= 1 - \lim_{t \rightarrow \infty} (0)((1 + \lambda)(0) - \lambda)$$

$$= 1$$



Figure(6.2): The CDF of THGPHRD in case $n=2$, where $a_0 = .038$; $a_1 = -.002$; $a_2 = 0.00003$.it is noted that the CDF starts from the value *zero* and then increasing until it reaches one number where $T \rightarrow \infty$.

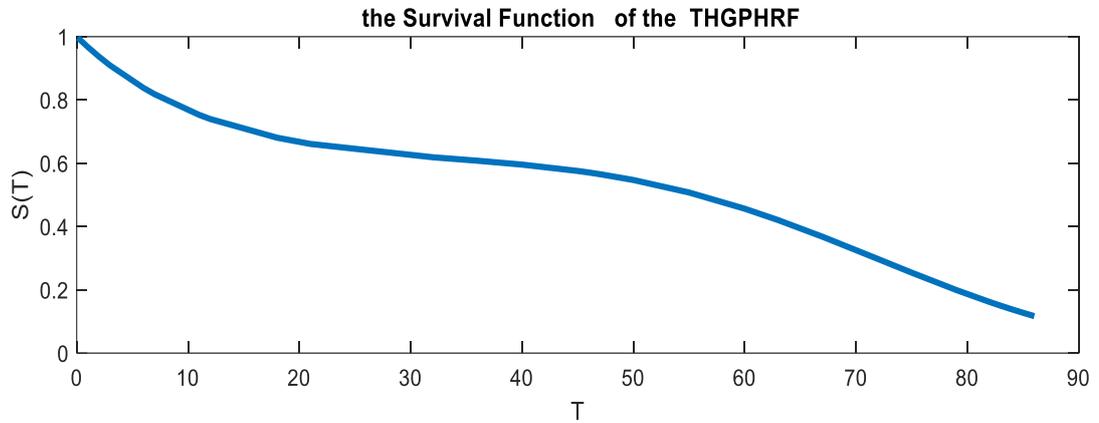
6.1.1.3 The Limit of Survival Function of THGPHRD

$$\lim_{t \rightarrow 0} S(t) = \lim_{t \rightarrow 0} e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left((1 + \lambda) e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right)$$

$$\lim_{t \rightarrow 0} S(t) = ((1 + \lambda) - \lambda) = 1$$

$$\lim_{t \rightarrow \infty} S(t) = \lim_{t \rightarrow \infty} e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left((1 + \lambda) e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right)$$

$$\lim_{t \rightarrow \infty} S(t) = 0((1 + \lambda)0 - \lambda) = 0$$



Figure(6.3): The Survival Function of THGPHRD in case n=2 , where $a_0 = .038$; $a_1 = -.002$; $a_2 = 0.00003$.it is noted that the survival function starts from the value *one number* and then decreasing until it reaches zero number where $T \rightarrow \infty$.

6.1.1.4 The Limit of Hazard Function of THGPHRD

$$\lim_{t \rightarrow 0} h_2(t) = \lim_{t \rightarrow 0} \left[\frac{(1+\lambda) \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}}{\left((1+\lambda) e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right)} + \sum_{i=0}^n a_i t^i \right] \dots\dots\dots(6.7)$$

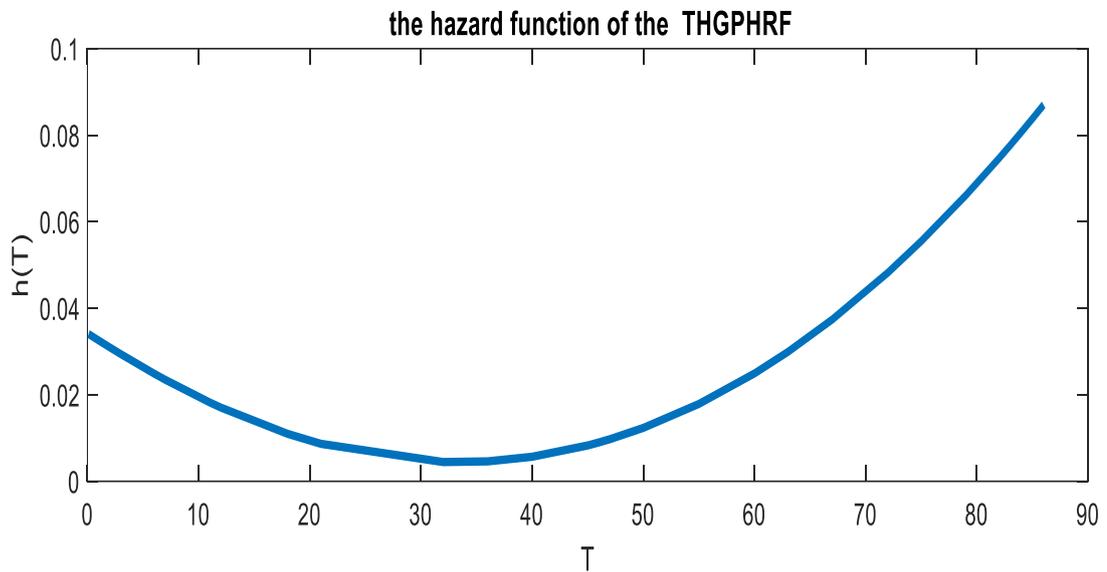
$$= \lim_{t \rightarrow 0} \left[\frac{(1 + \lambda)a_0}{((1 + \lambda) - \lambda)} + a_0 \right]$$

$$= [(2 + \lambda)a_0]$$

$$\lim_{t \rightarrow \infty} h_2(t) = \lim_{t \rightarrow \infty} \left[\frac{(1 + \lambda) \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}}}{\left((1 + \lambda) e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right)} + \sum_{i=0}^n a_i t^i \right]$$

$$= \left[\frac{(1 + \lambda)(a_0 + \infty)(0)}{((1 + \lambda)(0) - \lambda)} + (a_0 + \infty) \right]$$

$$= \left[\frac{(1 + \lambda)(a_0 + \infty)(0)}{(-\lambda)} + (a_0 + \infty) \right] = \infty$$



Figure(6.4): The Hazard Function of THGPHRD in case $n=2$, where $a_0 = .038; a_1 = -.002; a_2 = 0.00003$.it is noted that the hazard function starts from the value $a_0 = .038$ and then decreasing and increasing until it reaches ∞ where $T \rightarrow \infty$

Remark(6.1)

Another technique to show that $h_2(t)$ meets the requirements of the hazard function, using the density function as evidence. To prove this the third property of hazard function .From equations (6.6) and (6.7).It is get .

$$h_2(t) \geq 0 \text{ and } \lim_{t \rightarrow 0} f_2(t) = \lim_{t \rightarrow 0} h_2(t); f_2(t) \geq h_2(t) \forall t > 0$$

6.1.2 Statistical Properties

This section presents the r^{th} moment and moment generating function of the

Theorem 6.2

If $T \sim \text{THGPHRD} (a_i, \lambda)$ with then the r^{th} moment about the origin, and about the mean are given as

$$E(T^r)$$

$$= 2(1 + \lambda)$$

$$\begin{aligned} & * \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(r+i+\sum_{i=1}^n (i+1)j_i)!}{(2a_0)^{1+r+i+\sum_{i=1}^n (i+1)j_i}} \\ & - \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(r+i+\sum_{i=1}^n (i+1)j_i)!}{a_0^{(1+r+i+\sum_{i=1}^n (i+1)j_i)}} \quad (6.8) \end{aligned}$$

$$E(T - \mu)^r$$

$$= 2(1 + \lambda)$$

$$\begin{aligned} & * \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(k+i+\sum_{i=1}^n (i+1)j_i)!}{(2a_0)^{1+k+i+\sum_{i=1}^n (i+1)j_i}} \\ & - \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(k+i+\sum_{i=1}^n (i+1)j_i)!}{a_0^{(1+k+i+\sum_{i=1}^n (i+1)j_i)}} \end{aligned}$$

$$\left(2(1 + \lambda) * \right.$$

$$\left. \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n (i+1)j_i)!}{(2a_0)^{2+i+\sum_{i=1}^n (i+1)j_i}} - \right.$$

$$\lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \dots \left(\frac{a_n}{n+1}\right)^{j_n} \frac{(1+i+\sum_{i=1}^n (i+1) j_i)!}{a_0^{(2+i+\sum_{i=1}^n (i+1) j_i)}} \right)^{r-k} \quad (6.9)$$

Proof

$$E(T^r) = \int_0^{\infty} t^r \left(2(1 + \lambda) \sum_{i=0}^n a_i t^i e^{-2\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right) dt$$

$$E(T^r) = 2(1 + \lambda) \sum_{i=0}^n a_i \int_0^{\infty} t^{r+i} e^{-2\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} dt - \lambda \sum_{i=0}^n a_i \int_0^{\infty} t^{r+i} e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} dt$$

Suppose that

$$S_1 = \sum_{i=0}^n a_i \int_0^{\infty} t^{r+i} e^{-2\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} dt \text{ and } S_2 = \sum_{i=0}^n a_i \int_0^{\infty} t^{r+i} e^{-a_0 t} e^{-\sum_{i=1}^n a_i \frac{t^{i+1}}{i+1}} dt$$

$$S_1 = \sum_{i=0}^n a_i \int_0^{\infty} t^{r+i} e^{-2a_0 t} e^{-2\sum_{i=1}^n a_i \frac{t^{i+1}}{i+1}} dt$$

$$S_1 = \sum_{i=0}^n a_i \int_0^{\infty} t^{r+i} e^{-2a_0 t} e^{-2a_1 \frac{t^2}{2}} e^{-2a_2 \frac{t^3}{3}} \dots \dots e^{-2a_n \frac{t^{n+1}}{n+1}} dt \quad \dots(6.10)$$

As it is know that .

$$e^{-t} = \sum_{j=0}^{\infty} \frac{t^j}{j!} \dots \dots \dots (6.11)$$

By substituting the equation (6.11) into the function (6.10), it is get

$$S_1 = \sum_{i=0}^n a_i \int_0^{\infty} t^{r+i} e^{-2a_0 t} \sum_{j_1=0}^{\infty} \frac{(2a_1 \frac{t^2}{2})^{j_1}}{j_1!} \sum_{j_2=0}^{\infty} \frac{(2a_2 \frac{t^3}{3})^{j_2}}{j_2!} \dots \dots \sum_{j_n=0}^{\infty} \frac{(2a_n \frac{t^{n+1}}{n+1})^{j_n}}{j_n!} dt$$

=

$$\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \dots \left(\frac{2a_n}{n+1}\right)^{j_n}}{j_1! j_2! \dots j_n!} \int_0^{\infty} t^{r+i+2j_1+3j_2+\dots+n+1j_n} e^{-2a_0 t} dt$$

$$= \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \dots \left(\frac{2a_n}{n+1}\right)^{j_n}}{j_1! j_2! \dots j_n!} \frac{\Gamma(1+r+i+\sum_{i=1}^n (i+1) j_i)}{(2a_0)^{1+r+i+\sum_{i=1}^n (i+1) j_i}}$$

$$= \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} a_i \frac{\binom{2a_1}{2}^{j_1}}{j_1!} \frac{\binom{2a_2}{3}^{j_2}}{j_2!} \cdots \frac{\binom{2a_n}{n+1}^{j_n}}{j_n!} \frac{(r+i+\sum_{i=1}^n (i+1) j_i)!}{(2a_0)^{1+r+i+\sum_{i=1}^n (i+1) j_i}}.$$

And

$$S_2 = \sum_{i=0}^n a_i \int_0^{\infty} t^{r+i} e^{-a_0 t} e^{-\sum_{i=1}^n a_i \frac{t^{i+1}}{i+1}} dt$$

$$S_2 = \sum_{i=0}^n a_i \int_0^{\infty} t^{r+i} e^{-a_0 t} e^{-a_1 \frac{t^2}{2}} e^{-a_2 \frac{t^3}{3}} \cdots e^{-a_n \frac{t^{n+1}}{n+1}} dt$$

By using (6.11)

Thus

$$S_2 =$$

$$\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} (-1)^{j_1+\cdots+j_n} a_i \frac{\binom{a_1}{2}^{j_1}}{j_1!} \frac{\binom{a_2}{3}^{j_2}}{j_2!} \cdots \frac{\binom{a_n}{n+1}^{j_n}}{j_n!} \int_0^{\infty} t^{r+i+\sum_{i=1}^n (i+1) j_i} e^{-a_0 t} dt$$

=

$$\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} (-1)^{j_1+\cdots+j_n} a_i \frac{\binom{a_1}{2}^{j_1}}{j_1!} \frac{\binom{a_2}{3}^{j_2}}{j_2!} \cdots \frac{\binom{a_n}{n+1}^{j_n}}{j_n!} \frac{\Gamma(r+i+\sum_{i=1}^n (i+1) j_i + 1)}{a_0^{(1+r+i+\sum_{i=1}^n (i+1) j_i)}}$$

$$= \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} (-1)^{j_1+\cdots+j_n} a_i \frac{\binom{a_1}{2}^{j_1}}{j_1!} \frac{\binom{a_2}{3}^{j_2}}{j_2!} \cdots \frac{\binom{a_n}{n+1}^{j_n}}{j_n!} \frac{(r+i+\sum_{i=1}^n (i+1) j_i)!}{a_0^{(1+r+i+\sum_{i=1}^n (i+1) j_i)}}$$

$$E(T^r) = 2(1 +$$

$$\lambda) \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} (-1)^{j_1+\cdots+j_n} a_i \frac{\binom{2a_1}{2}^{j_1}}{j_1!} \frac{\binom{2a_2}{3}^{j_2}}{j_2!} \cdots \frac{\binom{2a_n}{n+1}^{j_n}}{j_n!} \frac{(r+i+\sum_{i=1}^n (i+1) j_i)!}{(2a_0)^{1+r+i+\sum_{i=1}^n (i+1) j_i}}$$

$$\lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} (-1)^{j_1+\cdots+j_n} a_i \frac{\binom{a_1}{2}^{j_1}}{j_1!} \frac{\binom{a_2}{3}^{j_2}}{j_2!} \cdots \frac{\binom{a_n}{n+1}^{j_n}}{j_n!} \frac{(r+i+\sum_{i=1}^n (i+1) j_i)!}{a_0^{(1+r+i+\sum_{i=1}^n (i+1) j_i)}}$$

Based on binomial theorem ,it is get

$$\begin{aligned}
 & E(T - \mu)^r \\
 &= \sum_{k=0}^n C_k^n 2(1 + \lambda) \\
 & * \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(k+i+\sum_{i=1}^n(i+1))}{(2a_0)^{1+k+i+\sum_{i=1}^n(i+1)}} \\
 & - \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(k+i+\sum_{i=1}^n(i+1))}{a_0^{(1+k+i+\sum_{i=1}^n(i+1))}} \\
 & \left(2(1 + \lambda) \right. \\
 & * \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n(i+1)j_i)!}{(2a_0)^{2+i+\sum_{i=1}^n(i+1)j_i}} \left. \right) \\
 & * \left(-\lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n(i+1)j_i)!}{a_0^{(2+i+\sum_{i=1}^n(i+1)j_i)}} \right)^{r-k}
 \end{aligned}$$

Corollary 6.3.

If $T \sim THGPHRD (a_i, \lambda)$ then the mean , variance , coefficients of variation ,skewness and kurtosis of general proposed nth degree hazard rate distribution are respectively as:

$$\begin{aligned}
 E(T) &= 2(1 + \\
 & \lambda) \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n(i+1)j_i)!}{(2a_0)^{2+i+\sum_{i=1}^n(i+1)j_i}} - \\
 & \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n(i+1)j_i)!}{a_0^{(2+i+\sum_{i=1}^n(i+1)j_i)}} \dots
 \end{aligned}$$

...(6.12)

$$\begin{aligned}
 Var(T) = & 2(1 + \lambda) * \\
 & \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(2+i+\sum_{i=1}^n (i+1) j_i)!}{(2a_0)^{3+i+\sum_{i=1}^n (i+1) j_i}} - \\
 & \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(2+i+\sum_{i=1}^n (i+1) j_i)!}{a_0^{(3+i+\sum_{i=0}^n (i+1) j_i)}} - \\
 & \left[2(1 + \right. \\
 & \lambda) \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n (i+1) j_i)!}{(2a_0)^{2+i+\sum_{i=1}^n (i+1) j_i}} - \\
 & \left. \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n (i+1) j_i)!}{a_0^{(2+i+\sum_{i=1}^n (i+1) j_i)}} \right] \\
 & \dots(6.13)
 \end{aligned}$$

$$\begin{aligned}
 SK = E(\mu - T)^3 = & \sum_{k=0}^3 C_k^3 \left(2(1 + \right. \\
 & \lambda) \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(k+i+\sum_{i=1}^n (i+1) j_i)!}{(2a_0)^{1+k+i+\sum_{i=1}^n (i+1) j_i}} - \\
 & \left. \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_1}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(k+i+\sum_{i=1}^n (i+1) j_i)!}{a_0^{(1+k+i+\sum_{i=1}^n (i+1) j_i)}} \right) * \\
 & \left(-2(1 + \lambda) * \right.
 \end{aligned}$$

$$\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} (-1)^{j_1+\cdots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n (i+1) j_i)!}{(2a_0)^{2+i+\sum_{i=1}^n (i+1) j_i}} +$$

$$\lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} (-1)^{j_1+\cdots+j_n} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_1}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n (i+1) j_i)!}{a_0^{(2+i+\sum_{i=1}^n (i+1) j_i)}} \Big)^{3-k}$$

..(6.14)

$$KU = E(\mu - T)^4 = \sum_{k=0}^4 C_k^4 \left(2(1 + \right.$$

$$\lambda) \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} (-1)^{j_1+\cdots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(k+i+\sum_{i=1}^n (i+1) j_i)!}{(a_0)^{1+k+i+\sum_{i=1}^n (i+1) j_i}} -$$

$$\lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} (-1)^{j_1+\cdots+j_n} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_1}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(k+i+\sum_{i=1}^n (i+1) j_i)!}{a_0^{(1+k+i+\sum_{i=1}^n (i+1) j_i)}} \Big)^* \Big)$$

$$\left(-2(1 + \lambda) * \right.$$

$$\sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} (-1)^{j_1+\cdots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n (i+1) j_i)!}{(a_0)^{2+i+\sum_{i=1}^n (i+1) j_i}} +$$

$$\lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} (-1)^{j_1+\cdots+j_n} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_1}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=0}^n (i+1) j_{(i+1)})!}{a_0^{(2+i+\sum_{i=1}^n (i+1) j_i)}} \Big)^{4-k}$$

.(6.15)

The Coefficient of Variation CV is given by:

$$CV = \frac{\sqrt{\text{var}(t)}}{E(t)}$$

Standard deviation

$$\sigma = \left(2(1 + \lambda) \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(2+i+\sum_{i=1}^n (i+1) j_i)!}{(a_0)^{3+i+\sum_{i=1}^n (i+1) j_i}} - \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_1}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(2+i+\sum_{i=0}^n (i+1) j_{(i+1)})!}{a_0^{(3+i+\sum_{i=1}^n (i+1) j_i)}} - \left[2(1 + \lambda) * \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n (i+1) j_i)!}{(a_0)^{2+i+\sum_{i=1}^n (i+1) j_i}} - \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_1}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=0}^n (i+1) j_{(i+1)})!}{a_0^{(2+i+\sum_{i=1}^n (i+1) j_i)}} \right]^{1/2} \tag{6.17}$$

Coefficient of Skewness

$$\gamma = \frac{SK}{E((T-\mu)^2)^{3/2}}$$

$$\gamma = \sum_{k=0}^3 C_k^3 \left(2(1 + \lambda) * \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(k+i+\sum_{i=1}^n (i+1) j_i)!}{(2a_0)^{1+k+i+\sum_{i=1}^n (i+1) j_i}} - \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_1}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(k+i+\sum_{i=1}^n (i+1) j_i)!}{a_0^{(1+k+i+\sum_{i=1}^n (i+1) j_i)}} \right) * \left(-2(1 + \lambda) * \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} (-1)^{j_1+\dots+j_n} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n (i+1) j_i)!}{(2a_0)^{2+i+\sum_{i=1}^n (i+1) j_i}} + \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_1}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=0}^n (i+1) j_{(i+1)})!}{a_0^{(2+i+\sum_{i=1}^n (i+1) j_i)}} \right)^{3-k} \left(2(1 + \lambda) * \dots \right)$$

$$\lambda * \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(2+i+\sum_{i=1}^n (i+1) j_i)!}{(2a_0)^{3+i+\sum_{i=1}^n (i+1) j_i}} -$$

$$\lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_1}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(2+i+\sum_{i=1}^n (i+1) j_i)!}{a_0^{(3+i+\sum_{i=1}^n (i+1) j_i)}} -$$

$$\left[2(1 + \lambda) \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(1+i+\sum_{i=1}^n (i+1) j_i)!}{(2a_0)^{2+i+\sum_{i=1}^n (i+1) j_i}} -$$

$$\lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_1}{3}\right)^{j_2}}{j_2!} \cdots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \sum_{i=1}^n (i +$$

$$1) j_i \frac{(1+i+\sum_{i=0}^n (i+1) j_{(i+1)})!}{a_0^{(2+i+\sum_{i=1}^n (i+1) j_i)}} \Bigg]^{-3/2} \dots (6.18)$$

6.1.3 Mode

$$\frac{\partial f(t)}{\partial t} = 0$$

$$\frac{\partial \left(2(1+\lambda) \sum_{i=0}^n a_i t^i e^{-2 \sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right)}{\partial t}$$

$$= 2(1 + \lambda) \left(\sum_{i=1}^n a_i i t^{i-1} e^{-2 \sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - 2 \left(\sum_{i=0}^n a_i t^i \right)^2 e^{-2 \sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right) -$$

$$\lambda \left(- \left(\sum_{i=0}^n a_i t^i \right)^2 e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} + e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \sum_{i=1}^n (a_i i) t^{i-1} \right) = 0$$

$$= 2(1 + \lambda) \left(\sum_{i=1}^n a_i i t^{i-1} e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - 2 \left(\sum_{i=0}^n a_i t^i \right)^2 e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right) =$$

$$\lambda \left(- \left(\sum_{i=0}^n a_i t^i \right)^2 + \sum_{i=1}^n (a_i i) t^{i-1} \right)$$

Which can be solved numerically .

6.1.4 Moments generating function of THGPHRD

$$M_T(z) = \sum_{r=1}^{\infty} \frac{z^r}{r!} \left[2(1 + \lambda) \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(r+i+\sum_{i=1}^n (i+1) j_i)!}{(a_0)^{1+r+i+\sum_{i=1}^n (i+1) j_i}} - \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_1}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(r+i+\sum_{i=1}^n (i+1) j_i)!}{a_0^{(1+r+i+\sum_{i=1}^n (i+1) j_i)}} \right] \dots (6.19)$$

6.1.5 Characteristic Function of THGPHRD

$$M_t(iz) = \sum_{r=1}^{\infty} \frac{(iz)^r}{r!} \left[2(1 + \lambda) \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{2a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{2a_2}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{2a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(r+i+\sum_{i=1}^n (i+1) j_i)!}{(2a_0)^{1+r+i+\sum_{i=1}^n (i+1) j_i}} - \lambda \sum_{i=0}^n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_n=0}^{\infty} a_i \frac{\left(\frac{a_1}{2}\right)^{j_1}}{j_1!} \frac{\left(\frac{a_1}{3}\right)^{j_2}}{j_2!} \dots \frac{\left(\frac{a_n}{n+1}\right)^{j_n}}{j_n!} \frac{(r+i+\sum_{i=1}^n (i+1) j_i)!}{a_0^{(1+r+i+\sum_{i=1}^n (i+1) j_i)}} \right] (6.20)$$

6.1.6 The Order statistic of THGPHRD

if $T_1 \leq \dots \leq T_n$ denotes the order statistics of the random sample with CDF and PDF have a THGPHRD .

The CDF of the min order statistic .

$$F_{Y_1}(t) = 1 - \left(e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left((1 + \lambda) e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right) \right)^n \dots (6.21)$$

Then the PDF of the min order statistic .

$$f_{Y_1}(t) = n \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left(2(1 + \lambda) e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right) \left(e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} * \left((1 + \lambda) * e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right) \right)^{n-1} \dots (6.22)$$

Then CDF of the max order statistic

$$F_{Y_n}(t) = \left(1 - e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left((1 + \lambda) e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right) \right)^n \dots (6.23)$$

Then PDF of the max order statistic

$$f_{Y_n}(t) = \sum_{i=0}^n a_i t^i e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \left(2(1+\lambda) e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda \right) \left(1 - \left((1+\lambda) e^{-2\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} - \lambda e^{-\sum_{i=0}^n a_i \frac{t^{i+1}}{i+1}} \right) \right)^{n-1} \dots (6.24)$$

6.1.7 Maximum Likelihood Estimators of THGPHRD

The likelihood function is

$$L = L(a_1, a_2, \dots, a_n, \lambda; t_1, t_2, \dots, t_m) = \prod_{j=1}^m f_2(a_1, a_2, \dots, a_n, \lambda; t_j) \\ = \prod_{j=1}^m f_2(t) = \prod_{j=1}^m \sum_{i=0}^n a_i t_j^i e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} \left(2(1+\lambda) e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} - \lambda \right)$$

$$L = e^{-\sum_{j=0}^m \sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} \prod_{j=1}^m \left[\sum_{i=0}^n a_i t_j^i \right] \left(2(1+\lambda) e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} - \lambda \right) \dots (6.25)$$

The log likelihood function is given by

$$\ln L =$$

$$= \left[-\sum_{j=0}^m \sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1} \right] + \left[\sum_{j=0}^m \ln \sum_{i=0}^n a_i t_j^i \right] + \sum_{j=0}^m \ln \left(2(1+\lambda) e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} - \lambda \right)$$

By taking the partial derivatives of L with respect to unidentified parameters (a_i)

$$\frac{\delta \ln L}{\delta a_0} =$$

$$= \left[-\sum_{j=0}^m t_j \right] + \sum_{j=0}^m \frac{1}{(a_0 + \sum_{i=1}^n a_i t_j^i)} - \sum_{j=0}^m \frac{\left(2(1+\lambda) t_j e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} \right)}{\left(2(1+\lambda) e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} - \lambda \right)} \dots (6.26)$$

$$\begin{aligned} \frac{\delta \ln L}{\delta a_1} &= \\ &= \left[-\sum_{j=0}^m \frac{t_j^2}{2} \right] + \sum_{j=0}^m \frac{t_j}{(a_0 + \sum_{i=1}^n a_i t_j^i)} - \sum_{j=0}^m \frac{\left((1+\lambda) t_j^2 e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} \right)}{\left(2(1+\lambda) e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} - \lambda \right)} \dots (6.27) \end{aligned}$$

$$\begin{aligned} \frac{\delta \ln L}{\delta a_2} &= \\ &= \left[-\sum_{j=0}^m \frac{t_j^3}{3} \right] + \sum_{j=0}^m \frac{t_j^2}{\ln(a_0 + \sum_{i=1}^n a_i t_j^i)} + \sum_{j=0}^m \frac{\left(\frac{2}{3}(1+\lambda) t_j^3 e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} \right)}{\left(2(1+\lambda) e^{-a_0 t_j} e^{-\sum_{i=1}^n a_i \frac{t_j^{i+1}}{i+1}} - \lambda \right)} \dots (6.28) \end{aligned}$$

$$\begin{aligned} \frac{\delta \ln L}{\delta a_i} &= \\ &= \left[-\sum_{j=0}^m \frac{t_j^{i+1}}{i+1} \right] + \frac{t_j^i}{\sum_{j=0}^m \ln(a_0 + \sum_{i=1}^n a_i t_j^i)} + \sum_{j=0}^m \frac{\left(\frac{2}{i+1}(1+\lambda) t_j^{i+1} e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} \right)}{\left(2(1+\lambda) e^{-a_0 t_j} e^{-\sum_{i=1}^n a_i \frac{t_j^{i+1}}{i+1}} - \lambda \right)} \dots (6.29) \end{aligned}$$

$$\frac{\delta \ln L}{\delta \lambda} = \sum_{j=0}^m \frac{2e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} - 1}{\left(2(1+\lambda) e^{-\sum_{i=0}^n a_i \frac{t_j^{i+1}}{i+1}} - \lambda \right)} \dots (6.30)$$

When $\frac{\partial \ln L}{\partial a_i} = 0$ There is no closed solution of (6. 26),(6.27),(6.28),(6.29) and (6.30) therefor, numerical technique (Newton- Raphson method) should be apply to solve.

6.2 The Transmuted Hazard Three Degree Hazard Rate Distribution THTHRD

Let T is positive random variable , It has a function $h(t)$ mentioned in the following formulas, As this function $h(t)$ satisfies the conditions of the hazard function in equation(4.1) .

$$h(t) = \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \quad t > 0 , a_{\varphi} \in \mathcal{R}^+ , \varphi \subseteq n, \forall \varphi = 0,1,2,3$$

Using formula of transmuted hazard in equation (5.5) which it is found earlier

$$h_2(t) = \left[\frac{(1+\lambda) \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}}}{\left((1+\lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right)} + \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \right] \dots(6.30)$$

Whereas , this function $h_2(t)$ satisfies the conditions of the hazard function , that is, its means

$$1- h_2(t) \geq 0 \quad \forall t > 0, \text{ and at } \frac{2e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}}}{\left(1-2e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right)} \geq \lambda$$

$$2- \int_0^{\infty} h_2(t) dt = \left[\int_0^{\infty} \frac{(1+\lambda) \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}}}{\left((1+\lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right)} dt + \sum_{\varphi=0}^3 a_{\varphi} \int_0^{\infty} t^{\varphi} dt \right] = \infty$$

The Cumulative Hazard Rate Function of THTHR

$$H_2(t) = \sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1} - \ln \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \dots(6.31)$$

The Density Function

The PDF of The New Distribution THTHR .

$$f_2(t) = \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \dots \dots(6.32)$$

a function $f_2(t)$ is a probability density function with the following conditions

the first condition is satisfied for all $t \geq 0$. and $(\lambda) \leq \frac{2e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}}}{\left(1 - 2e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right)}$

The second condition is shown below:

$$\int_0^{\infty} f_2(t) dt = 2(1 + \lambda) \sum_{\varphi=0}^3 a_{\varphi} \int_0^{\infty} t^{\varphi} e^{-2\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} dt - \lambda \int_0^{\infty} \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} dt$$

$$l_1 = \int_0^{\infty} \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-2\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} dt = \frac{1}{2}$$

And $l_2 = \int_0^{\infty} \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} dt = 1$

$$\int_0^{\infty} f_2(t) dt = 1$$

The Cumulative Function THTHR

$$F(t) = 1 - e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \dots\dots (6.33)$$

The Survival Function THTHR

$$S(t) = e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \dots\dots\dots(6.34)$$

6.2.1 Shape of THTHR :

6.2.1.1 The limit of PDF of THTHR .

$$\begin{aligned} \lim_{t \rightarrow 0} f_2(t) &= \sum_{\varphi=0}^3 a_{\varphi} \lim_{t \rightarrow 0} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \\ &= a_0 (2(1 + \lambda) - \lambda) = a_0 (2 + \lambda) \dots(6.35) \end{aligned}$$

$$\begin{aligned} \lim_{t \rightarrow \infty} f_2(t) &= \sum_{\varphi=0}^3 a_{\varphi} \lim_{t \rightarrow \infty} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \\ &= \infty \end{aligned}$$

6.2.1.2 The limit of CDF of THTHR .

$$\lim_{t \rightarrow 0} F(t) = 1 - \lim_{t \rightarrow 0} \left[e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \right] = 0$$

$$\lim_{t \rightarrow \infty} F(t) = 1 - \lim_{t \rightarrow \infty} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) = 1$$

6.2.1.3 The limit of Survival Function of THTHR .

$$\lim_{t \rightarrow 0} S(t) = \lim_{t \rightarrow 0} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) = 1$$

$$\lim_{t \rightarrow \infty} S(t) = \lim_{t \rightarrow \infty} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) = 0$$

6.2.1.4 The limit of hazard function of THTRD .

$$\lim_{t \rightarrow 0} h_2(t) = \left[\lim_{t \rightarrow 0} \frac{(1 + \lambda) \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}}}{\left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right)} + \lim_{t \rightarrow 0} \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \right]$$

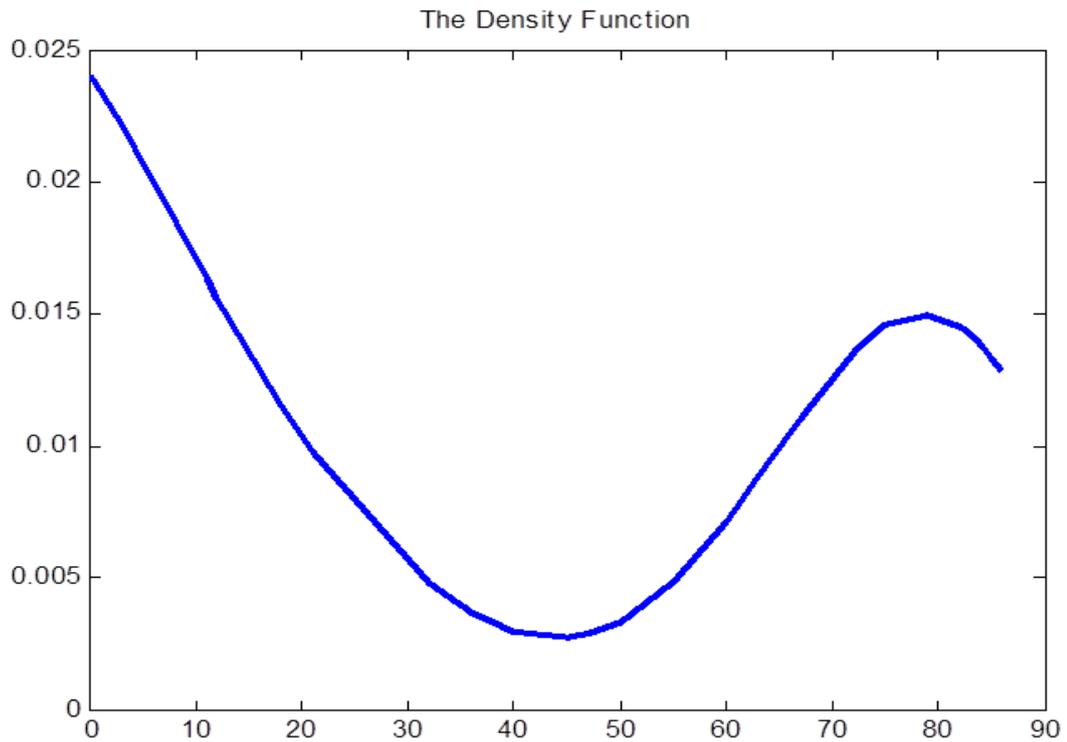
$$\lim_{t \rightarrow 0} h_2(t) = [(\lambda)a_0 + 2a_0] \dots(6.36)$$

Remark(6.4)

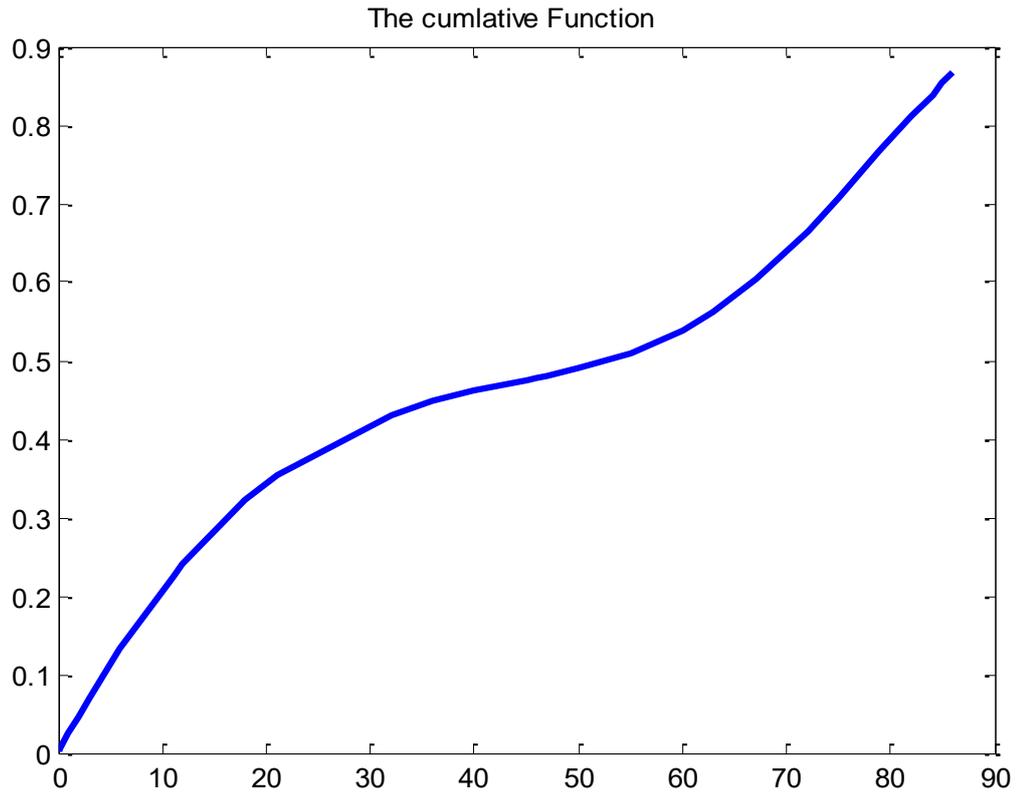
Another technique to show that $h_2(t)$ meets the requirements of the hazard function, using the density function as evidence. To prove this the third property of hazard function .From equations (6.35) and (6.36).It is get .

$$h_2(t) \geq 0 \text{ and } \lim_{t \rightarrow 0} f_2(t) = \lim_{t \rightarrow 0} h_2(t) ; f_2(t) \geq h_2(t) \forall t > 0$$

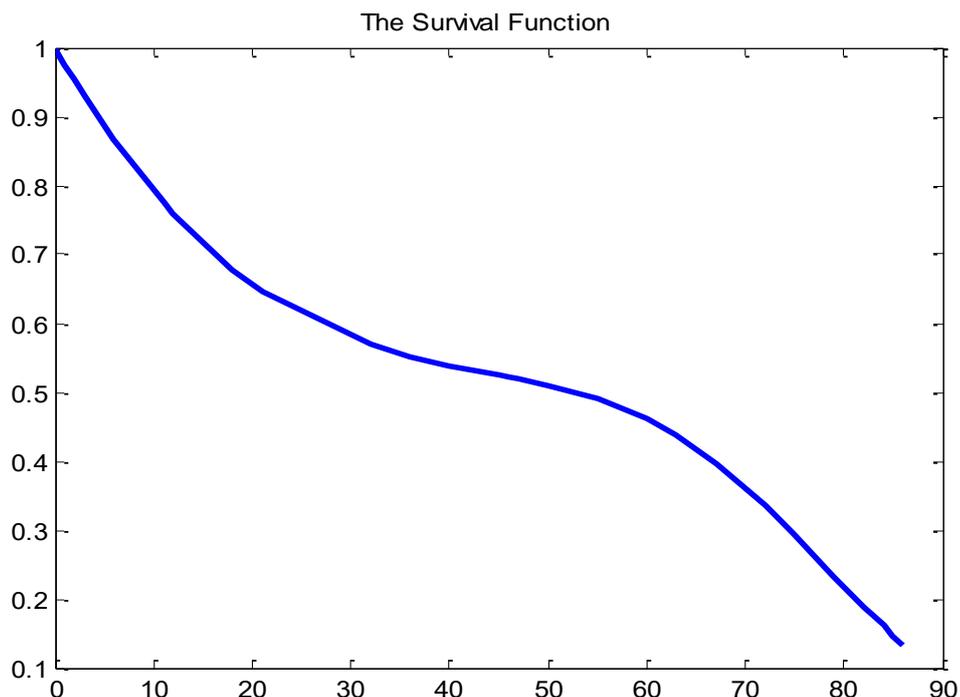
The following figure there are The shows of PDF , The Cumulative Function ,The Survival Function, The Hazard Function of the THTRD .



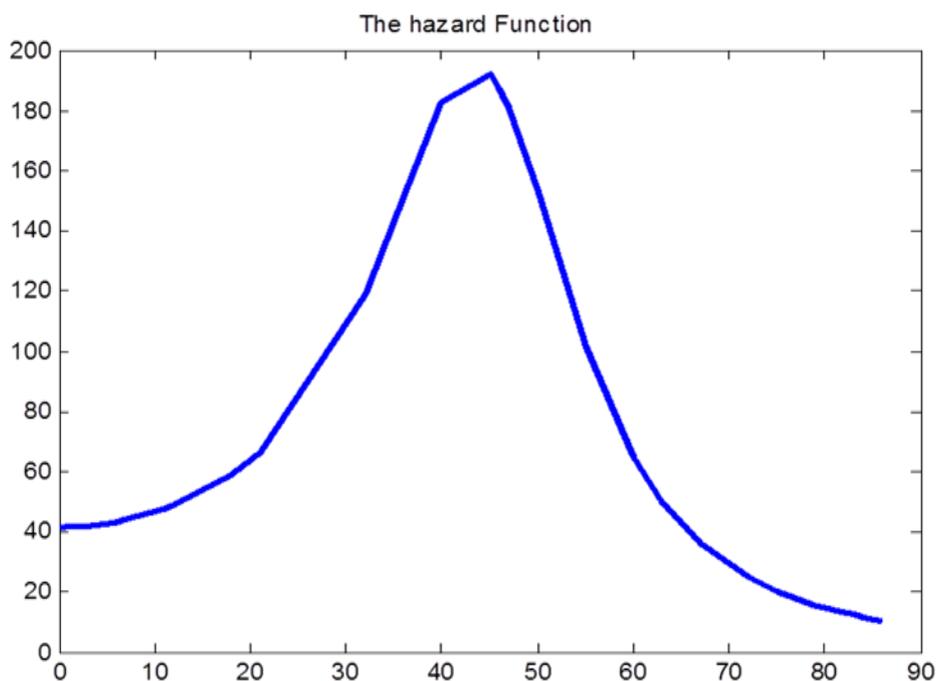
Figure(6.5): The Density Function of THTHRD , where $a_0 = .012$; $a_1 = 0.00000000000564$; $a_2 = 0.000015$, $a_3 = 0.00000023$; $\lambda = 009$; We note that the PDF starts from the value a_0 and then decreases until it reaches zero as long as increasing T.



Figure(6.6): The Cumulative Function of THTHRD, where $a_0 = .012$; $a_1 = 0.000000000000564$; $a_2 = 0.000015$, $a_3 = 0.00000023$; $\lambda = 009$; .we note that the cdf starts from the value 0 and then nondecreasing until it reaches 1 when it approaches t to ∞ .



Figure(6.7): The Survival Function of THTHRD, where $a_0 = .012$; $a_1 = 0.000000000000564$; $a_2 = 0.000015$, $a_3 = 0.00000023$; $\lambda = 0.09$; . we notice that this function starts from the value 1 and then nonincreasing until it reaches 0 when it approaches t to ∞ .



Figure(6.8): The Hazard Function of THTHRD, where $a_0 = 0.012; a_1 = 0.000000000000564; a_2 = 0.000015$, $a_3 = 0.00000023; \lambda = 0.009$; we note that the hazard function starts from the value $2a_0 + \lambda a_0$ and it increases to get closer ∞ when it t get close to ∞ . and we note that it is taking upside down the shape of a swimming pool .

6.2.2 Statistical Properties .

This section presents the r th moment and moment generating function of the

Theorem 6.5.

If $T \sim$ THTHRD (a_φ, λ) with then the r th moments about the origin and about the mean are given as

$$E(T^r) = \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_\varphi \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(r+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_\varphi)!}{(2a_0)^{1+r+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_\varphi}} - \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_\varphi \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(r+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_\varphi)!}{a_0^{(1+r+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_\varphi)}} \quad (6.37)$$

$$E(T - \mu)^r = \sum_{k=0}^r C_k^r \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_\varphi \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_\varphi)!}{(a_0)^{1+k+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_\varphi}} - \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_\varphi \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_\varphi)!}{a_0^{(1+k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_\varphi)}} \right) \left(- \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_\varphi \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_\varphi)!}{(2a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_\varphi}} - \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_\varphi \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_\varphi)!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_\varphi)}} \right) \quad r-k \quad (6.38)$$

Corollary 6.5.

If $T \sim THTHR D (a_i, \lambda)$ then the mean , variance , coefficients of variation ,skewness and kurtosis of general proposed nth degree hazard rate distribution are respectively as:

$$E(T) =$$

$$\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1) j_{\varphi})!}{(a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1) j_{\varphi}}} -$$

$$\lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \quad (6.39)$$

$$Var(T) =$$

$$\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda)(a_1)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{a_3}{2}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(2+\varphi+\sum_{\varphi=1}^3(\varphi+1) j_{\varphi})!}{(a_0)^{3+\varphi+\sum_{\varphi=1}^3(\varphi+1) j_{\varphi}}} -$$

$$\lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_1}{3}\right)^{j_2} \left(\frac{a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(3+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} -$$

$$\left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1) j_{\varphi})!}{(a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1) j_{\varphi}}} - \right.$$

$$\left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right)^2 .$$

(6.40)

$$LET SK = E(T - \mu)^3 =$$

$$E(T - \mu)^3 =$$

$$\sum_{k=0}^3 C_k^3 \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=1}^3(\varphi+1) j_{\varphi})!}{(a_0)^{1+k+\varphi+\sum_{\varphi=1}^3(\varphi+1) j_{\varphi}}} -$$

$$\lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(1+k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}}$$

$$\left(- \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} - \right. \\ \left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right)^{3-k} \quad (6.41)$$

And $E(T - \mu)^4 = KU$

$$E(T - \mu)^4 =$$

$$\sum_{k=0}^4 C_k^4 \left[\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{1+k+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} - \right. \\ \left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(1+k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right] \times$$

$$\left(- \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} - \right. \\ \left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right)^{4-k} \quad ($$

6.42)

The Coefficient of Variation CV of THTHRD

is given by:

$$CV = (V_2(t))^{1/2} (E_2(T))^{-1}$$

=

$$\left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{3+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} - \right. \\ \left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(3+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right. \\ \left. \right)^{-1}$$

$$\begin{aligned}
 & \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda)\left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} \right. \\
 & \left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right)^{\frac{1}{2}} * \\
 & \left(\left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda)\left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} \right. \right. \\
 & \left. \left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right) \right)^{-1} \\
 & (6.43)
 \end{aligned}$$

The Standard deviation OF THTRD is

$$\begin{aligned}
 \sigma = & \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda)\left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{3+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} \right. \\
 & \left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(3+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right. \\
 & \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda)\left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} \right. \\
 & \left. \left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right) \right)^{1/2} \\
 & (6.44)
 \end{aligned}$$

The Coefficient of Skewness of THTRD is

$$\gamma = \frac{E(T - \mu)^3}{E((T - \mu)^2)^{3/2}}$$

$$\begin{aligned}
 \gamma = & \sum_{k=0}^3 C_k^3 \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda)\left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{1+k+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} - \\
 & \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_1}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(1+k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} * \\
 & \left(- \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda)\left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} - \right. \\
 & \left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_1}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right)^{3-k} * \\
 & \left[\sum_{k=0}^2 C_k^2 \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda)\left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{1+k+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} - \right. \\
 & \left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_1}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(1+k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} * \right. \\
 & \left. \left(- \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda)\left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} - \right. \right. \\
 & \left. \left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_1}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right)^{2-k} \right]^{-3/2}
 \end{aligned}$$

(6.45)

6.2.3 Mode of THTRD

$$\frac{\partial f(t)}{\partial t} = 0$$

$$\frac{\partial \left(2(1+\lambda) \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-2 \sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right)}{\partial t}$$

$$\begin{aligned}
 &= 2(1 + \lambda) \left(\sum_{\varphi=1}^3 (a_{\varphi} \varphi) t^{\varphi-1} e^{-2 \sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \right. \\
 &2 \left(\sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \right)^2 e^{-2 \sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left. - \lambda \left(- \left(\sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \right)^2 e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} + \right. \right. \\
 &e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left. \sum_{\varphi=1}^3 (a_{\varphi} \varphi) t^{\varphi-1} \right) = 0 \\
 &= 2(1 + \lambda) \left(\sum_{\varphi=1}^3 a_{\varphi} \varphi t^{\varphi-1} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \right. \\
 &2 \left(\sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \right)^2 e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left. \right) = \lambda \left(- \left(\sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \right)^2 + \sum_{\varphi=1}^3 (a_{\varphi} \varphi) t^{\varphi-1} \right)
 \end{aligned}$$

.....(6.46)

Which can be solved numerically to for the need

Moments generating function of THTRD

$$\begin{aligned}
 M_T(z) = & \sum_{r=1}^{\infty} \frac{z^r}{r!} \left[\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{a_3}{2}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(r+\varphi+\sum_{\varphi=1}^3(\varphi+1) j_{\varphi})!}{(a_0)^{1+r+\varphi+\sum_{\varphi=1}^3(\varphi+1) j_{\varphi}}} - \right. \\
 & \left. \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\lambda \left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(r+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(1+r+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right] (6.47)
 \end{aligned}$$

Characteristic Function of THTRD

$$\begin{aligned}
 M_t(ix) = & \sum_{r=1}^{\infty} \frac{(ix)^r}{r!} \left[\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2}}{j_1! j_2!} \dots \frac{\left(\frac{2a_3}{3+1}\right)^{j_3}}{j_3!} \frac{(r+\varphi+\sum_{\varphi=1}^3(\varphi+1) j_{\varphi})!}{(a_0)^{1+r+\varphi+\sum_{\varphi=1}^3(\varphi+1) j_{\varphi}}} - \right.
 \end{aligned}$$

$$\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\lambda \left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_1}{3}\right)^{j_2}}{j_1! j_2!} \cdots \frac{\left(\frac{a_3}{3+1}\right)^{j_3}}{j_3!} \frac{(r+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(1+r+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \quad (6.48)$$

6.2.4 Order statistic of THTRD.

if $T_1 \leq \dots \leq T_3$ denotes the order statistics of the random sample from a continuous population with cdf and pdf then the pdf of distribution given by

the CDF of the min order statistic

$$F_{Y_1}(t) = 1 - \left(e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \right)^n \quad (6.49)$$

Then PDF of the min order statistic

$$f_{Y_1}(t) = n \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \left(e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \right)^{n-1} \quad (6.50)$$

Then CDF of the max order statistic

$$F_{Y_3}(t) = \left(1 - e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \right)^n \quad (6.51)$$

Then the max pdf order statistic

$$f_{Y_3}(t) = n \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \times$$

$$\left(1 - \left((1 + \lambda) e^{-2 \sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right) \right)^{n-1} \quad (6.52)$$

6.2.5 Maximum Likelihood Estimators of THTHRD

The likelihood function is

$$L = L(a_0, a_1, a_2, a_3, \lambda; t_1, t_2, \dots, t_m) = \prod_{j=1}^m f_2(a_0, a_1, a_2, a_3, \lambda; t_j)$$

$$= \prod_{j=1}^m \sum_{\varphi=0}^3 a_{\varphi} t_j^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)$$

$$= e^{-\sum_{j=0}^m \sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \prod_{j=1}^m \left[\sum_{\varphi=0}^3 a_{\varphi} t_j^{\varphi} \right] \left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right) \quad (6.53)$$

It can be presented the log likelihood function for the vector of parameters

$(a_{\varphi})^T$ as

$\ln L =$

$$= \left[- \sum_{j=0}^m \sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1} \right] + \left[\sum_{j=0}^m \ln \sum_{\varphi=0}^3 a_{\varphi} t_j^{\varphi} \right] + \sum_{j=0}^m \ln \left(2(1+\lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)$$

By taking the partial derivatives of L with respect to unidentified parameters (a_{φ}) as

$$\begin{aligned} \frac{\partial \ln L}{\partial a_0} &= \\ &= \left[- \sum_{j=0}^m t_j \right] + \sum_{j=0}^m \frac{1}{(a_0 + \sum_{\varphi=1}^3 a_{\varphi} t_j^{\varphi})} - \\ &\quad \frac{\left(2(1+\lambda) t_j e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \right)}{\sum_{j=0}^m \left(2(1+\lambda) e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)} \end{aligned} \quad (6.53)$$

$$\begin{aligned} \frac{\partial \ln L}{\partial a_1} &= \\ &= \left[- \sum_{j=0}^m \frac{t_j^2}{2} \right] + \sum_{j=0}^m \frac{t_j}{(a_0 + \sum_{\varphi=1}^3 a_{\varphi} t_j^{\varphi})} - \\ &\quad \frac{\left((1+\lambda) t_j^2 e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \right)}{\sum_{j=0}^m \left(2(1+\lambda) e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)} \end{aligned} \quad (6.54)$$

$$\frac{\partial \ln L}{\partial a_2} =$$

$$= \left[-\sum_{j=0}^m \frac{t_j^3}{3} \right] + \sum_{j=0}^m \frac{t_j^2}{(a_0 + \sum_{\varphi=1}^3 a_{\varphi} t_j^{\varphi})} + \sum_{j=0}^m \frac{\left(\frac{2}{3}(1+\lambda)t_j^3 e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \right)}{\left(2(1+\lambda)e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)} \quad (6.54)$$

$$\frac{\partial \ln L}{\partial a_3} =$$

$$= \left[-\sum_{j=0}^m \frac{t_j^4}{4} \right] + \sum_{j=0}^m \frac{t_j^3}{(a_0 + \sum_{\varphi=1}^3 a_{\varphi} t_j^{\varphi})} + \sum_{j=0}^m \frac{\left(\frac{2}{4}(1+\lambda)t_j^{\varphi+1} e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \right)}{\left(2(1+\lambda)e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)} \quad (6.54)$$

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{j=0}^m \frac{\left(2e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - 1 \right)}{\left(2(1+\lambda)e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)} \quad (6.55)$$

When $\frac{\partial \ln L}{\partial \lambda}, \frac{\partial \ln L}{\partial a_{\varphi}} = 0$ There is no closed solution of(6.51),(6.52) ,(6.53),(6.54) and (6.55) therefor, numerical technique (Newton- Raphson method) should be apply to solve.

6.3 Transmuted Hazard Linear Exponential Distribution THLED

Starting from an arbitrary parent cumulative density function $F(t)$, a random variable T is said to have a Transmuted distribution if its cdf is given by equation

(1.3)

Differentiating Equation (5.5) with respect to T gives;

$$h_2(t) = \left[\frac{(1+\lambda)h(t)e^{-\int_0^t h(x)dx}}{\left((1+\lambda)e^{-\int_0^t h(x)dx} - \lambda \right)} + h(t) \right]$$

It is good to note that if $\lambda = -1$; Equation (5.5) reduces to the parent distribution.

Hence, a random variable T is said to have THLED with parameters θ, b and λ if the hazard function is given by;

$$h_2(t) = \left[\frac{(1+\lambda)(\theta+bt)e^{-\theta t - b\frac{t^2}{2}}}{\left((1+\lambda)e^{-\theta t - b\frac{t^2}{2}} - \lambda \right)} + (\theta + bt) \right] \dots\dots\dots (6.55)$$

The corresponding cumulative Hazard function is given by;

$$H(t) = \int_0^t \frac{(1+\lambda)(\theta+bt)e^{-\theta t - b\frac{t^2}{2}}}{\left((1+\lambda)e^{-\theta t - b\frac{t^2}{2}} - \lambda \right)} dt + \int_0^t (\theta + bt) dt \dots\dots\dots (6.56)$$

$$H(t) = -\ln \left((1 + \lambda)e^{-\theta t - b\frac{t^2}{2}} - \lambda \right) + t \left(\theta + \frac{1}{2}bt \right) \dots\dots (6.57)$$

The Density Function

the PDF of the THLED from it is product of $h(t)$ and $e^{-H(t)}$

$$f_2(t) = \left[\frac{(1+\lambda)(\theta+bt)e^{-\theta t - b\frac{t^2}{2}}}{\left((1+\lambda)e^{-\theta t - b\frac{t^2}{2}} - \lambda \right)} + (\theta + bt) \right] e^{\ln \left((1+\lambda)e^{-\theta t - b\frac{t^2}{2}} - \lambda \right) - t \left(\theta + b\frac{1}{2}t \right)} \dots\dots (6.58)$$

$$f_2(t) = (\theta + bt)e^{-t(\theta + \frac{1}{2}bt)} \left[2(1 + \lambda)e^{-t(\theta + \frac{1}{2}bt)} - \lambda \right] \dots\dots (6.59)$$

As this function $f_2(t)$ is to verification two conditions of the density function is as follows .

1) the first property $f_2(t) \geq 0$ is satisfied for all $t > 0$, and at

$$\left[\frac{2e^{\left(-\theta t - b \frac{t^2}{2}\right)}}{\left(1 - 2e^{\left(-\theta t - b \frac{t^2}{2}\right)}\right)} \right] \geq \lambda .$$

The second property is shown below:

$$\int_0^\infty f_2(t) dt = \int_0^\infty (\theta + bt)e^{-\left(\theta t - b \frac{t^2}{2}\right)} \left[2(1 + \lambda)e^{\left(-\theta t - b \frac{t^2}{2}\right)} - \lambda \right] dt$$

$$\int_0^\infty f_2(t) dt = 2(1 + \lambda) \int_0^\infty (\theta + bt)e^{2\left(-\theta t - \frac{bt^2}{2}\right)} dt - \lambda \int_0^\infty \left(\theta + \frac{1}{b^2}t\right) e^{\left(-\theta t - \frac{bt^2}{2}\right)} dt$$

Let $L1 = 2(1 + \lambda) \int_0^\infty (\theta + bt)e^{-2t(\theta + bt)} dt$ and

$$L2 = \int_0^\infty (\theta + bt)e^{(\theta + bt)} dt$$

$$L1 = (1 + \lambda) \int_0^\infty 2(\theta + bt)e^{-t(2\theta + bt)} dt$$

Let $u = 2\left(\theta t + b \frac{t^2}{2}\right)$ $du = 2(\theta + bt)dt$

$$L1 = (1 + \lambda) \int_0^\infty e^{-u} du = (1 + \lambda)(-e^{-\infty} + e^{-0}) = (1 + \lambda)$$

And

$$L2 = \lambda \int_0^\infty (\theta + bt)e^{\left(-\theta t - \frac{bt^2}{2}\right)} dt$$

Let $u = \left(\theta t + b \frac{t^2}{2}\right)$ $du = (\theta + bt)dt$

$$L2 = \lambda \int_0^{\infty} ue^{-u} dt = \lambda(-e^{-\infty} + e^{-0}) = \lambda$$

$$\int_0^{\infty} f_2(t) dt = (1 + \lambda) - \lambda = 1$$

The Cumulative function of THLED .

The CDF of this distribution is:

$$F_2(t) = \int_0^t \left[2(1 + \lambda) (\theta + bt)e^{2\left(-\theta t - b\frac{t^2}{2}\right)} - \lambda (\theta + bt)e^{\left(-\theta t - b\frac{t^2}{2}\right)} \right] dt \quad (6.60)$$

$$\text{Let } L1 = 2(1 + \lambda) \int_0^t (\theta + bt)e^{2\left(-\theta t - b\frac{t^2}{2}\right)} dt$$

$$\text{And } L2 = \int_0^t (\theta + bt)e^{\left(-\theta t - b\frac{t^2}{2}\right)} dt$$

$$L1 = (1 + \lambda) \int_0^t 2(\theta + bt)e^{2\left(-\theta t - b\frac{t^2}{2}\right)} dt$$

$$\text{Let } u = 2\left(\theta t + b\frac{t^2}{2}\right) \quad du = 2(\theta + bt) dt$$

$$L1 = (1 + \lambda) \int_0^t e^{-u} du = (1 + \lambda)(-e^{-u} + e^{-0}) = (1 + \lambda)(1 - e^{-2(\theta + bt)})$$

$$L2 = \lambda \int_0^t (\theta + bt)e^{\left(-\theta t - b\frac{t^2}{2}\right)} dt$$

$$\text{Let } u = \left(\theta t + b\frac{t^2}{2}\right) \quad du = (\theta + bt) dt$$

$$L2 = \lambda \int_0^t e^{-u} du = \lambda(-e^{-u} + e^{-0}) = \lambda(1 - e^{-u})$$

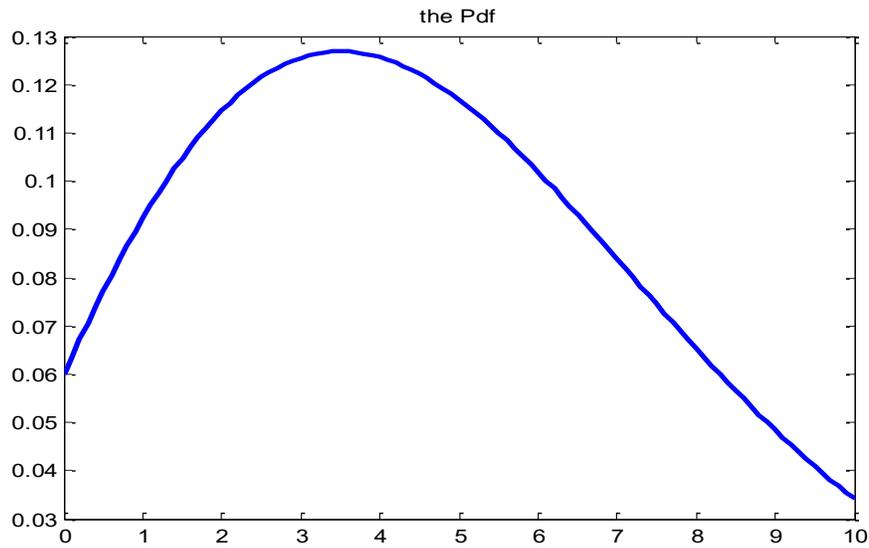
$$= \lambda \left(1 - e^{-\left(\theta t + \frac{bt^2}{2}\right)} \right)$$

$$F_2(t) = 1 - e^{-\left(\theta t + \frac{bt^2}{2}\right)} \left((1 + \lambda)e^{-\left(\theta t + \frac{bt^2}{2}\right)} - \lambda \right) \quad \dots\dots(6.61)$$

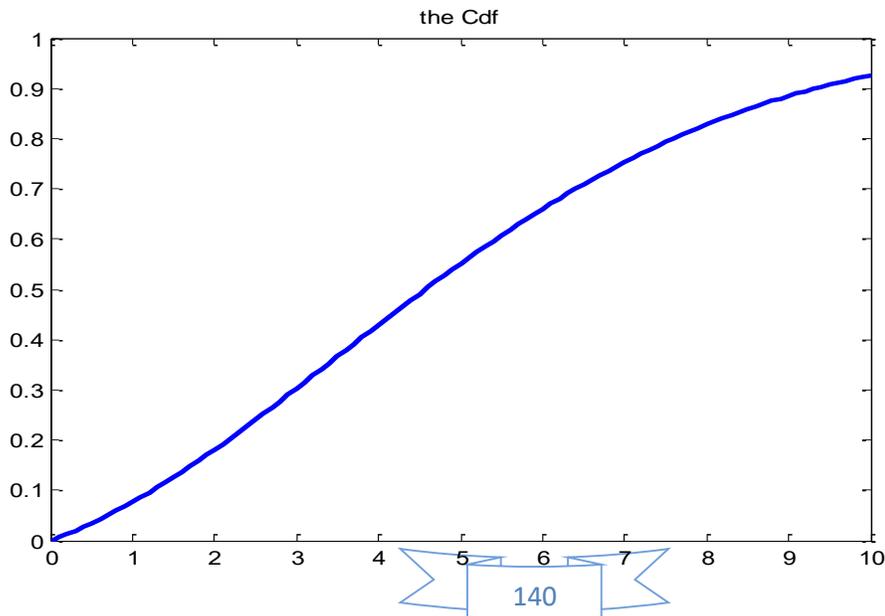
The Survival function of THLED

$$S_2(t) = e^{-\left(\theta t + \frac{bt^2}{2}\right)} \left((1 + \lambda)e^{-\left(\theta t + \frac{bt^2}{2}\right)} - \lambda \right) \dots\dots(6.62)$$

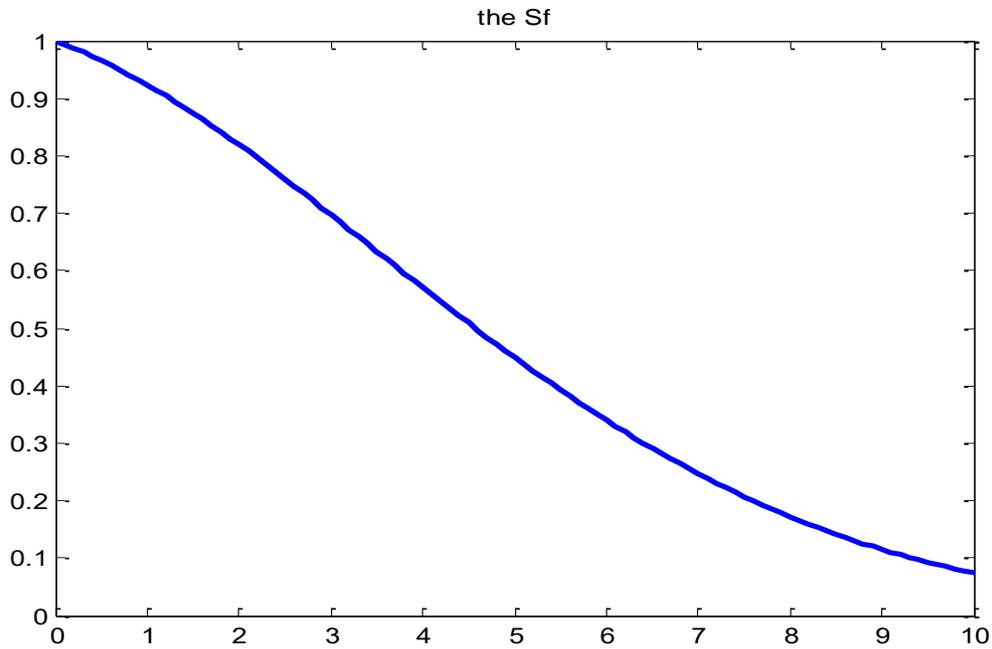
The following figure there are The shows of pdf , The Cumulative Function ,The Survival Function, The Hazard Function of the THLED.



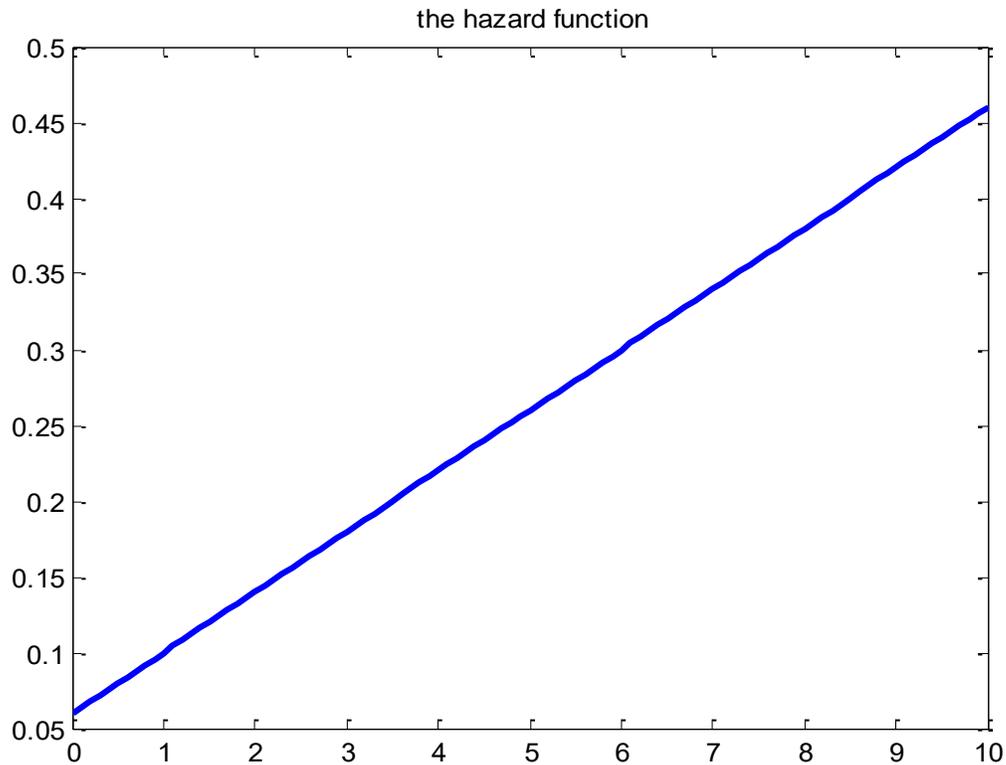
Figure(6.9): The Density Function of THLED ,where $\theta = 0.03$; $b = .02$; $\lambda = -0.0000000000002$; We note that the decreases until it reaches zero as long as increasing T.



Figure(6.10): The Cumulative Function of THLED, where $\theta = 0.03$; $b = .02$; $\lambda = -0.00000000000002$; . we note that the cdf starts from the value 0 and then nondcreasing until it reaches 1 when it approaches t to ∞ .



Figure(6.11): The Survival Function of THLED, where $\theta = 0.03$; $b = .02$; $\lambda = -0.00000000000002$; we notice that this function starts from the value 1 and then nonincreasing until it reaches 0 when it approaches t to ∞ .



Figure(6.12): The Hazard Function of THLED, where $\theta = 0.03; b = .02; \lambda = -0.00000000000002$; it increases to get closer ∞ when it gets close to ∞ . and we note that it is taking the shape of a swimming pool .

6.3.1 Moment of THLED .

Theorem 6.6

If $T \sim \text{THLED}(\theta, b, \lambda)$ then the r_{th} moment about the origin , and about the mean μ one help hey as of T, say μ'_r , is given as

$$E(T^r) = \sum_{j=0}^{\infty} \frac{2(-b)^j(1+\lambda)}{j!} \left[\frac{\theta(r+2j)!}{(2(\theta))^{r+2j+1}} + \frac{b(r+2j+1)!}{(2(\theta))^{r+2j+2}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(r+2j)!}{(\theta)^{r+2j}} + \frac{b(r+2j+1)!}{(\theta)^{r+2j+2}} \right) \right] \dots\dots(6.63)$$

$$E(T - \mu)^r = \sum_{k=0}^r C_k^r \left(\sum_{j=0}^{\infty} \frac{2(-b)^j(1+\lambda)}{j!} \left[\frac{\theta(r+2j)!}{(2(\theta))^{r+2j+1}} + \frac{b(r+2j+1)!}{(2(\theta))^{r+2j+2}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\frac{(r+2j)!}{(\theta)^{r+2j}} + \frac{b(r+2j+1)!}{(\theta)^{r+2j+2}} \right] \right) \left(- \sum_{j=0}^{\infty} \frac{2(-b)^j(1+\lambda)}{j!} \left[\frac{\theta(2j+1)!}{(2(\theta))^{r+2j+1}} + \frac{b(2j+2)!}{(2(\theta))^{r+2j+2}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\frac{(2j+1)!}{(\theta)^{r+2j}} + \frac{b(2j+2)!}{(\theta)^{r+2j+2}} \right] \right)^{r-k} \dots\dots(6.64)$$

Proof

$$E(T^r) = \int_0^{\infty} t^r (\theta + bt) e^{(-\theta t - b \frac{t^2}{2})} \left[2(1 + \lambda) e^{(-\theta t - b \frac{t^2}{2})} - \lambda \right] dt$$

$$E(T^r) = \int_0^{\infty} 2(1 + \lambda) e^{2(-\theta t - b \frac{t^2}{2})} t^r (\theta + bt) dt - \int_0^{\infty} \left[\lambda t^r (\theta + bt) e^{(-\theta t - b \frac{t^2}{2})} \right] dt$$

Let

$$L_1 = \int_0^{\infty} 2(1 + \lambda) e^{2(-\theta t - b \frac{t^2}{2})} t^r (\theta + bt) dt$$

And $L_2 = \int_0^{\infty} \left[\lambda t^r (\theta + bt) e^{(-\theta t - b \frac{t^2}{2})} \right] dt$

Note

$$L_1 = \int_0^{\infty} 2(1 + \lambda) t^r (\theta + bt) e^{2(-\theta t)} e^{2(-b \frac{t^2}{2})} dt$$

Using the following expansion of $e^{2(-b \frac{t^2}{2})}$

$$L_1 = \sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \int_0^{\infty} t^{r+2j} (\theta + bt) e^{2(-\theta t)} dt$$

$$L_1 = \sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\theta \int_0^{\infty} t^{r+2j} e^{2(-\theta t)} dt + b \int_0^{\infty} t^{r+2j+1} e^{2(-\theta t)} dt \right]$$

Let $y = 2(\theta t)$ then $2(\theta)dt = dy$

$$L_1 = \sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\theta \int_0^{\infty} \left(\frac{y}{2(\theta)} \right)^{r+2j} e^{-y} \frac{dy}{2(\theta)} + b \int_0^{\infty} \left(\frac{y}{2(\theta)} \right)^{r+2j+1} e^{-y} \frac{dy}{2(\theta)} \right]$$

$$L_1 = \sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\theta \frac{\Gamma(r+2j+1)}{(2(\theta))^{r+2j+1}} + b \frac{\Gamma(r+2j+2)}{(2(\theta))^{r+2j+2}} \right]$$

$$L_1 = \sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(r+2j)!}{(2(\theta))^{r+2j+1}} + \frac{b(r+2j+1)!}{(2(\theta))^{r+2j+2}} \right]$$

And

$$L_2 = \lambda \int_0^{\infty} \left(\theta t^r e^{(-\theta t)} e^{\left(-b \frac{t^2}{2}\right)} + b t^{r+1} e^{(-\theta t)} e^{\left(-b \frac{t^2}{2}\right)} \right) dt$$

$$e^{\left(-b \frac{t^2}{2}\right)} = \sum_{j=0}^{\infty} \frac{(-b)^j}{j!} t^{2j}$$

$$L_2 = \lambda \sum_{j=0}^{\infty} \frac{(-b)^j}{j!} \int_0^{\infty} \left(\theta t^{r+2j} e^{(-\theta t)} + b t^{r+2j+1} e^{(-\theta t)} \right) dt$$

Let $y = \theta t$ and $dy = \theta dt$

$$L_2 = \lambda \sum_{j=0}^{\infty} \frac{(-b)^j}{j!} \int_0^{\infty} \left(\theta \left(\frac{1}{\theta}\right)^{r+2j} (y)^{r+2j} e^{(-y)} + \frac{b}{(\theta)^{r+2j+1}} (y)^{r+2j+1} e^{(-y)} \right) \frac{dy}{\theta}$$

$$L_2 = \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{\Gamma(r+2j+1)}{(\theta)^{r+2j}} + \frac{\Gamma(r+2j+2)}{b^2(\theta)^{r+2j+2}} \right) \right]$$

$$E(T^r) = \sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(r+2j)!}{(2(\theta))^{r+2j+1}} + \frac{b(r+2j+1)!}{(2(\theta))^{r+2j+2}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(r+2j)!}{(\theta)^{r+2j}} + \frac{b(r+2j+1)!}{(\theta)^{r+2j+2}} \right) \right]$$

Based on part one from theorem (6.6), we get the expected value .

$$E(T) = \sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(2j+1)!}{(2(\theta))^{2j+2}} + \frac{b(2j+2)!}{(2(\theta))^{2j+3}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(2j+1)!}{(\theta)^{1+2j}} + \frac{b(2j+2)!}{(\theta)^{2j+3}} \right) \right] \dots\dots(6.65)$$

Proof of the second part of the theorem (6.6)

Depending on the equations (6.63) and (6.65) , it is get

$$\begin{aligned}
 E(T - \mu)^r &= \sum_{k=0}^r C_k^r E(T^k)(-\mu)^{r-k} \\
 E(T - \mu)^r &= \\
 &= \sum_{k=0}^r C_k^r \left(\sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(k+2j)!}{(2(\theta))^{k+2j+1}} + \frac{b(k+2j+1)!}{(2(\theta))^{k+2j+2}} \right] \right. \\
 &\quad \left. - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left(\frac{(k+2j)!}{(\theta)^{k+2j}} \right. \right. \\
 &\quad \left. \left. + \frac{b(k+2j+1)!}{(\theta)^{k+2j+2}} \right) \right) * \left(- \sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(2j+1)!}{(2(\theta))^{2j+2}} \right. \right. \\
 &\quad \left. \left. + \frac{b(2j+2)!}{(2(\theta))^{2j+3}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(2j+1)!}{(\theta)^{1+2j}} + \frac{b(2j+2)!}{(\theta)^{2j+3}} \right) \right] \right)^{r-k}
 \end{aligned}$$

Note Based on part two from theorem (6.6), it is get the variance , SK and KU .

$$\begin{aligned}
 Var(t) &= E(T - \mu)^2 \\
 &= \sum_{k=0}^2 C_k^2 \left(\sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(k+2j)!}{(2(\theta))^{k+2j+1}} + \frac{b(k+2j+1)!}{(2(\theta))^{k+2j+2}} \right] - \right. \\
 &\quad \left. \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(k+2j)!}{(\theta)^{k+2j}} + \frac{b(k+2j+1)!}{(\theta)^{k+2j+2}} \right) \right] \right) \left(- \sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(2j+1)!}{(2(\theta))^{2j+2}} + \right. \right. \\
 &\quad \left. \left. \frac{b(2j+2)!}{(2(\theta))^{2j+3}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(2j+1)!}{(\theta)^{1+2j}} + \frac{b(2j+2)!}{(\theta)^{2j+3}} \right) \right] \right)^{2-k} \dots\dots(6.66)
 \end{aligned}$$

Let

$$SK = E(T - \mu)^3 = \sum_{k=0}^3 C_k^3 \left(\sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(k+2j)!}{(2(\theta))^{k+2j+1}} + \frac{b(k+2j+1)!}{(2(\theta))^{k+2j+2}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(k+2j)!}{(\theta)^{k+2j}} + \frac{b(k+2j+1)!}{(\theta)^{k+2j+2}} \right) \right] \right) \left(- \sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(2j+1)!}{(2(\theta))^{2j+2}} + \frac{b(2j+2)!}{(2(\theta))^{2j+3}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(2j+1)!}{(\theta)^{1+2j}} + \frac{b(2j+2)!}{(\theta)^{2j+3}} \right) \right] \right)^{3-k} \dots\dots(6.67)$$

$$KU = E(T - \mu)^4 = \sum_{k=0}^4 C_k^4 \left(\sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(k+2j)!}{(2(\theta))^{k+2j+1}} + \frac{b(k+2j+1)!}{(2(\theta))^{k+2j+2}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(k+2j)!}{(\theta)^{k+2j}} + \frac{b(k+2j+1)!}{(\theta)^{k+2j+2}} \right) \right] \right) \left(- \sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(2j+1)!}{(2(\theta))^{2j+2}} + \frac{b(2j+2)!}{(2(\theta))^{2j+3}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(2j+1)!}{(\theta)^{1+2j}} + \frac{b(2j+2)!}{(\theta)^{2j+3}} \right) \right] \right)^{4-k} \dots\dots(6.68)$$

Note Based on theorem (6.6), it is get .

The Coefficient of Variation CV is

$$CV = \left(\sum_{k=0}^2 C_k^2 \left(\sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(k+2j)!}{(2(\theta))^{k+2j+1}} + \frac{b(k+2j+1)!}{(2(\theta))^{k+2j+2}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(k+2j)!}{(\theta)^{k+2j}} + \frac{b(k+2j+1)!}{(\theta)^{k+2j+2}} \right) \right] \right) \left(- \sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(2j+1)!}{(2(\theta))^{2j+2}} + \frac{b(2j+2)!}{(2(\theta))^{2j+3}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(2j+1)!}{(\theta)^{1+2j}} + \frac{b(2j+2)!}{(\theta)^{2j+3}} \right) \right] \right)^{2-k} \right)^{1/2} * \left(\sum_{j=0}^{\infty} \frac{2(1+\lambda)(-b)^j}{j!} \left[\frac{\theta(2j+1)!}{(2(\theta))^{2j+2}} + \frac{b(2j+2)!}{(2(\theta))^{2j+3}} \right] - \sum_{j=0}^{\infty} \frac{\lambda(-b)^j}{j!} \left[\left(\frac{(2j+1)!}{(\theta)^{1+2j}} + \frac{b(2j+2)!}{(\theta)^{2j+3}} \right) \right] \right)^{-1} \dots(6.69)$$

The Standard deviation is

$$\begin{aligned} \sigma &= \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{3+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} \right. \\ &- \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(3+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \\ &- \left(\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} \right. \\ &\left. \left. - \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right)^2 \right)^{1/2} \end{aligned}$$

... (6.70).

The Coefficient of Skewness of THTRD .

$\gamma =$

$$\begin{aligned} &\sum_{k=0}^3 C_k^3 \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{1+k+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} - \\ &\lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(1+k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} * \\ &\left(- \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} - \right. \\ &\left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right)^{3-k} * \end{aligned}$$

$$\left[\sum_{k=0}^2 C_k^2 \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{1+k+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} - \right.$$

$$\lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_1}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(1+k+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} * \left.$$

$$\left(- \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{2+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} - \right.$$

$$\left. \lambda \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_1}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(1+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(2+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right)^{2-k} \Big]^{-3/2} \dots (6.71)$$

6.3.2 Mode of THTRD .

$$\frac{\partial f(t)}{\partial t} = 0$$

$$\frac{\partial \left(2(1+\lambda) \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-2 \sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right)}{\partial t}$$

$$= 2(1+\lambda) \left(\sum_{\varphi=1}^3 (a_{\varphi} \varphi) t^{\varphi-1} e^{-2 \sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \right.$$

$$\left. 2 \left(\sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \right)^2 e^{-2 \sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right) - \lambda \left(- \left(\sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \right)^2 e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} + \right.$$

$$\left. e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \sum_{\varphi=1}^3 (a_{\varphi} \varphi) t^{\varphi-1} \right) = 0$$

$$= 2(1+\lambda) \left(\sum_{\varphi=1}^3 a_{\varphi} \varphi t^{\varphi-1} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - 2 \left(\sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \right)^2 e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right) =$$

$$\lambda \left(- \left(\sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} \right)^2 + \sum_{\varphi=1}^3 (a_{\varphi} \varphi) t^{\varphi-1} \right)$$

Which can be solved numerically .

6.3.3 Moments generating function of THTHRD

$M_T(z)$

$$\begin{aligned}
 &= \sum_{r=1}^{\infty} \frac{z^r}{r!} \left[\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{a_3}{2}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(r+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{1+r+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} \right. \\
 &\quad \left. - \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\lambda \left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_2}{3}\right)^{j_2} \left(\frac{a_3}{4}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(r+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(1+r+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right] \dots (6.72)
 \end{aligned}$$

6.3.4 Characteristic Function of THTHRD

$M_t(ix) =$

$$\begin{aligned}
 &\sum_{r=1}^{\infty} \frac{(ix)^r}{r!} \left[\sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_3=0}^{\infty} a_{\varphi} \frac{2(1+\lambda) \left(\frac{2a_1}{2}\right)^{j_1} \left(\frac{2a_2}{3}\right)^{j_2} \left(\frac{2a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(r+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi})!}{(a_0)^{1+r+\varphi+\sum_{\varphi=1}^3(\varphi+1)j_{\varphi}}} \right. \\
 &\quad \left. - \sum_{\varphi=0}^3 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_3=0}^{\infty} a_{\varphi} \frac{\lambda \left(\frac{a_1}{2}\right)^{j_1} \left(\frac{a_1}{3}\right)^{j_2} \left(\frac{a_3}{3+1}\right)^{j_3}}{j_1! j_2! j_3!} \frac{(r+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{(\varphi+1)})!}{a_0^{(1+r+\varphi+\sum_{\varphi=0}^3(\varphi+1)j_{\varphi})}} \right]
 \end{aligned}$$

(6.73)

6.3.5 Order statistic of THTHRD .

the CDF of the min order statistic

$$F_{Y_1}(t) = 1 - \left(e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \right)^n \quad (6.74)$$

Then PDF of the min order statistic

$$f_{Y_1}(t) = n \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \left(e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \right)^{n-1} \quad \dots\dots (6.75)$$

Then CDF of the max order statistic

$$F_{Y_3}(t) = \left(1 - e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left((1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \right)^n \quad (6.76)$$

Then the PDF of max order statistic

$$f_{Y_3}(t) = n \sum_{\varphi=0}^3 a_{\varphi} t^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda \right) \times \left(1 - \left((1 + \lambda) e^{-2\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} - \lambda e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t^{\varphi+1}}{\varphi+1}} \right) \right)^{n-1} \quad (6.77)$$

6.3.6 Maximum Likelihood Estimators of THTRD

$$\begin{aligned}
 L &= L(a_0, a_1, a_2, a_3, \lambda; t_1, t_2, \dots, t_m) = \prod_{j=1}^m f_2(a_0, a_1, a_2, a_3, \lambda; t_j) \\
 &= \prod_{j=1}^m \sum_{\varphi=0}^3 a_{\varphi} t_j^{\varphi} e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right). \\
 &= e^{-\sum_{j=0}^m \sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \prod_{j=1}^m \left[\sum_{\varphi=0}^3 a_{\varphi} t_j^{\varphi} \right] \left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)
 \end{aligned}
 \tag{6.77}$$

the log likelihood function for the parameters a_{φ} as

$\ln L =$

$$\begin{aligned}
 &= \left[-\sum_{j=0}^m \sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1} \right] + \left[\sum_{j=0}^m \ln \sum_{\varphi=0}^3 a_{\varphi} t_j^{\varphi} \right] \\
 &\quad + \sum_{j=0}^m \ln \left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)
 \end{aligned}$$

by taking the derivatives of L with respect to unidentified parameters (a_{φ}) as

$$\frac{\partial \ln L}{\partial a_0} = \left[- \sum_{j=0}^m t_j \right] + \sum_{j=0}^m \frac{1}{(a_0 + \sum_{\varphi=1}^3 a_{\varphi} t_j^{\varphi})} - \sum_{j=0}^m \frac{\left(2(1 + \lambda) t_j e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \right)}{\left(2(1 + \lambda) e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)}$$

(6.78)

$$\frac{\partial \ln L}{\partial a_1} =$$

$$\left[- \sum_{j=0}^m \frac{t_j^2}{2} \right] + \sum_{j=0}^m \frac{t_j}{(a_0 + \sum_{\varphi=1}^3 a_{\varphi} t_j^{\varphi})} - \sum_{j=0}^m \frac{\left((1 + \lambda) t_j^2 e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \right)}{\left(2(1 + \lambda) e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)}$$

$$\frac{\partial \ln L}{\partial a_2} =$$

$$= \left[- \sum_{j=0}^m \frac{t_j^3}{3} \right] + \sum_{j=0}^m \frac{t_j^2}{(a_0 + \sum_{\varphi=1}^3 a_{\varphi} t_j^{\varphi})} + \sum_{j=0}^m \frac{\left(\frac{2}{3} (1 + \lambda) t_j^3 e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \right)}{\left(2(1 + \lambda) e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)}$$

(6 .80)

$$\begin{aligned} \frac{\partial \ln L}{\partial a_3} &= \\ &= \left[-\sum_{j=0}^m \frac{t_j^4}{4} \right] + \sum_{j=0}^m \frac{t_j^3}{(a_0 + \sum_{\varphi=1}^3 a_{\varphi} t_j^{\varphi})} \\ &\quad + \sum_{j=0}^m \frac{\left(\frac{2}{4} (1 + \lambda) t_j^{\varphi+1} e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} \right)}{\left(2(1 + \lambda) e^{-a_0 t_j} e^{-\sum_{\varphi=1}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)} \end{aligned} \quad (6 .81)$$

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{j=0}^m \frac{\left(2e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - 1 \right)}{\left(2(1 + \lambda) e^{-\sum_{\varphi=0}^3 a_{\varphi} \frac{t_j^{\varphi+1}}{\varphi+1}} - \lambda \right)} \quad (6 .82)$$

When $\frac{\partial \ln L}{\partial \lambda}, \frac{\partial \ln L}{\partial a_{\varphi}} = 0$ There is no closed solution of(6 .78) (6 .79) (6 .80) ,(6 .81)and (6 .82)) therefor, numerical technique (Newton- Raphson method) should be apply to solve.

Conclusions

Important findings from this research include the following:

- 1- Depending on the previous tables, we can see that the distributions that we found are better than the original distributions.
- 2- The distributions obtained from the transmuted hazard formula are better than the distributions obtained from the first general form GPHRD.
- 3- When comparing the distributions that we obtained from the first formula GPHRD, we found the best distribution in the case of $n = 2$.
- 4- When comparing the distributions that we got from the formula we got from the previous formula THGPHRD, we found the best distributions in the case of $n = 2$.
- 5- The obtained distributions are characterized by high flexibility.
- 6- We took an example of the survival transformation of the Landley distribution and proved using the general possibility method that it is better than the original Lindley distribution.
- 7- Maximum likelihood sampling was employed to evaluate the distributions.
- 8- A formula was discovered that depends on the hazard function GPHRD, from which we can generate a finite number of distributions.
- 9- A formula was discovered that depends on the risk function, from which we can generate a finite number of distributions.
- 10- Likewise, with regard to the transmuted hazard formula that depends on the risk function, we can generate from it a finite number of distributions, which are characterized by high flexibility.

Future Work.

Important ideas and notes that may be utilized for further research include:

1. Learning the transmuted hazard formula by practicing it with different types of distributions.
- 2 .Studying the transmuted hazard formula and substitute any hazard function into it, so we will get a new distribution.
- 3 .Determining other methods of estimating these parameters.
- 4 .Using the best predicted parameters of each distribution to analyse real-world sample data, and then put that knowledge to use in a genuine experiment.
5. Generalized the transmuted hazard formula by increasing its parameters

Conclusion and Future Work

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