Republic of Iraq
Ministry of Higher Education
and Scientific Research
University of Babylon
College of Education for Pure Sciences

Using Some Concepts of Differential Geometry for Reliability Model with Application of Different Distributions

A Dissertation

Submitted to the Council of the College of Education for Pure Sciences in University of Babylon as a Partial Fulfillment of the Requirements for the Degree of Doctor of philosophy in Education / Mathematics

By

Nada Mohammed Abbas Jasem

Supervised by

Prof .Dr .Audie Sabri Abd Al-Razzak

2023 A.D. 1444 A.H

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿ اللَّهُ لَا إِلَـهَ إِلَّا هُوَ الْحَيُّ الْقَيُّومُ لَا تَأْخُذُهُ
سِنَةٌ وَلَا نَوْمٌ لَهُ مَا فِي السَّمَاوَاتِ وَمَا فِي
الْأَرْضِ مَنْ ذَا الَّذِي يَشْفَعُ عِنْدَهُ إِلَّا بِإِذْنِهِ
يَعْلَمُ مَا بَيْنَ أَيْدِيهِمْ وَمَا خَلْفَهُمْ وَلَا
يُحِيطُونَ بِشَيْءٍ مِنْ عِلْمِهِ إِلَّا بِمَا شَاءَ
وَسِعَ كُرْسِيُّهُ السَّمَاوَاتِ وَالْأَرْضَ وَلَا
يَؤُودُهُ حِفْظُهُمَا وَهُوَ الْعَلِيُّ الْعَظِيمُ ﴾

صدق الله العلي العظيم

سورة البقرة الآية 255

Dedication

I dedicate this work to my master, Al-Husayn Bin Ali (peace be upon him), who was slaughtered thirstily in Karbala, and to his brother, Al-Abass Bin Ali (peace be upon him).

Acknowledgments

Praise be to Allah, the Lord of the worlds. Thanks be to God, who gave me the ability to complete my studies and granted me success in them. I am very grateful to my supervisor, Prof. Dr. Audie Sabrie, for his continuous follow-up and support. I also extend my thanks to all the professors in the Mathematics Department at the College of Education for Pure Sciences for their assistance and encouragement. I can never adequately express my appreciation for them. I would like to express my deepest appreciation to my husband, Mr. Ali Falih, for his significant role in my life, and to my children for making me proud of what I have accomplished. Finally, I thank my family and friends for supporting and encouraging me throughout my studies.

Supervisor's Certification

I certify that the dissertation entitled **“Using some Concepts of Differential Geometry for Reliability Model with Application of Different Distributions** by **Nada Mohammed Abbas** has been prepared under my supervision in Babylon University/ College of Education for Pure Sciences as a partial requirement for the Degree of Doctor of philosophy in Education / Mathematics.

Signature :

Name: Audie Sabri Abd Al-Razzak

Title: Professor

Date: / / 2023

In view of the available recommendation, I forward this dissertation for debate by the examining committee.

Signature:

Name: Dr.Azal Jaafar Musa

Head of Mathematics Department

Title: Assistant Professor

Date: / / 2023

Certification of linguistic Expert

I certify that I have read this dissertation entitled **“Using some Concepts of Differential Geometry for Reliability Model with Application of Different Distributions"** and corrected its grammatical mistakes; therefore, it has qualified for debate.

Signature:

Name: Mais Flaieh Hasan Al-Jabbawi

Title:

Address:

Date: **/** **/** 2023

Certification of Scientific Expert

I certify that I have read the scientific content of this dissertation **“Using some Concepts of Differential Geometry for Reliability Model with Application of Different Distributions"** and I have approved this dissertation is qualified for debate

Signature:

Name: Inas Yahya Abdullah

Title: Professor

Date:

Certification of Scientific Expert

I certify that I have read the scientific content of this dissertation **“Using some Concepts of Differential Geometry for Reliability Model with Application of Different Distributions"** and I have approved this dissertation is qualified for debate

Signature:

Name: Dr.Reyadh D.Ali

Title: *Assistant Professor*

Date:

Examining Committee Certification

We certify that we have read the dissertation entitled **“Using some Concepts of Differential Geometry for Reliability Model with Application of Different Distributions"** by**" Nada Mohammed Abbas "** and as a committee examined the student in its contents and, according to our opinion, it is accepted as a dissertation for the degree for the Degree of Doctor of philosophy in Education / Mathematics .

Signature:
Name: Dr.Luay Abd Al haine Al swidi
Title: Professor
Date: 17 / 9 / 2023
Chairman

Signature:
Name: Kareema Abed AL-Kadim
Title: Professor
Date:17 / 9 / 2023
Member/

Signature:
Name: Dr.Yiezi Kadham Mahdi
Title: Professor
Date: 17/ 9 / 2023
Member /

Signature:
Name: Dr.Zahir Dobeas Al-Nafie
Title: Professor
Date: 17 / 9 / 2023
Member /

Signature:
Name: Dr.Azal Jaafar Musa
Title: Assistant Professor
Date: 17 / 9 / 2023
Member /

Signature:
Name: Audie Sabri Abd Al-Razzak
Title: Professor
Date: 17 / 9 / 2023
Member/Advisor

Approved by the Dean of college
Signature:
Name: Dr.Bahaa Hussein Salih Rabee
Title : Professor
Address: Dean of the College of Education for Pure Sciences
Date : / / 2023

Publications Arising from This Thesis

Journal of Positive School Psychology

ISSN: 2717-7564

Date: 07.05.2022

Paper Title: **Using Hellinger Method To Evaluate The Distance For Continuous Distributions And Discrete Distributions**

Dear **Nada Mohammed Abbas   Audie Sabrie**
Congratulations! As a result of reviews, we are pleased to inform that your manuscript has been accepted

for publication in **Journal of Positive School Psychology**.

On behalf of Editorial Board and publisher, thank you very much for your submission to our journals.

Regards,

Editorial Team,

Journal of Positive School Psychology

Journal of
Survey in Fisheries Sciences

ACCEPTANCE LETTER

Date: 21-02-2023.
Manuscript ID: JSFS-5143.2023.
Title: Fuzzy Triangular Exponential Two-Parameter Distribution for K-Out-Of-N System Reliability

Dear Authors: Nada Mohammed Abbas And Prof .Dr .Audie Sabrie

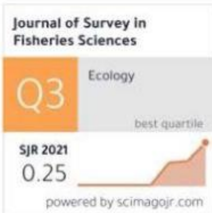
Education College for Pure Sciences , College of Basic Education2 University of Babylon, Iraq

It's our great pleasure to inform you that your above-mentioned manuscript has been reviewed and accepted for publication in the upcoming special issue of the **Journal of Survey in Fisheries Sciences (JSFS)** with **E-ISSN:7487-2368** . Your article will be published in the forthcoming **special Issue, 2023**. This letter of acceptance is to be considered as the official acceptance of your manuscript with no further amendments required. Use below link to find article formatting instruction to format article according to journal format. Author Instruction Link: http://sifisheriessciences.com/page/21/Submission-Instruction.

Thank you for your contribution to the Journal and we are looking forward to your future participation!

Kind regards

Editor-in-Chief
Prof. Dr. Selamoglu, Zeliha
Journal of Survey in Fisheries Sciences
Publisher: Green Wave Publishing of Canada
E-ISSN:7487-2368
http://sifisheriessciences.com/index.php?&slct_pg_id=10&sid=1&slc_lang=en
https://www.scopus.com/sourceid/21100905326



ندى محمد عباس <pure.nada.moh@uobabylon.edu.iq>

Regarding Acceptance

1 message

editor@nonlinear-analysis.com <editor@nonlinear-analysis.com> Thu, Jul 13, 2023 at 5:18 PM
To: pure.nada.moh@uobabylon.edu.iq, udaisabri51@gmail.com

Dear Author,

We are pleased to inform you that your article, "Parametric Models of Reliability functions" has been accepted for publication in the Results in Nonlinear Analysis(RNA).

We would like to commend you on the high quality of your research and the valuable contribution it makes to the field of Mathematics. Your paper was thoroughly reviewed by our team of experts and we believe it will be of great interest to our readers.

We look forward to working with you to bring your research to the wider academic community. Our team will be in touch with you shortly to provide more information about the next steps in the publication process.

Thank you again for choosing RNA, for your research, and we wish you all the best in your future endeavors.

Best regards,
The RNA Team

List of Contents

List of Figures

List of Symbols

X	An universal set
μ_A	membership function
$\widetilde{A}$	the fuzzy set A
$\widetilde{A}_\propto$	$\propto$ −cut set of $\widetilde{A}$
$\mathrm{supp}(\widetilde{A})$	support of $\widetilde{A}$
$P_r(x)$	Probability of element x
$E_{X\sim p}[f(x)]$	The expectation
p.d.f	Probability density function
p.m.f	Probability mass function
c.d.f	The cumulative distribution function
$R(t)$	Reliability function
$l(R(t,\lambda))$	log-likelihood function
$\gamma_j l(R(t,\lambda))$	Partial derivative function
RN	Reliability manifold
(U,φ)	Chart
$(U_i,\varphi_i)_{i\in I}$	An atlas
$J(\lambda^1 \ldots \lambda^n)$	Jacobian matrix
H	Transition map
$x^i(\bar{x}^1,\bar{x}^2 \ldots \bar{x}^n)$	Transformation of Coordinates
$\sum_{i=1}^{n} a^i e_i$	Einstein's summation convention.
δ^i_j	kronecker delta symbol
$T_{R(t,\lambda)}$	The tangent space
T_λ	Tangent space in the coordinate system λ
$T_\lambda^{(1)}$	1- representation
$\mathfrak{B}^\alpha_j(\lambda)$	Jacobin matrices of T_λ
∂_i	The transformation law of the bases vectors
V^i	The transformation law of tangent vector
A_{ij}	Contravariant tensor of rank 2
g_{ij}	Metric Tensor
И	Hessian matrix
$D(\lambda_1\|\lambda_2)$	Contrast function
$D_N(\varphi(\lambda_1)\|\varphi(\lambda_2))$	A contrast function on N
$D_f(p\|q)$	Divergence f −

$D_{fKL}(R_1(t,\lambda_1)\|R_2(t,\lambda_2))$	**Kullback-Leibler-divergence**
$H_f(p,q)$	**Hellinger divergence**
$D_{AN}(R_1(t,\lambda_1),R_2(t,\lambda_2))$	**A-N divergence**
$\widetilde{P_r}(A)$	**cut of the fuzzy probability**$\propto -$
$\widetilde{R}(t)$	**Fuzzy reliability**
$F\widetilde{MTT}F$	**Fuzzy mean time to failure**
$.\widetilde{\mathbf{Q}}(t)$	**fuzzy failure probability**
$\widetilde{h}(t)$	**The fuzzy hazard function**
$(FN,F\tau)$	**fuzzy topological space**
FRN	**fuzzy reliability manifold**
$\widetilde{\mathcal{A}}$	**fuzzy atlas**

Abstract

In our work many definitions, theorems and applications are produced related to it . We deals with a family of reliability functions which depend on many parameters which forms a surface where every point in it represent a reliability function whose it's parameters are real numbers and proofing following :

The family N in a parametric model with one dimension $N = \{e^{-\lambda t} : \lambda \in I\}$ and the family $N = \{e^{-\lambda(t-\theta)} : t \in (0,\infty), (\lambda,\theta) \in I\}$ is a parametric model with two-dimension .Depending on the parametric model , we construct a reliability manifold with n-dimensions such that parameters λ play the role of the coordination system .Proofing that $N = \{e^{-\lambda(t-\theta)} : t \in (0,\infty), (\lambda,\theta) \in I\}$ is a smooth reliability manifold . the tangent space $T_{R(t,\lambda)}$ of N at the point $R(t,\lambda)$ is formed an n-dimensional vector space if N is sufficiently smooth. proofing that a tangent vector at a point R(t, λ) is a directional derivative operator along a curve that passes through R(t, λ) , such that $V_{R(t,\lambda)}(f) = \partial_i(f)$. we notice that the partial derivatives $\{\partial_i : i = 1,2,\dots,n\}$ represent tangential vectors on the smooth reliability manifold at the point R(t, λ).Proof that $\{\partial_i : i = 1,2,\dots,n\}$ represents a base for the tangential vector space $T_{R(t,\lambda)}\,(N)$, and hence the tangential vector is an n-dimensional .Defining tangent space T_λ in the coordinate system λ is an n-dimensional vector space spanned by directional derivative ∂_i such that $T_\lambda = \{V : V = V^i \partial_i\}$,where V^i are the components of the tangent vector V with respect to the basis $\{\partial_i\}$ is called a tangent space with the coordinate system $\lambda = (\lambda^1,\dots,\lambda^n) \in R^n$.Defining 1- representation of the tangent space such that $T_\lambda{}^{(1)} = \{V(t) : V(t) = V^i \partial_i \ell(R(t,\lambda))\}$.

Show that the 1- representation $V(t)$ of V doesn't change for any coordinate system . Moreover , if the basis of the tangent space T_λ is

known, then it is possible to construct a tangent space ${T_\lambda}^{(1)}$ from it .In addition , we have given some definitions about the dual vector space to $T_{R(t,\lambda)}$ or the cotangent vector space of N at $R(t,\lambda)$, or the covariant vector space , which is denoted by ${T^*}_{R(t,\lambda)}$, and proof that the dual vector space ${T^*}_{R(t,\lambda)}$ is a vector space over R .generalizing the vectors into tensors the tensor studied the differential and integral for objects in definite space by coordinate systems .For studying the Riemannian reliability manifold, we return the basic concepts as a Riemannian metric to determine g_{ij} which is called the Riemannian metric tensor to use it in the Riemannian reliability manifold $(N, g_{ij}(\lambda))$, and we find the Riemannian metric ds^2 of a two-dimensional reliability with an exponential lifetime distribution such that $e^{-\lambda(t-\theta)}$in case $t = 1$. Using the contrast function , which is a smooth function on the Riemannian reliability manifold , induced by the contrast function, $D(\lambda_1 \| \lambda_2)$ which is important to measure the degree of separation between two reliability functions but is not a distance function , it is used only in manifolds since it measures the degree of local separation on a manifold . The contrast function by a convex function f which valid under certain conditions to get the measure of distance between two reliability functions, which is called the f −Divergence function and take some of its kinds as the Kullback-Leibler divergence and the Hellinger divergence . finding a new f −Divergence function, we call it the A-N divergence function . Several applications are presented for any f −Divergence functions and proofing that The contrast function $D_{f^*}((q\|p)$ is the dual of $D_f(p\|q)$ if $f^*(u) = uf(\frac{1}{u})$. which is devoted to fuzzy reliability by utilizing triangular fuzzy exponential lifetime distributions with one and two parameters . we calculate the reliability of systems where the system depends on fuzzy exponential distributions with one and two parameters are calculated , and a fuzzy exponential distribution with three parameters

.We notice that increasing parameters are led to raising the value of fuzzy reliability, which tends to be 1 . Furthermore , new definitions of fuzzy reliability manifolds are presented at the end of our dissertation.

Chapter One Introduction

1.1 INTRODUCTION :

The science that tackles problems by using the concepts of differential geometry, statistics and information theory called information geometry [9,13] . in brief , is the study of geometrical structure of families of probability distribution that focuses on the study of probability distributions from geometrical point of view information geometry is the study of statistical manifolds [16] .

Differential geometry is a mature field of mathematics and has many introductory texts; still, it is not an easy field to master [13]. However, in this dissertation we shall require only the fundamental ideas and methodologies of differential geometry[14] . Differential geometry began as the study of the local properties of curves and surfaces in 3-dimensional space.

Fundamental concepts as curvature, torsion of curves and the curvature of surfaces are studied [30]. Gaus was a pioneer in differential geometry and in statistics but made no connection between these two topics . Jeffreys in (1939) was probably the first to set out statistical interpretation to the squared distance between two neighbouring points p and (p+ dp) given by (a quadratic form) and considered the distance between the distributions corresponding to it (i.e. quadratic form). Rao (1945, 1949) considered a parametric family of distributions and defined distance through the fisher information matrix, also he studied the resulting Riemannian space and in particular determined the geodesic distance between distributions in some special cases.

many authors developing the theory of regression. For defining statistical curvature of an arbitrary one parameter-model whether curved exponential or not , For more detail see, Amari [9] . Barndorff. Nielsen, (1986) used the concept that the coefficients of the expected Fisher information matrix is equal to the coefficients of the first fundamental form. Lauritzen (1987) defined a statistical manifold and nothing but a Riemannian manifold with asymmetric covariant tensor of order 3 . In (1999) Chen W.W.S. supplied a wide and deep understanding the mean of Gaussian curvature using some more general alternative computational procedures. Gruber in (2003)used a formula in order to computing the Gaussian curvature for Gamma family of distributions and normal distribution .

Ariwinik . , Del R.L. and Dodsor C.T.J (2004) produced a formula for universal connections an curvature on exponential families and they presented an example for the manifold of Gamma distributions . A manifold is a geometric object with more than two dimensions that can be thought of as a space that is similar to Euclidean space in some places [42] . Since differentiation is a locally defined property, it can be defined on a manifold in the same way that it is defined in Euclidean space. Several sets of parameters, which are called "local coordinate systems," can be used to describe a point on a manifold [31] .

A parametric model is a set of reliability functions; it is a subset of all the possible reliability functions [31]. In particular, a parametric model usually forms a finite-dimensional manifold embedded in the set of all the possible reliability functions, for example, the reliability of the exponential lifetime distribution of one, two, or three parameters.

The structure of an n-dimensional manifold, with serving as a coordinate system, can be easily introduced into a parametric model N when $R(t,\lambda)$ is sufficiently smooth in $\lambda = (\lambda^1, \dots, \lambda^n)$. Calculus on manifolds cannot be performed without a structure that is continuous and smooth . In light of this, let us first define the term "smooth manifold," which refers to an ordinary manifold that possesses an additional structure .

Moreover , we introduce tangent space, which is a real vector space that includes all the directions in which one can pass through the point p in a way that is tangential to it. Differential geometry and the tensor were developed by Gauss, Riemann, and Christoffel. The tensor calculus, also known as tensor analysis, was developed by Ricci and Levi-Civita as a development of vector analysis, or absolute differential calculus. Also, we define a Riemannian reliability manifold, which is a manifold on which one is able to measure distances between points; it is a manifold endowed with a metric structure**.**

f-Divergence function one of the most essential contrast function classes for measuring probability distribution distance. A divergence represents the degree of separation between two probability distributions (or reliability functions) .Amari and others studied in connection with statistical inference (1985),(1987) . Equchi (1992) proved that a contrast function gives all the data of the statistical manifold if it exists . Takan M.(1992) find a contrast function which induces a given statistical manifold . Furthermore , we present special case of f $-$Divergences function **.**

In the end we shall discuss fuzzy reliability in $R^k, k = 1, \ldots, n$ and fuzzy reliability manifold .It is well known that the traditional reliability analysis has been found to be in adequate to handle uncertainty of failure data. To treat this problem Onisawa and Kacprzyk (1995) used fuzzy set theory to evaluate the reliability of a system. From a long period of time, tasks have been made in design and development of reliable large-scale systems. Cai, Wen and Zhang (1993)(1995) presented two fundamental assumptions in the conventional reliability theory

1. Binary state assumption

2. Probability assumption

It is noted that the assumption of the binary for depict the system failure and success may be no longer an appropriate, the following two assumption are considered :

1-Fuzzy-state assumption

2-Possibility assumption

Singer (1990) preset a fuzzy approach to fuzzy triangular and reliability for series and parallel systems. Chen (1991) produced a method for fuzzy system reliability analysis utilizing fuzzy number arithmetic operations. Huang (1994) proposed the fuzzy event T is bigger than t (i.e. X >t) where X is a random variable for the failure time and " > " is a fuzzy sense . Viertl and Gurber (1995) considered the fuzzy lifetime data.

1.2 **Dissertation Outlines**

There are five chapters in this dissertation:

The first chapter is where you'll find an introduction, objectives, dissertation contributions, and related works .

The second chapter contains some definitions and basic concepts .

Third chapter whereas is devoted to discuss the Reliability Manifold ,

Riemannian Reliability Manifold , f-divergences functions and special case of f $-$divergence function .

Chapter four includes a study of fuzzy reliability in $R^k, k = 1, \dots., n$ and fuzzy reliability manifold . The final chapter consists of conclusion and future works .

1.3 Objectives of this Dissertation

This dissertation addresses the problems related to solve it by :

1- Studying a family of parameters dependent reliability functions , which it can be represented as a parameters surfaces .
2- Presenting some definitions of reliability manifold and tangent space with some important theorems about them .

3- Utilizing Hillinger distance between two different distributions , continuous and discrete distributions .
4- Producing a new divergence named A-N divergence.
5- Introducing an exponential distribution with three parameters (3-dimension)
6- Investigating the fuzzy reliability in R^k , $k = 1, \ldots, n$, for some systems by using triangular fuzzy lifetime distribution which assumed to be fuzzy parameter according to uncertainly .
7- Using exponential distribution to evaluate fuzzy reliability for k-out-of-n model which is consider an important model.
8- Making use of a triangular fuzzy exponential life time distribution with one and two parameters for k-out-of-n model system reliability evaluation.
9- producing a definition of fuzzy reliability manifold .

1.4 Related Works

Recent reviews that have been written on subjects relating to the statistical manifold have focused on, among other things, the following:

In the year 1999, Pistone, Giovanni, and Maria Piera Rogantin[34] conducted the research titled "The exponential statistical manifold: mean parameters, orthogonality, and space transformations."

In 2000 , Henmi, Masayuki, and Ryoichi Kobayashi.Hooke's law was published by [29] in the statistical manifolds and divergences journal .

In 2001, Matsuzoe and Hiroshi [49] did research on the geometry of both semi-Weyl manifolds and Weyl manifolds using their website.

In the year 2002, Kurose, Takashi [59] was the first person to present a conformal-projective geometry of statistical manifolds.

In 2003, Srivastava, Anuj, and a number of other researchers [66] reported their findings from a study on advances in statistical modeling of natural images

In the year 2004, the concepts of estimators, escort probabilities, and phi-exponential families were originally introduced to the study of statistical physics by Naudts, Jan. [68].

In the year 2005, John Lafferty, Guy Lebanon, and Tommi Jaakkola wrote [71] on the subject of Diffusion Kernels on Statistical Manifolds. This was a topic that was discussed.

In the year 2006, Furuhata, Hitoshi, and a few others [74] introduced the world to the concept of Sasakian statistical manifolds for the very first time.

In 2007 , Cena, Alberto, and Giovanni Pistone [78] presented Exponential statistical manifold .

2008 saw the publication of "Statistical analysis on Stiefel and Grassmann manifolds with applications in computer vision" written by Turaga, Pavan, Ashok Veeraraghavan, and Rama Chellappa [83]. Carlo Cafaro has also conducted research on information-geometric markers of chaos in Gaussian models on statistical manifolds with negative Ricci curvature. This research has been carried out by Carlo Cafaro. [84].

In 2009, Research on hypersurfaces in statistical manifolds was being done by Furuhata, Hitoshi [97] at the time.

In the year 2010, Matsuzoe, Hiroshi, and a number of other researchers [98] carried out research on statistical manifolds and affine differential geometry.

In the year 2011, Noda, Tomonori [94] presented the symplectic structures that were found on statistical manifolds.

In the year 2012, Newton, Nigel J.[96] discussed A statistical manifold with an infinite number of dimensions that is modeled on Hilbert space .

In 2013, Zhang, Jun. [93] conducted in-depth study on nonparametric information geometry, ranging from divergence functions to referential-representational biduality on statistical manifolds .

In 2014, research on domain adaptation on the statistical manifold was carried out by Baktashmotlagh, Mahsa, and others [90].

on 2015, Aydin, Muhittin Evren, Adela Mihai, and Ion Mihai established several inequalities on submanifolds on statistical manifolds with constant curvature using the [89] reference.

2016 saw the publication of The introduction of a new measure and linkages inside statistical manifolds by Vigelis, Rui F., David C. de Souza, and Charles C. Cavalcante [87].

In 2017, Notions of the ergodic hierarchy for curved statistical manifolds were discussed by Gomez, Ignacio S. [86].

In 2018, Decu, Simona, and colleagues [84] explored the curvature invariants of statistical submanifolds in Kenmotsu statistical manifolds of constant cross-sectional curvature.

In the year 2019, Chakraborty, Rudrasis, and Baba C. Vemuri [83] carried out research in reference to the subject of Statistics on the Stiefel manifold: theory and applications

Sylvain Calinon gave a presentation entitled "Gaussians on Riemannian manifolds: Applications for Robot Learning and Adaptive Control in the Year 2020" [82].

In the year 2021, the researchers Hua, Xiaoqiang, et al.[81] used Target identification inside nonhomogeneous clutter via total bregman divergence-based matrix information geometry detectors .

In 2022, Han, Rungang, Rebecca Willett, and Anru R. Zhang [77] searched An optimal statistical and computational framework for generalized tensor estimation.

In the year 2023, Zhang, Jingyi, et al. [76] did research on projection-based techniques with the purpose of resolving high-dimensional optimal transport challenges.

The most recent studies that have been examined on subjects that are linked with fuzzy reliability have concentrated on, the following topics:

In the year 2000, Dodagoudar, G. R., and G. Venkatachalam [95] published a paper on the reliability analysis of slopes utilizing fuzzy sets theory.

In the year 2001, Knezevic, Jezdimir, and E. R. Odoom [75] conducted research on the topic of modeling the reliability of repairable systems by using Petri nets and the fuzzy Lambda–Tau technique; the results are currently being analyzed.

2002 was the year that Savoia, Marco [73] conducted research on the topic of Structural reliability analysis by fuzzy number technique, with application to stability.

In 2003, Chen, Shyi-Ming looked at analyzing fuzzy system reliability using ambiguous set theory. [72] His research was published in the journal Information Sciences.

In the year 2004, Wu, Hsien-Chung. [70] conducted research on fuzzy reliability estimate utilizing a Bayesian technique .

In 2005, Zhao, Ruiqing, and Baoding Liu. [69] presented Standby redundancy optimization problems with undetermined lifetimes.

In 2006, Huang, Hong-Zhong, Ming J. Zuo, and Zhan-Quan Sun.[67] Bayesian reliability analysis has been established for ambiguous lifetime data.. Olgierd Hryniewicz provided some insightful commentary as well in our conversation.

In 2007, Hryniewicz and Olgierd.[65] carried out study on the application of fuzzy sets in the context of the reliability evaluation.

In 2008, Kumar, Amit, Shiv Prasad Yadav, and Surendra Kumar.[64] presented a study that investigated the dependability of fuzzy systems by using a variety in a number of types of hazy sets.

In 2009, Analysis of a k-out-of-n System with Uncertain Lifetimes was supplied by Gao, Yuan. [37] .

2010 was the year that Gholizadeh, Ramin, and his co-authors [62] carried out research on the dependability of fuzzy Bayesian systems that were based on the Pascal distribution.

In 2011 , Baloui, Jamkhanehezzatallah It was reported in [60]., written by Kumar, Mohit, and Shiv Prasad Yadav, that an investigation of the dependability of the system was based on a fuzzy life expectancy distribution.

In the year 2012, Wang, Zhonglai, and several other researchers [58] conducted research and analysis on a method for analyzing system reliability that makes use of fuzzy random variables.

In the year 2013, Eryilmaz, Serkan [57] carried out research on reliability analysis of a k-out-of-n system with components having random weights

In 2014, Jamkhaneh, E. Baloui. investigated methods for analyzing system dependability using fuzzy Weibull lifespan distribution [56] .

In 2015, Bohra, Khushal Singh, and S. B. Singh [54] explored the idea of Assessing the reliability of a fuzzy system. using intuitionistic fuzzy Rayleigh lifespan distribution.

In 2016, Garg, Harish.[19] proposed A novel method for evaluating the dependability of series-parallel systems that makes use of credibility theory and a variety of different intuitionistic fuzzy number types.

In the year 2017, Kumar, Jitender, and Meenu Goel.[53] carried out research on the subject of fuzzy reliability analysis of a pulping system in the paper sector using general distributions for all random variables.

In 2018, the subject of "Uncertainty analysis of transmission line end-of-life failure model for bulk electric system reliability studies, [52].

In the year 2019, Negi, Seema, and S. B. Singh [51] looked for fuzzy reliability evaluation of linear m-consecutive weighted-k-out-of-r-from-n: F systems .

In the year 2020, research on time-dependent intuitionistic fuzzy system reliability analysis was carried out by Akbari, Mohammad Ghasem, and Gholamreza Hesamian [48].

In the year 2021, Huang, YuLei, et al.[39] conducted a search for "Reliability evaluation of distribution network based on fuzzy spiking neural P system with self-synapse .

In the year 2022, Alharbi, Yasser S., and Amr R. Kamel. research was carried out on fuzzy system reliability analysis for Kumaraswamy distribution, including bayesian and non-bayesian estimation using simulation, and an application on cancer data. set. [38]

Chapter Two

Basic Definitions and Concepts

2.1 Introduction

To begin our study, we must first define and describe some of the mathematical terms and concepts we will encounter. The concept of a fuzzy number will be discussed, along with some related concepts. The concept of the reliability function, the failure rate function, and a few other related functions. We also discuss elementary reliability systems (models), with a focus on the k-out-of-n system model**.** In addition , we introduce some definitions and concepts that we need for the manifolds .[2,3,8]

2.2 Fuzzy Set Theory and Some Fundamental Concepts

Definition 2.2.1.[33]: An universal set (universe of discourse) X is a set consisting of all possible elements related with the given problem.

For example: All technical universities' in the world. .

Definition 2.2.2. [33,21]: Let X a nonempty set be the universal set (universe of discourse) , and let a function $\mu_A: X \to [0,1]$,for $x \in X$, called a **membership function** , its value which represent the degree of belonging x to a set X . Then **the fuzzy set** $\boldsymbol{A}$ is defined on X to be a set of ordered pairs of a member (element) in the universal set and the value of a membership function at that member(element) which is called a membership degree and denoted by $\mu_{\tilde{A}}(x)$, for a member $x \in X$. So $\tilde{A}$ could be written as ;

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)); x \in X\} \qquad \ldots(2.1)$$

that is , it is characterized by its member function .

The elements of a fuzzy set are ordered pairs in form $(x, \mu_{\tilde{A}}(x))$,first side is member included in the set$\tilde{A}$ while second part refer to degree of this inclusion (value between 0 and 1). A fuzzy set can have a finite or an infinite number of elements depending on the universal set , if it is a set of real , integers , or other situations and a membership function. These reflect on the support of a fuzzy set . It is clear that if one only allowed the extreme membership values of 0 and 1 , that is would actually be equivalent to crisp set . this is wide in discrete membership function . So fuzzy set can be expressed as :

1- **Discrete case**: when the universe of discourse X is finite set :

$X = \{x_1, x_2, \dots\dots, x_n\}$. A fuzzy set $\tilde{A}$ on X can be written as ;

$$\tilde{A} = \frac{\mu_{\tilde{A}}(x_1)}{x_1} + \frac{\mu_{\tilde{A}}(x_2)}{x_2} + \cdots . + \frac{\mu_{\tilde{A}}(x_n)}{x_n}$$

$$= \sum_{i=1}^{n} \frac{\mu_{\tilde{A}}(x_i)}{x_i} \qquad \dots(2.2)$$

2- **Continuous case** : When the universe of discourse X is infinite set :A fuzzy set $\tilde{A}$ on X can be written as ;

$$\tilde{A} = \int_{x} \frac{\mu_{\tilde{A}}(x_i)}{x_i} \qquad \dots..(2.3)$$

There are several basic notations related to fuzzy sets which will be needed in our discussion [21]

Suppose $\tilde{A}\ and\ \tilde{B}$ are fuzzy sets defined on X. Then we can define new fuzzy sets on X as follows:

1) Complement $\tilde{A}^c$: $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)$ ….(2.4)

2) Union $\tilde{A} \cup \tilde{B}$: $\mu_{\tilde{A}\cup\tilde{B}}(x) = \max\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right)$ ….(2.5)

3) Intersection $\tilde{A} \cap \tilde{B}$: $\mu_{\tilde{A}\cap\tilde{B}}(x) = \min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right)$ ….(2.6)

4) Product $\tilde{A} \cdot \tilde{B} = \mu_{\tilde{A}.\tilde{B}}(x) = \mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)$ ….(2.7)

5) Product of fuzzy with a number: $\mu_{a.\tilde{A}}(x) = a.\mu_{\tilde{A}}(x)$ ….(2.8)

where a be a crisp number.

6) Power of a fuzzy set: $\mu_{\tilde{A}^{\propto}}(x) = (\mu_{\tilde{A}}(x))^{\propto}$ ……(2.9)

where $\propto$ be a crisp number.

7) Difference: $\widetilde{A} - \tilde{B} = (\tilde{A} \cap \tilde{B}^c)$. ……(2.10)

8) Disjunctive sum of two fuzzy sets:

$$\tilde{A} \oplus \tilde{B} = (\tilde{A}^c \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{B}^c) \quad \text{.....(2.11)}$$

Definition 2.2.3. [22] :The empty fuzzy set of universal (nonempty) set X is defined as the fuzzy subset as the fuzzy subset $\emptyset$ of X such that $\mu(x) = 0$ for each $x \in X$, or sometimes denoted as $\emptyset(x)$.

Definition 2.2.4. [43]: A fuzzy set $\tilde{A}$ in a universe of discourse X , is said to be **normal fuzzy set** if $\exists\ x^{'} \in X$, and $\mu_{\tilde{A}}(x^{'}) = 1$, that is max $\mu_{\tilde{A}}(x) = 1$, for $x \in X$.

Definition 2.2.5.[2]: A fuzzy set $\tilde{A}$ is **subnormal** if it is not normal , that is $\nexists x^{'} \in X$ and and $\mu_{\tilde{A}}(x^{'}) = 1$, that is max $\mu_{\tilde{A}}(x) \neq 1$, for $x \in X$.

Definition 2.2.6.[19]: A fuzzy set $\tilde{A}$ is **convex** if and only if for

all $x_1, x_2 \in \tilde{A}\ \ and\ \ \lambda \in [0,1]$, we have

$$\mu_{\tilde{A}}\{\lambda x_1 + (1-\lambda)x_2\} \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\} \quad \text{.....(2.12)}$$

Definition 2.2.7.[21]: For any $\propto \in [0,1], an\ \propto -$**cut set** $of \tilde{A}$ denoted by $\tilde{A}_{\propto}$,is a classic set defined by :

$$\tilde{A}_{\propto} = \{x\ \in X: \mu_{\tilde{A}}(x) \geq \propto\}. \quad \text{.... (2.13)}$$

Definition 2.2.8.[2]: Let $\tilde{A}$ be a fuzzy set on universal set X , then **support of $\tilde{A}$** , denoted by $supp(\tilde{A})$ or $\tilde{A}_{sup}$,is the crisp set of all members of the universal set X , for which a members have non zero membership degree in $\tilde{A}$, and written as ;

$$supp(\tilde{A}) = \{x \in X: \mu_A(x) > 0\ \} \quad \text{......(2.14)}$$

Definition 2.2.9.[3]: A fuzzy set $\tilde{A}$, defined on real line R , with membership function $\mu_{\tilde{A}}: R \to [0,1]$ is referred to as a **fuzzy number** iff it is normalized, convex and continuous membership function $\mu_{\tilde{A}}(.)$ of bounded support finite set .

Definition 2.2.10.[8]: For all $\propto \in [0,1]$, **the cut set $\tilde{A}_{\propto}$ of fuzzy number** $\tilde{A}$ is a non-fuzzy set defined as $\tilde{A}_{\propto} = \{x \in X; \mu_{\tilde{A}}(x) \geq \alpha\}$, is a closed , bounded , interval for $0 \leq \propto \leq 1$. Hence we have

$$\tilde{A}_{\propto} = [a_i^{(\propto)}, a_r^{(\propto)}] \qquad \text{.....(2.15)}$$

where $a_i^{(\propto)}$will be will be a continuous, monotonically increasing function of $\propto$ in [0,1] (i.e. $\frac{da_i^{(\propto)}}{d\propto} > 0$) hold , while $a_r^{(\propto)}$ will be a continuous, monotonically decreasing function of $\propto$ in [0,1] (i.e. $\frac{da_r^{(\propto)}}{d\propto} < 0$) with

$a_i^{(\propto)} \leq a_r^{(\propto)}$, and $a_i^{(\propto)} = min\{x \in X; \mu_{\tilde{A}}(x) \geq \alpha\}$,

$$a_r^{(\propto)} = max\{x \in X; \mu_{\tilde{A}}(x) \geq \alpha\}.$$

Let $\tilde{A}_{\propto} = [a_i^{(\propto)}, a_r^{(\propto)}]$ and $\tilde{B}_{\propto} = [b_i^{(\propto)}, b_r^{(\propto)}]$, some common operations on fuzzy numbers as follows [5]

Addition: $(\tilde{A} + \tilde{B})_{\propto} = [a_i^{(\propto)} + b_i^{(\propto)}, a_r^{(\propto)} + b_r^{(\propto)}]$ ….(2.16)

Subtraction: $(\tilde{A} - \tilde{B})_{\propto} = [a_i^{(\propto)} - b_i^{(\propto)}, a_r^{(\propto)} - b_r^{(\propto)}]$ …..(2.17)

Multiplication:

$$(\tilde{A}.\tilde{B})_{\propto} = \begin{bmatrix} min(a_i^{(\propto)}\ b_i^{(\propto)}, a_r^{(\propto)} b_i^{(\propto)}, a_i^{(\propto)} b_r^{(\propto)}, a_r^{(\propto)} b_r^{(\propto)}) , \\ max(a_i^{(\propto)}\ b_i^{(\propto)}, a_r^{(\propto)} b_i^{(\propto)}, a_i^{(\propto)} b_r^{(\propto)}, a_r^{(\propto)} b_r^{(\propto)}) \end{bmatrix} \qquad \text{…(2.18)}$$

Division: $(\tilde{A}/\tilde{B})_{\propto} = \left[a_i^{(\propto)}, a_r^{(\propto)}\right] . [\frac{1}{b_r^{(\propto)}}, \frac{1}{b_i^{(\propto)}}]$,If $0 \notin \left[b_i^{(\propto)}, b_r^{(\propto)}\right]$ ... (2.19) .

We note that if $\tilde{A}$ and $\tilde{B}$ are fuzzy numbers, the so are $\tilde{A}$+$\tilde{B}$and $\tilde{A}.\tilde{B}$

Definition2.2.11.[19]: If the membership function of fuzzy number $\tilde{A}$ is determined by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & ; x \leq a_1 \\ \frac{(x-a_1)}{(a_2-a_1)} & ; a_1 \leq x \leq a_2 \\ \frac{(a_3-x)}{(a_3-a_2)} & ; a_2 \leq x \leq a_3 \\ 0 & ; x \leq a_3 \end{cases} \qquad \text{.....(2.20)}$$

For all $x, a_1, a_2, a_3 \in R$.Then $\tilde{A}$ is said to be a **triangular fuzzy number**, denoted $\tilde{A} = (a_1, a_2, a_3)$

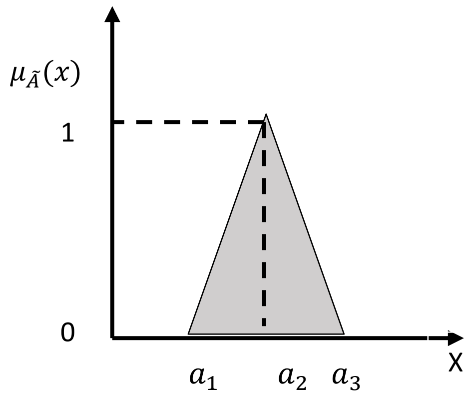


Figure 2.1: Triangular Fuzzy Number

Theorem 2.2.12 [43]: If $\tilde{A} = (a_1, a_2, a_3)$, then$\propto -$cuts

$$\tilde{A}_{\propto} = [a_1 + \propto (a_2 - a_1), a_3 - \propto (a_3 - a_2)]$$

Proof:-

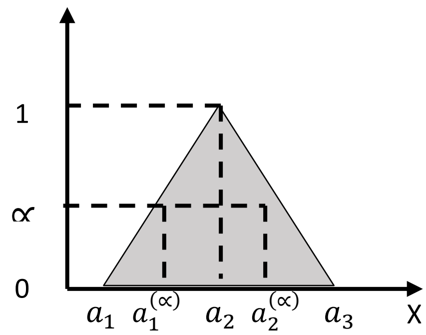


Figure 2.2: The Cut Set $\tilde{A}_{\propto}$ of Fuzzy Number $\tilde{A}$

Suppose $\tilde{A} = (a_1, a_2, a_3)$,and let $\propto \in [0,1]$

Then $\propto$ - cut of fuzzy set $\tilde{A}$ is $[a_1^{(\propto)}, a_2^{(\propto)}]$. Since

$$\mu_{\tilde{A}}(x) = \begin{Bmatrix} 0 & ; x < a_1 \\ \frac{(x-a_1)}{(a_2-a_1)} & ; a_1 \le x \le a_2 \\ \frac{(a_3-x)}{(a_3-a_2)} & ; a_2 \le x \le a_3 \\ 0 & ; x \ge a_3 \end{Bmatrix} \text{,and } a_1^{(\propto)} \in [a_1, a_2] \text{ ,}$$

then $\mu_{\tilde{A}}\left(a_1^{(\propto)}\right) = \propto = \frac{\left(a_1^{(\propto)} - a_1\right)}{(a_2-a_1)}$, and $a_1^{(\propto)} = a_1 + \propto (a_2 - a_1)$

Similarly since $a_3^{(\propto)} \in [a_2, a_3]$,then $\mu_{\tilde{A}}\left(a_3^{(\propto)}\right) = \propto = \frac{\left(a_3 - a_3^{(\propto)}\right)}{(a_3-a_2)}$,

and $a_3^{(\propto)} = a_3 - \propto (a_3 - a_2)$.

Hence $\tilde{A}_{\propto} = [a_1 + \propto (a_2 - a_1), a_3 - \propto (a_3 - a_2)]$

Theorem2.2.13[43]: Suppose $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$

Then $\tilde{A} + B$ *and* $\tilde{A} - \tilde{B}$ are both a triangular fuzzy number .Specifically

$$\tilde{A} + \tilde{B} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

and

$$\tilde{A} - \tilde{B} = [a_1 - b_1, a_2 - b_2, a_3 - b_3]$$

Proof:-

Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$

Since ,$\tilde{A}_{\propto} = [a_1 + \propto (a_2 - a_1), a_3 - \propto (a_3 - a_2)]$,and

$\tilde{B}_{\propto} = [b_1 + \propto (b_2 - b_1), b_3 - \propto (b_3 - b_2)]$, then

$$\tilde{A}_{\propto} + \tilde{B}_{\propto} = [(a_1 + b_1) + \propto \left((a_2 + b_2) - (a_1 + b_1)\right), (a_3 + b_3) - \propto ((a_3 + b_3) - (a_2 + b_2))]$$

Let $\tilde{C}_{\propto} = \tilde{A}_{\propto} + \tilde{B}_{\propto}$, If $\propto = 0$ then $\tilde{C}_0 = [a_1 + b_1, a_3 + b_3]$.

If $\propto = 1$, then $\tilde{C}_1 = [a_2 + b_2, a_2 + b_2]$

Therefore $\tilde{C}_{\propto} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$ .Similarly, $\tilde{A} - \tilde{B}$.

Definition 2.2.14.[8] If the membership function of fuzzy number $\tilde{A}$ is determined by

$$\mu_{\tilde{A}}(x) = \begin{Bmatrix} 0 & ; x \le a_1 \\ \frac{(x-a_1)}{(a_2-a_1)} & ; a_1 \le x \le a_2 \\ 1 & ; a_2 \le x \le a_3 \\ \frac{(a_4-x)}{(a_4-a_3)} & ; a_3 \le x \le a_4 \\ 0 & ; a_4 \le x \end{Bmatrix} \quad \text{....(2.21)}$$

Then $\tilde{A}$ is referred to as a **trapezoidal fuzzy number**, denoted $\tilde{A}$=(a_1, a_2, a_3, a_4).

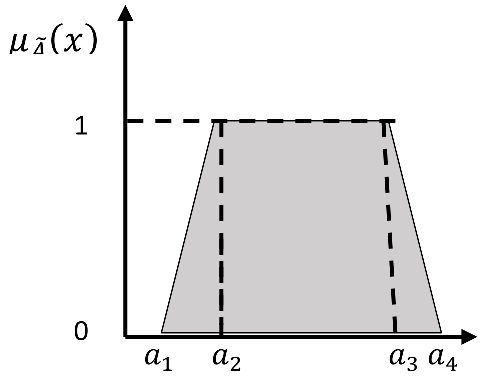


Figure 2.3: Trapezoidal Fuzzy Number

Suppose $\tilde{A}$=(a_1, a_2, a_3, a_4) and $\tilde{B}$=(b_1, b_2, b_3, b_4) .

Then $\tilde{A} + \tilde{B}$ and $\tilde{A} - \tilde{B}$ are both a trapezoidal fuzzy number.

Specifically, $\tilde{A} + \tilde{B} = [a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4]$

And,$\tilde{A} - \tilde{B} = [a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4]$.

Definition 2.2.15 [19]:The core of a fuzzy number is the set of values where the membership value equals one . If $\tilde{A} = (a_1, a_2, a_3)$, then the core of A is the single point a_2 . However ,if $\tilde{A}$=(a_1, a_2, a_3, a_4) ,then the core of $\tilde{A} = [a_2, a_3]$.

2.3 Mathematical Background of Probability

The set of random events that are used to figure out probabilities is only a part of the whole set of events for which probabilities can be worked out [4,7 39,62]. This part will not be expanded upon further. because we'll be introducing some of the important terms and ideas that we'll be using throughout our research.

Definition 2.3.1. [40] : Let P_r be a function that assigns a nonnegative real number to each event E of a sample space S i.e. $P_r: S \longrightarrow [0,1]$ We call P_r a **probability** if ,

Axiom 1: Non-negative

$$0 \leq P_r(E) \leq 1$$

Axiom 2: Total one

$$P_r(S) = 1 \text{ and } P_r(\emptyset) = 0$$

Axiom 3: : $P_r(E^c) = 1 - P_r(E)$, here E^c represent to complement of E

Axiom 4: For any pair of event $E_1, E_2 \subseteq S$,we have:

$$P_r(E_1 \cup E_2) = P_r(E_1) + P_r(E_2) - P_r(E_1 \cap E_2).$$

A probability distribution is a description of how likely a random variable or set of random variable is to take on each of its possible states. It is described by:[4]

- A probability mass function (p.m.f.)in the case of discrete variables.
- A probability density function (p.d.f.)in the case of continuous variables

Definition 2.3.2.[40]: The cumulative distribute function of a continuous random variable X is

$$F(x) = P_r(X \leq x) = \int_{-\infty}^{x} f(u)\ du \qquad \text{…..(2.22)}$$

And $$P_r(a < x < b) = F(b) - F(a)$$

Then the relationship can be deduced between pdf and cdf as follows:

$f(x) = \frac{dF(x)}{dx}$ if the derivative exists .

Definition 2.3.3. [62] : Two events E_1 and E_2 are **mutually exclusive or disjoint** if $E_1 \cap E_2 = \emptyset$ that is, if A and E_2 have no elements in common.

So , for any two disjoint E_1 and E_2 we have ;

$$P_r(E_1 \cup E_2) = P_r(E_1)+P_r(E_2). \qquad ….(2.23)$$

Definition2.3.4.[4,62]: The probability of an event E_2 under the knowledge that the outcome will be in event E_1 is denoted as P(E_2/E_1), and this is called the **conditional probability** of E_2 given E_1 . So, P(E_2/E_1) is given by

$$P_r(E_2/E_1) = \frac{P_r(E_1 \cap E_2)}{P_r(E_1)} \; , for \; P(E_1) > 0 \qquad ….(2.24)$$

In case The two event E_1 and E_2 are equally likely outcomes then

$$P_r(E_2/E_1) = \frac{n(E_1 \cap E_2)}{n(E_1)}$$

The conditional probability of E_2 given E_1 equals to zero if two events are mutually exclusive.

Definition2.3.5.[7,40] :Two event E_1 and E_2 are **independent** if any one of the following equivalent statements is true:

1) $P_r(E_2/E_1) = P_r(E_2)$
2) $P_r(E_1/E_2) = P_r(E_1)$
3) $P_r(E_1 \cap E_2) = P_r(E_1)P_r(E_2)$.

Definition 2.3.6.[62]: The expectation or expected value of a function $f(x)$ with respect to a probability distribution $p(x)$ is the average of $f(x)$ for $X{\sim}p(X)$. For continuous variable , it is computed with an integral :

$$E_{X\sim p}[f(x)] = \int_{-\infty}^{\infty} p(x)f(x) \; dx \qquad …..(2.25)$$

Definition 2.3.7. [3,10]:The Binomial distribution is one of the most important discrete probability distributions. The probability of exactly x success in n trials is

$$P_r(x) = C_x^n \, p^x \, q^{n-x} = \frac{n!}{(n-x)!x!} \, p^x \, q^{n-x}$$

Or:

$$b(x,n,p) = \binom{n}{x} \, p^x (1-p)^{n-x}. \qquad \text{….(2.26)}$$

The cumulate probability of Binomial distribution is given by:

$$B(x;n,p) = \sum_{k=0}^{x} b(k;n,F), for\ x = 0,1,2, \dots, n$$

Or:

$$B(x;n,p) = \sum_{k=0}^{x} \binom{n}{k} \, p^k, (1-p)^{n-k}$$

The mean or expected value of the Binomial random variable is given by:

$$\mu = E(x) = n\,P$$

And its variance is $\sigma^2 = n\,p\,(1-p)$

2.4 Mathematical Background of Reliability

In this section, the most important parts of reliability have been broken down, including some of its definition [32, 92].A block diagram of reliability that shows its parts, what they do , and the many different reliability systems.

Definition 2.4.1.[91]: **A component** is a chunk of equipment of system, that it is evaluated as a separated existence, that's mean the reliability of any component does not effect by another one .

Definition 2.4.2.[41]: A system is a configuration of components that react with each other, foreign components of another systems, and operators to implement of any intentional function

Definition 2.4.3.[25] The reliability of a component (or a system) at time t denoted by R(t) is defined as:

$$R(t) = P(T > t) = 1 - P(T \leq t)$$
$$= 1 - F(t) \qquad \ldots(2.27)$$

Therefore, we can write the reliability function in terms of p.d.f as follows:

$$R(t) = 1 - \int_0^t f(x)\,dx = \int_t^\infty f(x)\,dx \qquad \ldots(2.28)$$

Conversely, we can write the pdf in terms of R(t) as follows:

$$f(t) = -\frac{d(R(t))}{dt} \qquad \ldots.(2.29)$$

Where F (t) = $P(T < t) = \int_0^t f(x)\,dx$ is the failure distribution function or Failure Function.

The reliability R(t) at time t , has the following properties:

1- $R(t) \in [0,1]$

2- Since $F(0) = 0, F(\infty) = 1,$ therefore

$R(0) = 1 \text{ and } R(\infty) = 0 \text{ this implies that } 0 \leq R(t) \leq 1$

3- R(t) is a decreasing function of time t.

The probability of failure of a given system in a particular time interval $[t_1, t_2]$ can be written in terms of the reliability function as:

$$\int_{t_1}^{t_2} f(x)\;\; dx = \int_{t_1}^{\infty} f(x)\;\; dx - \int_{t_2}^{\infty} f(x)\;\; dx$$
$$= R(t_1) - R(t_2)$$

Using the exponential distribution, the pdf can be written in the form:

$$f(t) = \lambda e^{-\lambda t} \qquad \ldots..(2.30)$$

here λ is a parameter of the exponential distribution.

Therefore, the reliability function of the exponential distribution can be derived based on equation (2.28) as follows:

$$R(t) = 1 - \int_0^t \lambda e^{-\lambda x} dx$$

So, the reliability function becomes as follows:

$$R(t) = e^{-\lambda t} \quad \text{….(2.31)}$$

here λ is a failure rate.

Definition 2.4.4.[92]:The expected life or (expectation concept of probability theory), that is the expected time during which a component will survive and perform successfully, can be expressed as :

$$E(T) = \int_0^\infty t\, f(t)\, dt$$

$$= -\int_0^\infty t \frac{d\,R(t)}{dt}\, dt$$

$$= -t\, R(t)|_0^\infty + \int_0^\infty R(t)\, dt$$

Since $R(t = 0) = 1$ *and* $R(t = \infty) = 0$, therefore the expected life can be expressed as: $E(T) = \int_0^\infty R(t)\, dt$, when there is a constant failure rate,

$$E(T) = \int_0^\infty e^{-\lambda t}\, dt = \frac{1}{\lambda}. \quad \text{….(2.32)}$$

If the system is simply replaced by a good system (i.e no maintenance required or non-repairable), the E(T) useful life is also defined as the mean time to failure which is denoted by (MTTF) and represented by $MTTF = E(T) = \frac{1}{\lambda}$

In general, $E(T) = MTTF = \int_0^t t\, f(t)\ dt$

$$= \int_0^\infty R(t)\ dt \quad \text{….(2.33)}$$

2.5 Simple Reliability Systems (Models)

Let's say we need to figure out how reliable a system is that is made up of a lot of different parts. One way to figure out how reliable a system is is to figure out how reliable each individual part is and then combine those individual reliabilities in a way that takes into account how the parts are connected. To explain, this refers to whether they are connected in a series, parallel, series-parallel, or parallel connection.

Definition 2.5.1[18]: A Reliability Block Diagram is often used to depict the relationship between the functioning of a system and the functioning of its components. In a reliability block diagram ,a rectangle or a circle is often used to represent a component. A reliability block diagram does not necessarily represent how the components are physically connected in the system. It only indicates how the functioning of the components will ensure the functioning of the system. That is why a reliability block diagram represents the logic relationship between the functioning of the system and the functioning of its components . Reliability block diagram have been used to represent series structures, parallel structures, series-parallel structures, and parallel- series structures. The diagrams of these structures will be given when they are introduced. However, not all systems can be represented by reliability block diagram. For example, the k-out-f-m system cannot be represented by a reliability block diagram without duplicating components. In discussions of system structures, we often use "n" to indicate the number of components in the system and each components is given a unique label from set {1,2,…,n}. The set of components in a system is denoted by C.

Definition 2.5.2. [12]: In **series system** all components must be connected serially in order to make the system to perform flawlessly. Series system fails if any one of its component fails. Therefore, any weak (unreliable) component leads to the complete breakdown of the whole system . The block diagram of series system is given below in figure 2.3

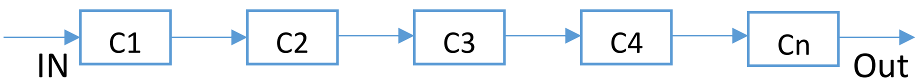


Figure 2.4: Block Diagram for Components in series

Suppose a system consists of total "n" components connected in series model. Further suppose that the ith component failure time follows the p.d.f $f(t,\lambda_i), i = 1,2,\dots,n$. The distribution of the entire system is defined by $F(t) = p(T < t)$ where T denotes the failure time of the system with $T = \min(T_1, T_2, T_3, \dots, T_n)$ (i.e) the lifetime of a series system is equal to the smallest lifetime among the lifetimes of all components, where T_i represents the failure time of the ith component. Then the cdf corresponding to the system lifetime is :

$$F_s(t) = P[T \le t] = P[\min(T_1, T_2, T_3, \dots, T_n) \le t]$$
$$= 1 - P[\min(T_1, T_2, T_3, \dots, T_n) > t]$$
$$= 1 - P[T_1 > t,\ T_2 > t\ , \dots, T_n > t]$$
$$= 1 - \prod_{i=1}^{n}(1 - P[T_i \le t])$$
$$= 1 - \prod_{i=1}^{n}(1 - F_{T_i}(t))$$

on assuming the series system with" n" i.i.d. components, then

$$F_s(t) = 1 - \{1 - F(t, \lambda)\}^n$$

Similarly the system reliability

$$R_s(t) = P[T_s > t] = P[\min(T_1, T_2, T_3, \dots, T_n) > t]$$
$$= P[T_1 > t,\ T_2 > t\ , \dots, T_n > t]$$
$$= \prod_{i=1}^{n} P[T_i > t] = \prod_{i=1}^{n} R_i(t)$$

on assuming the series system with "n" i.i.d. components then we have

$$R_s(t) = \{R(t)\}^n \qquad \text{….(2.34)}$$

Definition 2.5.3. [45]:A system is referred to as **parallel system** if its components are connected in such logic that system fails when all components have failed. Thus only one of the components is enough to operate the system satisfactorily, the block diagram is given in figure 2.4.

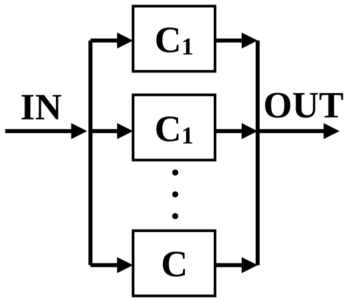


Figure 2.5: Block Diagram for Components in Parallel

Suppose, a parallel system consists of total "n" components and failure time of the ith component follows the pdf $f(t, \lambda_i), i = 1,2,3, \dots, n$. The distribution of the entire system is defined by $F(t) = P[T \leq t]$, where T denotes the failure time of the system with $T = \max(T_1, T_2, \dots, T_n)$, while T_i represents the failure time of the ith component. Then the C.D.F. corresponding to the system lifetime is,

$$F_s(t) = P[T_s \leq t] = P[\max(T_1, T_2, \dots, T_n) \leq t]$$

$$= P[T_1 \leq t\,, T_2 \leq t\,, \dots, T_n \leq t]$$

$$= \prod_{i=1}^{n} F_{T_i}(t)$$

On assuming the system with "n" i.i.d. components then,

$$F_s(t) = \{F_T(t)\}^n$$

Similarly, the system reliability

$$R_s(t) = P[T_s > t] = P[\max(T_1, T_2, \dots, T_n) > t]$$

$$= 1 - P[\max(T_1, T_2, \dots, T_n) \leq t]$$

$$= 1 - P[T_1 \leq t\,, T_2 \leq t\,, \dots, T_n \leq t]$$

$$= 1 - \prod_{i=1}^{n} P[T_i \leq t]$$

$$= 1 - \prod_{i=1}^{n}(1 - R_i(t))$$

On assuming the system with "n" i.i.d. components then we have

$$R_s(t) = 1 - \{1 - R(t)\}^n \qquad \text{….(2.35)}$$

Definition 2.5.4. [15]: Mixed Compositions

When designing a system, it is sometimes necessary to use a mix of series and parallel topologies to meet functional or reliability requirements called **Mixed Compositions** . The combinations produce configurations known as series–parallel and parallel–series. The reliability of these two types of systems is evaluated as follows :

Definition 2.5.5.[19] : A system in which "m" subsystem are connected in parallel where each subsystem has "n" components connected in series is said to be in **series-parallel configuration**

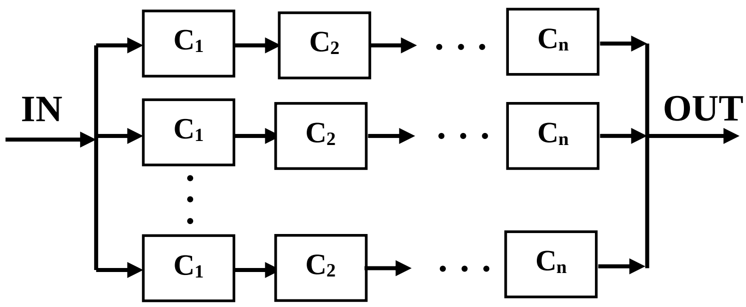


Figure 2.6: Block Diagram for Components in Series–Parallel

If R is the reliability of the individual component then the reliability of each of the subsystem equal to$(R(t))^n$.

Therefore the reliability of the whole system is

$$R_s = 1 - (1 - (R(t))^n)^m \qquad \text{…..(2.36)}$$

Definition 2.5.6 [26] : A system in which "m" subsystem are connected in series where each subsystem has "n" components connected in parallel is said to be in **parallel-series** configuration If R is the reliability of the individual component. The reliability of each of the subsystem with i.i.d. components equal to

$1-(1-R)^n$. Therefore the reliability of the whole system is

$$R_s = \{1-(1-R)^n\}^m \qquad \text{….(2.37)}$$

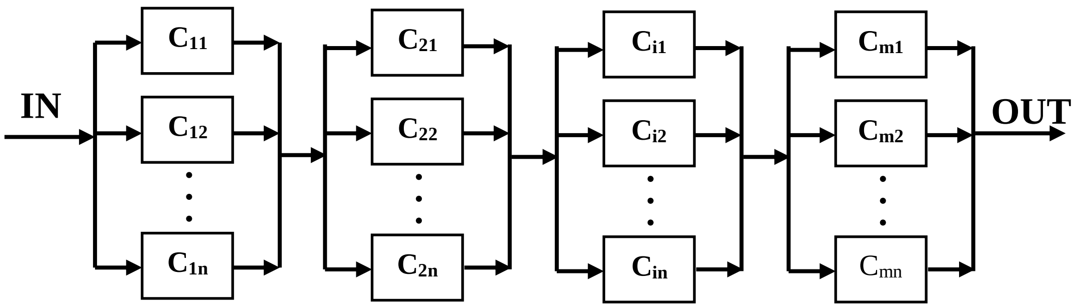


Figure 2.7: Block Diagram for Components in Parallel- Series

2.5.7 The k-out-of-n system Model

The k-out-of-n system structure is a very popular type of redundancy in Fault-tolerant system. It finds wide applications in both industrial and military systems[57]. . Fault-tolerant systems include the multi-display system in cockpit, the multi-engine system in an airplane, and the multi-pump system in hydraulic control system. For example, it may be possible to drive a car with a $V8$ engine if only for cylinders are firing. However if less than four cylinders fire, then the automobile cannot be driven. Thus, the functioning of the engine may be represented by a 4-out-of-8:G-system. The system is tolerant of failures of up to four cylinders for minimal functioning of the engine.

In this system we provide a kinds of the K-out-of-n system and reliability evaluation of the K-out-of-n systems. Essentially, there are two distinct types of this system :

1) An n- component system that works (or is good) if and only if at least k of n components work (or are good) is called a k-out-of-n: G system

2)An n-component system that fails if and only if at least k of n component fail is called a k-out-of-n: F system .

Based on these two kinds, we can note that a k-out-of-n : G system is equivalent to an(n-k+1)- out-of-n : F system, and the term k-out-of-n system is often used to indicate either a G system, an F system, or both. Further, both parallel and series systems are special cases of the k-out-of-n system. A series system is equivalent to a 1-out-of-n:F system and to an n-out-of-n:G system, while a parallel system is equivalent to an n-out-of-n:F system and 1-out-of-n:G system.

Definition 2.5.8.[37]: In **a k-out-of-n: G system** with i.i.d. components, the number of working components follows the binomial distribution with parameter n and p. Thus, we have

$$P_r[exactly\ i \text{ components work}] = \binom{n}{i}\ p^i q^{n-i}$$

$$i = 0,1,\dots,n\ and\ q = 1-p$$

When the components in a k-out-of-n: G system are i.i.d., then the reliability function of the system is equal to the probability that the number of working components is greater than or equal to k:

$$R_G(t;k,n) = \sum_{i=k}^{n} \binom{n}{i}\ (R(t))^i (F(t))^{n-i} \qquad \text{.... (2.38)}$$

The cdf of the system lifetime is given by:

$$F_s(t) = 1 - R_s(t) = \sum_{i=0}^{k-1} \binom{n}{i}\ (R(t))^i (F(t))^{n-i} \qquad \text{...(2.39)}$$

The pdf of the system lifetime is then

$$f_s(t) = \frac{d\,F_s(t)}{dt} = k\binom{n}{k}\ f(t)\ (F(t))^{n-k} (R(t))^{k-1} \qquad \text{...(2.40)}$$

The expected lifetime of the system, or meantime to failure, can be evaluated using the standard equation .

$$E(T) = MTTF_s = \int_0^\infty t \; f_s(t) \, dt = \int_0^\infty \; R_s(t) \, dt$$

For example, when the component follow the exponential distribution, then

$$R_s(t) = \sum_{i=k}^{n} \binom{n}{i} \; (e^{-\lambda t})^i (1 - e^{-\lambda t})^{n-i}$$

$$F_s(t) = \sum_{i=0}^{k=1} \binom{n}{i} \; (e^{-\lambda t})^i (1 - e^{-\lambda t})^{n-i}$$

If $k = 1$, then $E(T)$of a parallel system is $(\frac{1}{\lambda}) \sum_{j=1}^{n} (\frac{1}{j})$.

If $k = n$, then $E(T)$ of a series system is $\frac{1}{n\lambda}$

Example 2.1. : Consider a system having three identical coolants loops, each with two identical pumps connected in parallel. The coolant system requires that at least two out of three loops function successfully . The reliability of pump over the life of the plant is $R = 0.6$, then the reliability of the system is

$$R_s(t) = \sum_{i=2}^{3} \binom{3}{2} \; (R_L)^i (1 - R_L)^{n-i}$$

where

$$R_L = [1 - \left(1 - R_{pump1}\right)\left(1 - R_{pump2}\right)]$$

$$= R_{pump1} + R_{pump2} - R_{pump1} R_{pump2}$$

$$= 0.6 + 0.6 - (0.6)(0.6)$$

$$= 0.84$$

Then $R_s(t) = \binom{3}{2} \; (0.84)^2 (1 - 0.84) + \binom{3}{3} \; (0.84)^3 (1 - 0.84)^0$

$$= 3(0.84)^2 (0.16) + (0.84)^3$$

$= 0.931$

Definition 2.5.9.[51] :The reliability of a **k-out-of-n: F system** with independently and identically distributed (i.i.d.) components is equal to the probability that the number of failing components is less than or equal to k-1.

$$R_F(t;k,n) = \sum_{i=0}^{k-1} \binom{n}{i} (R(t))^{n-1} \big(F(t)\big)^{i} \qquad \ldots(2.41)$$

As a k-out-of-n: F system is equivalent to a (n-k+1)-out-of-n:G system, equation (2.55) is equivalent to

$$\sum_{j=n-k+1}^{n} \binom{n}{j} (R(t))^{j} \big(F(t)\big)^{n-j}$$

Then we have

$$R_G(t,k,n) = R_F(t,n-k+1,n)$$

2.6 Mathematical Background of Fuzzy Reliability

We investigates the fuzzy reliability of some system using triangular fuzzy lifetime distribution, in which the lifetime parameters are assumed to be fuzzy parameter due to uncertainty and inaccuracy of data.

Expressions for fuzzy reliability , fuzzy mean time to failure , fuzzy hazard function and their $\propto -cut$ have been discussed when systems follow triangular fuzzy lifetime distribution. We take , exponential and Binomial distribution**[1,3,4,6,10,17]**.

Furthermore, the fuzzy functions of series systems, parallel systems, series- parallel systems, parallel- series systems and k-out-of n systems are discussed respectively. Finally, some numerical examples are presented to illustrate how to calculate the fuzzy reliability characteristics and. their $\propto -cut$ set.

Definition 2.6.1.[5]: Fuzzy probability is a triangular fuzzy number $A = (a, b, c)$,where a is the mean of the a_i , b is the mean of b_i and c is the mean of c_i .

Definition 2.6.2.[5]: Let $A = \{x_1, x_2, \dots, x_k\}$ be (crisp) subset of $X = \{x_1, x_2, \dots, x_n\}$, and let P_r be a probability function defined on all subset of X with $P_r(\{x_i\}) = a_i$, $1 \le i \le n$, $0 \le a_i \le 1$ and $\sum_{i=1}^{n} a_i = 1$. We will substitute a fuzzy number $\tilde{a_i}$ for a_i to obtain a discrete fuzzy probability distribution . Then an $\propto -$**cut of the fuzzy probability** is define as :

$$\widetilde{P_r}(A)[\propto] = \left\{\sum_{i=1}^{k} a_i \,|a_i \in \tilde{a_i}[\propto], 1 \le \mathrm{i} \le \mathrm{n}\right\} \qquad \dots(2.42)$$

Definition 2.6.3.[5]: Let $A = \{x_1, x_2, \dots, x_k\}$, $B = \{x_l, x_2, \dots, x_m\}$ for $1 \le l \le k \le m \le n$ so that A and B are not disjoint .Then **the fuzzy conditional probability** is given by :

$$\widetilde{P_r}(A/B)[\propto] = \left\{\frac{\sum_{i=l}^{k} a_i}{\sum_{i=l}^{m} a_i} \mid a_i \in \tilde{a_i}[\propto], 1 \le i \le n\right\} \qquad \dots\dots(2.43)$$

The basic properties of fuzzy conditional probability which are :

1- $0 \le \widetilde{P_r}(A/B) \le 1$

2- $\widetilde{P_r}(B/B) = 1$

3- $\widetilde{P_r}(A_1 \cup A_2/B) \le \widetilde{P_r}(A_1/B) + \widetilde{P_r}(A_2/B)$, if $A_1 \cap A_2 = \emptyset$

4- $\widetilde{P_r}(A/B) = 1$, if $B \subseteq A$

5- $\widetilde{P_r}(A/B) = 0$, if $B \cap A = \emptyset$.

Definition 2.6.4.[5]: Two events A and B are **strongly independent** if

$$\widetilde{P_r}(A/B) = \widetilde{P_r}(A) \qquad \dots(2.44)$$

and

$$\widetilde{P_r}(B/A) = \widetilde{P_r}(B) \qquad \dots(2.45)$$

A and B are **weakly independent** if the membership values are one , then we have

$$\widetilde{P_r}(A/B)[1] = \widetilde{P_r}(A)[1] \qquad \dots\dots(2.46)$$

and

$$\widetilde{P_r}(B/A)[1] = \widetilde{P_r}(B)[1] \qquad .....(2.47)$$

Clearly, if they are strongly independent they are weakly independent .

Definition 2.6.5. [3,10]: Fuzzy reliability or fuzzy survival function $\tilde{R}(t)$ is the fuzzy probability in which a unit survives beyond time t.

Evidently, $\tilde{R}$ is a function of $\tilde{A}_i$. If the sign A denotes that a device performs its purpose adequately and the sign A_i denotes one performance subsets , then in terms of the definition of fuzzy conditional probability, we have the following:

$$\widetilde{P_r}\left(\mathrm{A} \wedge \tilde{A}_i\right) = P_r(A)\, P_r(\,\tilde{\mathrm{A}}_{\mathrm{i}}/\,A) \qquad ...(2.48)$$

where $\tilde{A}_i$ denotes discussing one of fuzzy performance subsets and the sing Λ denotes algebraic product .

According to the definitions of classical reliability R and fuzzy reliability $\tilde{R}$, we know

$$P_r(A) = R \qquad \widetilde{P_r}\left(A \wedge \tilde{A}_i\right) = \tilde{R} \qquad .....(2.50)$$

Substituting Eq. (2.50) in to Eq. (2.48) , we obtain

$$\tilde{R} = P_r\left(\,\tilde{A}_i/\,A\right)R \qquad ...(2.51)$$

Suppose that $\mu_{\tilde{A}_i}(R)$ is the degree of membership of R in $\tilde{A}_i$ and substitute $\mu_{\tilde{A}_i}(R)$ for $P_r\left(\,\tilde{A}_i/\,A\right)$, then

$$\tilde{R} = \mu_{\tilde{A}_i}(R)R \qquad ....(2.52)$$

Which represents the relation between classical reliability R and fuzzy reliability $\tilde{R}$.

Let the random variable X denote lifetime of a system components, and it has density function and cumulative distribution function f(x,θ) and $F_x(t,\theta) = P(x \le t)$ respectively, then the fuzzy reliability function at time t is defined as

$$\tilde{R}(t,\theta) = \tilde{P}(X > t) = 1 - F_X(t,\theta)\,, t > 0 \qquad ...(\,2.53)$$

The unreliability function $\tilde{Q}(t,\theta)$ is the probability of failure or the probability of an item failing in the time interval [0,t], is given by

$$\tilde{Q} = \tilde{P}(X \le t) = F_X(t,\theta)\,, t > 0 \qquad \ldots(2.54)$$

Definition 2.6.6.[10]: The fuzzy hazard function (fuzzy failure rate) $\tilde{h}(t)$ is the fuzzy conditional probability of an item failing in the interval t to $(t + \Delta t)$ given that is has not failed by time t. Mathematically, the fuzzy hazard function is defined as :

$$\tilde{h}(t)[\propto] = \lim_{\Delta t \to 0} \frac{\tilde{P}(t < x < t + \Delta t / X > t)}{\Delta t}$$

$$= \left\{\frac{f(t,\theta)}{R(t,\theta)} | \theta \in \tilde{\theta}[\propto]\right\} \qquad \ldots(2.55)$$

Definition 2.6.7.[3]: Fuzzy mean time to failure ($F\widetilde{MTTF}$) is the expected time to failure. It is defined as :

$$\mathrm{F}\widetilde{\mathrm{MTTF}}[\propto] = \int_0^\infty x\ f(x,\theta)\ dx\ \mid \theta \in \tilde{\theta}[\propto]$$

$$= \int_0^\infty R(t,\theta)\ dt\ \mid \theta \in \tilde{\theta}[\propto]$$

$$= [P^L[\propto], P^U[\propto]] \qquad \ldots(2.56)$$

where , $P_L[\propto] = \min\{\int_0^\infty R(t,\theta)\ dt\ \mid \theta \in \tilde{\theta}[\propto]\}$,

$$P_U[\propto] = \max\{\int_0^\infty R(t,\theta)\ dt\ \mid \theta \in \tilde{\theta}[\propto]\} .$$

Definition 2.6.8. [6] : Let (m) independent Bernoulli experiment, suppose $p = B(r)$ is the probability of (r) successes in the (m) experiments . So that p value is uncertain and we substitute $\tilde{p}$ for p and $\tilde{q}$ for q so that there is a $p \in \tilde{p}[1]$ *and* $q \in \tilde{q}\,[1]$ with $p + q = 1$. $\tilde{q}$ could equal $1 - \tilde{p}$. Now, let $\tilde{B}(r)$ be the fuzzy probability of (r) successes in (m) independent trials of the experiment. Under our restricted fuzzy arithmetic we obtain .

$$\tilde{b}(r)[\propto] = \{\binom{m}{r} p^r\, q^{m-r} / \mathrm{S}_\propto\} \qquad \ldots..(2.57)$$

for $0 \leq \propto \leq 1$ $, where\ S_{\propto} = \{(p,q): p \in \tilde{p}[\propto], q \in \tilde{q}[\propto], p + q = 1$

is **Fuzzy Binomial distribution** function . If $\tilde{b}(r)[\propto] = \left[b_L[\propto], b_U[\propto]\right]$, then

$$b_L[\propto] = \min\{\binom{m}{r} p^r q^{m-r} / S_{\propto}\}$$

and

$$b_U[\propto] = \max\{\binom{m}{r} p^r q^{m-r} / S_{\propto}\}$$

Example 2.2.

Let $p = 0.4, q = 0.6\ and\ m = 3$.Since $p\ and\ q$ are uncertain we use

$\tilde{p} = (0.3,0.4,0.5)$ for p such that 0.3 is the smallest possible value of p , 0.5 is the highest possible value of p , and 0.4 is the most likely value of p .

Similarly, we use $\widetilde{q} = (0.5,0.6,0.7)$ for q . Now, we will calculate the fuzzy number $\tilde{b}(2)$. If $p \in \tilde{p}[\propto] then\ q = 1 - p \in \tilde{q}[\propto]$, Since $\tilde{b}(r)[\propto] = \left[B_L[\propto], B_U[\propto]\right], then\ B_L[\propto] = \min\{\binom{3}{2} p^2\ q\ / S\ \} = \min\{3\, p^2\ q / S\}$.

And $b_U[\propto] = \max\{\binom{3}{2} p^2 q / S\} = \max\{3\, p^2 q / S\}$

So we obtain , $\tilde{b}(2)[\propto] = [3(b_L[\propto])^2 q_L, 3(b_U[\propto])^2 q_U]$, where

$\tilde{b}[\propto] = \left[b_L[\propto], b_U[\propto]\right] = [a_1 + (a_2 - a_1) \propto, a_3 - (a_3 - a_2) \propto]$

Definition 2.6.9.[6] : If $\tilde{b}[a,b]$ be the fuzzy probability of x successes so that $a \leq x \leq b$, then **The fuzzy Binomial reliability function of a component** is:

$$\tilde{R}([a,b])[\propto] = \{\textstyle\sum_{x=a}^{b} C_x^m\, p^x q^{m-x} / S_{\propto}\} \qquad \ldots(2.58)$$

and if $\widetilde{R(}[a,b])[\propto] = [R_L[a,b][\propto], R_U[a,b][\propto]]$, then

$$R_L[a,b][\propto] = \min\{\textstyle\sum_{x=a}^{b} C_x^m\, p^x q^{m-x} / S_{\propto}\} ,$$

and

$$R_U([a,b])[\propto] = \max\{\sum_{x=a}^{b} C_x^m\, P^x q^{m-x}/S_\propto\}$$

Definition 2.6.10. [1,4,6,17]: Let X be a random variable the Exponential distribution with parameter λ has pdf

$$f(x\,,\lambda) = \begin{cases} \lambda e^{-\lambda x} & For \quad x \geq 0 \\ 0 & For \quad x < 0 \end{cases} \qquad \text{….(2.59)}$$

The cumulative distribution function is

$$F_X(x) = \begin{cases} 1-e^{-\lambda x} & ; \quad if \ x>0 \\ 0 & ; \quad otherwise \end{cases} \qquad \text{…(2.60)}$$

The expected value of exponential distribution is $E(X) = \frac{1}{\lambda}$.

Now , if $\tilde{\lambda}$ be fuzzy parameters. Then the **fuzzy exponential distribution of one parameter** in the interval $[a,b], a \geq 0$ is given by

$\tilde{f}(X,\lambda) = \tilde{P}(a \leq X \leq b)[\propto]$, and compute its $\propto -cut$ as follows

$$\tilde{P}(a \leq X \leq b)[\propto] = \{\int_a^b \lambda\, e^{-\lambda x} \ \ dx : \ \lambda \in \tilde{\lambda}[\propto] \qquad \text{….(2.61)}$$

$$= \left[P_L[\propto], P_U[\propto]\right], \qquad forall \ \propto$$

where,

$$P_L[\propto] = \min\{\int_a^b \lambda\, e^{-\lambda x} \ \ dx : \ \lambda \in \tilde{\lambda}[\propto]\}$$

and

$$P_U[\propto] = \max\{\int_a^b \lambda\, e^{-\lambda x} \ \ dx : \ \lambda \in \tilde{\lambda}[\propto]\}$$

To calculate the fuzzy reliability of a component has distribution lifetime parameter $\tilde{\lambda}$ represent a triangular fuzzy number, $[a_1, a_2, a_3]$ whose membership function is defined as:

$$\mu_{\lambda}(x) = \begin{Bmatrix} \dfrac{x-a_1}{a_2-a_1} & ; a_1 \le x \le a_2 \\ \dfrac{a_3-x}{a_3-a_2} & ; a_2 \le x \le a_3 \\ 0 \quad , & 0.w \end{Bmatrix}$$

And $\propto -cut$ of $\tilde{\lambda}[\propto]$ defined by

$$\lambda_L = (a_2 - a_1) \propto + a_1, \lambda_U = a_3 - (a_3 - a_2) \propto$$

Then the fuzzy reliability function of component is given by

$$\tilde{R}(t) = \int_t^{\infty} \lambda\, e^{-\lambda x} \quad : \ \lambda \in \tilde{\lambda}[\propto]$$

$$= \{e^{-\lambda t} \quad : \ \lambda \in \tilde{\lambda}[\propto]\} \qquad \dots(2.62)$$

Since $e^{-\lambda t}$ is decreasing function, then we have

$$\tilde{R}(t) = [e^{-[\lambda_L]t} \quad , e^{-[\lambda_U]t}]$$

$$= [e^{-[(a_2-a_1)\propto + a_1]t}, e^{-[a_3-(a_3-a_2)\propto]t}] \qquad \dots(2.63)$$

Fuzzy reliability curve have the following properties:-

1) $\tilde{R}(0)[\propto] = \tilde{1}$

2) $\tilde{R}(\infty)[\propto] = \tilde{0}$

3) It is decrease monotonically, that is $\forall\ t_1, t_2, \tilde{R}(t_1)[\propto] \ge \tilde{R}(t_2)[\propto]$

The fuzzy mean time to failure ($F\widetilde{MTT}F$) is

$$\mathrm{F\widetilde{MTT}F}[\propto] = \int_0^{\infty} \lambda\, x e^{-\lambda x} \quad : \ \lambda \in \tilde{\lambda}[\propto]$$

$$= \{\frac{1}{\lambda} \ : \ \lambda \in \tilde{\lambda}[\propto]\}$$

$$= [\frac{1}{\lambda_L[\propto]}, \frac{1}{\lambda_U[\propto]}]$$

where $\tilde{\lambda}[\propto] = [\lambda_L[\propto], \lambda_U[\propto]]$ is the failure rate $\tilde{h}(t)[\propto]$.

Example 2.3.

Let lifetime of component is modeled by an exponential distribution with fuzzy parameter $\tilde{\lambda} = (0.3,0.5,0.7)$, then

1) $\propto -cut$ of fuzzy system reliability is

$\tilde{R}(t) = [e^{-[0.7-0.2\propto]t} \quad , e^{-[0.2\propto+0.3]t}]$, for all $\propto$

2) $\widetilde{\text{FMTTF}} = [\frac{1}{0.7-0.2\propto}, \frac{1}{0.2\propto+0.3}]$

3) $\propto -cut$ of fuzzy system reliability if $\propto = 0$, is $\tilde{R}(t) = [e^{-0.7t} \quad , e^{-0.3t}]$

4) If t=0.6 then fuzzy system reliability is , $\tilde{R}(0.6) = [e^{-(0.42-0,12\propto)t} \quad , e^{-(0.12\propto+0.18)t}]$.

2.7 Mathematical Background of Manifolds

Definition 2.7.1.[79] A function $f(x)$ is called **smooth function** if and only if it is a continuous and has a continuous derivative.

Definition 2.7.2.[63]: Given two points $x, y \in R^n$, **a convex combination** of them is any point of the form $z = \lambda x + (1-\lambda)y$, $\lambda \in [0,1]$.

Definition 2.7.3. [63] :A set of $S \subseteq R^n$ is **a convex set** if it contains all convex combinations of any two points within it . Or in equivalent : A set of points S is a convex set if the line segment joining any two points in S is wholly contained in S . As an example , any empty set and a single-point are both convex.

Definition 2.7.4.[63]: Let S be a convex set . The function $f(x): S \to R$ is said to be **convex function** if for any two points $x_1, x_2 \in S$, $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2), \lambda \in [0,1]$.

Definition 2.7.5.[9,16]:- A function $\psi: I \subseteq R^n \to \psi(I) \subseteq R^n$ is called **diffeomorphism function** if and only if it is one-to-one, onto, differentiable and has a differentiable inverse.

Example 2.3.

Let $\psi: (-3,-1) \to (1,9), \psi(x) = x^2$

(1) $D\psi_x = \left[\frac{\partial \psi}{\partial x}\right] = [2x]$, then $\frac{d\psi}{dx}|_{x=-2} = -4$

So ψ is differentiable function.

(2) ψ is one-to-one since if $\psi(x^1) = \psi(x^2)$, then $x^1 = x^2$.

So if $(x^1)^2 = (x^2)^2, then\, x^1 = \pm\, x^2$ are both in (-3,-1), so both x^1, x^2 are negative hence $x^1 = x^2$

(3) ψ is onto since if $y \in (1,9)$ such that $y = x^2\ then\ x = -\sqrt{y}$

(4) $\psi^{-1}: (1,9) \to (-3,-1)$, we want to show that $\psi^{-1}(y) = -\sqrt{y}$

Since $\psi^{-1}(\psi(x)) = \psi^{-1}(x^2) = -\sqrt{x^2} = -|x| = -x \in (-3,-1)$

$\psi(\psi^{-1}(y)) = \psi(-\sqrt{y}) = (-\sqrt{y})^2 = (-1)^2(\sqrt{y})^2 = y \in (1,9)$

So $\psi^{-1}(y) = -\sqrt{y}$

(5) ψ^{-1} is differentiable on (1,9) $\subseteq (0,\infty)$ Hence ψ is diffeomorphism function.

The parameterization of the parametric model is not unique. So we can change the coordinate system in the following ways:

Definition 2.7.6.[70,85]:- Let $U \subset R^n$ and $M \subseteq R^m$ be open set. A function

$f: U \to M$ is called a **smooth function** (or $C^\infty - function$) if and only if it is infinity differentiable, i.e. all it’s partial derivatives.

$$\partial^\ell f = \frac{\partial^{\ell_1+\ell_2+\cdots+\ell_n}}{(\partial x^1)^{\ell_1}(\partial x^2)^{\ell_2}\ldots(\partial x^n)^{\ell_n}} f \qquad \ldots(\mathbf{2}.64)$$

exist and are continuous for all positive integer ℓ.

Definition 2.7.7.[9,16] : Whenever an index (subscript or superscript) is repeated in a given term, summation should be done over that index from 1 to n unless otherwise specified; this is called the summation convention. i.e., let us consider the sum

$$\sum_{i=1}^{n} a^i e_i = a^1 e_1 + a^2 e_2 + \cdots + a^n e_n \text{ ,}$$

we write it as $a^i e_i$ without the summation $\sum$ such that $i = 1, \dots, n$, This is called **Einstein's summation convention.** This brings us to the definition of indices in Einstein notation:

1- Dummy Index:- If an index is summed over and repeated twice, replace it with any index if it satisfies both conditions:

a) Not already in the expression $a_{ii} b_i$ if the term is $a_{ij} b_j$ $(i.e. a_{ii} b_i \neq a_{ij} b_j$

b) That is over the same range

for example: $a_{ij} b_j = a_{ir} b_r \ni r = 1, \dots, n \text{ , } j = 1, \dots, n$ such index is called Dummy index or Umbral Index .

1- **Free Index**:- If an index is not summed over and occurs only once in any term and cannot be replaced by another index, such an index is called a free index.

   For example : The term $a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + \cdots + a_{in} b_n$,such that j is a Dummy index and i is Free index .

Definition 2.7.8.[11,16]: Since x^i $'s$ are independent of each other then we have

$$\frac{dx^i}{dx^j} = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases} \qquad \dots(2.65)$$

We define the **kronecker delta symbol** as :

$$\delta_j^i = \begin{cases} 0 & , \ if\ i \neq j \\ 1, & \ if\ i \equiv j \end{cases} \qquad \ldots(2.66)$$

So in Unbarred coordinate system we can say

$$\delta_j^i = \frac{dx^i}{dx^j} \qquad \ldots(2.67)$$

Similarly in Bared Coordinate system $\bar{x}^{\propto}\ 's$ are independent of each other and hence we can write,

$$\delta_{\beta}^{\propto} = \frac{d\bar{x}^{\propto}}{d\bar{x}^{\beta}}\,. \qquad \ldots(2.68)$$

A gain $x^i\ 's$ are function of $\bar{x}^{\propto}$, so

$$\frac{dx^i}{dx^j} = \frac{\partial x^i}{\partial \bar{x}^{\propto}} \frac{\partial \bar{x}^{\propto}}{\partial x^j}$$

Or

$$\delta_j^i = \frac{\partial x^i}{\partial \bar{x}^{\propto}} \frac{\partial \bar{x}^{\propto}}{\partial x^j} \qquad \ldots(2.69)$$

in Unbarred coordinate system .Similarly, in Barred coordinate system ,

$$\delta_{\beta}^{\propto} = \frac{\partial \bar{x}^{\propto}}{\partial x^i} \frac{\partial x^i}{\partial \bar{x}^{\beta}}\,. \qquad \ldots.(2.70)$$

Definition 2.7.9.[61,69] : The vector field Ṿ is said to be **contravariant of rank 1** if its component V^i in (x^i) coordinate system and $\bar{V}^i$ in $(\bar{x}^i)$ coordinate system are related by the following law of transformation :

$$\bar{V}^i = V^r \frac{\partial \bar{x}^i}{\partial x^r} \quad 1 \leq i \leq n\ . \qquad \ldots(2.71)$$

In other words, a vector whose components transform in a contravariant fashion under a change of coordinates is called a contravariant vector.

Definition 2.7.10.[80,90]:- The vector field U is said to be **a covariant vector of rank 1** if its components U_i in the (x^i) coordinate system and $\overline{U}_i$ in the $(\bar{x}^i)$ coordinate system are related by the following law of transformation

$$\bar{U}_i = U_r \frac{\partial x^r}{\partial \bar{x}^i} \quad 1 \le i \le n \qquad \ldots(2.72)$$

In other words, a vector whose components transform in a covariant fashion under a change of coordinates is called a covariant vector .

2.8. Mathematical Background of Tensors

Gauss, Riemann, and Christoffel were responsible for developing differential geometry and the tensor. Ricci and Levi-Civita created tensor calculus, also known as tensor analysis, which is a generalization of vector analysis and absolute differential calculus. Tensor analysis examines abstract objects called tensors, whose properties are independent of reference frames. A space has a tensor field if every point has a tensor. Basically, tensor calculus analyzes tensor fields. A vector is generalized to a tensor. In a given reference frame, a tensor is represented by a set of components, exactly like a vector. The law of change of a set of functions from one coordinate system to another determines whether it is a tensor. Tensor calculus studies objects in a coordinate system-approved space, where object components transform according to a law when we change coordinate systems. Scalars and vectors are particular examples of a more general object, a tensor of rank n, whose formulation in any coordinate system requires numbers called components. Scalars are zero-order tensors with components. Vectors are component-order-1 tensors. For a rank-n tensor component, we need its value at n basis vectors. In this section, some basic concepts are discussed that are helpful to understand tensors properly

This combination of nine components and nine sets of two basis vectors make it tensor of Rank two. Change of coordinates does not change this tensor (rank- 2 tensor) but only its components.

Now, in general to specify component of rank -n tensor need magnitude of component, n basis vectors.

Definition 2.8.1.[13]: A **Tensor** is an object that is invariant under a change of coordinate system with components that change a coordinate to a special set of mathematical formula .

Definition 2.8.2[20]:- Rank of a tensor is a number of basis vectors required to fully specify a component .

Examples 2.4.[20,30]:-

1) Scalars are often called tensors of rank zero, because scalars have no directional indicators and hence need no indices .Change of coordinate system does not change scalars.

2) A three dimensional vector can be represented as

$$A = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

A_x represents component a long i basis (or unit) vector, A_y represents components along j basis or (unit) vector and A_z represents components along k basis (or unit) vector .So we need one index for each component to represent one directional indicator (or basis Vector).

So vectors are often called tensor of rank one or first order tensor.

Note that change of coordinate system does not change displacement (vector).

3) Tensor of rank two can be represented

$$A = \begin{bmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{bmatrix}$$

nine components and nine set of two basis vectors. Each component has two indices referring two basis vectors .

Definition 2.8.3.[20,44]:- Let V is a matrix field composed of $n \times n$ scalar functions defined over a region U in R^n . Assume that in the (x^i) coordinate system the components of V are V^{ij}. Assume also that after a coordinate transformation

$$T = \bar{x}^i = \bar{x}^i(x^1, x^2, \dots, x^n)\,, 1 \leq i \leq n$$

that takes us to the $(\bar{x}^i)$ −coordinate system , the components of V become $\bar{V}^{\propto\beta}$. Then a matrix field V is said to be a **contravariant tensor of rank 2** if its components V^{ij} in the $(x^i)-$ coordinate system and $\bar{V}^{\propto\beta}$ in the $(\bar{x}^i)$ −coordinate system obey:

$$\overline{V}^{\propto\beta} = V^{ij}\frac{\partial\bar{x}^{\propto}}{\partial x^i}\frac{\partial\bar{x}^{\beta}}{\partial x^j}\,, 1 \leq \propto \beta \leq n \qquad \dots (2.73)$$

where i, j are dummy index while $\propto, \beta$ are free index.

Definition 2.8.4.[20,51]:- Suppose that W is a matrix field composed of $n \times n$ scalar functions defined over a region U in R^n. Assume that in the (x^i) coordinate system the components of W are W_{ij} .Assume also that after a coordinate transformation

$$T\colon \bar{x}^i = \bar{x}^i(x^1, x^2, \dots, x^n)$$

that takes us to the $(\bar{x}^i)$ −coordinate system , the components of W become $\overline{W}_{\propto\beta}$. Then a matrix field W is a **covariant tensor of rank 2** if its components W_{ij} in the $(x^i)-$ coordinate system and $\overline{W}_{\propto\beta}$ in the $(\bar{x}^i)$ −coordinate system obey:

$$\overline{W}_{\propto\beta} = W_{ij}\frac{\partial x^i}{\partial\bar{x}^{\propto}}\frac{\partial x^j}{\partial\bar{x}^{\beta}}\,, 1 \leq \propto, \beta \leq n \qquad \dots (2.74)$$

where i, j are dummy index while $\propto, \beta$ are free index. [20,51]

Definition 2.8.5. [13,51]:- if $A^{ij} = A^{ji}$ for all values of $i, j = 1, \dots, n.$ Then A^{ij} is said to be **symmetric contravariant tensor of rank 2**. Similarly, if $A_{ij} = A_{ji}$ for all values of $i, j = 1, \dots, n.$ Then A_{ij} is called **symmetric covariant tensor of Rank 2**

Definition 2.8.6. [11,13] :- If $A^{ij} = -A^{ji}$ for all values of $i\ and\ j$ between $1\ to\ n$, then A^{ij} is said to be **Anti-Symmetric contravariant tensor of rank 2** .Similarly, if $A_{ij} = -A_{ji}$ for all values of $i\ and\ j$ between $1\ to\ n$, then A_{ij} is called **Anti-Symmetric covariant tensor of rank 2 .**

Definition 2.8.7. [13,20] :- If we have a space of n-dimensions and a displacement vector dx^i ; $i = 1,2,\dots n$ determined by a pair of neighboring points $x^i$and $x^i + dx^i$. The quadratic formula

$$ds^2 = g_{ij}dx^i dx^j; i,j = 1,2,3,\dots n \qquad \dots(2.75)$$

calculates the distance ds^2 between two points that are close together with coordinates x^i and $x^i + dx^i$ in any system where

$$g_{ij} = \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta} \qquad \dots(2.76)$$

is the product of two partial derivatives and these are functions

of (x^i) such that $g = \ det(g_{ij}) \neq 0$. This quadratic differential form $g_{ij}dx^i dx^j$ which represents the square of the distance between two neighboring points is called **metric or Riemannian metric .** $g_{ij}\,'s$ are called the **metric Tensor or the fundamental metric tensor .**

Definition 2.8.8. [13,20] :- If g_{ij} be a metric tensor, then **the conjugate metric tensor** g^{ij} is given by

$$g^{ij} = \frac{cofactor\ of\ g_{ij}\ in\ g}{g} ; \qquad \dots(2.77)$$

provided $g = \det(g_{ij}) \neq 0$. then g^{ij} is called contravariant fundamental tensor, which is symmetric contravariant tensor of rank 2 and $g^{ij}\, g_{jk} = \delta^i_k$.

Chapter Three

Manifolds of reliability Function and Riemannian Manifolds

3.1 Introduction

One definition of an n-dimensional manifold is a topological space made up of a set or collection of points connected in such a way that it is locally flat, i.e., it looks likes R^n is in the neighborhood of every point, and can be built by stitching or gluing together portions of R^n. Therefore, all topological spaces and manifolds can be understood as variations on the Euclidean R^n. A manifold is locally Euclidean in the sense that every point has a neighborhood, which is a chart homeomorphic to an open subset of R^n, while topological space is concerned with properties that remain invariant under homeomorphism (continuous mappings with continuous inverses). Many ideas from R^n, including as differentiability, point-derivations, tangent spaces, and differential forms, transfer to a manifold since the coordinates on a chart allow one to carry out computations as though in a Euclidean space.

In this chapter, the reliability manifold is studied , which is a geometric abstractions started with the differential geometric study of manifold of reliability with a lifetime exponential distribution. Every point on this manifold is a reliability function, and any curve corresponds to a one-parameter subfamily of reliability functions.

Section two introduces the notion of a parametric model, which is a space of reliability functions, and contains important examples of parametric models, including the case of the reliability of a life-time exponential with one parameter and two parameters.

Section three contains definitions of n-dimensional manifolds, coordinate functions in coordinate neighborhoods, charts, and Atlas.

Section four discusses the smooth structure (differentiable structure). We want it to do calculus on a manifolds. Furthermore, smooth manifolds are defined and give examples of them. In the fifth section, the tangent space and 1-forms are discussed. [9,31,46,70]

3.2 Parametric Models:

This section deals with a family of reliability function which depends on several parameters and hence it can be organized as a parameterized surface each point of this surface represents a reliability function $R(t,\lambda)$ and can be parameterized on an open subset I ,by n-real valued variable $\lambda = (\lambda^1, \ldots, \lambda^n) \in I \subseteq R^n$ which is called failure rate. In this section we need log-likelihood function $l(R(t,\lambda))$ which is a useful mapping defined by :

$$l\big(R(t,\lambda)\big) = ln\big(R(t,\lambda)\big) \qquad \ldots(\mathbf{3}.1)$$

and its derivatives are

$$\partial_j l\big(R(t,\lambda)\big) = \frac{\partial\, ln\, R(t,\lambda)}{\partial\, \lambda^j} = \gamma_j l\big(R(t,\lambda)\big), \qquad 1 \le j \le n \qquad \ldots(\mathbf{3}.2)$$

Moreover , some basic definition and examples of parametric model and the parameterization of parametric model are presented .

Definition 3.2.1.:- Let $R(t,\lambda)$ be reliability function at time to failure t and failure rate λ .Then the family $N = \{R(t,\lambda), t \in (0,\infty), \lambda \in R^n > 0\}$ is said to be **parametric model**

1- if there exists a mapping $g: I \to N$ such that it is one-to-one mapping (i.e. if $g(\lambda^1) = g(\lambda^2)$ then $\lambda^1 = \lambda^2$

2- The Wronskian determinate $\begin{vmatrix} \gamma_1(t,\lambda) \ldots. & \gamma_n(t,\lambda) \\ \gamma_1'(t,\lambda) \ldots. & \gamma_n'(t,\lambda) \\ \gamma_1^{(n-1)}(t,\lambda) \ldots. & \gamma_1^{(n-1)}(t,\lambda) \end{vmatrix} \neq 0, \forall\lambda$

We can write it by

$$W(t, \gamma_1(t,\lambda), \ldots., \gamma_n(t,\lambda)) \neq 0 \qquad \ldots(\mathbf{3}.3)$$

where

$$\gamma_j(t,\lambda) = \frac{\partial R(t,\lambda)}{\partial \lambda^j} \qquad \ldots(\mathbf{3}.4)$$

and

$$\gamma^{(k)}(t,\lambda) = \frac{\partial^{(k)}\gamma_j(t,\lambda)}{\partial t} \qquad \ldots(\mathbf{3}.5)$$

The condition (2) states the regularity by the parametric model.

Definition 3.2.2.[50,27]:- The functions $\{\gamma_n(t,\lambda)\}$ are **linearly independent** if

$$W(t,\gamma_1(t,\lambda),\ldots,\gamma_n(t,\lambda)) \neq 0 \qquad \ldots(\mathbf{3}.6)$$

In else they are called linearly dependent .

Theorem 3.2.3. The condition of regularity (2) of the parametric model

$N = \{R(t,\lambda), t \in (0,\infty), \lambda \in R^n > 0\}$ holds if and only if for any $\lambda \in I$, the set $\{\partial_1 l(R(t,\lambda), \partial_2 l(R(t,\lambda), \ldots, \partial_n lR(t,\lambda)\}$ is a system of n linearly independent functions of t.

Proof:

Since $N = \{R(t,\lambda), t \in (0,\infty), \lambda \in R^n > 0\}$ be a parametric model, then $\{\gamma_n(t,\lambda)\}$ are independent functions.

Now, $\frac{\partial l(R(t,\lambda))}{\partial \lambda^j} = \frac{1}{R(t,\lambda)} \frac{\partial R(t,\lambda)}{\partial \lambda^j} = \frac{1}{R(t,\lambda)} \gamma_j(t,\lambda)$

Since the output is equal to the input multiplied by a constant, then we get the system $\left\{\frac{\partial l(R(t,\lambda))}{\partial \lambda^j}\right\}$ and $\left\{\gamma_j(t,\lambda)\}\right\}$ are proportional.

Since $\{\gamma_j(t,\lambda)\}$ are independent functions of time to failure t. Then $\left\{\frac{\partial l(R(t,\lambda))}{\partial \lambda^j}\right\}$ are independent functions of time to failure t .

Definition 3.2.4:- Let $N = \{R(t,\lambda), t \in (0,\infty), \lambda \in R^n > 0\}$ be the image of one-to-one mapping $g: I \to N$ such that $g(\lambda) = R(t,\lambda)$, and its inverse function $\varphi: N \to I \subseteq R^n$ such that $\varphi\big(R(t,\lambda)\big) = \lambda$.

Then the mapping φ assigns a failure rate parameter λ to each reliability $R(t,\lambda)$ called a **coordinate system for parametric model** we can denoted it by (I,φ)

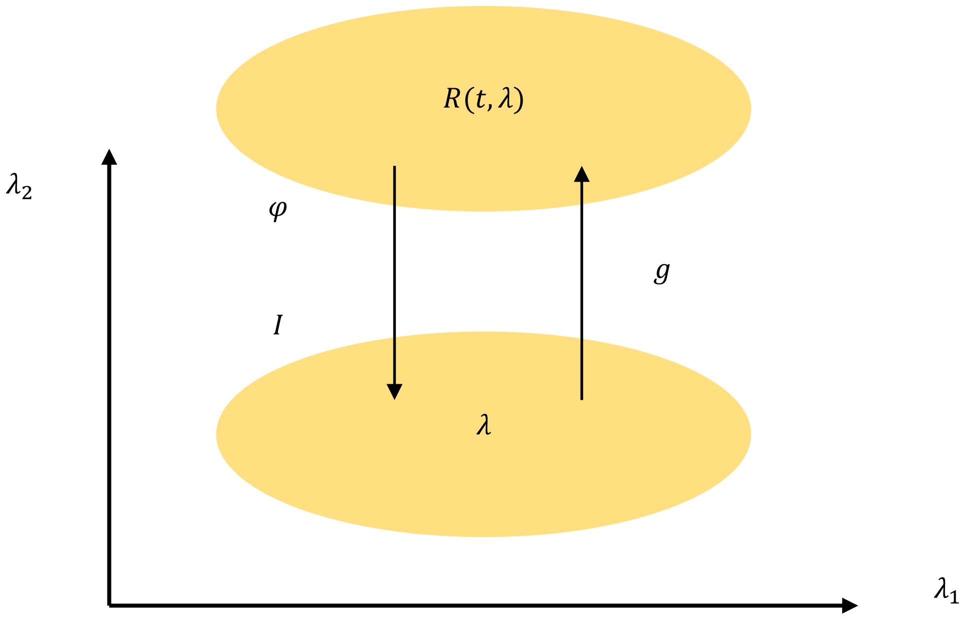


Figure **3**.1: Coordinate system for the parametric model

Definition 3.2.5:- Let I and $\psi(I)$ are two open subsets in R^n if $g: I \to N$ such that $g(\lambda) = R(t,\lambda)$ and $\psi: I \to \psi(I)$ be diffeomorphism function. Then the function $\psi_0 g^{-1}: N \to \psi(I)$ is **another coordinate system** and the parametric model can be written as:

$$N = \{R(t,\psi^{-1}(\rho)) : \rho \in \psi(I)\} \qquad \dots(\mathbf{3}.7)$$

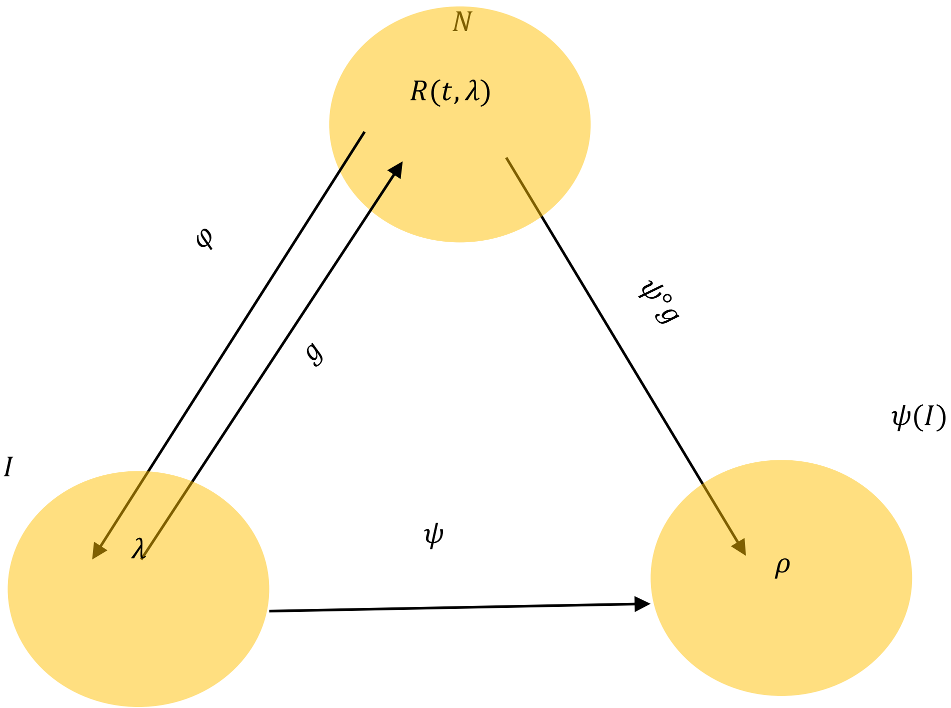


Figure **3**.2: Another coordinate system for parametric model

Theorem 3.2.6. Let $N = \{e^{-\lambda t}: \lambda \in I\}$ be a family of reliability functions with exponential lifetime distributions of one failure rate parameter .Then N is a parametric model of one-dimensional **.**

Proof:

Let $I = (0, \infty)$, be one-dimensional parameter space which is an open interval in R. The reliability of an exponential lifetime distribution with one failure rate parameter λ is given by the formula $R(t, \lambda) = e^{-\lambda t}$.

Now the mapping $g: I \to N$ is one-to-one for all time to failure $t \in (0, \infty)$, if $\lambda_1, \lambda_2 \in I$ such that $\lambda_1 = \lambda_2$, then $R(t, \lambda_1) = e^{-\lambda_1 t} = e^{-\lambda_2 t} = R(t, \lambda_2)$ and

$$\frac{\partial R(t,\lambda)}{\partial \lambda} = -t\, e^{-\lambda t} \neq 0, \forall\, t \in (0, \infty).$$

So, $N = \{e^{-\lambda t}: \lambda \in I\}$ is a parametric model of one dimension .

The diffeomorphism $\psi: (0,\infty) \to (0,\infty)$ such that $\psi(\lambda) = \frac{1}{\lambda}$, induces the new parametrization $R(t,\rho) = e^{\frac{-t}{\rho}}$

So, the family $N = \{e^{-\lambda t}: \lambda \in I\}$ is one-dimensional and hence can be considered a curve in the infinite dimensional space of functions $\{R(t,\lambda)\}$.

Theorem 3.2.7. The family of reliability functions with exponential lifetime distributions of two parameters is a two-dimensional surface parameterized by $(0,\infty) \times (0,\infty)$.

Proof :-

Let $I = (0,\infty) \times (0,\infty)$ be two-dimensional parameter space.

The reliability of an exponential lifetime distribution with two parameters given by

$$R(t,\lambda,\theta) = e^{-\lambda(t-\theta)}, \forall\ t\ \in (0,\infty), (\lambda,\theta) \in I = (0,\infty) \times (0,\infty)$$

where λ refer to the scale failure rate parameter and θ refer to the location failure rate parameter.

Let $g: I \to N$ and log-liklihood function is

$$\ell\ R(t,\lambda,\theta) = ln\ R(t,\lambda,\theta) = \ln(e^{-\lambda(t-\theta)}) = -\lambda(t-\theta)$$

and $\ell\ R(t,\lambda,\theta) = \ell\ R(t,\lambda',\theta')$, then $-\lambda(t-\theta) = -\lambda'(t-\theta')$

$$\lambda t - \lambda\theta = \lambda' t - \lambda'\theta'$$

Hence $\lambda = \lambda'\ and\ \theta = \theta'$.

Therefore, g is one-to-one mapping .

Now , Let $\gamma_1(t) = \partial_\lambda \ell\big(R(t,\lambda,\theta)\big) = \theta - t$,$\gamma_2(t) = \partial_\theta \ell\big(R(t,\lambda,\theta)\big) = \lambda$

$$\begin{vmatrix} \gamma_1(t) & \gamma_2(t) \\ \gamma_1'(t,\lambda) & \gamma_2'(t) \end{vmatrix} = \begin{vmatrix} \theta - t & \lambda \\ -1 & 0 \end{vmatrix} = \lambda \neq 0$$

So the family $N = \{e^{-\lambda(t-\theta)}\}$ be two- dimension surface parameterized by I.

3.3 Reliability manifold

When $R(t,\lambda)$ is sufficiently smooth in $\lambda = (\lambda^1, \dots, \lambda^n)$ it is natural to introduce in parametric model N , the structure of an n-dimensional manifold, where λ plays the role of a coordinate system. We need smooth structure when we want to do calculus on manifolds. Hence we will define smooth manifold, which is ordinary manifold with an additional structure.[9,11,55,46,70,85]

Definition 3.3.1: Let $N = \{R(t,\lambda), t \in (0,\infty), \lambda \in R^n > 0\}$ be a parametric model of dimension n with usual topology in which functions $R(t,\lambda)$ are continuous . Then **an n-dimensional reliability manifold (RN)** is a topological space that is Hausdorff, second countable and locally homeomorphic to an n-dimensional Euclidean space R^n.

Definition 3.3.2. Let U be an open subset of **RN** that is homeomorphic to R^n with a homeomorphism $\varphi: U \subset RN \to R^n$.Then a point $R(t,\lambda) \in U$ is transformed into a point $\lambda = (\lambda^1, \dots, \lambda^n) \in R^n$, $(i.e.)\ \varphi\big(R(t,\lambda)\big) = \lambda = (\lambda^1, \dots, \lambda^n)$. This mapping φ is called a coordinate function in the coordinate neighborhood U, such that each point $R(t,\lambda)$ in U is coordinates $\lambda = (\lambda^1, \dots, \lambda^n)$.

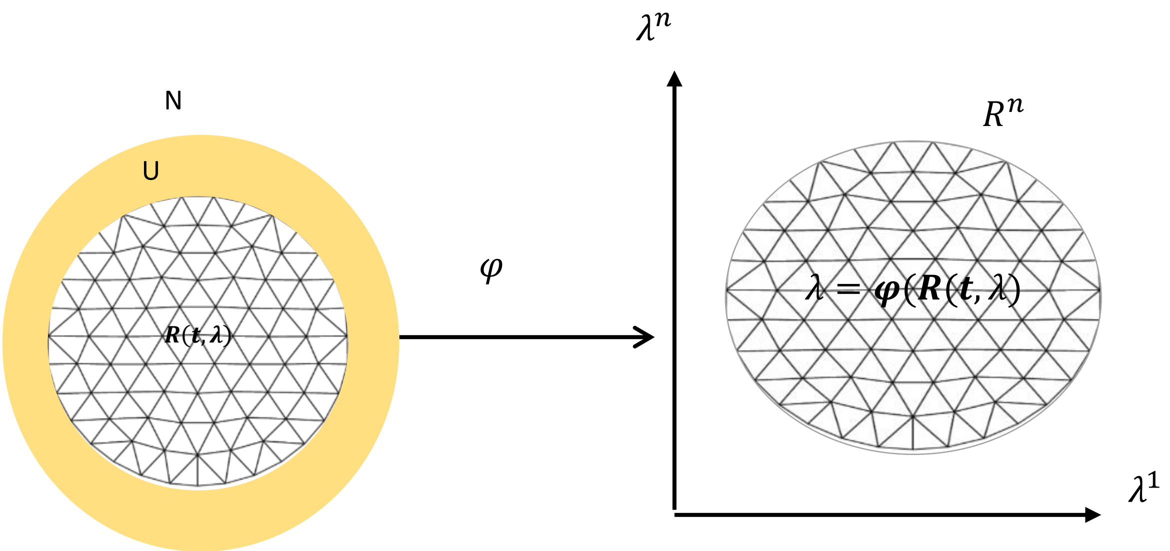


Figure 3.3: n-Dimensional Reliability Manifold

Definition 3.3.3.:- Let $U \subset RN$ be an open set and $\varphi: U \to \varphi(U) \subset R^n$ is a homeomorphism of the open set U in reliability manifold N onto open subset $\varphi(U)\ of\ R^n$. The pair (U, φ) is called a **chart** or (coordinate system). Chart (U, φ) gives us coordinates, which help us calculate on the reliability manifold, and in order to calculate on the whole reliability manifold, we need a lot of charts such that all charts cover the whole reliability manifold (RN). Hence, such a collection is called an atlas .

Definition 3.3.4- A collection of charts $(U_i, \varphi_i)_{i \in I}$ is called **an atlas** if $\cup_{i \in I} U_i = RN.$

Theorem (A-N) 3.3.5:- The graph of $R(t, \lambda) = e^{-\lambda t}$ is a topological 1- Reliability Manifold .

Proof : The graph of $R(t, \lambda) = e^{-\lambda t}$, is given by the set

$$\mathrm{RN} = \{(\lambda, e^{-\lambda t}): \lambda \in (0, \infty) = R^+\} \subset R^{+2}$$

Let (R^{+2}, τ) be topological space and $\mathrm{RN} \subset \mathrm{R}^{+2}$. Define the family τ_{RN} as a family of subset of RN as follow : $\tau_{RN} = \{RN \cap U: U \in \tau\}$.

Since $R^{+2} \in \tau$, and $\mathrm{RN} \subset \mathrm{R}^{+2}$, we have $RN \cap R^{+2} = AN$, so $RN \in \tau_{ANRN}$.

Since $\emptyset \in \tau$, and $\emptyset \subset R^{+2}$, then $\emptyset = RN \cap \emptyset$, so $\emptyset \in \tau_{RN}$.

Let $V_1, V_2 \in \tau_{RN}$,then there exist $U_{1\,,} U_2 \in \tau$, such that

$$V_1 = RN \cap U_1 \text{ and } V_2 = RN \cap U_2 \ .$$

Hence $V_1 \cap V_2 = (RN \cap U_1) \cap (RN \cap U_2) = RN \cap (U_1 \cap U_2) \in \tau$.

So , $V_1 \cap V_2 \in \tau_{RN}$. Let $V_\alpha \in \tau_{RN}; \alpha \in \aleph$. Then there exist $U_\alpha \in \tau$, such that $V_\alpha = RN \cap U_\alpha, \alpha \in \aleph$. Then $\cup_{\alpha \in \aleph} V_\alpha = \cup_{\alpha \in \aleph} (RN \cap U_\alpha) = RN \cap (\cup_{\alpha \in \aleph} U_\alpha) \in \tau$

Hence , $\cup_{\alpha \in \aleph} V_\alpha \in \tau_{RN}$. Therefore , τ_{RN} is a topology space on RN .

Let $(\lambda^1, e^{-\lambda^1 t})$, $(\lambda^2, e^{-\lambda^2 t}) \in RN$, then there exist $U, V \in R^{+2}$, such that $(\lambda^1, e^{-\lambda^1 t}) \in U$, and $(\lambda^2, e^{-\lambda^2 t}) \in V$,and $U \cap V = \emptyset$.

Since RN is subspace of R^{+2} , then $U^* = U \cap R^{+2}$ and $V^* = V \cap R^{+2}$ are two disjoint open sets in RN and containing $(\lambda^1, e^{-\lambda^1 t})$, $(\lambda^2, e^{-\lambda^2 t})$, respectively .

So , RN is Hausdorff .

Let $B = \{B_r(\lambda): \lambda \in Q^{+2}, r > 0, r \in Q^+\}$ is countable basis for R^{+2} .

Consequently , $B_{RN} = \{B_r(\lambda) \cap RN: \lambda \in Q^{+2}, r > 0, r \in Q^+\}$ is countable basis for the subspace RN . Let $\varphi: RN \to R^+$ define as $\varphi(\lambda, e^{-\lambda t}) = \lambda$, where $\lambda \in (0, \infty)$ and $e^{-\lambda t} \in R$. Let $(\lambda^1, e^{-\lambda^1 t})$, $(\lambda^2, e^{-\lambda^2 t}) \in RN$,then if $\varphi(\lambda^1, e^{-\lambda^1 t}) = \varphi(\lambda^2, e^{-\lambda^2 t})$,we have $(\lambda^1, e^{-\lambda^1 t}) = (\lambda^2, e^{-\lambda^2 t})$ .So φ is one-to-one . Let $\lambda \in R^+$, then there exists $(\lambda, e^{-\lambda t}) \in RN$ such that $\varphi(\lambda, e^{-\lambda t}) = \lambda$.

Hence φ is onto . Clearly φ is continuous as well as its invers $\varphi^{-1}(\lambda) = (\lambda, e^{-\lambda t})$. So it is homeomorphism .Thus RN is a topological 1- Reliability Manifold .

Definition 3.3.6. :-If $\varphi: U \to \lambda \; and \; \psi: U \to \theta$ be two coordinate functions to the same point $R(t, \lambda)$. Then there exists another homeomorphism.

$$\psi o \varphi^{-1}(\lambda): \lambda \to \theta \qquad \ldots(3.8)$$

such that ,

$$\psi o \varphi^{-1}(\lambda^i) = \theta^i(\lambda^1, \ldots, \lambda^n)$$

$$= (\theta^1(\lambda^1, \ldots, \lambda^n), \theta^2(\lambda^1, \ldots, \lambda^n), \ldots, \theta^n(\lambda^1, \ldots, \lambda^n)) \qquad \ldots.(3.9)$$

and its inverse

$$\varphi o \psi^{-1}(\theta): \theta \to \lambda \qquad \ldots(3.10)$$

such that ,

$$\varphi o \psi^{-1}(\theta^i) = \lambda^i(\theta^1, \ldots, \theta^n)$$

$$= (\lambda^1(\theta^1, \ldots, \theta^n), \lambda^2(\theta^1, \ldots, \theta^n), \ldots, \lambda^n(\theta^1, \ldots, \theta^n)) \qquad \ldots(3.11)$$

Hence $\lambda^i$and θ^i give a relation between the coordinates in the two coordinate functions and are called **coordinate transformations** or just changes of coordinates.

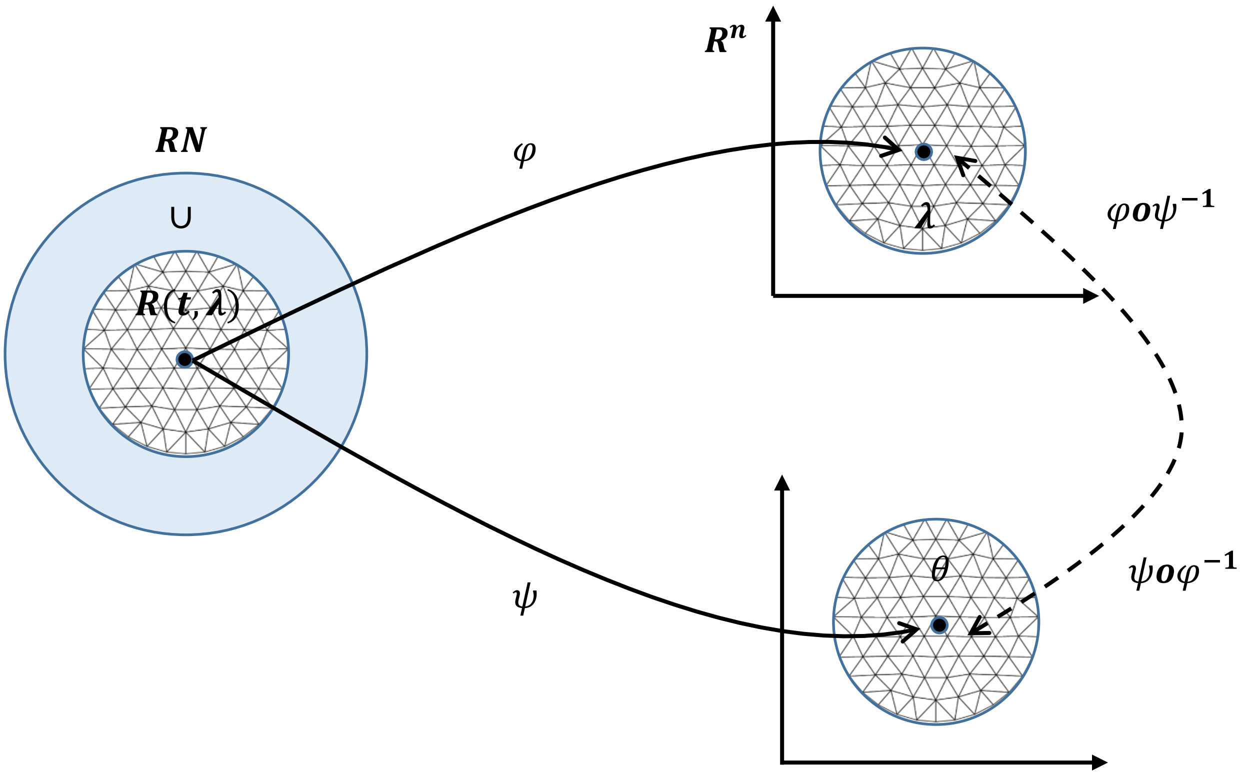


Figure 3.4 : Change of Coordinates.

3.4 The Smooth Reliability Manifolds

In order to apply calculus to manifolds, we require structures with a smooth structure. As a result, we are going to define smooth manifolds as ordinary manifolds that have an additional structure.

Definition 3.4.1:-If $\psi o\varphi^{-1}$ is a transformation from the $\lambda - plane$ to the $\theta - plane$ defined by the equations $\theta^i = \theta^i(\lambda^1 \ldots \lambda^n)$, then the Jacobian of $\psi o\varphi^{-1}$ which is denoted by $J(\lambda^1 \ldots \lambda^n)$ or $\frac{\partial\theta^i}{\partial\lambda^j}$ defined by

$$\det(J(\lambda^1 \ldots \lambda^n)) = \det\left[\frac{\partial\theta^i}{\partial\lambda^j}\right] = det\begin{bmatrix} \frac{\partial\theta^1}{\partial\lambda^1} & \cdots & \frac{\partial\theta^1}{\partial\lambda^j} \\ \vdots & & \vdots \\ \frac{\partial\theta^i}{\partial\lambda^1} & \cdots & \frac{\partial\theta^i}{\partial\lambda^j} \end{bmatrix} \qquad \ldots(3.12)$$

Definition 3.4.2:- The transformation $\psi o\varphi^{-1}$ which is transformation from $\lambda - plane$ to $\theta - plane$ is said to be a **diffeomorphism** if:

(1) The n-functions $\theta^i(\lambda^1 \ldots \lambda^n)$ are differentiable with respect to $\lambda^1 \ldots \lambda^n$

(2) The Jacobean of the transformation $\det\left[\frac{\partial\theta^i}{\partial\lambda^j}\right] \neq 0$.

Remarks 3.4.3.

(1) The inverse transformation from $\theta - plane$ to $\lambda - plane$ is also diffeomorphism.

(2) Since $\det(J) \neq 0$ then the n-functions $\theta^i(\lambda^1 \dots \lambda^n)$ are independent.

Definition 3.4.4:- Let U and M be an open sets in RN. If $\lambda = \varphi\big(R(t,\lambda)\big)$ and $\theta = \psi\big(R(t,\lambda)\big)$ are two charts. Then on the intersection of their domain $U \cap M$, there exists another homeomorphism **,**

$$H = \psi \text{ o } \varphi^{-1}: \varphi(U \cap M) \rightarrow \psi(U \cap M) \qquad \dots(3.13)$$

Which is called **transition map**,

and its invers is

$$\mathrm{H}^{-1} = \varphi o \psi^{-1} \qquad \dots(3.14)$$

.

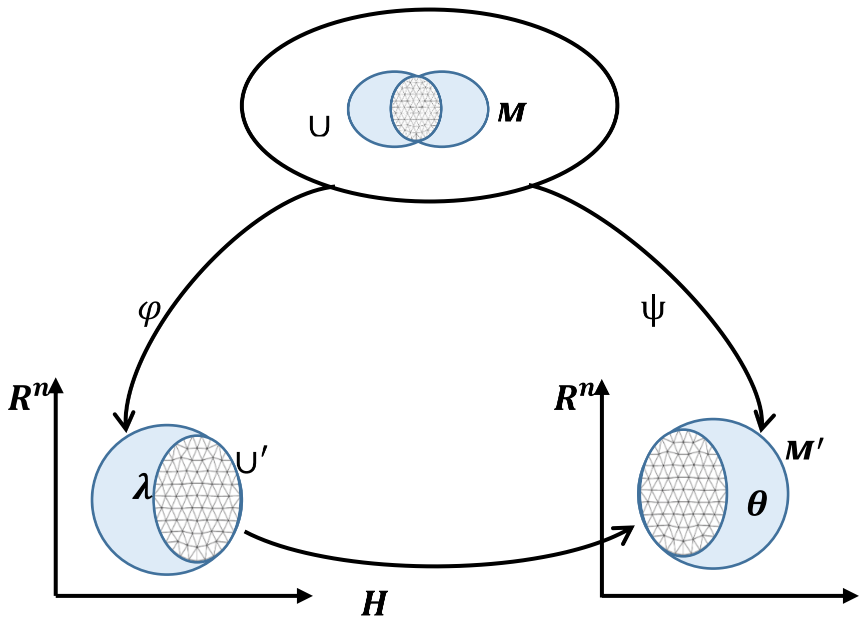


Figure 3.5 : Differential Manifold Correlated Charts.

Definition 3.4.5.:-The transition map H is called $\boldsymbol{C^k -}$ **diffeomorphism** if satisfy the following properties:

1- H is k-times continuously differentiable (i.e. partial derivatives up to the k-th order exist and are continuous).

2- H is bijective

3- $H^{-1} \in C^k\big(\psi(U \cap M)\big) such\ that\ k \in \{0,1, \dots\}\ or\ k = \infty$

Definition 3.4.6:-Two chart $\varphi\ and\ \psi$ are called $\boldsymbol{C^k -}$ **smoothly compatible** if the transition map H is a $C^k -$ diffeomorphism

Definition 3.4.7.:- An atlas $\{(U_i : \varphi_i)_{i\in I}\}$ is called $\boldsymbol{C^k - atlas}$ if any two chart are $C^k -$ smoothly compatible.

Definition 3.4.8.:-An atlas A is called **maximal** $\boldsymbol{C^k - atlas}$ or **complete atlas** if

1- A is a $C^k - atlas$

2- For any other $C^k - atlas$ β we have $\beta \not\supseteq A$

Definition 3.4.9.:- A **differentiable (smooth) reliability manifold** (SRN) is an n-dimensional (topological) manifold that has been equipped with a maximal C^k −atlas (C^k −smoothly structure) .

Theorem 3.4.10.: Let $(\lambda^1, \lambda^2) \in I = (0, \infty) \times (0, \infty)$ be two-dimensional parameter space of reliability function $e^{-\lambda^1(t-\lambda^2)}$. Then The set

$RN = \{R(t, \lambda^1, \lambda^2) : R(t, \lambda^1, \lambda^2) = e^{-\lambda^1(t-\lambda^2)}, t \in (0, \infty), (\lambda^1, \lambda^2) \in I\}$ is smooth manifold.

Proof :-

Let $RN = \left\{e^{-\lambda^1(t-\lambda^2)} : t \in (0, \infty), (\lambda^1, \lambda^2) \in I\right\}$ be reliability manifold defined by the reliability function $e^{-\lambda^1(t-\lambda^2)}$ with two parameters λ^1, λ^2 such that it is $C^k -$ diffeomorphism (partial derivatives exists and continuous to the k-th order.

Let $\ell\ R(t,\lambda^1,\lambda^2) = ln\ R(t,\lambda^1,\lambda^2) = \ln(e^{-\lambda^1(t-\lambda^2)})= -\lambda^1(t-\lambda^2)$.

let $\lambda = (\lambda^1,\lambda^2)$ be a coordinate system with chart $\varphi: U \subseteq RN \to \varphi(U) = \lambda \subset R^2$.

Then $RN = \{e^{-\lambda^1(t-\lambda^2)}: t \in (0,\infty), (\lambda^1,\lambda^2) \in I\}$ be two- dimensional reliability manifold that is diffeomorphic to the upper half-plane in R^2. The entire manifold is covered by only one atlas consists of only one chart φ from open subset U in RN onto open subset $\varphi(U)$ of R^2. There are other possible coordinate systems in particular, consider.

$$m^1 = E(T) = \frac{1}{\lambda^1} e^{\lambda^1\lambda^2}$$

$$m^2 = E(T^2) = \frac{2}{\lambda^1}$$

where E denotes the expectation of a random variable . Then m=(m^1,m^2) is a coordinate system define on the another chart $\psi: U \subseteq RN \to \psi(U) = m \subset R^2$ at the same point.

Then the transition map can be defined as $\psi \text{ o } \varphi^{-1}: \lambda \to$ m , and by using the Jacobean matrix of the transition map we get $\psi \text{ o } \varphi^{-1} = m^1m^2$ which is continuously differentiable of k-th order .

So $\psi \text{ o } \varphi^{-1} = m^1m^2$ is $C^k -$ diffeomorphism , for all $m^1m^2 > 0$, and bijective functions .

Its inverse is $\varphi o \psi^{-1} = \frac{(\lambda^1)^2}{2}\ e^{-\lambda^1\lambda^2}$ is also $C^k -$ diffeomorphism .

By definition we get , two charts φ and ψ are $C^k -$smoothly compatible .

An atlas (U,φ) is $C^k - atlas$ and can be extended to maximal $C^k - atlas$ which is smooth structure . So we get RN be smooth manifold SRN of the reliability function $e^{-\lambda^1(t-\lambda^2)}$.

3.5. Tangent space and 1- Forms in Reliability Manifolds

In the previous section, we discussed the concept of the differentiable manifold and its importance in allowing us to use the concepts of Calculus on the Manifold. Since the manifold does not represent a vector space, this is why we cannot take the sum of two points in the manifold to obtain the third point, and also it is not possible to compare tangent vectors between different points on the manifold. So the differentiable manifold allows us to define the concept of tangent space, that provides us with the possibility of combining and comparing vectors.

The manifold is a local Euclidean space, so at each of its points there is a tangent space. Hence the tangent space in the smooth manifold is a space of vectors on a point in the manifold, whose elements are the tangent vectors to the curves passing through that point, and the dimensions of the tangent space have the same dimension as the manifold.

The vector components of the tangent vector can change their coordinate system in one of two ways: contravariantly or covariantly. If the vector is a contravariant vector, then the components of this vector are calculated from parallel projection, as the component of contravariant vectors changes in the opposite manner to the basis. So since the indices are written in subscript on the basis that the indices in the components are written oppositely in the superscript, this vector is a contravariant vector, so it will be represented in the form of the components by, for example, $\vec{a} = a^1 e_1 + a^2 e_2$, where a^1 and a^2 vectors are the parallel projection of the vector $\vec{a}$, they are called contravariant components. If the vector is covariant, then the components of this vector are calculated from a perpendicular projection. Since the components and the basis of covariant vectors transform in the same manner, the indices in their components are written in the same subscript as the basis. Thus, if it is a covariant vector, it is represented in the form of components by representation. for example $\vec{a} = a_1 e_1 + a_2 e_2$ where a_1

and a_2 are perpendicular projection of the vector $\vec{a}$; there are called covariant components .

Contravariant vectors was used to define the tangent vector space and covariant vectors to define the cotangent vector space, or 1-Forms at point.

After we have talked about contravariant and covariant vector components, we now define invariant components, or invariants, as follows: Invariants are mathematical objects that have an intrinsic value and are fundamental significance in terms of coordinate transformations; invariant means as an object that does not change under a change of coordinates ; for example, tensors are invariant objects under a changed of coordinates but their components are not invariant in general but the tensors themselves are invariant. Below is how to convert a point between two coordinates in the same space:

If V_n be an n-dimensional space, let $(x^1, x^2 \dots x^n)$ be any set of coordinate in this space and $(\bar{x}^1, \bar{x}^2, \dots . \bar{x}^n)$ be another set of coordinates in the same space V_n. Each of the coordinates x^i will be a function of n other coordinates $\bar{x}^\propto$ and vice versa. This means we can write:

$x^i = x^i(\bar{x}^1, \bar{x}^2 \dots \bar{x}^n)$ where $i = 1,2 \dots n$, and $\bar{x}^\propto = \bar{x}^\propto(x^1, x^2 \dots x^n)$, where $\propto = 1,2, \dots n$. These relations are called **"transformation of coordinates from one frame of reference to another**.". We have, $x^i = x^i(\bar{x}^1, \bar{x}^2, \dots \bar{x}^n)$ so by partial differentiation we get,

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^1} d\bar{x}^1 + \frac{\partial x^i}{\partial \bar{x}^2} d\bar{x}^2 + \cdots + \frac{\partial x^i}{\partial \bar{x}^n} d\bar{x}^n$$

$$= \sum_{\propto=1}^{n} \frac{\partial x^i}{\partial \bar{x}^\propto} d\bar{x}^\propto \ , i = 1,2, \dots, n \qquad \dots(3.15)$$

Similarly,

$$d\bar{x}^\alpha = \sum_{i=1}^{n} \frac{\partial \bar{x}^\propto}{\partial x^i} dx^i \text{ where } \propto = 1,2 \dots, n \qquad \dots(3.16)$$

Thus unbarred coordinate system we use Latin letters (e.g. i,j,k,…) . Barred coordinate system use Geek letters (e.g. $\propto, \beta, \gamma$ ...) [16,31,46,70,80,90].

As a result of the fact that the tangent space of smooth manifolds is a generalization of the two-dimensional tangent lines to curves and the three-dimensional tangent planes to surfaces, the parametric representation of the curve in Euclidean space can be described as follows:

Definition 3.5. 1.:- Let $SRN = \{R(t, \lambda)\}$ is a smooth Reliability manifold, let $C(x): [a, b] \rightarrow SRN$ be a smooth curve on SRN such that ,

$C(x) = R(t, \lambda), \forall x \in [a, b]$ *and let* $\varphi: SRN \rightarrow R^n$ be a chart on SRN such that $\varphi^{-1}(\lambda) = R(t, \lambda)$. Then the parametric representation of the curve C in R^n is given by $\lambda(x) = \lambda = \varphi(R(t, \lambda)) = (\varphi o C)(x)$.

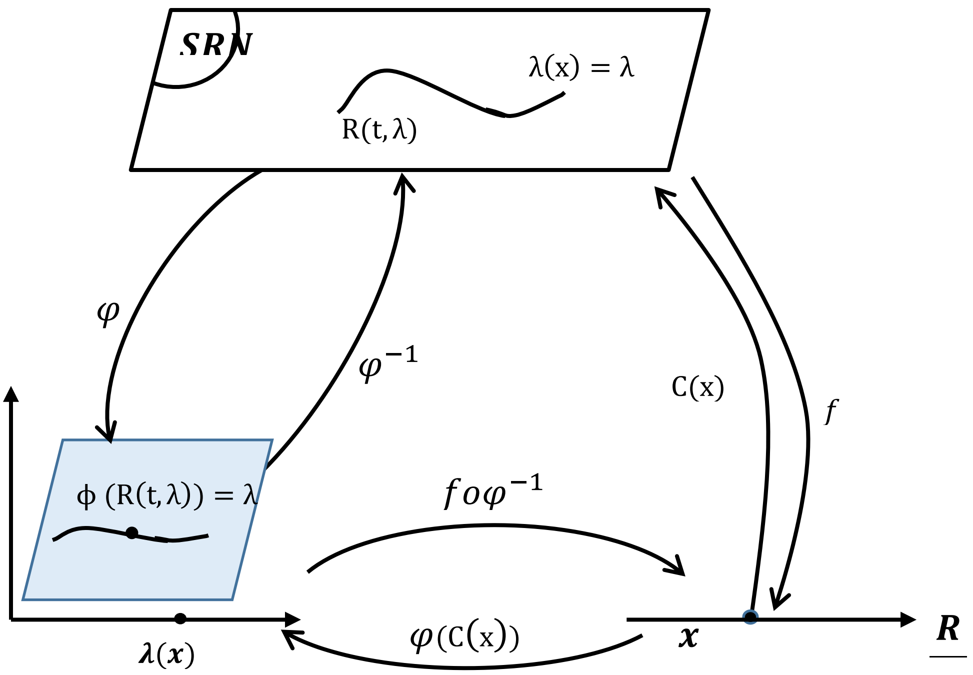


Figure 3.6 : The Parametric Representation of the Curve C in R^n.

Definition 3.5.2.: A function $f: SRN \rightarrow R$ is **differentiable on reliability smooth manifold SRN** provided the local coordinate representatives $f o\varphi^{-1}$: $\varphi(\lambda) \rightarrow$ R is differentiable for any given chart φ on SRN

Definition 3.5.3. :- A **smooth curve** C on the reliability smooth manifold SRN is a smooth map from [a,b] into N such that $C(x) = R(t,\lambda)$

Definition 3.5.4. : let F be the set of all the smooth real functions f on the smooth manifold SRN (i.e.) $f: SRN \to R$ .We define **tangent vector $\boldsymbol{V}$ at the point** $R(t,\lambda) \in SRN$ as a map $V_{R(t,\lambda)}: F \to R$ which is required to satisfy linearity and Leibniz rule:

(1) $V_{R(t,\lambda)}(af + bg) = aV_{R(t,\lambda)}(f) + bV_{R(t,\lambda)}(g)$. …(3.17)

(2) $V_{R(t,\lambda)}(fg) = f(R(t,\lambda)V_{R(t,\lambda)}(g) + g(R(t,\lambda)V_{R(t,\lambda)}(f)$ ...(3.18)

for all $a, b \in R \; and \; f, g \in F$.

Definition 3.5.5. : let $C : [a,b] \to SRN$ be differentiable curve on the smooth manifold SRN with $C(0) = R(t,\lambda)$and $f \in F$. If $foC = f\{\lambda(x)\} : [a,b] \to R$. Then the derivative of the function foC is called **a tangent vector to $\boldsymbol{C}$** where

$$V_{R(t,\lambda)}(f) = \frac{d(foC)}{dx} = \frac{d}{dx} f\{\lambda(x)\} = \sum_{i=1}^{n} \frac{d\lambda^i}{dx} \frac{\partial}{\partial \lambda^i} f \qquad \text{…(3.19)}$$

Note that $V_{R(t,\lambda)}(f)$ be the usual derivative of real valued function foC , and $\frac{d}{dx}$ is linear and satisfy leibmiz rule . So we can say that $V_{R(t,\lambda)}(f)$ is a tangent vector to $C[a,b]$.

Definition 3.5.6.:- The tangent space $T_{R(t,\lambda)}$ at a point $R(t,\lambda)$ of the smooth manifold SRN is the set of all tangent vectors V at $R(t,\lambda)$.The vectors of the vector space $T_{R(t,\lambda)}$ are celled Contravariant vectors.

Theorem 3.5.7:- Tangent space $T_{R(t,\lambda)}$ of SRN at the point $R(t,\lambda)$ form an n-dimensional vector space if SRN is sufficiently smooth ($C^\infty -$ manifold) .

Proof :-

Let SRN be $C^\infty -$ manifold , and $f: SRN \to R$ be a smooth functions .

If a coordinate system λ is given **,**so there exists n coordinate curves $C_1, \ldots, C_n$ passing through a point $R_0(t,\lambda)$.

Let coordinate curves C_1, C_2 be two curves passing through λ_0 , such that the curve C_1 is represented by

$\lambda_1(x) = \left(\lambda_o{}^1 + t, \lambda_o{}^2, \ldots, \lambda_o{}^n\right)$, where $\lambda_0 = \left(\lambda_o{}^1, \ldots, \lambda_o{}^n\right)$ is the coordinates of $R_0(t,\lambda)$ and the curve C_2 is represented by $\lambda_2(x) = \left(\lambda_o{}^1, \lambda_o{}^2 + t, \ldots, \lambda_o{}^n\right)$. Then the tangent vectors $V_1 \; of \; C_1$ and $V_2 \; of \; C_2$ are

$$\mathrm{V}_1(f) = \frac{d}{dx} f[\lambda_1(x)] = \frac{\partial}{\partial \lambda^1} f = \frac{\partial}{\partial \lambda^1} = \partial_1$$

and

$$\mathrm{V}_2(f) = \frac{d}{dx} f[\lambda_2(x)] = \frac{\partial}{\partial \lambda^2} f = \frac{\partial}{\partial \lambda^2} = \partial_2 \; .$$

Now we want to show that , for all $\partial_1 \; and \; \partial_2 \in T_{R(t,\lambda)} \, (SRN)$ we have

$(\partial_1 + r\partial_2)(f) \in T_{R(t,\lambda)} \, (N)$ for all $r \in R$.

1) Since ∂_1 and $\partial_2 \in T_{R(t,\lambda)} \, (SRN)$ then by definition of tangent vector ∂_1 and ∂_2 are R-linear .Hence $(\partial_1 + r\partial_2)$ are also R-linear .

2) Let $f, g \; \in F(SRN)$, then

$$\begin{aligned}(\partial_1 + r\partial_2)(fg) &= \partial_1(fg) + r\partial_2(fg) \\ &= \partial_1(f)g(R(t,\lambda)) + f(R(t,\lambda))\partial_1(g) + r[\partial_2(f)g(R(t,\lambda)) + \\ &\qquad f(R(t,\lambda))\partial_2(g) \\ &= (\partial_1 + r\partial_2)\,(f)\; g\big(R(t,\lambda)\big) + f(R(t,\lambda))(\partial_1 + r\partial_2)(g)\end{aligned}$$

It follows that , $(\partial_1 + r\partial_2)$ satisfies the Leibniz law .

Hence $(\partial_1 + r\partial_2) \in T_{R(t,\lambda)} \, (N)$. So $T_{R(t,\lambda)} \, (SRN)$ is a real vector space

Theorem 3.5.8.:- A tangent vector at a point R(t, λ) is a directional derivative operator along a curve that passes through R(t, λ).

Proof:-

Let $V_{R(t,\lambda)}$ be a tangent vector at a point R(t, λ) ∈ SRN, working on the smooth functions f that are on the manifold SRN , such that $V_{R(t,\lambda)}(f) = \frac{d}{dx}(foC(x))\backslash_{R(t,\lambda)}$.

Since $f \in F$, then we have the function $foC: [a, b] \to R$

Let φ be a chart at the point R(t, λ) will be given by $R(t, \lambda) = \varphi^{-1}(\lambda)$ where $\lambda = (\lambda^1, \dots, \lambda^n) \in R^n$.

So the point λ will be given with local coordinates .

Since $R(t, \lambda) = C(x)$, then $C(x) = \varphi^{-1}(\lambda)$ and the map C is mapped into a curve $\lambda(x)$ in R^n . We have immediately

$$V_{R(t,\lambda)}(f) = \frac{d}{dx}(fo\varphi^{-1}(\lambda))\backslash_{R(t,\lambda)}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial \lambda^i}(fo\varphi^{-1}(\lambda))\frac{d\lambda^i}{dx}\backslash_{R(t,\lambda)}$$

$$= \sum_{i=1}^{n} \partial_i(f)\frac{d\lambda^i}{dx}\backslash_{R(t,\lambda)}$$

So we have

$$\partial_i(f) = \frac{\partial}{\partial \lambda^i}(fo\varphi^{-1}(\lambda))$$

From theorem (3.5.7.) we get $\partial_i(f)$ satisfy linearity and Leibniz rule .

So $\partial_i(f)$ are directional derivatives or differential displacements, and they are locally derivative along the curve.

Hence these vectors ∂_i are tangent vectors to the manifold SRN at R(t, λ) and we may make the identification $\frac{\partial}{\partial\lambda^i}\left(fo\varphi^{-1}(\lambda)\right) = \partial_i$.

Theorem 3.5.9. :- let φ be a chart about $R(t,\lambda) \in SRN$. Then $\{\partial_i : i = 1,2,\dots,n\}$ is a basis of $T_{R(t,\lambda)}(N)$. In particular $T_{R(t,\lambda)}(SRN)$ has dimension n .

Proof:- Let F be the set of all the smooth real function $f(\lambda^1,\dots,\lambda^n)$ in λ.

Let $C = C(x) or\ \lambda(x)$ be smooth curve and a function $f \in F$.

Then $foC: [a,b] \to R$ which is written as $f\{\lambda(x)\}$.

Let V be a tangent vector such that $V = \frac{d(foc)}{dx}$.

In a given chart φ the point $R(t,\lambda)$ will be given by $R(t,\lambda) = \varphi^{-1}(\lambda)$

Hence $C(x) = \varphi^{-1}(\lambda)$.

Now,
$$V = \frac{d}{dx}\left(foC(x)\right)\backslash_{R(t,\lambda)}$$
$$= \frac{d}{dx}(fo\varphi^{-1}(\lambda))\backslash_{R(t,\lambda)}$$
$$= \sum_{i=1}^{n} \frac{\partial}{\partial\lambda^i}\left(fo\varphi^{-1}(\lambda)\right)\frac{d\lambda^i}{dx}\backslash_{R(t,\lambda)}$$

$$V = \sum_{i=1}^{n} \partial_i \frac{d\lambda^i}{dx}\backslash_{R(t,\lambda)} \qquad \dots(3.20)$$

The vectors ∂_i are maps acting on functions f on the manifold SRN defined by

$$\partial_i = \frac{\partial}{\partial\lambda^i}\left(fo\varphi^{-1}(\lambda)\right) \qquad \dots(3.21)$$

The vectors ∂_i satisfy linearity and the Leibniz rule and they are obviously directional derivatives or differential displacements

Hence these vectors ∂_i are tangent vectors to the manifold SRN at $R(t,\lambda)$

The fact that arbitrary tangent vectors can be expressed as linear combinations of the n vectors ∂_i shows that these vectors are linearly independent, span the vector space $T_{R(t,\lambda)}\ (SRN)$ and that the dimension of $T_{R(t,\lambda)}\ (SRN)$ is exactly n.

Hence the equation (3.20) can then be written as

$$V = \sum_{i=1}^{n} \partial_i V^i, V^i = \frac{d\lambda^i}{dx} \backslash_{R(t,\lambda)} \qquad \dots(3.22)$$

V^i are the components of the contravariant vector V with respect to the natural basis $\{\partial_i\}$.

Definition 3.5.10.: **Tangent space T_λ in the coordinate system $\boldsymbol{\lambda}$** is n-dimensional vector space spanned by directional derivative ∂_i such that

$$T_\lambda = \{V: V = V^i \partial_i\} \qquad \dots(3.23)$$

where V^i are the components of the tangent vector V with respect to the basis $\{\partial_i\}$ is called **a tangent space with the coordinate system $\boldsymbol{\lambda}$** $= (\lambda^1, \dots, \lambda^n) \in R^n$.

Definition 3.5.11.: Let n partial derivatives $\partial_i \ell\ (R(t,\lambda)), i = 1, \dots, n$, be linearly independent functions in the random variable t.

Then n-dimensional vector space spanned by $\partial_i \ell\ (R_\lambda(t))$ in t,

$$T_\lambda{}^{(1)} = \{V(t): V(t) = V^i \partial_i \ell(R(t,\lambda))\} \qquad \dots(3.24)$$

where V^i are the components of V(t) with respect to the basis $\partial_i \ell\ (R(t,\lambda))\ is$ called the **1- representation** of the tangent space .

Definition 3.5.12.:- Let λ= (λ^i) and $\theta = (\theta^\propto)$ be two coordinate system in the tangent space T_λ , $(i = 1, \dots, n, \propto = 1, \dots, n$

Let $\theta = \theta(\lambda), \lambda = \lambda(\theta)$ are diffeomorphism between $\lambda\ and\ \theta$

Then the **Jacobin matrices** of the two coordinate transformation are written as

$$\mathfrak{B}_j^{\alpha}(\lambda) = \frac{\partial\theta^{\propto}}{\partial\lambda^j} \quad \text{….(3.25)}$$

and its inverse given by

$$\ddot{\mathfrak{B}}_{\alpha}^{i}(\theta) = \frac{\partial\lambda^i}{\partial\theta^{\propto}} \quad \text{….(3.26)}$$

such that

$$\ddot{\mathfrak{B}}_{\alpha}^{i}\mathfrak{B}_j^{\alpha} = \delta_j^i = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ \ i = j \end{cases}$$

and similarly, we have

$$\mathfrak{B}_i^{\alpha}\ddot{\mathfrak{B}}_{\beta}^{i} = \delta_{\beta}^{\propto} = \begin{cases} 0 & if\ \ \propto \neq \beta \\ 1 & if\ \ \propto = \beta \end{cases}$$

where δ_j^i and $\delta_{\beta}^{\propto}$ $a$re the kroneeker delta

Definition 3.5.13..:- Let $\{\partial_i\}$ and $\{\partial_{\propto}\}$ be the natural bases of the tangent space with respect to $\lambda = (\lambda^i)$ and $\theta = (\theta^{\propto})$ respectively.

Then the relation

$$\partial_{\propto} = \ddot{\mathfrak{B}}_{\alpha}^{i}\partial_i \quad \text{…(3.27)}$$

is the transformation law of the base vectors under coordinate transformation $\lambda \to \theta$, and the relation.

$$\partial_i = \mathfrak{B}_i^{\alpha}\partial_{\propto} \quad \text{…(3.28)}$$

is the transformation law of the base vectors under coordinate transformation $\theta \to \lambda$. If V be a tangent vector in T_{λ} . So ,

$$V = V^i\partial_i = V^{\propto}\partial_{\propto} \quad \text{….(3.29)}$$

where $V^i and\ V^{\propto}$ are the components of vector V.

The transformation law of the tangent vector under the coordinate transformation $\theta \to \lambda$ is given by :

$$V^i = \ddot{\mathfrak{B}}_\alpha^i \, V^\propto \qquad \dots(3.30)$$

while the transformation law of the tangent vector under the coordinate transformation $\lambda \rightarrow \theta$ is given by:

$$V^\propto = \mathfrak{B}_i^\alpha V^i \qquad \dots(3.31)$$

Theorem 3.5.14.:- If the tangent vector $V \in T_\lambda$ corresponds to a random variables $V(t) \in {T_\lambda}^{(1)}$. Then the 1-repreentation $V(t)$ of V doesn't change for any coordinate system

Proof:-

Let $\{\partial_i\}$ and $\{\partial_\propto\}$ be the natural bases of the tangent space with respect to λ and θ respectively.

Since ∂_i and $\partial_\propto$ are partial derivatives , then the relations (3.27) and (3.28) holds . i.e.

$$\partial_\propto = \ddot{\mathfrak{B}}_\alpha^i \partial_i \ , \partial_i = \mathfrak{B}_i^\alpha \partial_\propto$$

Since V be a tangent vector , then we can be written as a linear combination of ∂_i and $\partial_\propto$ (i.e.)

$$V = \sum_{i=1}^{n} V^i \partial_i = \sum_{\propto=1}^{n} V^\propto \partial_\propto$$

with the components V^i and $V^\propto$

Since $\partial_\propto = \ddot{\mathfrak{B}}_\alpha^i \partial_i$, $\partial_i = \mathfrak{B}_i^\alpha \partial_\propto$,then the components

$$V^i = \ddot{\mathfrak{B}}_\alpha^i \, V^\propto , V^\propto = \mathfrak{B}_i^\alpha V^i$$

So the components of a vector are changed by the coordinate transformation. Therefore , the 1-representation $V(t)$ of V is invariant for any coordinate system

$$V(t) = \sum_{i=1}^{n} V^i \partial_i \, \ell\mathrm{R}(\mathrm{t},\lambda) = \sum_{\propto=1}^{n} V^{\propto} \partial_{\propto} \, \ell\mathrm{R}(\mathrm{t},\theta)$$

and only its components change in a contravariant manner as the basis changes .

Theorem 3.5.15.:- Let SRN $= \{\mathrm{R}(\mathrm{t},\lambda) : \mathrm{R}(\mathrm{t},\lambda) = \mathrm{e}^{-\propto(\mathrm{t}-\theta)}, t > 0, \propto, \theta > 0\}$, be reliability smooth manifold with exponential life time distribution . Let $\lambda = (\lambda_1, \lambda_2)$ be the parameter such that $\lambda_1 = \propto, \lambda_2 = \theta$. If the basis of the tangent space T_λ is known, then it is possible to construct a tangent space ${T_\lambda}^{(1)}$ from it.

Proof :- Let $\lambda = (\lambda_1, \lambda_2)$ be the parameter such that $\lambda_1 = \propto, \lambda_2 = \theta$.

Then the natural basis $\{\partial_i\}$ is $\partial_1 = \frac{\partial}{\partial \alpha}$, $\partial_2 = \frac{\partial}{\partial \theta}$,which is spanned the tangent vector T_λ .

From $l\big(R(t,\lambda)\big) = \ln R(t, \propto, \theta) = \ln(e^{-\propto(t-\theta)} = -\propto (t-\theta)$,

we get the basis $\partial_i \, l \, (R(t,\lambda)$ of the 1-representation is calculated as

$$\partial_1 l \left(R(t,\lambda)\right) = \frac{\partial}{\partial \alpha} \left(-\propto (t-\theta)\right) = \theta - t \qquad \text{....(3.32)}$$

$$\partial_2 l \left(R(t,\lambda)\right) = \frac{\partial}{\partial \theta} \left(-\propto (t-\theta)\right) = \propto \qquad \text{...(3.33)}$$

Then the space $T_\lambda^{(1)}$ is spanned by eq. (3.32)and eq.(3.33).

Let $\epsilon_1 = E(T) = \frac{1}{\propto} e^{\propto\theta} = \frac{1}{\lambda_1} e^{\lambda_1 \lambda_2}$, and $\epsilon_2 = E(T^2) = \frac{2}{\propto} = \frac{2}{\lambda_1}$

The parameter $\epsilon = (\epsilon^\gamma), \gamma = 1,2$ defines another coordinate system and the Jacobian matrix of the coordinate transformation is given by:

$$\mathfrak{B}_i^\gamma = \frac{\partial \epsilon^\gamma}{\partial \lambda_i} = \begin{bmatrix} \frac{\partial \epsilon^1}{\partial \alpha} & \frac{\partial \epsilon^1}{\partial \theta} \\ \frac{\partial \epsilon^2}{\partial \alpha} & \frac{\partial \epsilon^2}{\partial \theta} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\propto\theta}\left(\frac{\propto\theta-1}{\propto^2}\right) & e^{\propto\theta} \\ -\frac{2}{\propto^2} & 0 \end{bmatrix}$$

And its inverse is given by $\dddot{\mathfrak{B}}_{\gamma}^{i}$.

Since the $det(\mathfrak{B}_i^{\gamma}) = \frac{2}{\propto^2} e^{\propto\theta}$, then we have

$$\dddot{\mathfrak{B}}_{\gamma}^{i} = \frac{\propto^2}{2e^{\propto\theta}} \begin{bmatrix} 0 & -e^{\propto\theta} \\ \frac{2}{\propto^2} & e^{\propto\theta}\,[\frac{\propto\theta-1}{\propto^2}] \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{\propto^2}{2} \\ \frac{1}{e^{\propto\theta}} & \frac{\propto\theta-1}{2} \end{bmatrix}.$$

Then the natural basis vectors $\{\partial_\gamma\}$,

$$\partial_\gamma = \dddot{\mathfrak{B}}_{\gamma}^{i}\partial_i = [\partial_1\partial_2] \begin{bmatrix} 0 & \frac{-\propto^2}{2} \\ \frac{1}{e^{\propto\theta}} & \frac{\propto\theta-1}{2} \end{bmatrix},$$

where $\gamma = 1', 2', i = 1,2$.

$$[\partial_{1'} \quad \partial_{2'}] = \begin{bmatrix} \frac{\partial_2}{e^{\propto\theta}} & -\frac{\propto^2}{2}\partial_1 + \left(\frac{\propto\theta - 1}{2}\right)\partial_2 \end{bmatrix}$$

Then

$$\partial_{1'} \quad = \frac{\partial_2}{e^{\propto\theta}}$$

$$\partial_{2'} = -\frac{\propto^2}{2}\partial_1 + \left(\frac{\propto\theta-1}{2}\right)\partial_2$$

where$1'$ and $2'$ are used to denote the $\{\partial_\gamma\} - system$.

Their 1-representations are :

$$\partial_{1'}l\left(R(t,\lambda)\right) = \frac{\partial_2}{e^{\propto\theta}}\left(-\propto (t-\theta)\right)$$

$$=\frac{\partial_2}{e^{\propto\theta}}(-\propto t+\propto \theta)$$

$$=\partial_2\left(\frac{-\propto t}{e^{\propto\theta}}\right)+\left(\partial_2\frac{\propto\theta}{e^{\propto\theta}}\right)$$

$$=\propto^2 te^{\propto\theta-2\propto\theta}+\frac{\propto e^{\propto\theta}-\propto^2\theta e^{\propto\theta}}{e^{2\propto\theta}} \quad \ldots(3.34)$$

$$\partial_{2'}l\left(R(t,\lambda)\right)=\left[\frac{-\propto^2}{2}\partial_1+\left(\frac{\propto\theta-1}{2}\right)\partial_2\right][-\propto t+\propto\theta]$$

$$=\frac{\propto^3 t}{2}\partial_1-\frac{\propto^3\theta}{2}\partial_1-\left(\frac{\propto^2 t\theta+\propto t}{2}\right)\partial_2+\left(\frac{\propto^2\theta^2-\propto\theta}{2}\right)\partial_2$$

$$=\frac{3\propto^2 t}{2}-\frac{3\propto^2\theta}{2}-\frac{\propto^2 t}{2}+\frac{2\propto^2\theta-\propto}{2} \quad \ldots.(3.35)$$

Hence the space ${T_\lambda}^{(1)}$ is spanned by (3.34)and (3.35).

3.6. 1-FORMS

Definition 3.6.1:- Let $U: T_{R(t,\lambda)} \to R$ be a scalar function such that satisfies the linearity condition: $U(aV^1+bV^2)=aU(V^1)+bU(V^2)$ …..(3.36)

for all $a, b \in R$ and $V^1, V^2 \in T_{R(t,\lambda)}$.

Then the set of all such linear functions on $T_{R(t,\lambda)}$ is denoted by $T^*{}_{R(t,\lambda)}$ which is called **the dual vector space** to $T_{R(t,\lambda)}$ or **the cotangent vector space of SRN at $R(t,\lambda)$ or covariant vectors space.**

The element of $T^*{}_{R(t,\lambda)}$ are called **convectors or dual vectors or one-forms**

Theorem 3.6.2. : - The dual vector space $T^*{}_{R(t,\lambda)}$ is a vector space over R.

Proof :- from definition of dual vector space , we can define two binary operations**.**

(i)The binary operation "+" on $T^*{}_{R(t,\lambda)}$ as

$$(U_1+U_2)(V)=U_1(V)+U_2(V) \quad \ldots(3.37)$$

$\forall\ U_1, U_2, \in\ {T^*}_{R(t,\lambda)}\ and\ \ V \in T_{R(t,\lambda)}$

(ii) an operations of scalar multiplication by :

$$(aU)(V) = aU(V) \qquad \ldots(3.38)$$

$\forall\ U\ \in\ {T^*}_{R(t,\lambda)}\ and\ \ V \in T_{R(t,\lambda)}$. With this two operations we have

${T^*}_{R(t,\lambda)}$ is a vector space over R which is the dual space of $T_{R(t,\lambda)}$.

Definition 3.6.3.:- In local coordinates $(\lambda^1, \ldots, \lambda^n)$ we take $\{d\lambda^i\}$ is **the dual basis** of $\{\frac{\partial}{\partial \lambda^i} = \partial_i\}$ of $T_{R(t,\lambda)}$. (i.e.) $d\lambda^i(\frac{\partial}{\partial \lambda^j}) = \delta^i_j$,where δ^i_j denotes the Kronecker symbol. Then the **cotangent space** $\boldsymbol{{T^*}_{R(t,\lambda)}}$ of SRN at $R(t,\lambda)$ is spanned by $\{d\lambda^1, \ldots, d\lambda^n\}$.

Definition 3.6.4.:- Let $\{d\lambda^i\}$ and $\{d\lambda^\propto\}$ be the natural bases of the cotangent space with respect to $\beta = (\beta^i)$ and $\theta = (\theta^\propto)$ respectively .

If U be a tangent vector in ${T^*}_{R(t,\lambda)}$ .So

$$U_i = \mathfrak{B}^\propto_j U_\propto \qquad \ldots(3.39)$$

and

$$U_\propto = \dddot{\mathfrak{B}}^i_\propto U_i \qquad \ldots(3.40)$$

are **the transformation law of tangent vector** under the coordinate transformation and we have

$$d\lambda^i = \dddot{\mathfrak{B}}^i_\propto d\lambda^\propto \qquad \ldots(3.41)$$

$$d\lambda^\propto = \mathfrak{B}^\propto_j d\lambda^i \qquad \ldots(3.42)$$

3.7. Riemannian Reliability Manifold

A Riemannian reliability manifold, sometimes known as a Riemannian space, is a type of differentiable reliability manifold in which every tangent space is provided with a Riemannian metric tensor (also known as an inner product). In this chapter, we have discussed several fundamental ideas regarding tensors, which serve as the foundation for the Riemannian metric's definition. After that, we constructed the Riemannian reliability manifold and used the exponential life function to get the Riemannian metric distance for the two-dimensional reliability manifold. At the very end, we had a look at the f-divergence function, which evaluates the amount of difference that exists between two probability distributions as well as between two reliability functions. This is a measure of how different two functions are from one another. After that, we looked at several unique applications of f-divergence functions and applied some instances.. [11,13,20 ,30,44,51].

Definition 3.7.1. :- The inner product of two vectors $A = A^i \partial_i$ and $B = B^j \partial_j$ on the tangent space $T_{R(t,\lambda)}$ is given by

$$< A, B >=< A^i \partial_i, B^j \partial_j > = A^i B^j g_{ij} \qquad \dots (3.43)$$

On the other hand if $A(t)$ and $B(t)$ are vectors on the tangent space $T_\lambda{}^{(1)}$.

Then the inner product is given by :

$$< A, B >= E[A(t) * \ B(t)].$$

Definition 3.7.2. :- Let $A = A^i \partial_i$ and $B = B^j \partial_j$ be two tangent vectors on the tangent space $T_{R(t,\lambda)}$,if

$$< A, B >=< A^i \partial_i, B^j \partial_j > = A^i B^j g_{ij} = 0 \qquad \dots (3.44)$$

Then A and B are said to be **orthogonal tangent vectors .**

Definition 3.7.3. :- Riemannian metric tensor g_{ij} on differentiable reliability manifold DRN is an inner product of the two basis vector on the tangent space $T_{R(t,\lambda)}$, which is symmetric , positive definite matrix and covariant tensor of order 2 that depends on λ −coordinate system

$$g_{ij} =< \partial_i, \partial_j > \quad \text{....(3.45)}$$

The components transform as follows:

$$g_{\alpha\beta} =< \partial_\alpha, \partial_\beta > =< \ddot{\mathfrak{B}}_\alpha^i \partial_i, \ddot{\mathfrak{B}}_\beta^j \partial_j >$$

$$= \ddot{\mathfrak{B}}_\alpha^i \ddot{\mathfrak{B}}_\beta^j < \partial_i, \partial_j >$$

$$g_{\alpha\beta} = g_{ij} \ddot{\mathfrak{B}}_\alpha^i \ddot{\mathfrak{B}}_\beta^j \quad (3.46)$$

Definition 3.7.4.:- Riemannian reliability manifold $(RRN, g_{ij}(\lambda))$ is differentiable reliability manifold where each tangent space is equipped with Riemannian metric tensor $g_{ij}(\lambda)$ (or inner product) such that $g_{ij}(\lambda) \neq \delta_{ij}$, where $g_{ij}(\lambda)$ is called Fisher information matrix which is define as follows :

$$g_{ij}(\lambda) =< \partial_i, \partial_j >= E_\lambda\left[(\partial_{\lambda^i} \ell R(t,\lambda)) * (\partial_{\lambda^j} \ell\left(R(t,\lambda)\right)\right] \quad \text{... (3.47)}$$

or $g_{ij}(\lambda) = -E_\lambda\left[(\partial_{\lambda^i} \partial_{\lambda^j} \ell R(t,\lambda))\right]$ ... (3.48)

Definition 3.7.5. : Let $R(t,\lambda)$ and $R(t,\lambda + d\lambda)$ be two reliability functions which are "infinitesimally" close . Consider λ and $\lambda + d\lambda$ be respectively their coordinates .Then **The Riemannian metric ds^2** between two reliability functions $R(t,\lambda)$ and $R(t,\lambda + d\lambda)$ on any tangent space is given by :

$$ds^2 = g_{ij} d\lambda^i d\lambda^j \quad \text{.... (3.49)}$$

$$= g_{11} d\lambda^1 d\lambda^1 + g_{12} d\lambda^1 d\lambda^2 + g_{21} d\lambda^2 d\lambda^1 + g_{22} d\lambda^2 d\lambda^2$$

where g_{ij} be the Fisher information matrix .

Theorem 3.7.6.:- Consider the case of a two-dimensional reliability with an exponential lifetime distribution such that $e^{-\lambda(t-\theta)}$. Then The distance between two reliability functions is

$$ds^2 = \left[e^{-\lambda(1-\theta)}\left(\theta^2 + 2\theta + \frac{2\theta}{\lambda} - \frac{2}{\lambda} - \frac{2}{\lambda^2} - 1\right) - e^{\lambda(\theta)}\left(\theta^2 + \frac{2\theta}{\lambda} - \frac{2}{\lambda^2}\right)\right](d\lambda)^2 + d\lambda d\theta + d\theta d\lambda + \lambda^2(d\theta)^2$$

Proof :- The Fisher information matrix in equation (3.47) is

$$g_{ij}(\lambda,\theta) = E_\lambda\left[(\partial_{\lambda^i}\ell R(t,\lambda)) * (\partial_{\lambda^j}\ell\big(R(t,\lambda)\big)\right]$$

Since $f(t) = \lambda e^{-\lambda(t-\theta)}$ and the reliability function is $e^{-\lambda(t-\theta)}$

Then $\ell R(t,\lambda) = \theta\lambda - \lambda t$ and $\partial_\lambda \ell R(t,\lambda) = \theta - t$.

So $g_{11}(\lambda,\theta) = E_\lambda\left[\big(\partial_\lambda \ell R(t,\lambda)\big) * (\partial_\lambda \ell\big(R(t,\lambda)\big)\right] = E_\lambda[(\theta - t)^2]$

$$g_{11}(\lambda,\theta) = \int_0^\infty (\theta - t)^2 . f(t)dt = \int_0^\infty (\theta - t)^2 . \lambda e^{-\lambda(t-\theta)} dt$$

$$g_{11}(\lambda,\theta) = \left[e^{-\lambda(t-\theta)}(\theta^2 + 2t\theta + \frac{2\theta}{\lambda} - t^2 - \frac{2t}{\lambda} - \frac{2}{\lambda^2}\right]_0^\infty$$

Then , we have

$$g_{11}(\lambda,\theta) = e^{\lambda(\theta)}\left(\theta^2 + \frac{2\theta}{\lambda} - \frac{2}{\lambda^2}\right), \quad g_{12}(\lambda,\theta) = g_{21}(\lambda,\theta) = 1$$

and $\quad g_{22}(\lambda,\theta) = \lambda^2$

Hence (N, g_{ij}) be a Riemannian manifold with the metric ,

$$g_{ij} = \begin{bmatrix} e^{\lambda(\theta)}\left(\theta^2 + \frac{2\theta}{\lambda} - \frac{2}{\lambda^2}\right) & 1 \\ 1 & \lambda^2 \end{bmatrix}$$, where $t \in [0,1]$.

The distance between two reliability functions is :

$$ds^2 = g_{ij} d\lambda^i d\lambda^j$$

$$= \left[e^{\lambda(\theta)}\left(\theta^2 + \frac{2\theta}{\lambda} - \frac{2}{\lambda^2}\right)\right](d\lambda)^2 + d\lambda d\theta + d\theta d\lambda + \lambda^2 (d\theta)^2$$

Example 3.1. The p.d.f. of the Gamma distribution is

$p(x,\beta,\alpha) = \frac{\beta^\alpha}{\Gamma\alpha} e^{-\beta x} x^{\alpha-1}, x > 0, \beta, \alpha > 0$. Then the logarithm likelihood function is : $lnf = \alpha ln\beta + (\alpha - 1)lnx - x\beta - ln\Gamma\alpha$.

The partial derivatives of lnf with respect to β, α is :

$$\frac{\partial^2 lnf}{\partial\beta^2} = \frac{-\alpha}{\beta^2} \quad , \quad \frac{\partial^2 lnf}{\partial\beta\partial\alpha} = \frac{\partial^2 lnf}{\partial\alpha\partial\beta} = \frac{1}{\beta} \quad , \frac{\partial^2 lnf}{\partial\alpha^2} = -\left(\frac{(\acute{\Gamma}(\alpha))}{\Gamma(\alpha)}\right)' .$$

$$g_{11} = -E\left(\frac{\partial^2 lnf}{\partial\beta^2}\right) = \frac{\alpha}{\beta^2} \quad , \quad g_{12} = g_{21} = -E\left(\frac{\partial^2 lnf}{\partial\beta\partial\alpha}\right) = -\frac{1}{\beta}$$

and $g_{22} = -E\left(\frac{\partial^2 lnf}{\partial\alpha^2}\right) = \left(\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right)'$.

Hence the metric tensor is $g_{ij} = \begin{bmatrix} \frac{\alpha}{\beta^2} & -\frac{1}{\beta} \\ -\frac{1}{\beta} & \left(\frac{(\acute{\Gamma}(\alpha))}{\Gamma(\alpha)}\right)' \end{bmatrix}$.

3.8. f-Divergence functions

An asymmetry (or closeness) between two probability density functions on a manifold can be quantified by calculating their divergence using the divergence functions, also called contrast functions. A contrast function represents a degree of separation between two functions, but its square root is not distance since it does not satisfy the symmetry condition and also does not satisfy the gradient inequality . The contrast functions only make sense locally on a manifold; there may be no globally defined contrast functions because there may be no coordinate charts that include both probability density functions. f −Divergence function one of the most important classes of the contrast function use to measure of distance between two probability distributions. A divergence represents the degree of separation between two probability distributions (or reliability functions) , but its square root is not a distance .It does not necessarily satisfy the symmetry condition . So in general $D_f(R(t,\lambda_1)\|R(t,\lambda_2)) \neq D_f(R(t,\lambda_2)\|R(t,\lambda_1))$, also does not (in general) satisfy the triangle inequality. $D_f(R(t,\lambda_1)\|R(t,\lambda_2))$ we say the divergence from $R(t,\lambda_1)$ to $R(t,\lambda_2)$. We introduce some definitions which is related to our work . [47,53,63,79]

Definition 3.8.1 : Let $R(t,\lambda)$be a reliability function of T given some parameter $\lambda \in R^n > 0$ .If all of the second partial derivatives of $R(t,\lambda)$ are continuous over the domain , then **the Hessian matrix** И of $R(t,\lambda)$ is an $n \times n$ matrix:

$$И = \begin{bmatrix} \frac{\partial^2 R(t,\lambda)}{\partial \lambda_1^2} & \frac{\partial^2 R(t,\lambda)}{\partial \lambda_1 \partial \lambda_2} & \cdots & \frac{\partial^2 R(t,\lambda)}{\partial \lambda_1 \partial \lambda_n} \\ \frac{\partial^2 R(t,\lambda)}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 R(t,\lambda)}{\partial \lambda_2^2} & \cdots & \frac{\partial^2 R(t,\lambda)}{\partial \lambda_2 \partial \lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 R(t,\lambda)}{\partial \lambda_n \partial \lambda_1} & \frac{\partial^2 R(t,\lambda)}{\partial \lambda_n \partial \lambda_2} & \cdots & \frac{\partial^2 R(t,\lambda)}{\partial \lambda_n^2} \end{bmatrix} \quad \ldots (3.50)$$

This is a matrix of second derivatives, and can be thought of as the multivariate generalization of the second derivative.

Definition 3.8.2.: A square Hessian matrix И is **positive definite** if it is satisfy the following conditions :

1- И is symmetric matrix (i.e. $h_{ij} = h_{ji}$)

2- И is a non-singular matrix (i.e det(И)≠0).

3- Diagonal elements are positive (i.e. $h_{ij} > 0$)

4- $max_{\substack{1\le k\le n \\ 1\le j\le n}}|h_{kj}| \le max|h_{ii}|$

5- $(h_{ij})^2 < h_{ii}h_{jj}$, for all $i \neq j$.

Or equivalent , a square matrix H is positive definite if it is symmetric and for all non-zero vector v we have $v^T Иv > 0$.

Definition 3.8.3.: let I be an open set in R^n, and let $\lambda_1, \lambda_2 \in I$. Suppose SRN is a smooth reliability manifold. If there is any parameterization $\varphi: I \to SRN$, and a smooth function $D(.\|.): SRN \times SRN \to R$ on I such that :

$$D(\lambda_1\|\lambda_2) = D_N(\varphi(\lambda_1)\|\varphi(\lambda_2)) \qquad \text{... (3.51)}$$

is a **contrast function** on I satisfying the following properties :

1- Positive : $D(\lambda_1\|\lambda_2) \geq 0$, for all $\lambda_1, \lambda_2 \in I$,

2- Non-degenerate : $D(\lambda_1\|\lambda_2) = 0$, if and only if $\lambda_1 = \lambda_2$.

3- The first variation along the diagonal $\{\lambda_1 = \lambda_2\}$ vanishes : $\frac{\partial}{\partial \lambda_1^i} D(\lambda_1\|\lambda_2)_{|\lambda_1=\lambda_2} = \frac{\partial}{\partial \lambda_2^i} D(\lambda_1\|\lambda_2)_{|\lambda_1=\lambda_2} = 0$

4- The Hessian along the diagonal $\lambda_1 = \lambda_2$,

$$g_{ij}(\lambda_1) = \frac{\partial}{\partial \lambda_2^{\mathrm{i}}} \frac{\partial}{\partial \lambda_2^{\mathrm{j}}} D(\lambda_1\|\lambda_2)_{|\lambda_1=\lambda_2}$$

is positive definite and smooth with respect to λ_1 , where g_{ij} is a Riemannian metric tensor .

5- $D(\lambda_1\|\lambda_2)$ is a smooth function .

Then $\boldsymbol{D_N(\varphi(\lambda_1)\|\varphi(\lambda_2))}$ **is a contrast function on N** , which is smooth mapping .So $(\boldsymbol{SRN}, \ \boldsymbol{g_{ij}})$ **is called Riemannian manifold** induced by contrast function $D(\lambda_1\|\lambda_2)$.

Definition 3.8.4.: If $R(t, \lambda_1)$ and $R(t, \lambda_2)$ are sufficiently close and if g_{ij} be Riemannian metric tensor which is positive definite matrix , then the square of infinitesimal distance ds ,

$$ds^2 = D(\lambda \,\|\, \lambda + d\lambda) = \sum g_{ij}\,(\lambda) d\lambda^i d\lambda^j \qquad \text{... (3.52)}$$

Definition 3.8.5 :-Let $f:(0,\infty)\to R$ be a function such that $f(u)$ is convex , $f(1)=0$ and $f''(1)=1$. For each reliability functions $R_1(t,\lambda_1), R_2(t,\lambda_2)$,the contrast function on SRN associated with $f(u)$ is given by $D_f(R_1(t,\lambda_1)\|R_2(t,\lambda_2))=\int_0^\infty R_1(t,\lambda_1)f(\frac{R_2(t,\lambda_2)}{R_1(t,\lambda_1)})dt$,called ***f* −Divergence** function .

3.9. Special case of f −Divergences function

. **Definition 3.9.1.:** If we have two reliability function over the same random variable (T), $R_1(t,\lambda_1)$ and $R_2(t,\lambda_2)$, then the contrast function associated with the convex function $f(u)=-lnu$ is given by :

$$D_{fKL}(R_1(t,\lambda_1)\|R_2(t,\lambda_2))=$$

$$=\int_0^\infty R_1(t,\lambda_1)ln\left(\frac{R_1(t,\lambda_1)}{R_2(t,\lambda_2)}\right)dt \qquad ...(3.53)$$

referred to **as the Kullback-Leibler-divergence** of continuous distribution . On the other hand , If $R_1(t,\lambda_1)$, and $R_2(t,\lambda_2)$ represent the reliability functions with discrete life time distribution ,

then Kullback-Leibler divergence is calculated as a summation $D_{fKL}(R_1(t,\lambda_1)\|R_2(t,\lambda_2))=\sum_{t\in T}R_1(t,\lambda_1)ln\left(\frac{R_1(t,\lambda_1)}{R_2(t,\lambda_2)}\right)$ ….(3.54)

Example 3.2. Let two exponential distributions of two parameters ,

$p(x)=\lambda_1 e^{-\lambda_1(x-\theta_1)}$ and $q(x)=\lambda_2 e^{-\lambda_2(x-\theta_2)}$,

$x\geq 0, \lambda_i, \theta_i>0$, $i=1,2$.

Since $ln\left(\frac{p(x)}{q(x)}\right)=ln\frac{\lambda_1 e^{-\lambda_1(x-\theta_1)}}{\lambda_2 e^{-\lambda_2(x-\theta_2)}}=ln\frac{\lambda_1}{\lambda_2}+x(\lambda_2-\lambda_1)+\lambda_1\theta_1+\lambda_2\theta_2$ We find

$D_{fKL}(p\|q)=\int_0^\infty p(x)ln\left(\frac{p(x)}{q(x)}\right)dx$

$$=ln\frac{\lambda_1}{\lambda_2}\int_0^\infty p(x)dx+(\lambda_2-\lambda_1)\int_0^\infty xp(x)dx+\lambda_1\theta_1\int_0^\infty p(x)dx$$

$$+\lambda_2\theta_2\int_0^\infty p(x)dx$$

$$= \frac{\lambda_2}{\lambda_1} - ln\frac{\lambda_2}{\lambda_1} + \lambda_1\theta_1 + \lambda_2\theta_2 - 1 .$$

Example 3.3. If we have two reliability function $R_1(t,\lambda_1,\theta_1) = e^{-\lambda_1(t-\theta_1)}$ and $R_2(t,\lambda_2,\theta_2) = e^{-\lambda_2(t-\theta_2)}$ then the Kullback-Leibler-divergence of reliability is

$$D_{fKL}(R_1(t,\lambda_1,\theta_1)\|R_2(t,\lambda_2,\theta_2)) = \int_0^\infty R_1(t,\lambda_1,\theta_1) ln\left(\frac{R_1(t,\lambda_1,\theta_1)}{R_2(t,\lambda_2,\theta_2)}\right) dt$$

Since $ln\left(\frac{R_1(t,\lambda_1,\theta_1)}{R_2(t,\lambda_2,\theta_2)}\right) = ln\frac{e^{-\lambda_1(t-\theta_1)}}{e^{-\lambda_2(t-\theta_2)}} = t(\lambda_2 - \lambda_1) + \lambda_1\theta_1 - \lambda_2\theta_2$.

Then

$$D_{fKL} = \frac{t(\lambda_2 - \lambda_1)}{\lambda_1} e^{-\lambda_1(t)+2\lambda_1\theta_1} + \theta_1 e^{\lambda_1\theta_1} - \frac{\lambda_2\theta_2}{\lambda_1} e^{\lambda_1\theta_1}$$

Definition 3.9.2 : Let $R_1(t,\lambda_1)$, and $R_2(t,\lambda_2)$ are any two reliability functions with continuous lifetime distributions ,then the contrast function connected with the function $f(u) = 4(1-\sqrt{u})$ is given by :

$$H_f\big(R_1(t,\lambda_1), R_2(t,\lambda_2)\big) = \sqrt{4 - 4\int_0^\infty \sqrt{R_1(t,\lambda_1)R_2(t,\lambda_2)}\,dt} \quad .... \quad (3.55)$$

which is called **Hillinger Reliability Divergence** .

the other hand , If $R_1(t,\lambda_1)$, and $R_2(t,\lambda_2)$ represent the reliability functions with discrete lifetime distributions , then Hillinger reliability divergence is :

$$H_f\big(R_1(t,\lambda_1), R_2(t,\lambda_2)\big) = \Big(1 - \sum_{t\in T}\sqrt{R_1(t,\lambda_1)R_2(t,\lambda_2)}\Big)^{\frac{1}{2}} \quad .....(3.56)$$

Examples 3.4. For continuous distributions :-

Let two Gamma distribution $p(x) = \frac{1}{\Gamma\alpha\beta^\alpha} e^{-\frac{x}{\beta}} x^{\alpha-1}$,

$q(x) = \frac{1}{\Gamma a b^a} e^{-\frac{x}{b}} x^{a-1}$, for all $x \geq 0, \alpha, \beta, a, b > 0$. if we assume that $\alpha = a = 1$, then the Hillinger divergence given by

$$H_f(p,q) = \sqrt{4 - \frac{8\sqrt{\alpha\beta}}{\alpha+\beta}} .$$

Example 3. 5. For discrete distributions:-

Consider two Poisson distributions where the p.d.f

$p(x) = \frac{e^{-\lambda_1}(\lambda_1)^x}{x!}$, and $q(x) = \frac{e^{-\lambda_2}{\lambda_2}^x}{x!}$ for all $\lambda_i > 0$, $i = 1,2$, $x = 0,1,..$

If $\lambda_1 = 10$, and $\lambda_2 = 8$, then $H_f(p,q) = 2\left(1 - \sum_{x\in X}\sqrt{p(x)q(x)}\right)^{\frac{1}{2}}$

$= 2\sqrt{1 - e^{\sqrt{180}-9}}$

Example 3.6. Let two reliability function with $R_1(t,\lambda_1) = e^{-\lambda_1 t}$ and $R_2(t,\lambda_2) = e^{-\lambda_2 t}$, then the Hellinger divergence

$$H^2{}_f\left(R_1(t,\lambda_1), R_2(t,\lambda_2)\right) = 4 - 4\int_0^\infty \sqrt{R_1(t,\lambda_1)R_2(t,\lambda_2)}\ \mathrm{dt}$$

$$= 4 - \frac{8}{\lambda_1 + \lambda_2}$$

So

$$H_f\left(R_1(t,\lambda_1), R_2(t,\lambda_2)\right) = 2\sqrt{1 - \frac{2}{\lambda_1+\lambda_2}}$$

Definition 3.9.3.: A-N divergence derived from a convex function

$f(u) = \frac{1}{2}(u^2 - u)$ be in the form

$$D_{AN}(R_1(t,\lambda_1), R_2(t,\lambda_2)) = \frac{1}{2}\int_0^\infty \left[\frac{\left(R_2(t,\lambda_2)\right)^2}{R_1(t,\lambda_1)} - R_2(t,\lambda_2)\right] dt \quad \text{….(3.57)}$$

Which is measured divergence between two points on a manifold of reliability .

Example 3.7. Consider two reliability functions of lifetime exponential distributions , such that $R_1(t,\lambda_1) = e^{-\lambda_1 t}$ and $R_2(t,\lambda_2) = e^{-\lambda_2 t}$,

where $t \geq 0, \lambda_1, \lambda_2 > 0$. Then

$$D_{AN}\left(R_1(t,\lambda_1), R_2(t,\lambda_2)\right) = \frac{1}{2}\int_0^\infty \left[\frac{\left(R_2(t,\lambda_2)\right)^2}{R_1(t,\lambda_1)} - R_2(t,\lambda_2)\right] dt$$

$$D_{AN}(e^{-\lambda_1 t}, e^{-\lambda_2 t}) = \frac{1}{2}\int_0^\infty \left[\frac{(e^{-\lambda_2 t})^2}{e^{-\lambda_1 t}} - e^{-\lambda_2 t}\right] dt$$

$$= \frac{1}{-4\lambda_2+2\lambda_1} - \frac{1}{\lambda_2}$$

Theorem 3.9.4.:- The contrast function $D_{f^*}((R_2(t,\lambda_2)\|R_1(t,\lambda_1))$ is the dual of $D_f(R_1(t,\lambda_1)\|R_2(t,\lambda_2))$ if $f^*(u) = uf(\frac{1}{u})$.

Proof:- Let $D_f^*(R_2(t,\lambda_2)\|R_1(t,\lambda_1)) = D_f(R_1(t,\lambda_1)\|R_2(t,\lambda_2))$, then $D_{f^*}(R_2(t,\lambda_2)\|p) = \int_0^\infty R_2(t,\lambda_2) f^*\left(\frac{R_1(t,\lambda_1)}{R_2(t,\lambda_2)}\right) dt$

$$D_{f^*}(R_2(t,\lambda_2)\|R_1(t,\lambda_1)) = \int_0^\infty R_2(t,\lambda_2)\frac{R_1(t,\lambda_1)}{R_2(t,\lambda_2)} f\left(\frac{R_2(t,\lambda_2)}{R_1(t,\lambda_1)}\right) dt$$

$$= \int_0^\infty R_1(t,\lambda_1) f\left(\frac{R_2(t,\lambda_2)}{R_1(t,\lambda_1)}\right) dt$$

$$= D_f(R_1(t,\lambda_1)\|R_2(t,\lambda_2))$$

Chapter Four

Fuzzy Reliability Models With Exponential Fuzzy Distribution and Fuzzy Reliability Manifolds

4.1. Introduction

The Exponential distribution is often concerned with the amount of time until some specific event occur. Exponential Distributions are commonly used in calculation of product reliability, or length of time a product lasts.An important property of the Exponential distribution is that it is memoryless .The chance of an event does not depend on past trials. Therefore, the occurrence rate remains constant. The memoryless property indicates that the remaining life of a component is independent of its current age. The Exponential distribution is often used to model the longevity of an electrical or mechanical device. For example, the lifetime of a certain computer part has the Exponential distribution with a mean of ten years $X \sim Exp(0.1)$ .The memoryless property says that knowledge of what has occurred in the past has no effect on future probabilities. In this case it means that an old part is not any more likely to break down at any particular time than a brand new part. In other words, the part stays as good as new until suddenly breaks. In this distribution we use three parameters are :

1- Location parameters(or shift): Shift a distributions along the time axis
2- Scale parameters, used to expand or contract the time axis by factor $\propto$
3- Shape parameters, affect the shape of the probability density function (p.d.f).

We present the following definitions and examples using a fuzzy exponential distribution with one parameter two parameters and three parameters .

4.2 Evaluating Fuzzy Reliability System with Triangular Fuzzy Exponential Distribution of One Parameter

The fuzzy reliability function $\tilde{R}_s(t)[\propto]$ with triangular fuzzy exponential distribution is given by :

1) Fuzzy Series Model :

If n-components are connected in series, then the $\propto -cut$ of fuzzy reliability function $\tilde{R}_s(t)[\propto]$ is given by :

$$\tilde{R}_s(t)[\propto] = \left(\tilde{R}(t)\right)^n[\propto]$$

$$= \{e^{-n[\lambda_L]t} \quad , e^{-n[\lambda_U]t}\} \qquad \ldots(4.1)$$

2) Fuzzy Parallel Model :

Let n-components, be connected in parallel, and then the $\propto -cut$ of fuzzy reliability function $\tilde{R}_p(t)[\propto]$ is given by

$$\tilde{R}_p(t)[\propto] = \{1 - (1 - \tilde{R}(t))^n\}$$

$$= [1 - \left\{1 - e^{-[\lambda_L]t}\right\}^n, 1 - \left\{1 - e^{-[\lambda_U]t}\right\}^n] \quad \ldots(4.2)$$

3) Fuzzy Series - Parallel Model :

Let n-components be in series , and m-components are in parallel, then the $\propto -cut$ of fuzzy reliability function $\tilde{R}_{sp}(t)[\propto]$ is given by

$$\tilde{R}_{sp}(t)[\propto] = 1 - (1 - (R(t))^n)^m$$

$$= [1 - (1 - e^{-n[\lambda_L]t})^m, 1 - (1 - e^{-n[\lambda_U]t})^m] \quad \ldots(4.3)$$

4) Fuzzy Parallel - Series Model :

Let n-components be in series , and m-components are in parallel, then the $\propto -cut$ of fuzzy reliability function $\tilde{R}_{ps}(t)[\propto]$ is given by

$$\tilde{R}_{PS}(t) = \{1-(1-\left(\tilde{R}(t)\right)^{n}\}^{m}$$

$$= [\{1-\left(1-e^{-[\lambda_L]t}\right)^{n}\}^{m}, \{1-\left(1-e^{-[\lambda_U]t}\right)^{n}\}^{m}] \quad \ldots.(4.4)$$

5) Fuzzy k-out-of-n Model :

A k-out-of-n system have (n) component such that system works if at least k out of m components is working. Each component (i) can be represented by a Bernoulli random variable γ, with fuzzy reliability (fuzzy survival probability) $\widetilde{\mathrm{R}}(t)$ and unreliability (fuzzy failure probability) $\widetilde{\mathrm{Q}}(t)$.

That is :

$$\gamma_i = \begin{Bmatrix} 1 & , with\ fuzzy \text{ probability } \widetilde{\mathrm{R}}(t) \\ 0 & , with\ fuzzy \text{ probability } \widetilde{\mathrm{Q}}(t) \end{Bmatrix}$$

Then, $Z = \sum_{i=1}^{m} \gamma_i$, the number of survivors at time t has the fuzzy Binomial distribution. The fuzzy system reliability function is given by :

$$\widetilde{\mathrm{R}}(t)[\propto] = \widetilde{\mathrm{P}}(z \geq k) = \{\textstyle\sum_{j=k}^{m} \binom{m}{j} \left(R(t)\right)^{j} \left(Q(t)\right)^{m-j} | R(t)[\propto]\} \ldots.(4.5)$$

for $0 \leq \propto \leq 1 \quad and\ t > 0$, where

$$R(t)[\propto] = \{\left(R(t), Q(t)\right) | R(t) \in \tilde{R}(t)[\propto], Q(t) \in \tilde{Q}(t)[\propto], R(t) + Q(t) = 1\}$$

There are two special case of the k-out-of-m series model which is equivalent to an m-out-of-m model , the $\propto -cut$ fuzzy reliability function with modeled fuzzy exponential is as

$$\tilde{R}(t) = [e^{-m[\lambda_L]t} \quad , e^{-m[\lambda_U]t}]$$

or parallel system which is equivalent to 1-out-of-m model , the $\propto -cut$ fuzzy reliability function is

$$\tilde{R}(t) = [1-\left(1-e^{-[\lambda_L]t}\right)^{m} \quad , 1-(1-e^{-[\lambda_U]t})^{m}]$$

The $\widetilde{FMTTF}$ of k-out-of-m model is

$$F\widetilde{MTTF} = \{\sum_{i=k}^{m} \frac{1}{\lambda_i} = \frac{1}{\lambda}\sum_{i=k}^{m} \frac{1}{i} : \lambda \in \tilde{\lambda}\,(\propto)\} \qquad \ldots(4.6)$$

Example 4.1. An exponential distribution fuzzy parameter $\tilde{\lambda}$ is used to model the lifetime of two out of three systems which are independent and have identical components such that $\tilde{\lambda} = (0.3,0.5,0.7)$ then,

A. Fuzzy system reliability is

$$\tilde{R}(t) = \{\sum_{i=k}^{m} \binom{m}{i} \left(\tilde{R}(t)\right)^{i} \left(\tilde{Q}(t)\right)^{m-i}\}$$

$$= 3\left(\tilde{R}(t)\right)^{2} \tilde{Q}(t) + \left(\tilde{R}(t)\right)^{3}$$

$$= [e^{-(1.4-0.4\propto)t} \quad \left(3 - 2\, e^{-(0.7-0.2\propto)t}\right), e^{-(0.4\propto+0.6)t} \left(3 - 2\, e^{-(0.2\propto+0.3)t}\right)]$$

B. $\tilde{R}(0.6)[\propto] = [e^{-(0.84-0.2\propto)t} \quad \left(3 - 2\, e^{-(0.42-0.12\propto)}\right),$

$$e^{-(0.24\propto+0.36)} \left(3 - 2\, e^{-(0.12\propto+0.18)t}\right)]$$

C. $\tilde{R}(t)[0] = [e^{-1.4t} \quad (3 - 2\, e^{-0.7}), e^{0.6t}\, (3 - 2\, e^{0.3t})]$

Definition 4.2.1. :- The probability density function of two parameters exponential is defined by:

$$f(x) = \lambda e^{-\lambda(x-\theta)} \qquad , x > \theta, \lambda, \theta > 0 \qquad \ldots(4.7)$$

where λ is scale parameters, θ is location parameter, and it's cumulative distribution is

$$F_X(t) = P(X \leq t) = 1 - e^{-\lambda(t-\theta)} \qquad \ldots(4.8)$$

Because of the uncertainty and inaccuracy of data, the estimation of precise value of lifetime parameters becomes very difficult.

So that we take the lifetime of exponential distribution λ is replaced by triangular fuzzy number $\tilde{\lambda}$. In this case, the fuzzy probability of event

$X \in [c,d], c \geq 0$, $(\tilde{P}(c \leq X \leq d))$ and its $\propto -cut$ set is given by :

$$\tilde{P}(c \leq X \leq d)[\propto] = \{\int_c^d \lambda\, e^{-\lambda(x-\theta)}\ dx : \lambda \in \tilde{\lambda}[\propto]\} \qquad ....(4.9)$$

$$= [P_L[\propto], P_U[\propto]]$$

for all $\propto$, where

$$P_L[\propto] = \min\{\int_c^d \lambda\, e^{-\lambda(x-\theta)}\ \ dx :\ \lambda \in \tilde{\lambda}[\propto]\}$$

and

$$P_U[\propto] = \max\{\int_c^d \lambda\, e^{-\lambda(x-\theta)}\ \ dx :\ \lambda \in \tilde{\lambda}[\propto]\}\,.$$

Fuzzy reliability function of two parameters or fuzzy survival function $\tilde{R}(t)$ is the fuzzy probability in which a unit survives beyond time t. Let the lifetime parameter of a component has fuzzy exponential distribution and lifetime parameter $\tilde{\lambda}$ represent a triangular fuzzy number as $\tilde{\lambda} = (a_1, a_2, a_3)$, then we can describe a membership function $\mu_{\tilde{\lambda}}(x)$ in the following manner.

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{(x-a_1)}{(a_2-a_1)} & ; a_1 \leq x \leq a_2 \\ \dfrac{(a_3-x)}{(a_3-a_2)} & ; a_2 \leq x \leq a_3 \\ 0 & ; 0.w \end{cases}$$

Then the $\propto -cut$ of fuzzy reliability function is defined by:

$$\tilde{R}(t) = \{\ e^{-\lambda(t-\theta)}\ :\ \lambda \in \tilde{\lambda}\ [\propto]\}$$

$$= [e^{-[\lambda_L](t-\theta)}, e^{-[\lambda_U](t-\theta)}] \qquad ....(4.10)$$

Where

$$\lambda_L = (a_2 - a_1) \propto + a_1$$

and

$$\lambda_U = a_3 - (a_3 - a_2) \propto$$

So ,

$$\tilde{R}(t) = [\, e^{-[(a_2-a_1)\propto+a_1](t-\theta)}, e^{-[a_3-(a_3-a_2)\propto](t-\theta)}] \quad \ldots(4.11)$$

$\tilde{R}(t)$ is two dimensional function in terms of $\propto$ and t ($0 \leq\propto\leq 1$ and $t > 0$)

For every $\propto -cut$, reliability curve is like a band whose width depends on the ambiguity parameter of λ. The lesser uncertainty value results in less bandwidth, and if the parameter gets a crisp value, the lower and upper bounds will become equal, which means that reliability curve is in a classic state. This reliability band has properties as follow:

1) $\tilde{R}(0)[\propto] = \tilde{1}$, i.e. no one starts off dead.

2) $\tilde{R}(\infty)[\propto] = \tilde{0}$, i.e. everyone dies eventually.

3) $\tilde{R}(t_1)[\propto] \geq \tilde{R}(t_2)[\propto]$ if and only if $t_1 \leq t_2$, i.e. band of $\tilde{R}(t)[\propto]$ declines monotonically, ($\tilde{R}(t_1)[\propto] \tilde{\geqq} \tilde{R}(t_2)[\propto]$ if and only if $R_L(t_1)[\propto] \geq R_L(t_2)[\propto]$ and $R_U(t_1)[\propto] \geq R_U(t_2)[\propto]$ for all $\propto\in [0,1], where$ $\tilde{\geqq}$ means "fuzzy greater than or equal to")

4) For any fixed $\propto$ and $\theta \leq 1$ reliability band have convex functions.

Theorem 4.2.2 Fuzzy mean time to failure ($F\widetilde{MTTF}$) is the expected time to failure. then the ($F\widetilde{MTTF}$) of any component with a triangular fuzzy number is $[\frac{1}{\lambda_L} + \theta \; , \frac{1}{\lambda_U} + \theta]$.

Proof :

Since $F\widetilde{MTTF}[\propto] = \int_0^\infty x\, f(x) \; dx \colon \lambda \in \tilde{\lambda}[\propto]$

Then $F\widetilde{MTTF}[\propto] = \int_0^\infty x\, \lambda e^{-\lambda(x-\theta)} \; : \lambda \in \tilde{\lambda}[\propto]$

$$= \{\frac{1}{\lambda} + \theta \ : \lambda \in \tilde{\lambda}[\propto]\}$$

$$= \{\frac{1}{\lambda_L} + \theta \ , \frac{1}{\lambda_U} + \theta\}$$

$$= [\frac{1}{(a_2-a_1)\propto + a_1} + \theta \ , \frac{1}{a_3-(a_3-a_2)\propto} + \theta]$$

Theorem 4.2.3. If the lifetime follows fuzzy exponential distribution , then the fuzzy hazard function of any component with a triangular fuzzy number is given by $\tilde{h}(t)[\propto] = [\lambda_L, \lambda_U]$.

Proof :-

Since $\tilde{h}(t)[\propto] = \left\{\frac{f(t)}{R(t)} : \lambda \in \tilde{\lambda}[\propto]\right\}$.

Then we have $\tilde{h}(t)[\propto] = \left\{\frac{\lambda e^{-\lambda(t-\theta)}}{e^{-\lambda(t-\theta)}} \ : \lambda \in \tilde{\lambda}[\propto]\right\}$

$$= \left\{\lambda : \lambda \in \tilde{\lambda}[\propto]\right\} = [\lambda_L, \lambda_U]$$

$$= [(a_2 - a_1) \propto + a_1, a_3 - (a_3 - a_2) \propto]$$

It is to be noted that $\tilde{h}(t)[\propto]$ is a two dimensional function in terms of $\propto$ and t ($0 \leq \propto \leq 1$ and $t > 0$). So, for every $\propto -cut$, hazard rate curve is like a band.

4.2.4 Evaluating Fuzzy Reliability Systems Using Exponential Distribution of Two Parameters

a. **Fuzzy Series Model :**

If n-component are connected in series, then the $\propto -cut$ of fuzzy reliability function $\tilde{R}_s(t)[\propto]$ with triangular fuzzy exponential distribution is given by

$\tilde{R}_s(t)[\propto] = [e^{-n[(a_2-a_1)\propto + a_1](t-\theta)}, e^{-n[a_3-(a_3-a_2)\propto](t-\theta)}]$ …(4.12)

b. Fuzzy Parallel Model:-

Let n-component be connected in parallel, and then the $\propto -cut$ of fuzzy reliability function $\tilde{R}_P(t)[\propto]$ with triangular fuzzy exponential distribution is given by **:**

$$\tilde{R}_P(t)[\propto] = \begin{bmatrix} 1 - \left\{1 - e^{-[(a_2 - a_1)\propto + a_1](t-\theta)}\ \right\}^{n} \\ , \\ 1 - \left\{e^{-[a_3 - (a_3 - a_2)\propto](t-\theta)}\right\}^{n} \end{bmatrix} \qquad \ldots(4.13)$$

c. Fuzzy Series-Parallel Model :

If n-component are connected in series and m-component are in parallel, then the $\propto -cut$ of fuzzy reliability function

$$\tilde{R}_{SP} = \begin{bmatrix} 1 - \left\{1 - e^{-n[(a_2 - a_1)\propto + a_1](t-\theta)}\ \right\}^{m} \\ , \\ 1 - \left\{1 - e^{-n[a_3 - (a_3 - a_2)\propto](t-\theta)}\right\}^{m} \end{bmatrix} \qquad ....(4.14)$$

d. Fuzzy Parallel-Series Model :

Let n-component be in series and m-component are in parallel, then the $\propto -cut$ of fuzzy reliability function $\tilde{R}_{SP}(t)[\propto]$ with triangular fuzzy exponential distribution is given by:

$$\tilde{R}_{SP}(t)[\propto] = \begin{bmatrix} \{1 - \left(1 - e^{-[(a_2 - a_1)\propto + a_1](t-\theta)}\ \right)^{m}\}^{n} \\ , \\ \{1 - \left(1 - e^{-[a_3 - (a_3 - a_2)\propto](t-\theta)}\right)^{m}\}^{n} \end{bmatrix} \qquad \ldots.(4.15)$$

e. Fuzzy k-out-of-n Model :

$$\tilde{R}(t) = \sum_{j=k}^{m} \binom{m}{j} \left(\tilde{R}(t)\right)^{j} \left(\tilde{Q}(t)\right)^{m-j}$$

$$= \sum_{j=k}^{m} \binom{m}{j} \left(\tilde{R}(t)\right)^{j} \left(1 - \tilde{R}(t)\right)^{m-j} \qquad \ldots(4.16)$$

Example 4.2.

If we have 2-out-of -3 system which is independent and identical components lifetime is modeled by an exponential distribution fuzzy parameter $\tilde{\lambda}$ such that $\tilde{\lambda} = (0.3,0.5,0.7)$. Then fuzzy reliability is

$$\tilde{R}_S(t)[\propto] = \sum_{j=k}^{m} \binom{m}{j} \left(\tilde{R}(t)\right)^j \left(1 - \tilde{R}(t)\right)^{m-j}$$

$$= 3\left(\tilde{R}(t)\right)^2 \left(1 - \tilde{R}(t)\right) + \left(\tilde{R}(t)\right)^3$$

Since $\tilde{R}(t) = \left[e^{-[0.7-0.2\propto]t}, e^{-[0.2\propto+0.3]t}\right]$

Then $\widetilde{R}_S(t)[\propto] = 3(\left[e^{-[0.7-0.2\propto](t-\theta)}, e^{-[0.2\propto+0.3]t}\right])^2\big(1 - \left[e^{-[0.7-0.2\propto](t-\theta)}, e^{-[0.2\propto-0.3](t-\theta)}\right]\big)$

$+(\left[e^{-[0.7-0.2\propto](t-\theta)}, e^{-[0.2\propto+0.3]t}\right])^3$

$$= [e^{-[1.4-0.4\propto](t-\theta)}, \left(3 - 2e^{-[0.7-0.2\propto](t-\theta)}\right), e^{-[0.4\propto+0.6](t-\theta)} \times \left(3 - 2e^{-[0.2\propto+0.3](t-\theta)}\right)]$$

If we assume $\propto = 0$, $t = 0.6$, $\theta = 0.2$

$$\tilde{R}(t)[\propto] = [e^{-1.4(0.4)} \left(3 - 2e^{-(0.7)(0.4)}\right), e^{-(0.6)(0.4)}, \left(3 - 2e^{-(0.3)(0.4)}\right)]$$

$$= [e^{-0.56} (3 - 2e^{-0.28}), e^{-0.24} (3 - 2e^{-0.12})]$$

$$= [3e^{-0.56} - 2e^{-0.84}, 3e^{-0.24} - 2e^{-0.36}]$$

$$\approx [3(0.57) - 2(0.43), 3(0.79) - 2(0.69)]$$

$$\approx [1.71 - 0.86, 2.37 - 1.38] \approx [0.85,0.99]$$

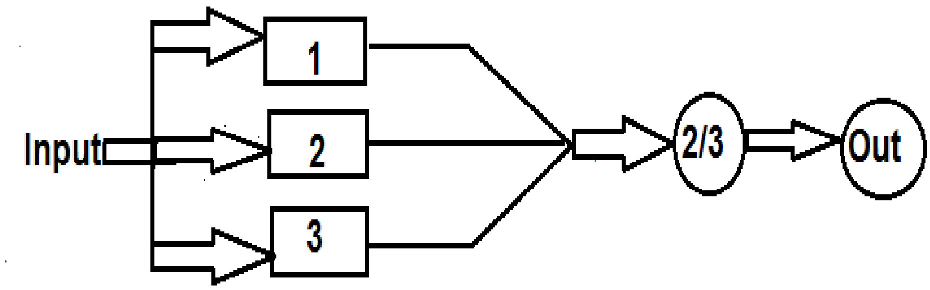


Figure 4.1 : The 2-out-of -3 system

Theorem 4.2.5. If The Fuzzy Exponential Reliability Function of two Parameters be $\tilde{R}(t) = e^{-\lambda(t-\theta)}$.Then the fuzzy reliability function of three parameters is $R^*(t,\lambda,\beta,\theta) = 1 - \left[1 - e^{-\lambda(t-\theta)}\right]^{\beta}$.

Proof : From equation (4.8) taking ${F^*}_X(t) = [F_X(t)]^{\beta}$, $\beta > 0$, then for all $t > 0, \lambda, \beta, \theta > 0$, we get :

$$ {F^*}_X(t) = \left[1 - e^{-\lambda(t-\theta)}\right]^{\beta} \qquad \ldots(4.17)$$

which represent an Exponential distribution with three parameters λ, β and θ.

Now , from the relation $f^*(x) = ({F^*}_X(t))'$ we get :

$$ f^*(x) = \beta\lambda e^{-\lambda(t-\theta)}\left[1 - e^{-\lambda(t-\theta)}\right]^{\beta-1} \qquad \ldots(4.18)$$

and

$$ R^*(t,\lambda,\beta,\theta) = 1 - {F^*}_X(t) = 1 - \left[1 - e^{-\lambda(t-\theta)}\right]^{\beta} \qquad \ldots(4.19)$$

The $\propto -cut$ of fuzzy reliability function ${\tilde{R}^*}_s(t,\lambda,\beta,\theta)[\propto]$ with triangular fuzzy exponential distribution is given by :

$$ {\tilde{R}^*}_s(t,\lambda,\beta,\theta)[\propto] = \{1 - \left[1 - e^{-\lambda(t-\theta)}\right]^{\beta} : \lambda \in \tilde{\lambda}\,[\propto]\} \qquad \ldots(4.20)$$

4.3. Fuzzy Reliability Manifold

The concept of a fuzzy manifold arises as a logical extension of the concepts of a fuzzy topological vector space and a fuzzy derivative of a fuzzy continuous mapping between fuzzy topological vector spaces. Despite the existence of established theories on relevant fuzzy structures such as fuzzy topological spaces, fuzzy topological vector spaces, and fuzzy derivatives, the development of a satisfactory structure for fuzzy differentiable manifolds has been a relatively unexplored area. The objective of this specific section is to build a structure for fuzzy reliability manifolds .[176,177, 179]

Definition 4.3.1 : let $N = \{R(t,\lambda), t \in (0,\infty), \lambda \in R^n > 0\}$ be nonempty crisp set and $\mu_N: N \to [0,1]$ be membership function on N. Then a fuzzy set FN is a subset of the space $N \times [0,1]$,such that

$FN = \left\{\tilde{R}\,(t) = \int_t^\infty \lambda e^{-\lambda t} dt : \lambda \in \tilde{\lambda}\,[\alpha]\right\} = \left\{e^{-\lambda t} : \lambda \in \tilde{\lambda}\,[\alpha]\right\}$, with $F\tau$ as a collection of fuzzy sets of N **be a fuzzy topological space** if $F\tau$ as a collection of fuzzy sets of N , satisfies the following conditions:

(1) $\tilde{1}, \tilde{0} \in F\tau$

(2) $\{\tilde{A}_i\}_{i\in I} \subseteq F\tau\,, then\ U_{i\in Z}\, A_i \in F\tau$

(3) $\tilde{A}, \tilde{B} \in F\tau, then\ \tilde{A} \cap \tilde{B} \in F\tau$

each element of $F\tau$ is called $F\tau -$ open fuzzy sets or simply open fuzzy sets.

Definition 4.3.2 : let $(FN, F\tau)$ be a fuzzy topological space consider

$\tilde{V} \subseteq FN, p = \tilde{R}(t,\lambda)$ and $p \in supp\ FN$

If there exist an $F\tau -$ open subset $\tilde{U}$ of FN such that $p \in supp\ \tilde{U}$ and

$\tilde{U} \subseteq \tilde{V}$, then $\tilde{V}$ is called an $\boldsymbol{F\tau}$ **−neighborhood of $\boldsymbol{p}$ in $\boldsymbol{FN}$**, denoted it by $FN(p)$.

Definition 4.3.3: A fuzzy set $\tilde{A}$ is $\boldsymbol{F\tau -}$ **open** iff for each fuzzy set $\tilde{B}$ continued in $\tilde{A}, \tilde{A} \in FN(p)$.

Definition 4.3.4: Let $(FN, F\tau)$ be an fuzzy topological space and for all

$p_1, p_2 \in suppFN, p_1 \neq p_2$ there exists two $F\tau$ −neighbohoods

$U_{p_1} \in FN(p_1), U_{p_2} \in FN(p_2)$ such that $U_{p_1} \cap U_{p_2} = 0$ then $(FN, F\tau)$ is called **Hausdroff fuzzy topological space.**

Definition 4.3.5: A family β of fuzzy open sets in FN is called a **base** for an fuzzy topological space $(FN, F\tau)$ iff members of $F\tau$ can be written as unions of members of β.

Definition 4.3.6 : Let $(FN, F\tau)$ be an fuzzy topological space if there are exists a countable basis for $F\tau$, then $(FN, F\tau)$ is called **a second countable fuzzy topological space.**

Definition 4.3.7 : Let $(FN_1, F\tau_1), (FN_2, F\tau_2)$ be two fuzzy topological space. A mapping $f$of $(FN_1, F\tau_1)$ into $(FN_2, F\tau_2)$ is said to be **fuzzy continuous** if for each open fuzzy set $\tilde{B}$ in $F\tau_2$ the inverse image $f^{-1}(\tilde{B})$ is in $F\tau_1$. conversely, f is said to be **fuzzy open** if for each open fuzzy set $\tilde{A}$ in $F\tau_1$ the image $f[\tilde{A}]$ is in $F\tau_2$, consequently the composition of two fuzzy continuous is fuzzy continuous .

Definition 4.3.8: An one-to-one , onto mapping g of fuzzy topological space $(FN_1, F\tau_1)$ onto a fuzzy topological space $(FN_2, F\tau_2)$ is called **a fuzzy homeomorphism** if it is fuzzy continuous and fuzzy open.

Consequently, the composition of two fuzzy continuous (resp. fuzzy open) mapping is a fuzzy continuous (resp. fuzzy open) mapping.

Definition 4.3.9: Let $FN = \{\tilde{R}(t, \lambda), t \in (0, \infty)\, \lambda \in \tilde{\lambda}[\propto]\}$ be a fuzzy parametric model of n parameter and the parameter $\tilde{\lambda}$ represent a triangular fuzzy number . Then an n-dimensional **fuzzy reliability manifold FN** is a fuzzy topological space which is fuzzy Hausdroff , fuzzy second countable and fuzzy locally homeomorphism to an $[0,1]$.

Definition 4.3.10: Let FN be fuzzy reliability manifold and $\emptyset: \widetilde{U} \to \emptyset(\widetilde{U}) \subseteq [0,1]$ is a fuzzy homeomorphism defined on support of $\widetilde{U}, \{p \in FN : \mu_{\widetilde{U}}(p) > 0\}$ onto an open fuzzy set $\emptyset(\widetilde{U})$. Then the pair $(\widetilde{U}, \emptyset)$ is said to be a **fuzzy chart** at p.

Definition 4.3.11: A **fuzzy atlas $\widetilde{\mathcal{A}}$** on FN is a collection of pairs $(\widetilde{U}_j, \emptyset_j)$ which satisfies the following conditions.

(i) Each $\widetilde{U}_j$ is fuzzy open set in FN and $sup_j \left\{\mu_{\widetilde{U_j}}(p)\right\} = 1$ for all $p \in FN, j$ in the index set.

(ii) Each $\emptyset_j: \widetilde{U}_j \to \emptyset_j[\widetilde{U}_j]$ be a bjective defined on the support of $\widetilde{U}_j$,

$\{p \in FN: \mu_{\widetilde{U_j}}(p) > 0\}$ and $\emptyset_j[\widetilde{U}_j \cap \widetilde{U}_i]$ is an open fuzzy set in FN , for each pair of indices j, i .

(iii) The mapping $\emptyset_i o\, {\emptyset_j}^{-1}: \emptyset_j[\widetilde{U}_j \cap \widetilde{U}_i] \to \emptyset_i[\widetilde{U}_j \cap \widetilde{U}_i]$ is a fuzzy diffeomorphism for each pair of indices j, i .

Definition 4.3.12:- Let FN_1 , FN_2 be fuzzy Reliability manifolds and let

$f: FN_1 \to FN_2$. Then f is said to be **fuzzy differentiable** at a point

$p \in FN_1$, if there is a fuzzy chart $(\widetilde{U}, \emptyset)$ at $p \in FN_1$ and fuzzy chart $(\check{V}, \psi)$ at $f(p) \in FN_2$ such that the mapping $\psi\, ofo\, \emptyset^{-1}: \emptyset[\widetilde{U} \cap f^{-1}[\check{V}] \to \psi[\check{V}]$ is fuzzy differentiable at $\emptyset(p)$.

The mapping f is fuzzy differentiable if it is fuzzy differentiable at every point of FN_1.

Definition 4.3.13 : Let FN_1, FN_2 be fuzzy Reliability manifolds . A bijection f of FN_1, onto FN_2 is said to be **fuzzy diffeomorphism** if it and its inverse f^{-1} are fuzzy differentiable, and f' , $(f^{-1})'$ are fuzzy continuous . In other words: f is fuzzy diffeomorphism if $\psi\ ofo\ \emptyset^{-1}$ is a fuzzy diffeomorphism.

Definition 4.3.14: Let FN be a fuzzy Reliability manifold . Let $\breve{U}$ be an open fuzzy set in FN , and $\emptyset: \breve{U} \rightarrow \emptyset(\breve{U})$ is a fuzzy homeomorphism defined on the support of $\breve{U}$. Then $(\breve{U}, \emptyset)$ is said to be **compatible** with atlas $\{(\breve{U}_\iota, \emptyset_i)_{i\in I}\}$ if each transition map $\emptyset_j o\ \emptyset^{-1}: \emptyset(\breve{U} \cap \breve{U}_j) \rightarrow \emptyset_j(\breve{U} \cap \breve{U}_j)$is a fuzzy diffeomorphism . Two fuzzy atlases are compatible if each fuzzy chart of one atlas is compatible with each fuzzy chart of the other atlas

Chapter Five

Conclusion and Future Work

5.1 Conclusions

The goal of the study focused on the following points:

1- Several definitions of smooth manifolds with related theorems are needed in order to extract a reliability manifold N and prove that it is a parametric model in 2-dimension .

2- A tangent space includes all the directions in which one can pass via p in a way that is tangential to it.

- The tangent vectors at a point p are another name for the components that make up the tangent space at p.
- Every point on a connected manifold has the same dimension as the tangent space.
- For a tangent vector, its vector components can change their coordinate system in a contravariant or covariant way.
- If the vector is contravariant, then its components are evaluated from a parallel projection, and if the vector is covariant, then its components are evaluated from a perpendicular projection.
- Using contravariant vectors to define the tangent vector space versus using covariant vectors to define the cotangent vector space.
- Defining invariants with an illustrative example .

3- Many geometrical procedures from different researches are presented to measure the dissimilarity between two probability density functions (p.d.f). In current study Hillingers approach is used to evalute the distance between two continuous distributions and two discrete distributions respectively as an application . Hellingers distance depend on the concept of contrast function on space of density functions , since a contrast functional on statistical manifolds are a natural extensions of Kullback-Leibler relative entropy from statistical models.

4- Memoryless electronic devices have several uses triangular fuzzy exponential lifetime distribution can assess fuzzy reliability for various systems, such as the k-out-of-n model, by using triangle-cut and fuzzy number operations. As an extension of crisp reliability, fuzzy reliability is evaluated for different types of systems because it is an advantage and produces more accurate results than crisp reliability, which ignored uncertainties in data and the fuzziness of equipment endurance defects. We can conclude that the increasing of parameters is raising the value of system reliability since in the case of a one parameter exponential distribution, the reliability value for a k-out-of-n model is between [0.73 , 0.91], whereas in the case of a two-parameter exponential distribution, the reliability value for the same model becomes [0.85 , 0.99], which is near the maximum value. This means that an improvement in reliability has occurred.

5.2 Future Works

We hope that the suggested works for graduate studies in mathematics are as follows :

1- Applying soft fuzzy multistate in reliability analysis .

2- Using neutrosophic sets in uncertainty applications as reliability.

3- More advanced studies are needed about the relation between statistics and differential geometry since this title is still limited in research .

4- Employing fuzzy reliability manifolds in a wide range of sciences.

5- Utilizing fuzzy exponential three-dimension parameter distributions for k-out-of-n system reliability analysis .

6- Using the type-2 fuzzy set concept with differential geometry in reliability and safety applications .

References

References

[1] Aly H.M.,"The Exponential Distribution", lectures on state-613 "Reliability",lecture3, Cairo university , 2009/2010.

[2] Z. Li, Fuzzy Chaotic Systems , Springer- Verlag Berlin Heidelberg , Netherlands, 2006, pp.275-283.

[3] A .Luay al swidi , S. A . Audi ALsaeed " Soft fuzzy and Fuzzy soft set Theory with Application " ALnoor publishment – Germany (2020).

[4] Geiss,C.and Geiss, S.,"An Introduction to Probability Theory", Department of Mathematics and Statistics , University of Jyvaskyla , Jun 3,2009 .

[5] Buckley , J.J , Fuzzy probability and statistics , Springer-Verlag Berlin Heidelberg 2006 .

[6] Baloui, Jamkhaneh Ezzatallah. "An evaluation of the systems reliability using fuzzy lifetime distribution." (2011): 73-80 .

[7] Banhadli ;K.and other,"Probability" ,Mosule university, Iraq,1990.

[8] Dubois , D. , Prade , H :Operations of fuzzy number , Int. J . syst. (1978).

[9] Amari, Shun-ichi. "Differential geometrical theory of statistics." Differential geometry in statistical inference 10 (1987): 19-94.

[10] Li Tingjie and Gao He ,"Fuzzy Reliability",Beijing Institute of Aeronautics and Astronautics ,1988.

[11] Renteln, Paul. Manifolds, tensors, and forms: an introduction for mathematicians and physicists. Cambridge University Press, 2013.

[12] Alghamdi, S. M. and Percy, D. F., Reliability equivalence factors for a series-parallel system of components with exponentiated Weibull lifetimes, IMA Journal of Management Mathematics, (2015), vol. 28.3, pp. 339-358.

[13] Nayak, Prasun Kumar. Textbook of tensor calculus and differential geometry. PHI Learning Pvt. Ltd., 2011.

[14] Amari, Shun-ichi, and Hiroshi Nagaoka. Methods of information geometry. Vol. 191. American Mathematical Soc., 2000.

[15] Beichelt, F. and Tittmann, P., Reliability and maintenance: networks and systems, CRC Press, (2012).

[16] Amari, Shun-ichi. Differential-geometrical methods in statistics. Vol. 28. Springer Science & Business Media, 2012.

[17] YUN, Yong-Sik; SONG, Jae-Choong; RYU, Sang-Uk. On the exponential fuzzy probability. Communications of the Korean Mathematical Society, 2006, 21.2: 385-395.

[18] Chelson, P. O. and Eckstein, E., Reliability computation from reliability block diagrams, IEEE Transactions on Reliability, vol. 25 (1975), pp. 283-283.

[19] Garg, Harish. "A novel approach for analyzing the reliability of series-parallel system using credibility theory and different types of intuitionistic fuzzy numbers." Journal of the Brazilian Society of Mechanical Sciences and Engineering 38.3 (2016): 1021-1035..

[20] Islam, Nazrul. Tensors and their applications. New Age International, 2006.

[21] Zadeh . L. A ,Probability Measures of Fuzzy Event, Department of Electrical Engineering and Computer Sciencees University of California ,Berkeley , California, August 1968.

[22] L. Reznik , Fuzzy Controllers ,Great Britain by Biddles Ltd ,Guildford and King´s Lynn,pp.(1-57)(63-72)(124-152), 1997.

[23] Dhillon, B. S., Design Reliability Fundamentals and Applications, 1st ed., CRC Press, Taylor, Francis group, Boca Ration, Florida (1999).

[24] Gaston, Peter C., and Tomas Lozano-Perez. "Tactile recognition and

localization using object models: The case of polyhedra on a plane." IEEE transactions on pattern analysis and machine intelligence 3 (1984).

[25] Chaturvedi, Sanjay Kumar. "Network reliability: measures and evaluation".Jhon Wiley Sons, 2016.

[26] Dhillon, Balbir S. Maintainability, maintenance, and reliability for engineers. CRC press, (2006).

[27] Barndorff-Nielsen, Ole E. Parametric statistical models and likelihood. Vol. 50. Springer Science & Business Media, 2012.

[28] Pennec, Xavier. "Probabilities and statistics on Riemannian manifolds: Basic tools for geometric measurements." NSIP. Vol. 3. 1999.

[29] Henmi, Masayuki, and Ryoichi Kobayashi. "Hooke's law in statistical manifolds and divergences." Nagoya mathematical journal 159 (2000).

[30] Ibragimov, Nail H. "Tensors and Riemannian geometry with applications to differential equations and relativity." (2008).

[31] Lauritzen, Stefan L. "Statistical manifolds." Differential geometry in statistical inference 10 (1987): 163-216.

[32] Faraci, V., Calculating failure rates of series/parallel networks, The Journal of Alion s, System Reliability Center, vol. First Quarter (2006), pp. 1-3.

[33] Leondes C. T., Fuzzy Logic and Expert Systems Applications, ACADEMIC PRESS, 1998, pp.(1-5)(40-51)(60-77) .

[34] Pistone, Giovanni, and Maria Piera Rogantin. "The exponential statistical manifold: mean parameters, orthogonality and space transformations. " Bernoulli (1999): 721-760.

[35] Ferraro, Mario, and David H. Foster. "C1 fuzzy manifolds." *Fuzzy* sets and systems 54.1 (1993): 99-106.

[36] Ashbourn , J. "Advanced Identity Verification." The Complete Guide (2000).

[37] Gao, Yuan. "Analysis of k-out-of-n System with Uncertain Lifetimes." Order 1 (2009): 1.

[38] Alharbi, YASSER S., and Amr R. Kamel. "Fuzzy System Reliability Analysis for Kumaraswamy Distribution: Bayesian and Non-Bayesian Estimation with Simulation and an Application on Cancer Data Set." WSEAS Transactions on Biology and Biomedicine 19 (2022): 118-139.

[39] Huang, YuLei, et al. "Reliability evaluation of distribution network based on fuzzy spiking neural P system with self-synapse." Journal of Membrane Computing 3 (2021): 51-62.

[40] Fine, Terrence L. "Theories of probability: An examination of foundations." Academic press, (2014).

[41] Mellal , Mohamed Arezki and Enrico Zio. "System reliability – Redundancy optimization with cold-standby strategy by enhanced Nest cuckoo optimization algorithm."Reliability Engineering and System Sfety 201 (2020):106973.

[42] Shima, Hirohiko, and Katsumi Yagi. "Geometry of Hessian manifolds." Differential geometry and its applications 7.3 (1997):

277-290.

[43] Sha Fu, Zhongli Liu, Hangjun Zhou, Dan Song and Yezhi Xiao , "Trapezoidal Fuzzy Number Attitude Indicators Group Decision Making Approaches Based on Fuzzy Language", Asian network for scientific Information ,Journal of Applied Sciences, 14:2304-2308,2014.

[44] Ahsan, Zafar. "Tensors: Mathematics of differential geometry and relativity". PHI Learning Pvt. Ltd., 2015.

[45] Hassan, Z. A. H. and Udriste, C., Geometry of Reliability Models, Ph.D. Thesis, University Politehnica of Bucharest , (2016).

[46] Lee, John. Introduction to topological manifolds. Vol. 202. Springer Science & Business Media, 2010.

[47] Calin, Ovidiu, and Constantin Udrişte. Geometric modeling in probability and statistics. Vol. 121. Berlin, Germany:: Springer, 2014.

[48] Akbari, Mohammad Ghasem, and Gholamreza Hesamian. "Time-dependent intuitionistic fuzzy system reliability analysis." Soft Computing 24 (2020): 14441-14448.

[49] Matsuzoe, Hiroshi. "Geometry of semi-Weyl manifolds and Weyl manifolds." Kyushu Journal of Mathematics 55.1 (2001): 107-117.

[50] Calin, Ovidiu, and Constantin Udrişte. Geometric modeling

in probability and statistics. Vol. 121. Berlin, Germany: Springer, 2014 .

[51] Negi, Seema, and S. B. Singh. "Fuzzy reliability evaluation of linear m-consecutive weighted-k-out-of-r-from-n: F systems." International Journal of Computing Science and Mathematics 10.6 (2019): 606-621.

[52] Teh, Jiashen. "Uncertainty analysis of transmission line end-of-life failure model for bulk electric system reliability studies." IEEE Transactions on Reliability 67.3 (2018): 1261-1268.

[53] Kumar, Jitender, and Meenu Goel. "Fuzzy reliability analysis of a pulping system in paper industry with general distributions for all random

[54] Bohra, Khushal Singh, and S. B. Singh. "Evaluating fuzzy system reliability using intuitionistic fuzzy Rayleigh lifetime istribution." Mathematics In Engineering, Science and AerospaCE 6.2 (2015): 245-254.

[55] Lee, John M., and John M. Lee. Smooth manifolds. Springer New York, 2012.

[56] Jamkhaneh, E. Baloui. "Analyzing system reliability using fuzzy Weibull lifetime distribution." International Journal of Applied 4.1 (2014): 93-102.

[57] Eryilmaz, Serkan. "On reliability analysis of a k-out-of-n system with components having random weights." Reliability Engineering & System Safety 109 (2013): 41-44.

[58] Wang, Zhonglai, et al. "An approach to system reliability analysis with fuzzy random variables." Mechanism and Machine Theory 52 (2012): 35-46.

[59] Kurose, Takashi. "Conformal-projective geometry of statistical manifolds." Interdisciplinary Information Sciences 8.1 (2002): 89-100.

[60] Baloui, Jamkhaneh Ezzatallah. "An evaluation of the systems reliability using fuzzy lifetime distribution." (2011): 73-80.

[61] Lee, John M. Introduction to Riemannian manifolds. Vol. 2. Cham: Springer, 2018.

[62] Gholizadeh, Ramin, et al. "Fuzzy Bayesian system reliability assessment based on Pascal distribution." Structural and Multidisciplinary Optimization 40 (2010): 467-475.

[63] Eguchi, Shinto. "Geometry of minimum contrast." Hiroshima Mathematical Journal 22.3 (1992): 631-647.

[64] Kumar, Amit, Shiv Prasad Yadav, and Surendra Kumar. "Fuzzy system reliability using different types of vague sets." International Journal of Applied Science and Engineering 6.1 (2008): 71-83.

[65] Hryniewicz, Olgierd. "Fuzzy sets in the evaluation of reliability." Computational Intelligence in Reliability Engineering: New Metaheuristics, Neural and Fuzzy Techniques in Reliability (2007).

[66] Srivastava, Anuj, et al. "On advances in statistical modeling of natural images." Journal of mathematical imaging and vision 18 (2003): 17-33.

[67] Huang, Hong-Zhong, Ming J. Zuo, and Zhan-Quan Sun. "Bayesian reliability analysis for fuzzy lifetime data." Fuzzy Sets and Systems 157.12 (2006): 1674-1686.

[68] Naudts, Jan. "Estimators, escort probabilities, and phi-exponential families in statistical physics." arXiv preprint math-ph/0402005 (2004).

[69] Zhao, Ruiqing, and Baoding Liu. "Standby redundancy optimization problems with fuzzy lifetimes." Computers & Industrial Engineering 49.2 (2005): 318-338.

[70] Wu, Hsien-Chung. "Fuzzy reliability estimation using Bayesian approach." Computers & Industrial Engineering 46.3 (2004).

[71] Lafferty, John, Guy Lebanon, and Tommi Jaakkola. "Diffusion kernels on statistical manifolds." Journal of Machine Learning Research 6.1 (2005).

[72] Chen, Shyi-Ming. "Analyzing fuzzy system reliability using vague set theory." International Journal of Applied Science and Engineering 1.1 (2003): 82-88.

[73] Savoia, Marco. "Structural reliability analysis through fuzzy number approach, with application to stability." Computers & Structures 80.12 (2002): 1087-1102.

[74] Furuhata, Hitoshi, et al. "Sasakian statistical manifolds." Journal of Geometry and Physics 117 (2017): 179-186.

[75] Knezevic, Jezdimir, and E. R. Odoom. "Reliability modelling of repairable systems using Petri nets and fuzzy Lambda–Tau methodology." Reliability Engineering & System Safety 73.1 (2001) .

[76] Zhang, Jingyi, et al. "Projection-based techniques for high-dimensional optimal transport problems." Wiley Interdisciplinary Reviews: Computational Statistics 15.2 (2023): e1587.

[77] Han, Rungang, Rebecca Willett, and Anru R. Zhang. "An optimal statistical and computational framework for generalized tensor estimation." The Annals of Statistics 50.1 (2022): 1-29.

[78] Cena, Alberto, and Giovanni Pistone. "Exponential statistical manifold." Annals of the Institute of Statistical Mathematics 59 (2007) .

[79] Lauritzen, Stefan L. "Statistical manifolds." Differential geometry in statistical inference 10 (1987): 163-216.

[80] Jänich, Klaus, and Leslie Kay. Vector analysis. Vol. 5. Berlin: Springer, 2001.

[81] Hua, Xiaoqiang, et al. "Target detection within nonhomogeneous clutter via total bregman divergence-based matrix information geometry detectors." IEEE Transactions on Signal Processing 69 (2021) .

[82] Calinon, Sylvain. "Gaussians on Riemannian manifolds: Applications for robot learning and adaptive control." IEEE Robotics & Automation Magazine 27.2 (2020): 33-45.

[83] Chakraborty, Rudrasis, and Baba C. Vemuri. "Statistics on the Stiefel manifold: theory and applications." (2019) .

[84] Decu, Simona, et al. "Curvature invariants of statistical submanifolds in Kenmotsu statistical manifolds of constant ϕ-sectional curvature." Entropy 20.7 (2018) .

[85] Lee, John M. "Introduction to Smooth Manifolds." Changes (2007).

[86] Gomez, Ignacio S. "Notions of the ergodic hierarchy for curved statistical manifolds." Physica A: Statistical Mechanics and its Applications 484 (2017) .

[87] Vigelis, Rui F., David C. de Souza, and Charles C. Cavalcante. "New metric and connections in statistical manifolds." Geometric Science of Information: Second International Conference, GSI 2015, Palaiseau, France, October 28-30, 2015, Proceedings. Cham: Springer International Publishing, 2016.

[88] Wang, Wen, et al. "Discriminant analysis on Riemannian manifold of Gaussian distributions for face recognition with image sets." Proceedings of the IEEE conference on computer vision and pattern recognition. 2015.

[89] Aydin, Muhittin Evren, Adela Mihai, and Ion Mihai. "Some inequalities on submanifolds in statistical manifolds of constant curvature." Filomat 29.3 (2015) .

[90] Baktashmotlagh, Mahsa, et al. "Domain adaptation on the statistical manifold." Proceedings of the IEEE conference on computer vision and pattern recognition. 2014.

[91] Thoft-Cristensen, P. and Baker, M. J., Structural reliability theory and its applications, Springer Science and Business Media, (2012).

[92] Todinov, M. T., Risk-based reliability analysis and generic principles for risk reduction, Elsevier, (2006).

[93] Zhang, Jun. "Nonparametric information geometry: From divergence function to referential-representational biduality on statistical manifolds." Entropy 15.12 (2013) .

[94] Noda, Tomonori. "Symplectic structures on statistical manifolds." Journal of the Australian Mathematical Society 90.3 (2011): 371-384.

[95] Dodagoudar, G. R., and G. Venkatachalam. "Reliability analysis of slopes using fuzzy sets theory." Computers and Geotechnics 27.2 (2000) .

[96] Newton, Nigel J. "An infinite-dimensional statistical manifold modelled on Hilbert space." Journal of Functional Analysis 263.6 (2012) .

[97] Furuhata, Hitoshi. "Hypersurfaces in statistical manifolds." Differential Geometry and its Applications 27.3 (2009) .

[98] Matsuzoe , Hiroshi, et al. Statistical manifolds and affine differential geometry. Adv. Stud. Pure Math, 2010, 57: 303-321.

الخلاصة

في هذا العمل قدمنا العديد من التعاريف و النظريات والتطبيقات المتعلقة به . نتعامل مع عائلة من دوال الموثوقية التي تعتمد على العديد من المعلمات التي تشكل سطحا حيث ان كل نقطة في هذا السطح تمثل دالة موثوقية التي معلماتها هي اعداد حقيقية وقد برهنا مايلي : ان عائلة $N=\left\{e^{-\lambda t}:\lambda\in I\right\}$ هي نموذج برامتري ذو بعد واحد و ان العائلة $N=\left\{e^{-\lambda(t-\theta)}:t\in(0,\infty),(\lambda,\theta)\in I\right\}$ هي نموذج برامتري ذو بعدين وهكذا .بالاعتماد على النموذج البرمتري قمنا بانشاء متشعب ذو بعد n حيث المعلمة λ تلعب دور النظام الاحداثي . برهنا ان $N=\left\{e^{-\lambda(t-\theta)}:t\in(0,\infty),(\lambda,\theta)\in I\right\}$ هي متشعب سلس . برهنا ان الفضاء المماسي للمتشعب عند النقطة $R(t,\lambda)$ هو فضاء متجهات ذو بعد n .برهنا ان اي متجه مماسي عند نقطة يمكن اعتنباره عامل اشتقاق اتجاهي على طول منحني يمر من خلال تلك النقطة بحيث ان $V_{R(t,\lambda)}(f)=\partial_i(f)$.المشتقات الجزئية $\{\partial_i: i=1,2,\dots,n\}$ تمثل متجهات مماسية على متشعب معولي سلس عند نقطة $R(t,\lambda)$. نلاحظ ان المشتقات الجزئية $\{\partial_i: i=1,2,\dots,n\}$ تشكل قاعدة لفضاء المتجهات المماسية . عرفنا فضاء مماسي اخر T_λ على النظام الاحداثي λ وهو يمثل فضاء متجهات ذو بعد n متولد بواسطة ∂_i التي تمثل مشتقات اتجاهية .عرفنا تمثيل-1 لفضاء مماسي حيث ان $T_\lambda{}^{(1)}=\left\{V(t):V(t)=\partial_i\ell\big(R(t,\lambda)\big)\right\}.$ برهنا ان تمثيل-1 لاتتغير متجهاته لا تتغير بتغيير النظام الاحداثي . قدمنا بعض التعاريف الخاصة ب dual vector space واثبتنا انه كذلك فضاء مماسي .تعميم المتجهات الى تنسورات حيث يدرس التنسور التفاضل والتكامل في الفضاء المحدد بانظمة احداثيات .استخدام دالة التباين التي هي دالة سلسة على متشعب معولي ريماني محتث بواسطة دالة التباين .قيدنا دالة التباين بدالة محدبة تحقق شروط معينه لنحصل على قياس المسافة بين دالتين احتمالتين .درسنا الضبابية المعولية عندما يكون النظام يتبع توزيع اوسي ضبابي ثلاثي مع بعض الامثلة العددية حساب موثوقية الانظمة في حالة التوزيع الأسي بمعلمة واحدة وبمعلمتين وثلاثة معلمات .قدمنا تعاريف جديدة لمتشعب ضبابي معولي .

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

تطبيق بعض مفاهيم الهندسة التفاضلية على نمط للمعولية وتطبيق باستعمال توزيعات مختلفة

رسالة

مجلس كلية التربية للعلوم الصرفة في جامعة بابل كجزء من متطلبات نيل شهادة

الدكتوراه فلسفة في التربية / الرياضيات

من قبل

ندى محمد عباس جاسم

باشراف

أ. د. عدي صبري عبد الرزاق

1444هـ 2023