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Generalization of Topological Concepts via Cluster Spaces

A Dissertation

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بسم الله الرحمن الرحيم

((إن الله وملائكته يصلون على

النبي يا أيها اللذين امنوا

صلوا عليه وسلموا تسليما))

صدق الله العلي العظيم

الأحزاب آية 56

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Dedication

To the Great Prophet of Good the Seal of Prophets "**Mohammed**"

To the Lady of the Worlds Women **"Fatima AL-Zahra**"

To Madam and Moulatee "**Sharifa Bint AL- Hassan**"

To the Martyr of the AL-taf "**Bakr Bin Ali**"

To the Person, I Carry His Name, my Dear **Father**.

To the Mountain of Patience and Optimism and Hope my Dear **Mother**.

To the Soul that Inhabited my Soul

To the Price of my Happiness in life my **Husband**.

To their Presence are a Great Support, my **Brothers** and **Sisters**.

To all my **Teachers** who Taught me to come to this stage of learning.

To everyone who Supports me and Stay beside me in my Life.

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2023

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Contents

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Symbol	Description
X	**The universal set**
(X, δ)	**Proximity space**
$A\delta B$	***A* near *B***
$A\bar{\delta}B$	***A* far *B***
σ	**Cluster Family**
$(X, \tau, \delta, \sigma)$	**Proximity cluster topological space**
$(X, \delta, \tau_\delta, \sigma)$	**Cluster topological proximity space**
$(X, \delta, \sigma, \tau_\sigma)$	**$\sigma -$ Topological Proximity Space.**
P_{t_σ}	**Takeoff set**
P_{f_σ}	**Follower set**
P_{O_σ}	**Cluster outer set**
$D_\delta(P)$	**Cluster disputed set**
$B_\delta(P)$	**Cluster brim set**
$\mathcal{S}$	**Sporadic Family**
$H_{\mathcal{S}_f}$	**Sporadic follower set**
$H_{\mathcal{S}_t}$	**Sporadic takeoff set**

Abstract

Due to the ease of connecting mathematics to real-life situations within the proximity space, this study aims to find new mathematical concepts in the proximity space. The study examines its properties and their relationship to the agreed upon mathematical concepts. Thus, the study attempts to obtain useful results to solve some of the mathematical problems to cope with the increasing requirements of daily life.

The dissertation consists of five parts:

Part one presents the concept of σ-Topological Proximity based on a binary relationship built in this study with the assistance of the *cluster concept*. This family fulfills the conditions of the topological family, except for the feature of union and intersection. Hence, it will give us the opportunity to study the concepts of internal, external, and boundary points, as well as the limit points within this family, and to clarify the difference between them and the study of these concepts in terms of topology. As a result, we have three spaces in this study: $(X, \tau, \delta, \sigma)$,$(X, \delta, \tau_\delta, \sigma)$, and $(X, \delta, \sigma, \tau_\sigma)$, where X is a non-empty set. All the mathematical concepts presented in this dissertation are studied in terms of these three spaces.

Part two presents two types of *follower* and *takeoff* points. The set of all these points are named *follower set* and *takeoff set*. Through these sets the researchers managed to generate topologies based on Kuratowsk theory. Additionally, via the follower set, the researchers were able to provide a parallel definition for density, which we called *bushy set*, and the space that contains at least one bushy set, it called *Co bushy space*, where many properties and results can only be achieved within this space.

Part three is dedicated to the concept of bushy in studying the ability of *dismountable space* by two disjoint bushy sets, which represents a reduction of the concept of resolvable space that every dismountable space is resolvable. The concepts of *Bushy space* and *Attached space* and the relationship between them and submaximal, and hyper connected space were also introduced.

In part four three types of disjoint sets that generate space were studied. These were called *cluster outer, cluster brim, and cluster disputed sets*. They were built based on *follower* and *takeoff points* that focus on studying the most important properties that were obtained based on these sets and their relationship to the space to dismountable ability. Thus, the ability of space can be resolvable. In addition, a special type of open and closed sets were presented within the proximity space. Therefore, they can be exploited in constructing finer or coarser topological spaces than the topology generated in the proximity space.

Part Five of this dissertation deals with a synonymous concept of the cluster family, where the complement of all the sets of the cluster family were taken to generate a family that we called *Sporadic* family. It became possible to study all the previous concepts with this family. Although the study of this family was brief, we were able to obtain some important results that relate this family with the clusters as well as other results that distinguish it from the clusters family.

In part sixth, the researchers provide some applications about proximity space and link the results reached by the researchers to the healthy reality.

Introduction

Topological ideas emerged in the nineteenth century, leading to the classic definition of a topological space. This definition was established either through the closure operator Kartowisky [1] or through open sets, which is the more popular approach. Consequently, contemporary researchers have focused on studying open or closed sets through of metric spaces.

Metric spaces are important in various fields of engineering, applied and pure sciences. Although they determine the distance between points and sets, some problems have confused scientists and researchers including when the distance between points or sets is zero. Scientists provide different interpretations of this issue, and different views on its importance.

The scientist Riesz solved these problems and described this situation by finding a definition that is in line with metric spaces, namely the relation of proximity. Proximity is an easy concept to understand even for non-mathematicians as people use the words near and far in everyday life. Scientist Lagrange said that proximity can be explained to the first person we meet on the street.

Riesz [2] put forward the concepts of proximity spaces in the "Theory of enchainment". In 1909, at a conference in Switzerland, however, his idea did not receive the attention of researchers at that time.

In 1952, Efremovic reintroduced proximity and gave a definition of the proximity zone based on distance [3]. He stated that two sets in a metric space are close if the gap between them is zero. Based on this idea, mathematicians turned to the study of the concept of proximity, the likes of Smirnov, Leader, Lodeto and other researchers who discovered many important results by applying them to many life problems.

Smirnov in 1952 has shown that EF-proximity can be used to generate all T_2- copactifiction of a given Tychonov space [4]. Additionally, Leader studied the non-symmetric case and his student Lodeto studied the symmetric in 1966 [5]. Moreover, in 1959 Leader introduced the concept of Cluster, which is a generalization of ultrafilter [6]. We know that $\mathcal{L}$ is called ultrafilter if and only if $\mathcal{L}$ satisfy three conditions:

1) $H, D \in \mathcal{L}$ if and only if $H \cap D \neq \emptyset$.

2) $H \cup D \in \mathcal{L}$ if and only if $H \in \mathcal{L}$ or $D \in \mathcal{L}$.

3) $H \cap D \neq \emptyset$ for every $D \in \mathcal{L}$, then $H \in \mathcal{L}$.

If the above conditions, intersect is replaced by near one gets a cluster in proximity space. The cluster is one of the important concepts and has many applications in proximity space, which forms a fundamental pillar within this work [7].

All these results have been inspiring researchers to study this space to this day. Thus, scientists and researchers were inclined to study these new spaces on the one hand, and to develop mathematical concepts on the other. [8, 9, 10, 11].

It is important to mention some contemporary researchers who have used these spaces to study and create new sets or concepts. Among them is the researcher Dargham [12], who used proximity space in a different direction by creating new sets called Center Set and building the theory of (σ-Algebra) set. Another researcher, Ghassan [13], in 2022, constructed functions and new spaces within central topological spaces using the theory of proximity. He studied all the topological concepts that can be explored within those spaces, as well as the relationships between them within this space.

Other contemporary researchers have attempted to enrich this space with many interesting concepts. In 2022, the researcher Yiezi [14] introduced a family of sets in proximity spaces called focal set, which relies on the relationship of ideality with the open set within i-topology space.

i- topology space presented by Irina Zvina in 2006 [15], which used the symmetry relationship with ideal to provide a binary relationship called $\approx$ as that $A \approx B$ if and only if $(A - B) \cup (B - A) \in I$ where I is ideal.

During this relationship he was able to build a family called i-topology and if the following conditions are satisfies:

1) $X, \emptyset \in$ T;

2) For any collection $\mathcal{U}$ of T, there exists $\mathcal{V} \in$ T such that $\bigcup\mathcal{U} \approx \mathcal{V}$;

3) For any $\mathcal{U},\mathcal{V} \in$ T, there exists $\mathcal{W} \in$ T such that $(\mathcal{U} \cap \mathcal{V}) \approx \mathcal{W}$;

4)T $\cap I = \{\emptyset\}$.

Then (X, T, I) is called i – topological space.

In this work, symmetry was also used, but within the cluster in the proximity space, to present a binary relationship called $\approx_\sigma$ where $A \approx_\sigma B$ if and only if $(A - B) \cup (B - A) \in \sigma$ where σ is a cluster. We created a parallel family to the i-topology family within the proximity space, based on the concept of the cluster. We studied most of the mathematical concepts within this family and their relationship with topology and whether this family can satisfy the conditions of topology and under what conditions. All of this and more is explained in the dissertation.

The dissertation is divided into six chapters:

Chapter I: Consists of four sections. The first section includes the foundational definitions required for this work. The second section focuses on the generated topology within the proximity space and some necessary theorems. The third section introduces two binary relationships ($\approx_\sigma$), and ($\propto_\sigma$) based on the symmetry relation and the difference between two sets, along with studying the important properties of these relationships. The fourth section builds a family based on the binary relationship ($\approx_\sigma$) called $\sigma-$Topological Proximity (τ_σ). We are studying this family, mentioning its important properties, and its relationship with the topology family, and exploring which one implies the other and whether this family can form a topology or not, and under what conditions.

Chapter II: Focuses on three sections. The first section introduces the concept of *takeoff set*, which is parallel to the concept of an operator, ψ, studying its important properties. The second section introduces the concept of *follower set*, which is parallel to the concept of the local function, studying its properties. The follower set is then employed to construct topology based on the theory of Kuratowsk. The third section presents a special type of closed and open sets, studying their important properties and the relationships between them.

Chapter III: Focuses on three sections. The first section introduces the concept of the *bushy set* and studies its relationship with dense sets and the important results obtained when each bushy set is an open set or vice versa. It also examines its relationship with hyper connected and submaximal concepts. The second section introduces the *co-bushy space* and studies its important properties. The third section presents the dismountable space and non-dismountable space and their relationship with resolvable and irresolvable spaces.

Chapter IV: Centered on two sections. The first section disjoint sets aggregates that divide the space into separate aggregates was presented, we called them: cluster outer, cluster brim, and cluster disputed sets with the study of the characteristics of each set. As for the second section, the definition of cluster too intense, clutter semi intense, and cluster intense sets was presented and their properties and their relationship to open and closed sets were studied.

Chapter V: In this chapter, a family synonymous with the cluster family was presented. It called the *Sporadic family*, and all the results and characteristics that were studied on the cluster family were re-studied on the sporadic family. This is clarifying the difference between the two studies and if there is a relationship between these two families. It also focuses on the most important spaces, and the most results and characteristics.

Chapter VI: Contains the application of proximity space, and conclusions that we have reached as well as future work.

Publications Associated with this dissertation

1. R. Almohammed, and L. A. AL-Swidi, "Focal Cluster via $\sigma-$ Topological Proximity Space", accepted for publication in ACCEPTED in AIP Conference proceeding Journal.
2. R. Almohammed, and L. A. A. Jabar, "Follower and Takeoff points in proximity space", Iraqi Journal of Science, Published on October, 64(2023).
3. R. Almohammed, and L. A. A. Jabar, "cluster outer points via proximity spaces", Iraqi Journal of Science, Published on February, 65(2024).
4. R. Almohammed, and L. A. A. Jabar, "Focal Follower points in proximity space", accepted of the IEEE international conference for physics and mathematics.
5. R. Almohammed, and L.A.A. Jabar, "Dismountable and non-Dismountable Spaces via Proximity Space", Baghdad Science Journal, Published on September, 2023.
6. R. Almohammed, and L.A.A. Jabar, "Too Intense in Cluster Topological Proximity Space", accepted for publication in 9th International Conference and Worksshops on Basic and Applied Sciences (ICOWOBAS-2023)
7. L. A. AL-Swidi, and R. Almohammed, "The Bushy Sets in Cluster Topological Proximity Space", accepted of the IEEE 4th international science 2023.
8. L. A. AL-Swidi, and R. Almohammed, "New Concepts via Topological Proximity Space", accepted in AIP Conference Proceeding (ISSN).

1. 1 Proximity space

Proximity is an intuitive everyday notion of life. For instance, we become skilled in identifying the similarities among things in our environment at an early age and we can quickly evaluate the degree of similarity. Phrases such as "those cars look alike" or "those pictures have the same frame" serve to illustrate how frequently we make analogies to things that are not exactly identical but share some common properties. In this section we will recall some of the basic definitions related to proximity space, cluster and some theorems need in this work.

Definition 1. 1. 1 [16]: Let X be a nonempty set. A relation δ on the family $\mathcal{P}$(X) of all subsets of a set X is called a proximity on X if δ satisfies the following conditions:

[P1] $A\delta B$, then $B\delta A$;

[P2] $A\delta B$, then A $\neq \emptyset$ and B $\neq \emptyset$;

[P3] $A \cap B \neq \emptyset$, then $A\delta B$;

[P4] $A\delta(B \cup C)$ if and only if $A\delta B$, or $A\delta C$;

[P5] $A\delta B$ and $\{b\}\delta C$ for each $b \in B$, then $A\delta C$;

[P6] $A\bar{\delta}B$, then there exists $E \in \mathcal{P}$(X) such that A$\bar{\delta}$E and X $- E\bar{\delta}$B;

[P7] $\{x\}\delta\{y\}$, then $x = y$.

Strictly speaking, one should use the notation $(A, B) \in \delta$ or $(A, B) \notin \delta$ when the sets A and B are either near each other or not, but we shall simply write $A\delta B$ or $A\overline{\delta}B$. The pair (X, δ) is called a proximity space.

A Basic proximity δ or Cech's axioms is one that satisfies from [P1] to [P4]. A Lodato proximity δ is one that satisfies from [P1] to [P5]. Efremovic or EF- proximity δ is one that satisfies from [P1] to [P4], and

[P6]. Further proximity δ is separated if it satisfies [P7]. In this work we using a basic proximity.

Example 1. 1. 2: Let X $= \{1,2,3\}$, $\delta = \{(\{1\},\{1\})$, $(\{1\},\{1,2\}), (\{1\},\{1,3\}), (\{1\},X), (\{1\},\{3\}), (\{1\},\{2,3\}), (\{2\},\{2\})$, $(\{2\},\{1,2\}), (\{2\},\{2,3\}), (\{2\},X), (\{3\},\{3\}), (\{3\},\{1,3\}), (\{3\},\{2,3\})$, $(\{3\},X), (\{3\},\{1,2\}), (\{1,2\},\{1,2\}), (\{1,2\},\{1,3\})$, $(\{1,2\},\{2,3\}), (\{1,2\},X)$, $(\{1,3\},\{2,3\})$, $(\{1,3\},\{1,3\}), (\{1,3\},X), (\{2,3\},\{2,3\})$, $(\{2,3\},X), (X,X)$, $(\{1,2\},\{1\})(\{1,3\},\{1\}), (X,\{1\}), (\{3\},\{1\})$, $(\{2,3\},\{1\}), (\{1,2\},\{2\})$, $(\{2,3\},\{2\}), (X,\{2\}), (\{1,3\},\{3\}), (\{2,3\},\{3\}), (X,\{3\}), (\{1,2\},\{3\})$, $(\{1,3\},\{1,2\}), (\{2,3\},\{1,2\}), (\{X,\{1,2\}), (\{2,3\},\{1,3\}, (\{X,\{1,3\})$, $(\{1,3\},\{1,3\}), (\{2,3\},\{2,3\}), (\{2,3\},X)\}$.We see that δ satisfies from [P1] to [P6], thus δ is proximity define on X.

Example 1. 1. 3 [16]: Let $X = \{\mathrm{a},\mathrm{b},\mathrm{c}\}$, and δ_1 define by$A\delta$B if and only if $A \neq \emptyset$ and $B \neq \emptyset$.Then

$\delta_1 = \{(\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{a,b\}), (\{a\},\{b,c\}), (\{a\},\{a,c\})$, $(\{a\},\{a\}), (\{b\},\{b\}), (\{b\},\{c\}), (\{b\},\{a,b\}), (\{b\},\{a,c\}), (\{b\},\{b,c\})$, $(\{b\},X), (\{c\},\{c\}), (\{c\},\{a,b\})$, $(\{c\},\{b,c\}), (\{c\},\{a,c\}), (\{c\},\{a,c\})$, $(\{c\},X), (\{a,b\},X), (\{a,b\},\{b,c\})$, $(\{a,b\},\{a,c\}), (\{a,b\},\{b,a\})$, $(X,\{b,c\}), (\{b,c\},\{a,c\}), (\{b,c\},X), (\{a,c\},\{a,c\}), (\{a,c\},X), (\{b\},\{a\})$, $(\{c\},\{a\}), (\{a,b\},\{a\}), (\{b,c\},\{a\}), (\{a\},X), (X,\{a\}), (\{c\},\{b\}), (\{a,b\},\{b\})$, $(\{a,c\},\{b\}), (\{b,c\},\{b\})$, $(X,\{b\}), (\{a,b\},\{c\}), (\{b,c\},\{c\}), (\{a,c\},\{c\})$, $(\{a,c\},\{c\}), (X,\{c\}), (\{b,c\},\{a,b\}), (\{a,c\},\{a,b\}), (\{a,c\},\{a\}), (X,\{a,b\})$, $(\{a,c\},\{b,c\}), (\{b,c\},\{b,c\}), (\{a,c\},X)\}$.

δ_1 satisfies the conditions [P1] - [P6]. Hence δ_1 is proximity say indiscrete proximity.

Example 1. 1. 4 [16]: Let $X = \{a, b, c\}$. Let δ_D define by $A\delta_D B$ if and only if $A \cap B \neq \emptyset$. Then

$\delta_D = \{ (\{a\},\{a,b\}), (\{a\},\{a,c\}), (\{c\},\{c\}), (\{c\},\{b,c\}), (\{c\},\{a,c\}),$ $(\{c\},\{a,c\}), (\{c\},X), (\{a,b\},\{a,b\}), (\{a,b\},\{b,c\}), (\{a,b\},\{a,c\}),$ $(\{a,b\},X), (\{b,c\},\{b,c\}), (\{b,c\},\{a,c\}), (\{b,c\},X), (\{a,c\},\{a,c\}),$ $(\{a,c\},X), (\{a,b\},\{a\}), (\{b,c\},\{a\}), (\{a,c\},\{a\}), (X,\{a\}), (\{a,b\},\{b\}),$ $(\{b,c\},\{b\}), (X,\{b\}), (\{a\},X), (\{b,c\},\{c\}), (\{a,c\},\{c\}), (\{a,c\},\{c\}),$ $(X,\{c\}), (\{b,c\},\{a,b\}), (\{a,c\},\{a,b\}), (X,\{a,b\}), (\{a,c\},\{b,c\}),$ $(X,\{b,c\}), (\{a,c\},X) \}$. We see that δ_D satisfies the conditions [P1] - [P7]. Thus δ_D is a separated proximity say discrete proximity.

But if $\delta = \{(\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{b,c\}), (\{a\},X), (\{a\},\{a\}),$ $(\{b\},\{a,c\}), (\{b\},X), (\{c\},\{c\}), (\{c\},\{a,b\})(\{c\},X), (\{a,b\},\{a,b\}),$ $(X,\{a,b\}), (\{a,b\},X), (\{b,c\},\{b,c\}), (\{b,c\},X)(X,\{a\}), (\{c\},\{b\}),$ $(\{a,c\},\{b\}), (X,\{b\}), (X,\{c\}), (\{a,c\},X), (\{b\},\{a\}), (\{c\},\{a\}),$ $(\{b,c\},\{a\}), (\{a,b\},\{c\}), (X,\{b,c\}), (\{b\},\{b\}), (\{b\},\{c\}), (\{a,c\},X)\}$, then δ is not proximity because $\{a,b\} \cap \{a,c\} \neq \emptyset$, but $\{a,b\}\bar{\delta}\{a,c\}$.

Definition 1. 1. 5 [16]: If δ_1 and δ_2 are two proximities, we define $\delta_1 > \delta_2$ if and only if $A\delta_1 B$ implies $A\delta_2 B$. In this case we say that δ_1 is finer then δ_2, or δ_2 is coarser than δ_1.

By above Examples we see that $\delta_D > \delta_I$. Hence a discrete proximity finer then indiscrete proximity. Therefore it is obvious that $\delta_D > \delta > \delta_I$ for any proximity δ on X.

Examples 1. 1. 6 [17]: Let X be a nonempty set. δ is a proximity define by:

1. AδB if and only if clA $\cap$ $clB \neq \emptyset$.(fine Lodato proximity)
2. AδB if and only if clA $\cap$ $clB \neq \emptyset$ or both A and B are infinite. (Coarsest Lodato proximity).
3. AδB if and only if clA $\cap$ $clB \neq \emptyset$ or both A and B are uncountable
4. AδB if and only if $d(A, B) = 0$. (Metric proximity).

Theorem 1. 1. 7 [16]: Let (X, δ) be a proximity space. A, B nonempty subsets of X. Then

1. If $A\bar{\delta}B$, then $B\bar{\delta}A$.
2. If $A\delta B$, and $B \subset C$,then AδC .
3. If A$\bar{\delta}$B, and $C \subset B$,then A$\bar{\delta}$C .
4. If there exist a point $x \in$ X such that A$\delta\{x\}$ and $\{x\}\delta$B, then AδB.
5. A$\bar{\delta}\emptyset$ for every A $\subseteq$ X.
6. $\{x\}\delta\{x\}$ for each $x \in$ X.
7. If $A\bar{\delta}B$, then $A \cap B = \emptyset$.
8. If $A\bar{\delta}B$, then $\{x\}\bar{\delta}B$ for each $x \in A$.
9. $A\bar{\delta}C$ and $B\bar{\delta}C$ if and only if $(A \cup B)\bar{\delta}C$.

Definition 1. 1. 8 [14]: The proximity δ defined on a nonempty universal X is called σ – proximity if for any arbitrary family $\{\mu_\lambda ;\ \lambda \in \beta\}$ of subsets of X, it has the following feature $B\ \delta(\cup_{\lambda\in\beta}\ \mu_{\lambda_0})$ if and only if $B\ \delta\ \mu_{\lambda_0}$ for some $\lambda_0 \in \beta$.

Definition 1. 1. 9 [16]: Let (X, δ) be a proximity space, and A , B are subset of X . Then B is called a proximity neighborhood or δ – neighborhood of A if and only if $A\bar{\delta}(X - B)$ and is written $A \ll B$.

Theorem 1. 1. 10 [16]: Let (X, δ) be a proximity space, and A, B, C, D are subset of X. Then the relation $\ll$ satisfies the following properties:

1. $A \ll X$,and $\emptyset \ll A$ for any subset A of X;
2. If $A \ll B$, then $A \subset B$;
3. $A \subset B \ll C \subset D$ implies $A \ll D$;
4. $A \ll B$, implies $(X - B) \ll (X - A)$;
5. $A \ll B_i$ is true for $i = 1, 2, \dots, n$ if and only if $A \ll \cap_{i=1}^{n} B_i$;
6. If $A \ll B$, then there exists a set $C \subset X$ such that $A \ll C \ll B$;
7. If $\{x\} \ll A$ then $x \in A$;
8. If $A \ll B$, then $\{x\} \ll B$ for all $x \in A$.

Definition 1. 1. 11 [16]: Let (X, δ) be a proximity space, and A, B subset of X. The family $\mathcal{F}(A) = \{B \subseteq X; A \ll B\}$ is all $\delta - neighberhood$ of a set A in a proximity space (X, δ).

Proposition 1. 1. 12 [16]: Let(X, δ) be a proximity space. Then

1. If $B \in \mathcal{F}(A)$, then $A \subset B$.
2. If $B \in \mathcal{F}(A)$, then $X - A \in \mathcal{F}(X - B)$.
3. If $B \subset A$, then $\mathcal{F}(A) \subset \mathcal{F}(B)$.
4. If $B \in \mathcal{F}(A)$, then there exists a $C \in \mathcal{F}(A)$ such that $B \in \mathcal{F}(C)$.

Proposition 1. 1. 13 [18]: Let (X, δ) be a proximity space. Let Y a nonempty subset of X. $A, B \subset Y$, and $A \delta_Y B$ if and only if $A \delta_X B$. Then (Y, δ_Y) is a proximity subspace.

Definition 1. 1. 14 [6] A nonempty family of nonempty subsets of X denoted by σ is called a cluster in proximity space (X, δ) if σ satisfying the conditions:

[C1] For all $A, B \in \sigma \Longrightarrow A\delta B$;

[C2] $(A \cup B) \in \sigma \Leftrightarrow A \in \sigma$ or $B \in \sigma$;

[C3] $A\delta B$ for each $B \in \sigma \Rightarrow A \in \sigma$.

Proposition 1. 1. 15 [6]: Let (X, δ) be a proximity space. A, B nonempty subsets of X. Then

1. For any $E \subseteq X$ either $E \in \sigma$ or $X - E \in \sigma$.
2. If $A \in \sigma$ and $A \subset B$, then $B \in \sigma$.
3. $\sigma_x = \{A \subseteq X;\ A\delta\{x\}\}$ is cluster.
4. If $\{x\} \in \sigma$ for some $x \in X, then\ \sigma = \sigma_x$ is called point cluster.
5. If δ is indiscrete proximity, then $\sigma = \{A \subseteq X;\ A \neq \emptyset\}$ is cluster.

Example 1. 1. 16: Let $X = \{a, b, c\}$, δ be a discrete proximity defined on X, then $\sigma_a = \big\{\{a\}, \{a, b\}, \{a, c\}, X\big\}$, $\sigma_b = \big\{\{b\}, \{a, b\}, \{b, c\}, X\big\}$, and $\sigma_c = \{\{c\}, \{c, b\}, \{a, c\}, X\}$ are point clusters.

We can see that if δ defined on X is an indiscrete proximity, then there is only one cluster: $\sigma = \{\{a\}, \{a, b\}, \{a, c\}, \{b\}, \{c\}, \{b, c\}, X\}$, say point cluster.

Notes 1. 1. 17: Let (X, δ) be a proximity space. A, B nonempty subsets of X. Then

1st If $A, B \in \sigma$, then $A \cup B \in \sigma$, but $A \cap B$ is not necessary belong in cluster.

Because $A, B \subset A \cup B$ by Proposition 1. 1. 15 part 2, we get $A \cup B \in \sigma$. Therefore by Example 1. 1. 3, $\{a\}$ and $\{b\} \in \sigma$, but $\{a\} \cap \{b\} = \emptyset \notin \sigma$.

2nd If $A \cap B \in \sigma$, then $A\ and\ B \in \sigma$. Immediate consequence of Proposition 1. 1. 15 part 2.

3rd $\emptyset \notin \sigma$. Because $\emptyset \bar{\delta} A$, for every $A \subseteq X$.

4th By Proposition 1. 1. 15 part 1, if $A \notin \sigma$, then $(X - A) \in \sigma$. But convers not always is true, that is, if $(X - A) \in \sigma$ that not necessary $A \notin \sigma$ because if we take δ is indiscrete proximity, then A and $(X - A) \in \sigma$ for every nonempty subset A of X.

5th The union and intersection of any two families of clusters are not necessary to be a cluster. For example 1. 1. 16, σ_a and σ_b are clusters, then we see that $\sigma_a \cap \sigma_b$ and $\sigma_a \cup \sigma_b$ are not clusters.

6th Every proximity space has a family of cluster at most equal to the number of space elements or it has at least one.

Lemma 1. 1. 18 [18]: Let σ_1, σ_2 be a two clusters define on (X, δ). If $\sigma_1 \subset \sigma_2$, then $\sigma_1 = \sigma_2$.

Theorem 1. 1. 19 [17]: Let (X, δ) be a proximity space. If $A\delta B$, then there exists a cluster σ containing A and B.

Theorem 1. 1. 20 [17]: Let (X,δ) be a proximity space, and σ is a cluster in X. Let $A\subset X$ and $A\in\sigma$. Then the cluster $\sigma_A=\{H\subset A; H\in\sigma\}$ is the only cluster in (A,δ_A) contained in σ.

Example 1. 1. 21: Let (X,δ) be a proximity space. Let δ is a discrete proximity and $\mathrm{X}=\{1,2,3\}$. Then $\sigma=\{\{1\},\{1,2\},\{1,3\},X\}$. $\{1,3\}\in\sigma$ and $\{1,3\}\subset X$ we have that $\sigma_{\{1,3\}}=\{\{1\},\{1,3\}\}$. And $\sigma_{\{1\}}=\{\{1\}\}$, $\sigma_{\{1,2\}}=\{\{1\},\{1,2\}\}$.$\sigma_X=\{\{1\},\{1,2\},\{1,3\},X\}=\sigma$.

Definition 1. 1. 22 [18]: A mapping $f:(X,\delta_X)\to(Y,\delta_Y)$ is called a proximity or δ – continuous if $A\delta_X B$, then $f(A)\delta_Y f(B)$ for each $\mathrm{A},B\subseteq X$.

Proposition 1. 1. 23 [18]: A mapping $f:(X,\delta_X)\to(Y,\delta_Y)$ is δ – continuous if and only if for every $D,H\subset Y$, $D\bar{\delta}_Y H$, then $f^{-1}(D)\bar{\delta}_X f^{-1}(H)$.

Theorem 1. 1. 24 [18]: Let f be a proximity mapping from (X,δ_X) to (Y,δ_Y). Then for each cluster σ_X in X, there corresponds a cluster σ_Y in Y such that $f(\sigma_X)=\{A\subset Y;\ A\delta_Y\ f(B)$ for every $B\ in\ \sigma_X\}$.

1.2 Proximity Topology

In this section, we introduce the definition of topology in the proximity space and the most important characteristics that we need in this work. And discuss some properties of the proximity space.

Definition 1. 2. 1 [16]: Let (X, δ) be a proximity space. A subset $F \subset X$ is defined to be $\tau_\delta - closed$ if and only if $\{x\}\delta F$ implies $x \in F$. Thus if $x \notin F$ implies $\{x\}\overline{\delta}F$, then F is $\tau_\delta - closed\ set$.

By τ_δ denote the family of complements of all the sets defined in such a way. It is easy to notice that, X and $\emptyset$ are $\tau_\delta - closed\ set$.

Example 1. 2. 2: Let $X = \{1, 2, 3\}$, $\delta = \{(\{1\},\{1\}),$
$(\{1\},\{1,2\}), (\{1\},\{1,3\}), (\{1\},X), (\{2\},\{2\}), (\{2\},\{1,2\}),$
$(\{2\},\{2,3\}), (\{2\},X), (\{3\},\{3\}), (\{3\},\{1,3\}), (\{3\},\{2,3\}), (\{3\},X),$
$(\{1,2\},\{1,2\}), (\{1,2\},\{1,3\})$, $(\{1,2\},\{2,3\}), (\{1,2\},X), (\{1,3\},\{2,3\}),$
$(\{1,3\},\{1,3\}), (\{1,3\},X), (\{2,3\},\{2,3\})$, $(\{2,3\},X), (X,X), (\{1,2\},\{1\}),$
$(\{1,3\},\{1\}), (X,\{1\}), (\{1,2\},\{2\}), (\{2,3\},\{2\}), (X,\{2\}), (\{1,3\},\{3\}),$
$(\{2,3\},\{3\}), (X,\{3\}), (\{1,3\},\{1,2\}), (\{2,3\},\{1,2\}), (\{X,\{1,2\}), (\{2,3\},\{1,3\}),$
$(\{X,\{1,3\}), (\{1,3\},\{1,3\}), (\{2,3\},\{2,3\}), (\{2,3\},X)(\{2\},\{3\}), (\{3\},\{2\}),$
$(\{2\},\{1,3\}), (\{1,3\},\{2\}), (\{1,2\},\{3\}), (\{3\},\{1,2\})\}$.

We see that , $X, \emptyset, \{2,3\}, \{1\}$ are $\tau_\delta - closed$ sets. Because

$\{1\}\delta\{1\}$ implies $1 \in \{1\}$,

$\{2\}\delta\{2,3\}$ implies $2 \in \{2,3\}$,

$\{3\}\delta\{2,3\}$ implies $3 \in \{2,3\}$.

1, 2 and $3 \notin \emptyset$ implies $\{1\}$, $\{2\}$ and $\{3\}\overline{\delta}\emptyset$.

But , $\{2\}, \{3\}, \{1,3\}, \{1,2\}$ are not $\delta -$ closed sets. Because

$\{3\}\delta\{2\}$ but $3 \notin \{2\}$,

$\{2\}\delta\{3\}$ but $2 \notin \{3\}$,

$\{2\}\delta\{1,3\}$ but $2 \notin \{1,3\}$,

$\{3\}\delta\{1,2\}$ but $3 \notin \{1,2\}$.

Thus $\tau_\delta = \{ X, \emptyset, \{1\}, \{2,3\}\}$.

Proposition 1. 2. 3 [16]: If G is a subset of a proximity space(X,δ), then G is τ_δ $-$open in topology τ_δ if and only if $\{x\}\overline{\delta}X - G$ for every $x \in G$.

By Example 1. 1. 4, τ_{δ_D} is a discrete topology because every subset of X is $\tau_{\delta_D} - open.$

That is, $a \in \{a\}$ implies $\{a\}\overline{\delta_D}\{b,c\}$,

$b,\ c \in \{b,c\}$ implies $\{b\}\overline{\delta_D}\{a\} and\ \{c\}\overline{\delta_D}\{a\}$,

$a, c \in \{a,c\}$ implies $\{a\}\overline{\delta_D}\{b\} and\ \{c\}\overline{\delta_D}\{b\}$,

$a, b \in \{a,b\}$ implies $\{a\}\overline{\delta_D}\{c\} and\ \{b\}\overline{\delta_D}\{c\}$,

$b \in \{b\}$ implies $\{b\}\overline{\delta_D}\{a,c\}$,

$c \in \{c\}$ implies $\{c\}\overline{\delta_D}\{a,b\}$,

a, b and $c \in X$ implies $\{a\}$, $\{b\}$ and $\{c\}\ \overline{\delta_D}\emptyset$,

a, b and $c \notin \emptyset$ implies a, b and $\{c\}\ \delta_D X$.

In the same way we note that, by Example 1. 1. 3, τ_{δ_I} is indiscrete topology, hence $\tau_{\delta_I} = \{ X, \emptyset\}$.

Theorem 1. 2. 4 [16]: If (X,δ) is a proximity space, then the family τ_δ is a topology on the set X.

Moreover, if (X, δ_i) is a proximity space, then there exists a unique topology τ_{δ_i} generated by δ_i. For example δ_I generated indiscrete topology, and δ_D generated discrete topology.

Proposition 1. 2. 5 [16]: Let δ_1 and δ_2 be a two proximity defined on X. If $\delta_2 > \delta_1$, then $\tau_{\delta_1} \subset \tau_{\delta_2}$.

Proposition 1. 2. 6 [16]: Let (X, δ) be a proximity space. Then the τ_δ −closure A of a set A is given by: $\tau_\delta - cl(A) = \{x; \{x\}\delta A\}$.

Proposition 1. 2. 7 [16]: Let (X, δ) be a proximity space. Then $\tau_\delta - cl(A)$ is a Kuratowski closure operator.

Proposition 1. 2. 8: Let (X, δ) be a finite proximity space. Then $\mathcal{U}\overline{\delta}(X - \mathcal{U})$ if and only if $\mathcal{U}$ is $\tau_\delta - open$.

Proof.

Let $\mathcal{U}$ is τ_δ −open set. Then $\{x\}\,\overline{\delta}(X - \mathcal{U})$ for every $x \in \mathcal{U}$. By axiom [P4], $(\cup_{x\in\mathcal{U}} \{x\})\overline{\delta}(X - \mathcal{U})$ for every $x \in \mathcal{U}$. But $(\cup_{x\in\mathcal{U}} \{x\}) = \mathcal{U}$, thus $\mathcal{U}\,\overline{\delta}(X - \mathcal{U})$. Conversely, Let $\mathcal{U}\,\overline{\delta}(X - \mathcal{U})$, then by Theorem 1. 1. 7 part 3, $\{x\}\,\overline{\delta}(X - \mathcal{U})$, for every $x \in \mathcal{U}$ by Proposition 1. 2. 7, $\mathcal{U}$ is τ_δ −open set■

Proposition 1. 2. 9 [16]: $\mathcal{U}$ is $\tau_\delta - open$ set if and only if $\mathcal{U} \in \tau_\delta$. Also F is $\tau_\delta - closed$ set if and only if $X - F \in \tau_\delta$.

Remark 1. 2. 10 [16]: Let $\mathcal{U}$ be a $\tau_\delta - open$ of a point x. Then $\tau_\delta(x)$ denoted of all $\tau_\delta - open$ set of the point x. Thus $\tau_\delta(x) = \{\mathcal{U} \in \tau_\delta; x \in \mathcal{U}\}$.

Proposition 1. 2. 11 [16]: The topology τ_δ generated by a proximity relation δ in a space X is regular.

Proposition 1. 2. 12: Let τ_δ be a topology generated by a proximity relation δ on a finite nonempty set X. Then every $\tau_\delta - open$ set is $\tau_\delta - closed$ set.

Proof.

Let $\mathcal{U} \in \tau_\delta$. Then for every $x \in \mathcal{U}$, $\{x\}\overline{\delta}(X - \mathcal{U})$. By Theorem 1. 1. 7 part 3, $\{x\}\, \overline{\delta}\{y\}$ for every $y \in (X - \mathcal{U})$. Hence $\{y\}\overline{\delta}\{x\}$ for every $x \in \mathcal{U}$ By Theorem 1. 1. 7 part 9, we have that $\{y\}\overline{\delta} \cup_{x\in\mathcal{U}}\{x\}$ thus $\{y\}\overline{\delta}\mathcal{U}$ for every $y \in (X - \mathcal{U})$. In other word for every $y \in (X - \mathcal{U})$, $y \ll X - \mathcal{U}$ By Proposition 1. 2. 7, $(X - \mathcal{U}) \in \tau_\delta$. That is $\mathcal{U}$ is $\tau_\delta - closed$ set■

Note 1. 2. 13: If δ defined on a nonempty universal X is σ – proximity, then every $\tau_\delta - open$ set is $\tau_\delta - closed$ set.

Theorem 1. 2. 14 [17]: Let (X, δ) be a proximity space. $x \in X$. Then $\mathcal{F}(\{x\})$ is induced topology by the proximity δ.

Proposition 1. 2. 15: Let(X, δ) be a proximity space, and $A \subseteq X$.
$x \in \tau_\delta - cl(A)$ if and only if $\mathcal{U}\delta A$ for every $\mathcal{U} \in \tau_\delta(x)$.

Proof.

Let $x \in \tau_\delta - cl(A)$. Then $\{x\}\delta A$, by Theorem 1. 1. 7 part 2, $\mathcal{U}\delta A$ for every $\mathcal{U} \in \tau_\delta(x)$.

Conversely, let $\mathcal{U}\delta A$ for every $\mathcal{U} \in \tau_\delta(x)$. If possible $x \notin \tau_\delta - cl(A)$. Then $\{x\}\overline{\delta}A$, but $(X - A)$ is δ – neighborhood of x, by Theorem 1. 2. 14 there exists $\mathcal{V} \in \tau_\delta(x)$ such that $\{x\} \ll \mathcal{V} \ll X - A$, that is, $\mathcal{V}\overline{\delta}A$ which is a contradiction with hypothesis. Hence $x \in \tau_\delta - cl(A)$■

Definition 1. 2. 16: The quadruple $(X, \delta, \tau_\delta, \sigma)$ is called cluster topological proximity space, where (X, δ) is a proximity space.

That means the topology proximity and cluster depend upon the proximity space.

Definition 1. 2. 17: The quadruple $(X, \tau, \delta, \sigma)$ is called proximity cluster topological space, where (X, τ) is a topological space and (X, δ) is a proximity space.

That means the topology independent upon the proximity space, but cluster depend upon the proximity space.

1. 3 Binary relation $\approx_\sigma$ and Binary relation $\propto_\sigma$:

In this section, we introduce binary relation $\approx$ and binary relation $\propto_\sigma$. First, the symmetry relationship needs to be defined, which depends on the cluster set. Also, the properties of this relation are examined.

Definition 1. 3. 1: Let σ be a cluster in (X, δ), and A, B are subsets of X. The binary relation $\approx_\sigma$ defined on (X, δ) as follows:
$A \approx_\sigma B \iff (A - B) \cup (B - A) \in \sigma$. We denoted by $A \not\approx_\sigma B \iff (A - B) \cup (B - A) \notin \sigma$.

Example 1. 3. 2: Let $X = \{1,2,3\}$ and δ is a discrete proximity, $\sigma = \{\{1\}, \{1,2\}, \{1,3\}, X\}$, then $\{1\} \approx_\sigma \{2\}$. Because $(\{1\} - \{2\}) \cup (\{2\} - \{1\}) = \{1\} \cup \{2\} = \{1, 2\} \in \sigma$. But $\{1\} \not\approx_\sigma \{1,2\}$ because $(\{1\} - \{1, 2\}) \cup (\{1, 2\} - \{1\}) = \emptyset \cup \{2\} = \{2\} \notin \sigma$.

Proposition 1. 3. 3: Let (X, δ) be a proximity space. A, B are subsets of X and σ is cluster define on X. Then

1. $A \approx_\sigma X$, for every $A \notin \sigma$;
2. $A \not\approx_\sigma A$, for every $A \subseteq X$;
3. $A \approx_\sigma \emptyset$, for every $A \in \sigma$;
4. If $A \approx_\sigma B$, then $B \approx_\sigma A$;
5. For every $A \in \sigma$, there exsist $B \notin \sigma$, such that $A \approx_\sigma B$;
6. For every $A \notin \sigma$, there exsist $B \in \sigma$, such that $A \approx_\sigma B$;
7. If $A \notin \sigma$ and $B \notin \sigma$, then $A \not\approx_\sigma B$.

Proof.

1) Let $A \notin \sigma$. Then by proposition 1. 1. 15 part 1, $X - A \in \sigma$. But $X - A = \emptyset \cup (X - A) = (A - X) \cup (X - A)$, that is,$(A - X) \cup (X - A) \in \sigma$. Hence $A \approx_{\sigma} X$.

2) By Remark 1. 1. 17, $\emptyset \notin \sigma$. But $\emptyset = (A - A) \cup (A - A)$, hence $A \overline{\approx}_{\sigma} A$ for every $A \subseteq X$.

3) Since $A \in \sigma$ and $A = A \cup \emptyset = (A - \emptyset) \cup (\emptyset - A)$, we get $(A - \emptyset) \cup (\emptyset - A) \in \sigma$, hence $A \approx_{\sigma} \emptyset$.

4) That is clear by part 2 $A \approx_{\sigma} B$ if and only if $(A - B) \cup (B - A) \in \sigma$ if and only if $(B - A) \cup (A - B) \in \sigma$ if and only if $B \approx_{\sigma} A$.

5) This is an immediate consequence of part 3, because $\emptyset \notin \sigma$.

6) This is an immediate consequence of part 1, because X $\in \sigma$.

7) Let $A \notin \sigma$, $B \notin \sigma$, and let us suppose that $A \approx_{\sigma} B$. Then $(A - B) \cup (B - A) \in \sigma$. By axiom [C2] either $A - B \in \sigma$ or (B $-A) \in \sigma$. Since $(A - B) \subseteq A$ and $(B - A) \subseteq B$, by Proposition 1.1 15 part 2, we get $A \in \sigma$ or $B \in \sigma$ this contradiction , thus $A \overline{\approx}_{\sigma} B$■

Definition 1. 3. 4: Let σ a cluster in a proximity space(X, δ), the binary relation $\propto_{\sigma}$ defined on (X, δ) as follows:

$A \propto_{\sigma} B \Leftrightarrow A - B \in \sigma$. We denoted $A \overline{\propto_{\sigma}} B \Leftrightarrow A - B \notin \sigma$

By Example 1. 3. 2 we see that, $\{1,2\} \propto_{\sigma} \{2,3\}$ because $\{1, 2\} - \{2, 3\} = \{1\} \in \sigma$. But $\{2\} \overline{\propto_{\sigma}} \{3\}$.

Proposition 1. 3. 5: Let (X, δ) be a proximity space, and A, B are subsets of X, σ is a cluster define on X. Then

1. $A \overline{\propto_\sigma} X$, for every $A \subseteq X$;
2. $A \propto_\sigma \emptyset$, for every $A \in \sigma$;
3. $X \propto_\sigma A$, for every $A \notin \sigma$;
4. For every $A \in \sigma, there\ exsist\ B \notin \sigma$, such that $A \propto_\sigma B$;
5. For every $A \notin \sigma, then\ A \overline{\propto_\sigma} B$, for every $B \subseteq X$;
6. If $A \subset H\ and\ A \propto_\sigma B, then\ H \propto_\sigma B$;
7. If $D \subset B\ and\ A \propto_\sigma B, then\ A \propto_\sigma D$.

Proof.

1) This is an immediate, because $(A - X) = \emptyset$, and $\emptyset \notin \sigma$.

2) Let $A \in \sigma$. Then $(A - \emptyset) = A \in \sigma$, hence $A \propto_\sigma \emptyset$.

3) Let $A \notin \sigma$. Then by Proposition 1. 1. 15 part 1, $(X - A) \in \sigma$. Hence $X \propto_\sigma A$.

4) Let $A \in \sigma$, and suppose $B = \emptyset$.Then $\emptyset \notin \sigma$, thus by part 2, $A \propto_\sigma B$.

5) If $A = \emptyset$, then $\emptyset \overline{\propto_\sigma} B$ for every $B \subseteq X$. If $A \neq \emptyset$, then suppose that $A \propto_\sigma B$ this impels that $(A - B) \in \sigma$ but $(A - B) \subseteq A \in \sigma$ this contradiction, because $A \notin \sigma$. Thus $A \overline{\propto_\sigma} B$ for every $B \subseteq X$.

6) Let $A \propto_\sigma B$. Then $(A - B) \in \sigma$. Since $A \subset H$, $(A - B) \subset (H - B)$. By Proposition 1. 1. 15 part 2, $(H - B) \in \sigma$, hence $H \propto_\sigma B$.

7) Let $A \propto_\sigma B$. Then $(A - B) \in \sigma$. Since $D \subset B$, $(A - B) \subset (A - D)$. By Proposition 1. 1. 15 part 2, $(A - D) \in \sigma$, hence $A \propto_\sigma D$.

1. 4. σ −Topological Proximity

In this section, we will build a family relying on the binary relationship $\approx_\sigma$ with the cluster. And study its most important characteristics and its relationship with topology.

Definition 1. 4. 1: Let (X, δ) be a proximity space, σ is cluster define on X, then σ −Topological Proximity denoted by τ_σ the family nonempty subsets of X satisfies the conditions:

1. $X, \emptyset \in \tau_\sigma$;
2. For every sub collection $\mathcal{U}$ of τ_σ, there exists $\mathcal{V} \in \tau_\sigma$ such that $\cup \mathcal{U} \approx_\sigma \mathcal{V}$;
3. For every $\mathcal{U}, \mathcal{V} \in \tau_\sigma$, there exists $\mathcal{W} \in \tau_\sigma$ such that $(\mathcal{U} \cap \mathcal{V}) \approx_\sigma \mathcal{W}$;
4. $\tau_\sigma \cap \sigma = \{X\}$.

Thus the quadruple pair $(X, \delta, \sigma, \tau_\sigma)$ is denoted of σ− Topological Proximity Space.

Definition 1. 4. 2: $\mathcal{U}$ *is* $\tau_\sigma -$ open if and only if $\mathcal{U} \in \tau_\sigma$ and it's τ_σ −closed if and only if the complement is $\tau_\sigma -$ open set.

Examples 1. 4. 3: Let $X = \{1,2,3\}$. Then

I) Let δ be a discrete proximity. If $\sigma = \{\{1\}, \{1,2\}, \{1,3\}, X\}$, then $\{X, \emptyset, \{3\}, \{2\}\}$, $\{X, \emptyset, \{3\}\}$, and $\{X, \emptyset, \{3\}, \{2,3\}\}$ are σ- Topological Proximity.

In order to clarify more, let's take the collection $\tau_\sigma = \{X, \emptyset, \{2\}, \{3\}\}$ to show that τ_σ is a σ- Topological Proximity.

1- $X, \emptyset \in \tau_\sigma$.

2- For every sub collection $\{2\}, \{3\}, X$ of τ_σ there exists $\{2\}$ and $X \in \tau_\sigma$ such that $\{2,3\} \approx_\sigma X$, $\{2\} \approx_\sigma X$, $\{3\} \approx_\sigma X$ and $X \approx_\sigma \{2\}$.

3- For every $\{2\}, \{3\}, X$ and $\emptyset \in \tau_\sigma$ then there exists $\{2\}$ and $X \in \tau_\sigma$ such that $\emptyset \approx_\sigma X$ and $\{2\} \approx_\sigma X$, $\{3\} \approx_\sigma X$ and $X \approx_\sigma \{3\}$.

4- $\tau_\sigma \cap \sigma = \{X\}$. Hence τ_σ is $\sigma -$ Topological Proximity.

II) If δ is an indiscrete proximity, then $\sigma = \{\{1\}, \{2\}, \{3\}, \{2,3\}, \{1,2\}, \{1,3\}, X\}$ and $\tau_\sigma = \{X, \emptyset\}$ is only σ- Topological Proximity.

Examples 1. 4. 4: Let $X = \{a, b, c\}$, and δ be a discrete proximity. If $\sigma = \{\{c\}, \{a,c\}, \{b,c\}, X\}$, then $\tau_{1_\sigma} = \{X, \emptyset, \{a\}, \{b\}\}$, $\tau_{2_\sigma} = \{X, \emptyset, \{b\}\}$, and $\tau_{3_\sigma} = \{X, \emptyset, \{a\}, \{a,b\}\}$ are σ- Topological Proximity.

The Definition indicates that $\sigma -$ Topological Proximity can be topology as in $\tau_{2_\sigma}, \tau_{3_\sigma}$, also we see that τ_{1_σ} is $\sigma -$ Topological Proximity but not topology and discrete topology is not $\sigma -$ Topological Proximity. Therefore, it is concluded that τ_σ is an independent concept than the classical topology. Therefore, $\sigma -$ Topological Proximity is not amortizable topology, that is, it is not generated from a metric space.

According to above, it is noted that the intersection of any two $\tau_\sigma - open$ sets is not necessarily an $\tau_\sigma - open$ set, i.e., the power set is divided into three parts, a part containing the $\tau_\sigma - open$ sets, a part containing the cluster family, and a part that does not belong to either of the two classes

However, if the power set is separated into two parts only, one part will certainly contain the open sets and the other part will contain the cluster so that the common set between them is only X, then the intersection of any two $\tau_\sigma - open$ sets is an $\tau_\sigma - open$ set, that is, if $\mathcal{U}$ and $\mathcal{V} \in \tau_\sigma$, then $\mathcal{U} \cap \mathcal{V} \in \tau_\sigma$. Because if possible $\mathcal{U} \cap \mathcal{V} \notin \tau_\sigma$, then there exists σ such that $\mathcal{U} \cap \mathcal{V} \in \sigma$ this mean $\mathcal{U} \in \sigma$ and $\mathcal{V} \in \sigma$ which is a contradiction, hence $\mathcal{U} \cap \mathcal{V} \in \tau_\sigma$.

According to earlier results, we get the following remarks:

Remarks 1. 4. 5: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space. A, B nonempty subsets of X. Then

1st For any subset $\mathcal{U}$ of X such that $\mathcal{U} \notin \sigma$,then $\tau_\sigma = \{ X , \emptyset, \mathcal{U} \}$ is $\sigma -$ Topological Proximity

Proof.

$1 - X , \emptyset \in \tau_\sigma$

2-For every sub collection $\mathcal{U}$ of τ_σ such that $\mathcal{U} \notin \sigma$, $X \notin (\cup \mathcal{U})$ because $X \in \sigma$. Then there exists $X \in \tau_\sigma$ such that $(\cup \mathcal{U}) \approx_\sigma X$;

3-For every $\mathcal{U}, \mathcal{V} \in \tau_\sigma$, $(\mathcal{U} \cap \mathcal{V}) \notin \sigma$ because $\mathcal{V} = X$ or $\mathcal{V} = \mathcal{U}$ or $\mathcal{V} = \emptyset$, then there exists $X \in \tau_\sigma$ such that $(\mathcal{U} \cap \mathcal{V}) \approx_\sigma X$;

4- $\tau_\sigma \cap \sigma = \{X\}$. Hence $\tau_\sigma = \{ X , \emptyset, \mathcal{U} \}$ is $\sigma -$ Topological Proximity.

2nd Let $\tau_\sigma = \{ X , \emptyset, \mathcal{U}_i ; \mathcal{U}_i \notin \sigma, i \in \mathcal{N}\}$. Then τ_σ is $\sigma -$ Topological Proximity

Proof.

$1 - X, \emptyset \in \tau_\sigma$;

2-For every sub collection $\mathcal{U}_i$ of τ_σ, $(\bigcup_{i=1}^{n} \mathcal{U}_i) \notin \sigma$. If not, there exists sub collection $i, \dots\dots k$ such that $(\cup_{i=1}^{k} \mathcal{U}_i) \in \sigma$ by axiom [C2] there exists $\mathcal{U}_j \in \cup_{i=1}^{k} \mathcal{U}_i$ such that $\mathcal{U}_j \in \sigma$ which is a contradiction with hypothesis, thus $(\bigcup_{i=1}^{n} \mathcal{U}_i) \notin \sigma$ by Part 1 there exists $X \in \tau_\sigma$ such that $(\bigcup_{i=1}^{n} \mathcal{U}_i) \approx_\sigma X$;

3-For every $\mathcal{U}, \mathcal{V} \in \tau_\sigma$, $\mathcal{U} \cap \mathcal{V} \notin \sigma$. If not, $\mathcal{U} \cap \mathcal{V} \in \sigma$. By Note 1. 1. 17 part 2, $\mathcal{U}, \mathcal{V} \in \sigma$ which is a contradiction with hypothesis, thus $\mathcal{U} \cap \mathcal{V} \notin \sigma$ by Part 1 there exists $X \in \tau_\sigma$ such that $(\mathcal{U} \cap \mathcal{V}) \approx_\sigma X$;

4- $\tau_\sigma \cap \sigma = \{X\}$. Hence $\tau_\sigma = \{ X, \emptyset, \mathcal{U}_i ; \mathcal{U}_i \notin \sigma, i \in \mathcal{N}\}$. is $\sigma -$ Topological Proximity.

3rd In the case of indiscrete proximity space, τ_σ is a topology, in addition is τ_δ.

That is clear by Example 1. 1. 3 and Example 1. 4. 3 part 2, because $\tau_\sigma = \{X, \emptyset\}$, $\tau = \{ X, \emptyset\}$ and $\tau_\delta = \{X, \emptyset\}$.

4th Let δ be a non-indiscrete proximity space. Then topology proximity τ_δ is not τ_σ.

That is clear by Proposition 1. 2. 12, because $\mathcal{U}$ *and* $X - \mathcal{U}$ belong in τ_δ but by Proposition 1. 1. 15 part 1, $\mathcal{U}$ *or* $X - \mathcal{U}$ belong in σ, this implies that $\tau_\delta \cap \sigma \neq \{X\}$.

5th If (X, τ) be a topological space with $\tau \cap \sigma = \{X\}$, then τ is $\sigma -$ Topological Proximity.

Proof.

For every sub collection $\mathcal{U}$ of τ $then(\cup\,\mathcal{U}) \in \tau$, thus $(\cup\,\mathcal{U}) \notin \sigma$. Then by part 2, there exists $X \in \tau$ such that $(\cup\,\mathcal{U}) \approx_{\sigma} X$. Also, if $\mathcal{U}, \mathcal{V} \in \tau$, then $\mathcal{U} \cap \mathcal{V} \in \tau$ thus $\mathcal{U} \cap \mathcal{V} \notin \sigma$. Then by part 2, there exists $X \in \tau$ such that $(\mathcal{U} \cap \mathcal{V}) \approx_{\sigma} X$.

(Hint: If $(\cup\,\mathcal{U}) = X$, that not problem because by proposition 1.3.3 part 3 there exist $\emptyset \in \tau$ such that $X \approx_{\sigma} \emptyset$).

6th If $\mathcal{U}, \mathcal{V} \in \tau_{\sigma}$, then it is not necessary $\mathcal{U} \cap \mathcal{V}$ and $\mathcal{U} \cup \mathcal{V}$ belong in τ_{σ}. That evident by Examples 1. 4. 3.

7th Every proper member of cluster is not $\tau_{\sigma} - open$.

That is evident by axiom 4 of Definition 1. 4. 1, because if $C \in \sigma$, then $C \notin \tau_{\sigma}$.

8th For every $\sigma -$ Topological Proximity there exists a topology τ on X contains it.

The reason of that when using a $\sigma -$ Topological Proximity, we are not bound by achieving intersection and unions between sets, which is not the same case in topology.

More clarification let $X = \{1, 2, 3,4\}, \delta$ is discrete proximity and $\sigma = \{\{2\}, \{1, 2\}, \{2, 3\}, \{2,4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 4\}, X\}$ we see that $\tau_{1_{\sigma}} = \{X, \emptyset, \{3\}\}$, $\tau_{2_{\sigma}} = \{X, \emptyset, \{3\}, \{4\}\}$, $\tau_{3_{\sigma}} = \{X, \emptyset, \{1, 3\}, \{3, 4\}\}$, and $\tau_{4_{\sigma}} = \{X, \emptyset, \{3\}, \{1, 3\}, \{3, 4\}, \{1,4\}\}$ are $\sigma -$ Topological Proximity. There exists $\tau_1 = \{X, \emptyset, \{3\}\}$, $\tau_2 = \{X, \emptyset, \{3\}, \{4\}, \{3,4\}\}$, $\tau_3 = \{X, \emptyset, \{3\}, \{1, 3\}, \{3, 4\}, \{1, 3, 4\}$, and $\tau_4 =$

$\{X, \emptyset, \{3\}, \{4\}, \{1,3\}, \{3,4\}, \{1,4\}, \{1,3,4\}\}$ sequentially, are topologies contain τ_σ.

9th If τ_{1_σ} , τ_{2_σ} are two $\sigma-$ Topological Proximity define on same cluster σ, then $\tau_{1_\sigma} \cap \tau_{2_\sigma}$ and $\tau_{1_\sigma} \cup \tau_{2_\sigma}$ are also $\sigma-$ Topological Proximity.

Proof.

Let τ_{1_σ} , τ_{2_σ} be a two $\sigma-$ Topological Proximity. Let $\mathcal{U} \in (\tau_{1_\sigma} \cap \tau_{2_\sigma})$. Then $\mathcal{U} \in \tau_{1_\sigma}$ and $\mathcal{U} \in \tau_{2_\sigma}$ by Definition 1. 4. 1, $\mathcal{U} \notin \sigma$ thus by part 2, $(\tau_{1_\sigma} \cap \tau_{2_\sigma})$ is $\sigma-$ Topological Proximity. So that for every $\mathcal{U} \in (\tau_{1_\sigma} \cup \tau_{2_\sigma})$, then $\mathcal{U} \in \tau_{1_\sigma}$ or $\mathcal{U} \in \tau_{2_\sigma}$ by Definition 1. 4. 1 $\mathcal{U} \notin \sigma$ thus by part 2, $(\tau_{1_\sigma} \cup \tau_{2_\sigma})$ is $\sigma-$ Topological Proximity■

Definition 1. 4. 6: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma-$ Topological Proximity Space. A point $x \in A \subseteq X$ is called $\tau_\sigma - interior$ point of A if and only if there exists $\tau_\sigma - open$ set G such that $x \in G \subseteq A$ and the set of all $\tau_\sigma - interior$ point of A is denoted by $\tau_\sigma - int(A)$.

Proposition 1. 4. 7: Let $(X, \tau_\sigma, \sigma, \delta)$ be a $\sigma-$ Topological Proximity Space and A, B nonempty subset of X then:

1. $\tau_\sigma - int\ (A) \subseteq A$.
2. $A \in \tau_\sigma$, then $\tau_\sigma - int\ (A) = A$.
3. $\tau_\sigma - int\ (X) = X\ and\ \tau_\sigma - int\ (\emptyset) = \emptyset$.
4. $\tau_\sigma - int\ (A) = \cup \{G \in \tau_\sigma; G \subseteq A\}$.

5. If $A \subseteq B$, then $\tau_\sigma - int(A) \subseteq \tau_\sigma - int(B)$.
6. $\tau_\sigma - int(A \cap B) \subseteq \tau_\sigma - int(A) \cap \tau_\sigma - int(B)$.
7. $\tau_\sigma - int(A) \cup \tau_\sigma - int(B) \subseteq \tau_\sigma - int(A \cup B)$.

Proof.

1) Evident, by Definition 1. 4. 1.

2) By part 1, $\tau_\sigma - int\ (A) \subset A$. Let $x \in A$. Since $A \in \tau_\sigma$, for every $x \in A$ there exists $\tau_\sigma - open$ set A such that $x \in A \subseteq A$ thus $x \in \tau_\sigma - int\ (A)$. So that $\tau_\sigma - int\ (A) = A$.

3) It is an immediate consequence of part 2.

4) $x \in \tau_\sigma - int\ (A)$ if and only if there exists $\tau_\sigma - open$ set G such that $x \in G \subseteq A$ if and only if $x \in G$ for some $G \in \tau_\sigma$ if and only if $x \in \cup \{G \in \tau_\sigma; G \subseteq A\}$.

5) Let $x \in \tau_\sigma - int(A)$. Then there exists $\tau_\sigma - open$ set G such that $x \in G \subseteq A$. But $A \subseteq B$, that is, $x \in G \subseteq B$. Hence $x \in \tau_\sigma - int(B)$.

6) It is an immediate consequence of part 5.

7) It is an immediate consequence of part 5■

The following example shows that $\tau_\sigma - int(A)$ is not necessary $\tau_\sigma - open$ set and also if $A = \tau_\sigma - int(A)$, then it is not necessary that $A \in$

τ_σ as in the example below .Also the converse of cases (6) and (7) are explained.

Example 1. 4. 8: Let $X = \{1,2,3,4\}, \delta$ is discrete proximity, $and\ \tau_\sigma = \{X, \emptyset, \{1,2\}, \{2,4\}\}$, where $\sigma = \{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, X\}$. If $A = \{1,2,4\}$, then $A = \tau_\sigma - int(A)$ but $\tau_\sigma - int(A) \notin \tau_\sigma$, and when we take $H = \{1,2\}$, $B = \{2,4\}$,then $\tau_\sigma - int(H) \cap \tau_\sigma - int(B) = \{2\} \not\subset \tau_\sigma - int(H \cap B) = \emptyset$. Therefore, let $\tau_\sigma = \{X, \emptyset, \{1\}, \{4\}\}$. If we take $H = \{1,2\}$, $C = \{3,4\}$, then $\tau_\sigma - int(H \cup C) = X \not\subset \tau_\sigma - int(H) \cup \tau_\sigma - int(C) = \{1,4\}$.

Definition 1. 4. 9: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space, and $A\ is\ \tau_\sigma - open$ set if and only if $\tau_\sigma - int(A) = A$. Then A is called $\sigma - open$ set.

Definition 1. 4. 10: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space.A point $x \in A$ is called $\tau_\sigma - exterior$ point of $A \subseteq X$ if and only if there exists $\tau_\sigma - open$ set H such that $x \in H \subseteq X - A$ and the set of all $\tau_\sigma - exterior$ point of A is denoted by $\tau_\sigma - ext(A)$.

By Example 1. 4. 8 $ext(\{1,2,4\}) = \emptyset, ext(\{4\}) = \{1,2\}$.

Proposition 1. 4. 11: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space and let A, B are subset of X. Then

1. $\tau_\sigma - ext\ (A) = \tau_\sigma - int\ (X - A)$.
2. $A \cap \tau_\sigma - ext\ (A) = \emptyset$.
3. $\tau_\sigma - ext(X) = \emptyset\ and\ \tau_\sigma - ext\ (\emptyset) = X$.

4. $\tau_\sigma - ext\ (A) = \cup\{H \in \tau_\sigma; H \subseteq X - A\}$.
5. If $A \subseteq B$ then $\tau_\sigma - ext\ (B) \subseteq \tau_\sigma - ext\ (A)$.
6. $\tau_\sigma - ext(A \cap B) \supseteq \tau_\sigma - ext(A) \cap \tau_\sigma - ext(B)$.
7. $\tau_\sigma - ext(A \cup B) \subseteq \tau_\sigma - ext(A) \cup \tau_\sigma - ext(B)$.
8. $\tau_\sigma - ext(A \cup B) \subseteq \tau_\sigma - ext(A) \cap \tau_\sigma - ext(B)$.

Proof.

1) $x \in \tau_\sigma - ext\ (A)$ if and only if there exists $\tau_\sigma - open$ set H such that $x \in H \subseteq X - A$ if and only if $x \in \tau_\sigma - int\ (X - A)$.

2) If possible $A \cap \tau_\sigma - ext\ (A) \neq \emptyset$. Then there exists $x \in A\ and\ x \in \tau_\sigma - ext\ (A)$. Thus there exists $G \in \tau_\sigma$ such that $x \in G \subseteq X - A$, that is, $x \in X - A$ which is a contradiction, hence$A \cap \tau_\sigma - ext\ (A) = \emptyset$.

3) $\tau_\sigma - ext\ X = \tau_\sigma - int(X - X) = \tau_\sigma - int(\emptyset) = \emptyset$. So that, $\tau_\sigma - ext\ (\emptyset) = \tau_\sigma - int(X - \emptyset) = \tau_\sigma - int(X) = X$.

4) By Part 1. 4. 10, $\tau_\sigma - ext\ (A) = \tau_\sigma - int(X - A)$. But $\tau_\sigma - int(X - A) = \cup\{G \in \tau_\sigma; G \subseteq X - A\}$. Hence $\tau_\sigma - ext\ (A) = \cup\{G \in \tau_\sigma; G \subseteq X - A\}$.

5) Let $x \in \tau_\sigma - ext(B)$. Then there exists $G \in \tau_\sigma$ such that $x \in G \subseteq X - B$. Since $A \subseteq B$, $x \in G \subseteq X - A$, hence $x \in \tau_\sigma - ext(A)$.

6) It is an immediate consequence of part 5.

7) It is an immediate consequence of part 5.

8) $\tau_\sigma\, –ext(A \cup B) = \tau_\sigma - int\big(X - (A \cup B)\big) \subseteq \tau_\sigma - int(X - A) \cap \tau_\sigma - int(X - B) = \tau_\sigma\, –ext(A) \cap \tau_\sigma\, –ext(B)$■

By Example 1. 4. 8 if we take $A = \{4\}$ and $B = \{1, 2, 4\}$, then
$\tau_\sigma\, –ext(A) = \{1, 2\}$, $\tau_\sigma\, –ext(B) = \emptyset$ and so $\tau_\sigma\, –ext(A \cup B) = \emptyset$.
But $\tau_\sigma\, –ext(A) \cup \tau_\sigma\, –ext(B) = \{1, 2\}$.
Also, we noted that $\tau_\sigma\, –ext(\{3, 4\}) = \{1, 2\}$, $\tau_\sigma\, –ext(\{1, 3\}) = \{2, 4\}$, and so $\tau_\sigma\, –ext(\{3, 4\} \cap \{1, 3\}) = \tau_\sigma\, –ext(3) = \{1, 2, 4\}$.
But $\tau_\sigma\, –ext(\{3, 4\}) \cap \tau_\sigma\, –ext(\{1, 3\}) = \{2\}$. Also, $\tau_\sigma\, –ext(\{3, 4\} \cup \{1, 3\}) = \tau_\sigma\, –ext(\{1, 3, 4\}) = \emptyset$.

Definition 1. 4. 12: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space and $A \subseteq X$,then x is called τ_σ– limit point of A if and only if for each τ_σ – open set G of x such that $G \cap (A/\{x\}) \neq \emptyset$ .The set of all τ_σ −limit points of A is called the τ_σ – derived set and denoted by $\tau_\sigma - D(A)$.

Example 1. 4. 13: Let $X = \{a, b, c, d\}$, δ is discrete proximity . Let $\sigma = \{X, \{b, c\}, \{a, c\}, \{c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$, $\tau_\sigma = \{X, \emptyset, \{a, b\}, \{a, d\}\}\}$. If we take $A = \{a, b, d\}$, then $\tau_\sigma - D(\{A\}) = X$ and $\tau_\sigma - D(\{a, c\}) = \{b, c, d\}$, but $\tau_\sigma - D(\{c\}) = \emptyset$.

Proposition 1. 4. 14: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space and A, B are subset of X, then each of the following are achieved:

1. $A \subseteq B$, then $\tau_\sigma - D(A) \subseteq \tau_\sigma - D(B)$.

2. $\tau_\sigma - D(A \cup B) \supseteq \tau_\sigma - D(A) \cup \tau_\sigma - D(B)$.
3. $\tau_\sigma - D(A \cap B) \subseteq \tau_\sigma - D(A) \cap \tau_\sigma - D(B)$.
4. $\tau_\sigma - D(A) \subseteq A$, for each τ_σ -closed set A of X.

Proof.

1) Evident, by Definition 1. 4. 12 and by hypothesis that $A \subseteq B$.

2) It is an immediate consequence of Definition 1. 4. 12 and part 1.

3) It is an immediate consequence of Definition 1. 4. 12 and part 1.

4) Let $x \in \tau_\sigma - D(A)$. Then $G \cap A/\{x\} \neq \emptyset$, for each $G \in \tau_\sigma(x)$. If $x \notin A$, then $x \in X - A$. Since $X - A$ is τ_σ –open set, $X - A \cap A/\{x\} \neq \emptyset$ which is a contradiction■

The converse of case (3) and (4) is not true as in the following example:

Example 1. 4. 15: Let $X = \{a, b, c, d\}$, let δ is discrete proximity $\sigma = \{X, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, c\}, \{a, d, c\}, \{a, d\}\}$ and $\tau_\sigma = \{X, \emptyset, \{b, d\}, \{c, d\}\}$. If $A = \{a, b\}$, $B = \{c\}$, then $\tau_\sigma - D(A) \subseteq A$ but A is not τ_σ –closed set and $\tau_\sigma - D(A) = \{a\}, \tau_\sigma - D(B) = \{a\}$. But $\tau_\sigma - D(A \cup B) = \tau_\sigma - D(\{a, b, c\}) = \{a, d\}$,thus $\tau_\sigma - D(A \cup B) \supset \tau_\sigma - D(A) \cup \tau_\sigma - D(B)$.

Definition 1. 4. 16: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space and A is a subset of X ,the τ_σ – closure of A is a intersection of all τ_σ – closed sets contain A, denoted by $\tau_\sigma - cl(A)$, i.e , $\tau_\sigma - cl(A) = \cap \{F: F$ is τ_σ – closed set , $A \subseteq F\}$.

Proposition 1. 4. 17: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space and A, B are subset of X . Then

1. $A \subseteq \tau_\sigma - cl(A)$ for each subset A of X.
2. If A is $\tau_\sigma - closed$ set, then $A = \tau_\sigma - cl(A)$.
3. $\tau_\sigma - cl(X) = X$ and $\tau_\sigma - cl(\emptyset) = \emptyset$.
4. If $A \subseteq B$, then $\tau_\sigma - cl(A) \subseteq \tau_\sigma - cl(B)$.
5. $\tau_\sigma - cl(A \cup B) \supseteq \tau_\sigma - cl(A) \cup \tau_\sigma - cl(A)$.
6. $A \cup (\tau_\sigma - D(A)) = \tau_\sigma - cl(A)$.

Proof.

1) It is an immediate consequence of Definition 1. 4. 16.

2) Let A is $\tau_\sigma - closed$. Then A itself is a smallest $\tau_\sigma - closed$ set containing A. Thus $A = \tau_\sigma - cl(A)$.

3) By fact $X \ and \ \emptyset \ are \ \tau_\sigma - closed \ sets$.

4) Evident.

5) It is an immediate consequence of part 4.

6) Let $x \in (A \cup \tau_\sigma - D(A))$. Then $x \in A$ or $x \in \tau_\sigma - D(A)$. If $x \in A$ by Part 1, $x \in \tau_\sigma - cl(A)$. If $x \notin A$ and $x \in \tau_\sigma - D(A)$. Then for each $\tau_\sigma -$open set G of x, $G \cap A/\{x\} \neq \emptyset$, hence $G \cap A \neq \emptyset$. If possible $x \notin \tau_\sigma - cl(A)$, then $x \notin F$, for some $F \ is \ \tau_\sigma - closed$ contain A. By Proposition 1. 4. 14 part 4, $x \notin \tau_\sigma - D(F)$, that is, $\mathcal{U} \cap F/\{x\} = \emptyset$ for some $\mathcal{U} \in \tau_\sigma(x)$, but $A \subseteq F$, this implies to $\mathcal{U} \cap A/\{x\} = \emptyset$ which is a contradiction. Conversely, let x $\in \tau_\sigma - cl(A)$, if possible $x \notin (A \cup \tau_\sigma - D(A))$, then $x \notin A$ and $x \notin \tau_\sigma - D(A)$, hence $G \cap A/\{x\} = \emptyset$, for some τ_σ −open set G of x. Since $x \notin A$, $G \cap A = \emptyset$, thus $A \subset X - G$ but $X - G$ is τ_σ −closed set contain A and $x \notin X - G$ by Definition 1. 4. 16 $x \notin \tau_\sigma - cl(A)$ which is a contradiction. (Since $\tau_\sigma - cl(A) \subset X - G$)■

Proposition 1. 4. 18: Let $(X, \delta, \sigma, \tau_\sigma)$ be a σ − Topological Proximity Space. $x \in \tau_\sigma - cl(A)$ if and only if $G \cap A \neq \emptyset$ for each τ_σ −open set G of x.

Proof.

Let $x \in \tau_\sigma - cl(A)$ and suppose that $G \cap A = \emptyset$, for some $G \in \tau_\sigma(x)$, hence $A \subseteq X - G$. Since $\tau_\sigma - cl(A)$ is the intersection of all τ_σ – closed set containing A, $\tau_\sigma - cl(A) \subseteq X - G$, then $x \in X - G$ which is a contradiction. Conversely if possible that $x \notin \tau_\sigma - cl(A)$, then Proposition 1. 4. 17 part 6, $x \notin (A \cup \tau_\sigma - D(A))$, then $x \notin A$ and $x \notin \tau_\sigma - D(A)$, hence $G \cap A/\{x\} = \emptyset$, for some τ_σ −open set G of x. Since $x \notin A$, $G \cap A = \emptyset$ which is a contradiction■

Proposition 1. 4. 19: Let $(X, \delta, \sigma, \tau_\sigma)$ be a σ − Topological Proximity Space, and let A is a subset of X. Then

1. $X - \left(\tau_\sigma - cl(A)\right) \subseteq \tau_\sigma - cl(X - A)$.
2. τ_σ – $ext\,(A) = X - (\tau_\sigma - cl(A))$.
3. τ_σ – $int\,(A) = X - (\tau_\sigma - cl(X - A))$.

Proof.

1) Let $x \in X - (\tau_\sigma - cl(A))$, that is, $x \notin \tau_\sigma - cl(A)$. Then there exist τ_σ-open set G of x such that $G \cap A = \emptyset$, hence $G \subseteq (X - A)$, and then $x \in X - A$. But $X - A \subset \tau_\sigma - cl(X - A)$. Hence $x \in \tau_\sigma - cl(X - A)$.

2) $x \in \tau_\sigma - ext(A)$ if and only if $x \in \tau_\sigma - int(X - A)$ if and only if there exists $G \in \tau_\sigma$ such that $x \in G \subseteq X - A$ if and only if $G \cap A = \emptyset$ by Proposition 1. 4. 18 if and only if $x \notin \tau_\sigma - cl(A)$if and only if $x \in X - (\tau_\sigma - cl(A))$.

3) By part 2 and by Proposition 1. 4. 11 part 1 we have that $\tau_\sigma - int\ (A) = \tau_\sigma - ext\ (X - A) = X - cl(X - A)$■

Example1. 4. 20: Let $X = \{1, 2, 3, 4\}$, δ is a discrete proximity, $\sigma = \{\{3\}, \{1, 3\}, \{2, 3, 4\}, \{1, 2, 3\}, \{2, 3\}, \{3, 4\}, \{1, 3, 4\}, , X\}$, and $\tau_\sigma = \{X, \emptyset, \{1, 2\}, \{2, 4\}\}$. Then we can see that:

$\tau_\sigma - closuer$ is not necessary is $\tau_\sigma - closed$ set. Because if we take A = $\{3\}$, then $\tau_\sigma - cl(\{3\}) = \{3\}$, which is not $\tau_\sigma - closed$ set.
Also, if $H = \{3, 4\}$ and $C = \{1, 3\}$, then $\tau_\sigma - cl(H) = \{3, 4\}$ and $\tau_\sigma - cl(C) = \{1,3\}$, but $\tau_\sigma - cl(H \cup C) = X$, hence $\tau_\sigma - cl(H) \cup \tau_\sigma - cl(C) \subset \tau_\sigma - cl(H \cup C\ )$. Also, if $D = \{1\}$, then $\tau_\sigma - cl(D) = \{1, 3\}$. $\tau_\sigma - cl(A) \cap \tau_\sigma - cl(D) = \{3\}$, but $\tau_\sigma - cl(A \cap D) = \emptyset$. Thus $\tau_\sigma - cl(A) \cap \tau_\sigma - cl(D) \supset \tau_\sigma - cl(A \cap D)$.

Moreover, if $\mathcal{U} \in \tau_\sigma$, that not necessary $(\mathcal{U} \cap \tau_\sigma - cl(A)) \subseteq \tau_\sigma - cl(\mathcal{U} \cap A)$. Because if $X = \{1, 2, 3, 4\}, \delta$ is a discrete proximity, $\sigma = \sigma_2,\ and\ \tau_\sigma = \{X, \emptyset, \{1, 3\}, \{1, 4\}\}$, then $X, \emptyset, \{2,4\}, \{2, 3\}$ are $\tau_\sigma - closed$ sets, and if we take $A = \{2,3,4\}$ and $\mathcal{U} = \{1,3\}$, then $\tau_\sigma - cl(A) = X$, $\tau_\sigma - cl(\mathcal{U} \cap A) = \tau_\sigma - cl(\{3\} = \{2,3\}$, but $\mathcal{U} \cap \tau_\sigma - cl(A) = \{1,3\}$.

Therefore, the relation is true in the two cases:

1. If A is $\tau_\sigma - closed$, then $(\mathcal{U} \cap \tau_\sigma - cl(A)) \subseteq \tau_\sigma - cl(\mathcal{U} \cap A)$.
2. If $(\mathcal{U} \cap G) \in \tau_\sigma$ for every $G \in \tau_\sigma$, then $\left(\mathcal{U} \cap \tau_\sigma - cl(A)\right) \subseteq \tau_\sigma - cl(\mathcal{U} \cap A)$.

Definition 1. 4. 21: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space, and $F\ is\ \tau_\sigma - closed$ if and only if $\tau_\sigma - cl(F) = F$. Then F is called $\sigma - closed$ set.

We can see that, if $\mathcal{U}, \mathcal{V}$ are $\sigma - open$ sets, then that it is not necessary $\mathcal{U} \cap \mathcal{V}\ and\ \mathcal{U} \cup \mathcal{V}$ are $\sigma - open$ that is clear by Example 1. 4. 8, because $\{1,2\}$ and $\{2,4\}$ are $\sigma - open$ but $\{1,2\} \cap \{2,4\} = \{2\}\ and\ \{1,2\} \cup \{2,4\} = \{1,2,4\}$ are not $\sigma - open$ sets. Also, If F, K are $\sigma - closed$ sets, then that not necessary $F \cap K\ and\ F \cup K$ are $\sigma - closed$ that is clear by Example 1. 4. 8, because $\{1,3\}$ and $\{3,4\}$ are $\sigma - closed$ but $\{1,3\} \cap \{3,4\} = \{3\}\ and\ \{1,3\} \cup \{3,4\} = \{1,3,4\}$ are not $\sigma - closed$ sets.

Proposition 1. 4. 22: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space, and every $\tau_\sigma - int(A)$ is $\tau_\sigma - open$ set and every $\tau_\sigma - cl(A)\ is\ \tau_\sigma - closed$ sets then the following statements are hold:

1. G is $\sigma - open$ set if and only if $G\ is\ \tau_\sigma - open$.
2. F is $\sigma - closed$ set if and only if $F\ is\ \tau_\sigma - closed$.
3. If $\mathcal{U}, \mathcal{V} \in \tau_\sigma, then\ \mathcal{U} \cup \mathcal{V} \in \tau_\sigma$.
4. If $F, K\ are\ \tau_\sigma - closed, then\ F \cap K\ is\ \tau_\sigma - closed$.

Proof.

1) Let G is $\sigma - open$ set if and only if $G = \tau_{\sigma} - int$(G) if and only if $G \in \tau_{\sigma}$ if and only if G is $\tau_{\sigma} - open$.

2) Let F is $\sigma - closed$ set if and only if $F = \tau_{\sigma} - cl$(F) if and only if F is $\tau_{\sigma} - closed$ set.

3) Let $\mathcal{U}\ and\ \mathcal{V} \in \tau_{\sigma}$. By Proposition 1. 4. 7 part 2, $\mathcal{U} = \tau_{\sigma} - int(\mathcal{U})$ and $\mathcal{V} = \tau_{\sigma} - int(\mathcal{V})$.Thus $\mathcal{U} \cup \mathcal{V} = (\tau_{\sigma} - int(\mathcal{U})) \cup (\tau_{\sigma} - int(\mathcal{V})) \subset \tau_{\sigma} - int(\mathcal{U} \cup \mathcal{V})$. By Proposition 1. 4. 7 part 1, $\tau_{\sigma} - int(\mathcal{U} \cup \mathcal{V}) \subset \mathcal{U} \cup \mathcal{V}$. Hence $\mathcal{U} \cup \mathcal{V} = \tau_{\sigma} - int(\mathcal{U} \cup \mathcal{V})$. By hypothesis $\mathcal{U} \cup \mathcal{V}$ is τ_{σ} −open set, thus $\mathcal{U} \cup \mathcal{V} \in \tau_{\sigma}$.

4) Let F, $K\ are\ \tau_{\sigma} - closed$. Then F= $\tau_{\sigma} - cl(F), and\ K = \tau_{\sigma} - cl(K)$. Then $F \cap K = \tau_{\sigma} - cl(F) \cap \tau_{\sigma} - cl(K) \supset \tau_{\sigma} - cl(F \cap K)$,by Proposition 1. 4. 17 part 1, $F \cap K \subset \tau_{\sigma} - cl(F \cap K)$, hence $F \cap K = \tau_{\sigma} - cl(F \cap K)$. By hypothesis $F \cap K$ is $\tau_{\sigma} - closed$ set■

Definition 1. 4. 23: σ − Topological Proximity Space satisfies the conditions of Proposition 1. 4. 22 is called σ − $weekly$ Topological Proximity denoted by $\tau_{\sigma W}$.

It is to be noted that, according to earlier results, if $\mathcal{U}\ , \mathcal{V} \in \tau_{\sigma W}$, then that not necessary $\mathcal{U} \cap \mathcal{V}$ is belong in $\tau_{\sigma W}$, also if $F, K\ are\ \tau_{\sigma} - closed$, then that not necessary $F \cup K$ is $\tau_{\sigma W} - closed$. That is clear by Example 1. 4. 20 if we take $\tau_{\sigma W} = \{X, \emptyset, \{1,2\}, \{2, 4\}, \{1, 2, 4\}\}$, then$\{1,2\} and\ \{2, 4\} \in \tau_{\sigma W}$, but $\{1,2\} \cap \{2,4\} = \{2\} \notin \tau_{\sigma W}$. Also, $\{3,4\}$ and $\{1,3\}$ are $\tau_{\sigma W}$- closed, but $\{3,4\} \cup \{1,3\} = \{1,3,4\} is\ not \tau_{\sigma W} - closed.$

Lemma 1. 4. 24: Let $(X, \delta, \sigma, \tau_\sigma)$ be a σ − Topological Proximity Space. Then $\tau_{\sigma W}$ is τ_σ.

Proof.

It is an immediate consequence of Definition 1. 4. 23■

Proposition 1. 4. 25: Let $(X, \delta, \sigma, \tau_{\sigma W})$ be a $\sigma - weekly$ Topological Proximity Space, and $\mathcal{U} \cap \mathcal{V} \in \tau_{\sigma W}$, for every $\mathcal{U}, \mathcal{V} \in \tau_{\sigma W}$. Then the following statements are hold:

1. $\tau_{\sigma W} - int(A \cap B) = \tau_{\sigma W} - int(A) \cap \tau_{\sigma W} - int(A)$.
2. $\tau_{\sigma W} - cl(A \cup B) = \tau_{\sigma W} - cl(A) \cup \tau_{\sigma W} - cl(A)$.
3. $\tau_{\sigma W}$ is topology.

Proof.

1) By Proposition 1. 4. 7 part 5 and Lemma 1. 4. 24, $\tau_{\sigma W} - int(A \cap B) \subset \tau_{\sigma W} - int(A) \cap \tau_{\sigma W} - int(B)$. Again, let $x \in \tau_{\sigma W} - int(A) \cap \tau_{\sigma W} - int(B)$. Then there exists $\mathcal{U}, \mathcal{V} \in \tau_{\sigma W}(x)$ such that $\mathcal{U} \subset A$ and $\mathcal{V} \subset B$, then $\cap \mathcal{V} \subset A \cap B$, by hypothesis $\mathcal{U} \cap \mathcal{V} \in \tau_{\sigma W}(x)$ thus $x \in \tau_{\sigma W} - int(A \cap B)$, that is, $\tau_{\sigma W} - int(A) \cap \tau_{\sigma W} - int(B) = \tau_{\sigma W} - int(A \cap B)$.

2) Let $x \in \tau_{\sigma W} - cl(A \cup B)$. Then by Proposition 1. 4. 18, $G \cap (A \cup B) \neq \emptyset$ for each $\tau_{\sigma W}$ −open set G of x. if possible $x \notin (\tau_{\sigma W} - cl(A) \cup \tau_{\sigma W} - cl(A))$, then $x \notin \tau_{\sigma W} - cl(A)$ and $x \notin \tau_{\sigma W} - cl(B)$.

Thus there exists $\mathcal{U},\mathcal{V} \in \tau_{\sigma W}(x)$ such that $\mathcal{U} \cap A = \emptyset$ *and* $\mathcal{V} \cap B = \emptyset$ hence $(\mathcal{U} \cap A) \cup (\mathcal{V} \cap B) = \emptyset$. But $\mathcal{U},\mathcal{V} \in \tau_{\sigma W}$ by hypothesis $\mathcal{U} \cap \mathcal{V} \in \tau_{\sigma W}$ and $\mathcal{U} \cap \mathcal{V} \subset \mathcal{U}, \mathcal{U} \cap \mathcal{V} \subset \mathcal{V}$, then $\emptyset = (\mathcal{U} \cap A) \cup (\mathcal{V} \cap B) \supset \big((\mathcal{U} \cap \mathcal{V}) \cap A\big) \cup \big((\mathcal{U} \cap \mathcal{V}) \cap B\big) = ((\mathcal{U} \cap \mathcal{V}) \cap (A \cup B)$ which is a contradiction thus $x \in (\tau_{\sigma W} - cl(A) \cup \tau_{\sigma W} - cl(A))$ and so $\tau_{\sigma W} - cl(\mathrm{A} \cup \mathrm{B}) = \tau_{\sigma W} - cl(A) \cup \tau_{\sigma W} - cl(A)$.

3) $i.$ X *and* $\emptyset \in \tau_{\sigma W}$.

$ii.$ Let $\mathcal{U}$ *and* $\mathcal{V} \in \tau_{\sigma W}$ by hypothesis $\mathcal{U} \cap \mathcal{V} \in \tau_{\sigma W}$.

$iii.$ Let $\{\mathcal{U}_i\}_{i \in \Delta}$ be an arbitrary collection of the $\tau_{\sigma W} - open$ subsets of X. Then $\mathcal{U}_i \in \tau_{\sigma W}$ for every $i \in \Delta$, thus $\mathcal{U}_i = \tau_{\sigma W} - int(\mathcal{U}_i)$, and so $\cup_{i \in \Delta} \mathcal{U}_i = \cup_{i \in \Delta} (\tau_{\sigma W} - int(\mathcal{U}_i))$ but by Proposition 1. 4. 7 part 7, $\cup_{i \in \Delta} (\tau_{\sigma W} - int(\mathcal{U}_i)) \subset \tau_{\sigma W} - int(\cup_{i \in \Delta} \mathcal{U}_i)$, by Proposition 1. 4. 7 part 1, $\cup_{i \in \Delta} \mathcal{U}_i = \tau_{\sigma W} - int(\cup_{i \in \Delta} \mathcal{U}_i)$this means $\cup_{i \in \Delta} \mathcal{U}_i \in \tau_{\sigma W}$, hence $\tau_{\sigma W}$ is topology■

Definition 1. 4. 26: Let $(\mathrm{X}, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space. A point $x \in A$ is called $\tau_\sigma - Frontir$ or $\tau_\sigma - boundary$ point of $A \subseteq X$ if and only if it is neither an interior and exterior of A and the set of all $\tau_\sigma - Frontir$ point of A is denoted by $\tau_\sigma - Fr(A)$. Hence $x \notin \tau_\sigma - int(A)$ *and* $x \notin \tau_\sigma - ext(A)$.

Proposition 1. 4. 27: Let $(\mathrm{X}, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space and A, B are subset of X then:

1. $\tau_\sigma - Fr(A), \tau_\sigma - ext\ (A),$ *and* $\tau_\sigma - int\ (A)$ pairwise disjoint.
2. $\tau_\sigma - Fr(A) \cup \tau_\sigma - ext\ (A) \cup \tau_\sigma - int\ (A) = X$.

Proof.

1) $\emptyset = A \cap X - A \supseteq \tau_\sigma - int\ (A) \cap \tau_\sigma - int\ (X - A) = \tau_\sigma - int\ (A) \cap \tau_\sigma - ext\ (A)$.So that**,** $x \in \tau_\sigma - Fr\ (A)$ if and only if $x \notin \tau_\sigma - int\ (A)$ and $x \notin \tau_\sigma - ext\ (A)$ if and only if $x \notin (\tau_\sigma - int\ (A) \cup \tau_\sigma - ext\ (A))$if and only if $x \in X - (\tau_\sigma - int\ (A) \cup \tau_\sigma - ext\ (A))$ if and only if $\tau_\sigma - Fr(A) = X - (\tau_\sigma - int\ (A) \cup \tau_\sigma - ext\ (A))$. Thus $\tau_\sigma - Fr\ (A) \cap \tau_\sigma - int\ (A) = \emptyset$ and $\tau_\sigma - Fr\ (A) \cap \tau_\sigma - ext\ (A) = \emptyset$.

2) By Definition 1. 4. 26, $Fr(A) = X - (\tau_\sigma - int(A) \cup \tau_\sigma - ext\ (A))$. Thus $\tau_\sigma - Fr(A) \cup \tau_\sigma - ext\ (A) \cup \tau_\sigma - int\ (A) = X$■

Proposition 1. 4. 28: Let $(X, \delta, \sigma, \tau_\sigma)$be a $\sigma -$ Topological Proximity Space and A, B are subset of X then:

1. $\tau_\sigma - Fr\ (X) = \emptyset$, and $\tau_\sigma - Fr\ (\emptyset) = \emptyset$.
2. $\tau_\sigma - Fr(X) \subseteq \tau_\sigma - Fr(A)$ for every $A \subseteq X$.
3. $\tau_\sigma - Fr(A) = (\tau_\sigma - cl(A)) - (\tau_\sigma - int(A))$
4. $\tau_\sigma - Fr(X - A) = Fr(A)$.

Proof.

1) It is an immediate consequence of Proposition 1. 4. 27.

2) It is an immediate consequence of part 1.

3) By Definition 1. 4. 26, we get $\tau_\sigma - Fr(A) = X - ((\tau_\sigma - int(A) \cup (\tau_\sigma - ext(A))) = X - (\tau_\sigma - int(A)) \cap X - (\tau_\sigma - ext(A))$.

By Proposition 1. 4. 19 part 3 and part 4, $\tau_\sigma - ext(A) = X - (\tau_\sigma - cl(A))$

and $\tau_\sigma - int(A) = X - (\tau_\sigma - cl(X - A)$. Then

$\tau_\sigma - Fr(A) = X - (X - (\tau_\sigma - cl(X - A)) \cap X - (X - \left(\tau_\sigma - cl(A)\right) =$

$\tau_\sigma - cl(X - A) \cap \tau_\sigma - cl(A) = (\tau_\sigma - cl(A)) - (X - (\tau_\sigma - cl(X - A))$

$= (\tau_\sigma - cl(A)) - (\tau_\sigma - int(A))$.

4) By part 3, $\tau_\sigma - Fr(X - A) = \tau_\sigma - cl(X - A) \cap X - (\tau_\sigma - int(X - A))$

$= \tau_\sigma - cl(X - A) \cap \tau_\sigma - cl(A) = \tau_\sigma - cl(A) - (X - \left(\tau_\sigma - cl(X - A)\right)$

$= \tau_\sigma - cl(A) - (\tau_\sigma - int(A) = \tau_\sigma - Fr(A)$■

By Example 1. 4. 8 $\tau_\sigma = \left\{X, \emptyset, \{1, 2\}, \{2, 4\}\right\}$. If we take $A = \{1, 3\}$ and $B = \{3, 4\}$, then τ_σ $-Fr(A) = \{1,3\}$, τ_σ $-Fr(B) = \{3, 4\}$, hence τ_σ $-Fr(A \cup B) = \tau_\sigma -Fr(\{1, 3, 4\}) = X \supset \tau_\sigma -Fr(A) \cup \tau_\sigma -Fr(B) = \{1, 3, 4\}$.

But, if $A = \{1\}$ and $B = \{2\}$, then $\tau_\sigma -Fr(A) = \{1,3\}$, $\tau_\sigma -Fr(B) = X$, and $\tau_\sigma -Fr(A \cup B) = \tau_\sigma -Fr(\{1, 2\}) = \{3, 4\} \subset \tau_\sigma -Fr(A) \cup \tau_\sigma -Fr(B) = X$.

Also, we noted that $\tau_\sigma -Fr(\{1, 4\}) = X$, $\tau_\sigma -Fr(\{2, 3\}) = X$,

thus τ_σ $-Fr(\{1, 4\} \cap \{2, 3\}) = \tau_\sigma$ $-Fr(\emptyset) = \emptyset \subset \tau_\sigma$ $-Fr(\{1, 4\}) \cap$, τ_σ $-Fr(\{2, 3\}) = X$.

But, $\tau_\sigma -Fr(\{1, 3\}) = \{1, 3\}$, $\tau_\sigma -Fr(\{1, 2, 3\}) = \{3, 4\}$,

then τ_σ $-Fr(\{1, 3\} \cap \{1,2, 3\}) = \{1,3\} \supset \tau_\sigma$ $-Fr(\{1, 3\}) \cap \tau_\sigma$ $-Fr(\{1, 2, 3\}) = \{3\}$.

We see also, $\tau_\sigma - Fr(A)$ not necessary is $\tau_\sigma - closed$. That is clear because $\tau_\sigma -Fr(\{3\}) = \{3\}$, but $\{3\}$ not $\tau_\sigma - closed$.

Definition 1. 4. 29: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space, and $A \subseteq X$. Then A is called τ_σ – *dense* set if $\tau_\sigma - cl(A) = X$.

In the indiscrete τ_σ – topological space every subset of X is τ_σ– *dense* set.

Proposition 1. 4. 30: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space, and A, B are subset of X . Then

1. $A \subseteq B$ and A is τ_σ– dense set ,then B is τ_σ – dense set .
2. $A \cap B$ is τ_σ – dense set ,then A and B are τ_σ – dense set .
3. If A or B are τ_σ – dense set, then $A \cup B$ is τ_σ – dense set.

Proof.

1) Proposition 1. 4. 17 part 4, $\tau_\sigma - cl(A) \subset \tau_\sigma - cl(B)$.Since A is τ_σ-dense, $X \subset \tau_\sigma - cl(B)$, hence B is τ_σ-dense set.

2) Evident by part 1.

3) Evident by part 1■

Proposition 1. 4. 31: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space, then A is τ_σ – *dense* if and only if $G \cap A \neq \emptyset$, for each τ_σ – open set G.

Proof.

Let A is τ_σ – *dense* . Then $\tau_\sigma - cl(A) = X$. By Proposition 1. 4. 18, $G \cap A \neq \emptyset$ for each $G \in \tau_\sigma(x)$. Conversely, let $G \cap A \neq \emptyset$ for each $G \in \tau_\sigma$, that mean for every $x \in X$ and each $G \in \tau_\sigma(x)$, $G \cap A \neq \emptyset$ hence by Proposition 1. 4. 18, $\tau_\sigma - cl(A) = X$■

Definition 1. 4. 32: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space, then a subset A of X is called τ_σ – nowhere dense set if $\tau_\sigma - int(\tau_\sigma - cl(A)) = \emptyset$.

Proposition 1. 4. 33: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space, and A, B are subset of X, then the following statements are hold:

1. $A \subseteq B$ and B is τ_σ– nowhere dense ,then A is τ_σ – nowhere dense.
2. If A or B is τ_σ – nowhere dense, then $A \cap B$ is τ_σ – nowhere dense set.
3. If $A \cup B$ is τ_σ – nowhere dense set, then A and B are τ_σ – nowhere dense set.

Proof.

Evident by Definition 1. 4. 32 and Proposition 1. 4. 7 part 5■

Proposition 1. 4. 34: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space. If A is τ_σ – nowhere dense set, then $\tau_\sigma - int(A) = \emptyset$.

Proof.

Evident, because $\tau_\sigma - int(A) \subseteq \tau_\sigma - int(\tau_\sigma - cl(A))$■

2.1 Takeoff Points

Jankovic, D. and T. R.Hamlet [19] in 1990 defined the concept of ideal, which it is a nonempty of sets closed by hereditary property and Limited union. The ideal is a basic concept in topological spaces and it plays an important role in the study of topological problems. The concept of local function given by: $A^*(\tau, I) = \{x \in X;\ \forall\, \mathcal{U} \in \tau(x),\quad \mathcal{U} \cap A \notin I\}$. Moreover, further Hamlett and Jankovic in [20] and [21] studied the properties of ideal topological spaces and they have introduced another operator called ψ -operator given by: $\psi(A) = \{x \in X;\ \exists\, \mathcal{U} \in \tau(x),\ \mathcal{U} \cap (X - A) \in I\}$.

In this chapter, we use the idea of focal function and operator, to find follower points, and takeoff points, which will be an introduction to many of the mathematical concepts that this dissertation offers.

Definition 2. 1. 1: Let $(\mathrm{X}, \delta, \sigma,\ \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space. A point $x \in X$ is said to be *takeoff point* of a subset P of X if and only if there exist $\mathcal{U} \in\ \tau_\sigma(x)$ such that $(\mathcal{U} \cap\ X - P)\overline{\delta}C$ for some $C \in \sigma$.

All the takeoff points of a set P is denoted by P_{t_σ} . Thus $P_{t_\sigma}(\ \tau_\sigma, \sigma) = \{x \in X;\ \exists\, \mathcal{U} \in\ \tau_\sigma(x)\ s.t\ (\mathcal{U} \cap\ X - P)\overline{\delta}C$ for some $C \in \sigma\ \}$.

Example 2. 1. 2: Let $X = \{1, 2, 3, 4\}$, δ is a discrete proximity, $\sigma = \sigma_1$, and let $\tau_\sigma =\ \{X, \emptyset, \{2\}, \{4\}, \{3, 4\}\}$, then

X is $\tau_\sigma - open$ of 1,

$X, \{2\}$ are $\tau_\sigma - open$ of 2,

$X, \{3, 4\}$ are $\tau_\sigma - open$ of 3,

$X, \{4\}, \{3, 4\}$ are $\tau_\sigma - open$ of 4, If we take $P_1 = \{2,3\}$, then $X - P_1 = \{1,4\}$ thus $X \in \tau_\sigma(1), and\ X \cap \{1,4\}\delta C$ for every $C \in \sigma$, Hence $1 \notin P_{1_{t_\sigma}}$.

But there exists $\{2\} \in \tau_\sigma(2)$ such that $(\{2\} \cap \{1,4\})\overline{\delta}C$ for every $C \in \sigma$. Hence $2 \in P_{1_{t_\sigma}}$.

And there exists $\{3,4\} \in \tau_\sigma(3)$ and $\{1\} \in \sigma$ such that $(\{3,4\} \cap \{1,4\})\overline{\delta}\{1\}$. Hence $3 \in P_{1_{t_\sigma}}$.

Also, there exists $\{4\} \in \tau_\sigma(4)$ and $\{1,2\} \in \sigma$ such that $(\{4\} \cap \{1,4\})\overline{\delta}\{1,2\}$. Hence $4 \in P_{1_{t_\sigma}}$.

It follows that $P_{1_{t_\sigma}} = \{2,3,4\}$. In the same way, let $P_2 = \{1,2\}, P_3 = \{2,4\}$. Then $P_{2_{t_\sigma}} = X$, and $P_{3_{t_\sigma}} = \{2,3,4\}$.

The following proposition shows the most important characteristics of this set.

Proposition 2. 1. 3 Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma-$ Topological Proximity Space, and P_1, P_2 subsets of X. Then

1. If $P_1 \subset P_2$ $then$ $P_{1_{t_\sigma}} \subset P_{2_{t_\sigma}}$.
2. $(P_1 \cup P_2)_{t_\sigma} \supseteq P_{1_{t_\sigma}} \cup P_{2_{t_\sigma}}$.
3. $(P_1 \cap P_2)_{t_\sigma} = P_{1_{t_\sigma}} \cap P_{2_{t_\sigma}}$.
4. $P_{t_\sigma} = \cup \{\mathcal{U} \in \tau_\sigma, (\mathcal{U} \cap (X - P))\overline{\delta}\ C$ for some $C \in \sigma\}$.
5. If $G \in \tau_\sigma$, then $G \subseteq G_{t_\sigma}$.
6. $P_{t_\sigma} \subseteq (P_{t_\sigma})_{t_\sigma}$.
7. $X_{t_\sigma} = X$.
8. If $(X - P) \notin \sigma$, then $P_{t_\sigma} = X$.
9. $\tau_\sigma - int(P) \subseteq P_{t_\sigma}$.
10. $(\emptyset)_{t_\sigma} \subseteq P_{t_\sigma}$ for every $P \subseteq X$.

Proof.

1) Let $x \in P_{1_{t_\sigma}}$. Then there exists $\mathcal{U} \in \tau_\sigma(x)$ such that $(\mathcal{U} \cap (X - P_1))\overline{\delta}\, C$ for some $C \in \sigma$. By hypothesis $(X - P_2) \subseteq (X - P_1)$. By Theorem 1. 1. 7 part 3,$(\mathcal{U} \cap (X - P_2))\overline{\delta}\, C$ for some $C \in \sigma$, Hence $x \in P_{2_{t_\sigma}}$.

2) By part 1.

3) Let $x \in P_{1_{t_\sigma}} \cap P_{2_{t_\sigma}}$. Then $x \in P_{1_{t_\sigma}}$ *and* $x \in P_{2_{t_\sigma}}$, that is, there exist $\mathcal{U}$ *and* $\mathcal{V} \in \tau_\sigma(x)$, such that $(\mathcal{U} \cap (X - P_1)\,)\overline{\delta}\, C_1$, and $(\mathcal{V} \cap (X - P_2)\,)\overline{\delta} C_2$ for some $C_1, C_2 \in \sigma$ …..(1)
If possible, $x \notin (P_1 \cap P_2)_{t_\sigma}$,then for every $\mathcal{W} \in \tau_\sigma(x)$, $(\mathcal{W} \cap X - (P_1 \cap P_2)\,)\delta C$ for every $C \in \sigma$.
Thus $(\mathcal{W} \cap (X - P_1)\,) \cup (\mathcal{W} \cap (X - P_2)\,)\delta C$, that is, $(\mathcal{W} \cap (X - P_1))\delta C$ or $(\mathcal{W} \cap (X - P_2))\delta C$ which is a contradiction with (1). Hence $x \in (P_1 \cap P_2)_{t_\sigma}$.

4) $x \in P_{t_\sigma}$ if and only if there exists $\mathcal{U} \in \tau_\sigma(x)$ such that $(\mathcal{U} \cap (X - P)\,)\overline{\delta} C$ for some $C \in \sigma$ if and only if $x \in \cup \{\mathcal{U} \in \tau_\sigma(x), (\mathcal{U} \cap (X - P)\,)\overline{\delta}\mathrm{C}$ for some $C \in \sigma\}$.

5) Let $x \in G$. Since $G \in \tau_\sigma$, G is $\tau_\sigma - open$ of x, but $G \cap (X - G) = \emptyset$, and $(G \cap (X - G))\overline{\delta}\, C$for every $C \in \sigma$. Hence $x \in G_{t_\sigma}$.

6) Let $x \notin (P_{t_\sigma})_{t_\sigma}$. Then for every $\mathcal{U} \in \tau_\sigma(x)$, $(\mathcal{U} \cap (X - P_{t_\sigma}))\delta C$ for every $C \in \sigma$, this mean $\mathcal{U} \cap (X - P_{t_\sigma}) \neq \emptyset$, thus there exist $y \in (\mathcal{U} \cap (X - P_{t_\sigma}))$ such that $y \in \mathcal{U}$ *and* $y \in (X - P_{t_\sigma})$, thus $y \notin P_{t_\sigma}$, that is, for every $\mathcal{V} \in \tau_\sigma(y)$, $(\mathcal{V} \cap (X - P))\delta C$ for every $C \in \sigma$. Since $\mathcal{U}$ is also $\tau_\sigma - open$ of y, $(\mathcal{U} \cap (X - P))\delta C$ for every $C \in \sigma$, its follows that $x \notin P_{t_\sigma}$, hence $P_{t_\sigma} \subseteq \left(P_{t_\sigma}\right)_{t_\sigma}$.

7) For every $x \in X$ and every $\mathcal{U} \in \tau_\sigma(x)$, $(\mathcal{U} \cap (X - X)) = \emptyset\bar{\delta}\, C$ for every $C \in \sigma$. Hence $X_{t_\sigma} = X$.

8) For every $x \in X$, $X \in \tau_\sigma(x)$. Since $X - P \notin \sigma$, $X - P\bar{\delta}\, C$ for some $C \in \sigma$. Thus $(X \cap X - P)\bar{\delta}\, C$ for some $C \in \sigma$, that is, $P_{t_\sigma} = X$.

9) Let $x \in \tau_\sigma - int(P)$. Then there exists $\mathcal{U} \in \tau_\sigma(x)$, such that $\mathcal{U} \subseteq P$, thus $(\mathcal{U} \cap \mathrm{X} - \mathrm{P}) = \emptyset$, by theorem 1. 1. 7 part 5, $(\mathcal{U} \cap \mathrm{X} - \mathrm{P})\bar{\delta}C$ for every $C \in \sigma$, that is, $x \in P_{t_\sigma}$.

10) If $(\emptyset)_{t_\sigma} = \emptyset$ the prove is done. Otherwise Let $x \in (\emptyset)_{t_\sigma}$. Then there exists $\mathcal{U} \in \tau_\sigma(x)$, such that $(\mathcal{U} \cap X - \emptyset)\bar{\delta}C$ for some $C \in \sigma$ thus $\mathcal{U}\bar{\delta}C$ for some $C \in \sigma$. Since $(\mathcal{U} \cap \mathrm{X} - \mathrm{P}) \subseteq \mathcal{U}$ by Theorem 1. 1. 7 part 3, $(\mathcal{U} \cap \mathrm{X} - \mathrm{P})\bar{\delta}C$ for some $C \in \sigma$, That is true for every $P \subseteq X$. Hence $x \in P_{t_\sigma}$ ■

The following examples shows that the converse of cases (2), (5), and (9) is not always is true.

Examples 2. 1. 4:

1. Let $X = \{1, 2, 3, 4\}$, δ is proximity define by: $A\delta B\ \forall A \neq \emptyset, B \neq \emptyset$ except the relation: $\{1\}\bar{\delta}\{2,3\}$, $\{4\}\bar{\delta}\{2,3\}$, $\{2\}\bar{\delta}\{1,4\}$, $\{3\}\bar{\delta}\{1,4\}$, $\{1\}\bar{\delta}\{2\}$, $\{2\}\bar{\delta}\{1\}$, $\{1\}\bar{\delta}\{3\}$, $\{3\}\bar{\delta}\{1\}$, $\{4\}\bar{\delta}\{2\}$, $\{2\}\bar{\delta}\{4\}$, $\{4\}\bar{\delta}\{3\}$,

$\{3\}\overline{\delta}\{4\}$, $\{2,3\}\overline{\delta}\{1\}$, $\{2,3\}\overline{\delta}\{4\}$, $\{1,4\}\overline{\delta}\{2\}$, $\{1,4\}\overline{\delta}\{3\}$, $\{1,4\}\overline{\delta}\{2,3\}$, $\{2,3\}\overline{\delta}\{1,4\}$.

Then $\sigma = \{\{2\}, \{1,2\}, \{2,3,4\}, \{1,2,3\}, \{2,3\}, \{3,4\}, \{1,3,4\}, \{3\}, \{1,3\}, \{1,2,3\}, \{2,3\}, X\}$, and let $\tau_\sigma = \{X, \emptyset, \{1\}, \{4\}\}$, if we take $\{1,2\}, \{3,4\}$, then $\{1,2\}_{t_\sigma} = \{1,4\}, \{3,4\}_{t_\sigma} = \{1,4\}$. We see that $\{1,2\}_{t_\sigma} \cup \{3,4\}_{t_\sigma} = \{1,4\} \subset (\{1,2\} \cup \{3,4\})_{t_\sigma} = X_{t_\sigma} = X$. Let $P = \{2\}$. then $\tau_\sigma - int(\{2\}) = \emptyset$, but $\{2\}_{t_\sigma} = \{1,4\}$, thus $\tau_\sigma - int(\{2\}) \subset \{2\}_{t_\sigma}$.

2. Let $X = \{1,2,3\}$, if δ is a discrete proximity, then $\sigma = \{\{2\}, \{1,2\}, \{2,3\}, X\}$, and $\tau_\sigma = \{X, \emptyset, \{1\}, \{3\}\}$, then $\{3\}, \{1\} \in \tau_\sigma$ but $\{3\}_{t_\sigma} = \{1\}_{t_\sigma} = \{1,3\}$. Thus $G \subset G_{t_\sigma}$, and so that $\{1,3\}$ is not $\tau_\sigma - open$ set.

Through the Proposition 2. 1. 3, it is noted that: Takeoff set is not always an τ_σ −open set. That is clear by Example 2. 1. 4 part 2, $\{3\}_{t_\sigma} = \{1,3\} \notin \tau_\sigma$. Also, In case of $X - P \in \sigma$ of Proposition 2. 1. 3 part 8 that not meaning $P_{t_\sigma} = \emptyset$, but that means $P_{t_\sigma}$is equal $\emptyset$ or P_{t_σ} is less than X.

According to earlier results, we get the following remarks:

Remarks 2. 1. 5: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space. P nonempty subsets of X. Then

1st $\emptyset_{t_\sigma} = \emptyset$ if and only if X is only $\tau_\sigma - open$ of x.

Proof.

Suppose X is only $\tau_\sigma - open$ of x. Then $(X \cap X - \emptyset)\delta C$ for every $C \in \sigma$, thus $\emptyset_{t_\sigma} = \emptyset$. Conversely, let $\emptyset_{t_\sigma} = \emptyset$. Then for every $G \in \tau_\sigma$, $(G \cap X - \emptyset)\delta C$ for every $C \in \sigma$, by axiom [C3] its follows that $G = (G \cap X - \emptyset) \in \sigma$ which is a contradiction because $\tau_\sigma \cap \sigma = \{X\}$, thus that is true only in case of $G = X$.

2nd If δ is an indiscrete proximity defined on X, then $P_{t_\sigma} = \emptyset$ for every proper subset P of X.

Proof.

If possible $P_{t_\sigma} \neq \emptyset$, then there exists $x \in P_{t_\sigma}$. Since X is only τ_σ −open of each point, $(X \cap (X - P))\overline{\delta}\, C$ for some $C \in \sigma$, it follows that $X - P\overline{\delta}\, C$ thus $X - P \notin \sigma$, which is a contradiction because $X - P \neq \emptyset$. This means only $X_{t_\sigma} \neq \emptyset$. Moreover, $X_{t_\sigma} =$ X, because $X - X = \emptyset \notin \sigma$.

3rd If δ is a non-indiscrete proximity on X, then $P_{t_\sigma} \neq \emptyset$ for every P subset of X.

Proof.

If possible $P_{t_\sigma} = \emptyset$, then for every $\mathcal{U} \in \tau_\sigma(x)$, $(\mathcal{U} \cap X - P)\delta C$ for every $C \in \sigma$. By axiom[C3] we have that $\mathcal{U} \cap X - P \in \sigma$, by Proposition 1. 1. 15 part 2, $\mathcal{U} \in \sigma$ which is a contradiction because $\tau_\sigma \cap \sigma = \{X\}$■

Definition 2. 1. 6: Let $(X, \tau, \delta, \sigma)$is proximity cluster topological space. A point $x \in X$ is said to be *takeoff point* of a subset P of X if and only if there exist $\mathcal{U} \in \tau(x)$, such that $(\mathcal{U} \cap X - P)\overline{\delta}C$ for some $C \in \sigma$. Thus $P_{t_\sigma}(\tau, \sigma) = \{x \in X; \exists\, \mathcal{U} \in \tau(x)\ s.t\ (\mathcal{U} \cap X - P)\overline{\delta}C$ for some $C \in \sigma\}$.

Note 2. 1. 7: Let $(X, \tau, \delta, \sigma)$be a proximity cluster topological space. Then

1. All the parts of the Proposition 2. 1. 3 are satisfied.

2. Takeoff set is an $\tau -$open set. That is clear by part 1 all the parts of Proposition 2. 1. 3 are satisfied and by part 4 P_{t_σ} is $\tau - open$ because the union of $\tau - open$ sets is $\tau - open$ set.

3. P_{t_σ} forms a base, and therefore, by adding to unions, $\tau_{P_{t_\sigma}} \cup \{\emptyset\}$ is a coarse topology generated from P_{t_σ}.

Note 2. 1. 8: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. Then all the above results are achieved because every cluster topological proximity space is proximity cluster topology.

Examples 2. 1. 9: Let $X = \{1, 2, 3\}$, δ is proximity define by: $A\delta B\ \forall A \neq \emptyset, B \neq \emptyset$ except the relation: $\{1\}\overline{\delta}\{2,3\}$, $\{1\}\overline{\delta}\{2\}$, $\{2\}\overline{\delta}\{1\}$, $\{1\}\overline{\delta}\{3\}$, $\{3\}\overline{\delta}\{1\}$, $\{2, 3\}\overline{\delta}\{1\}$.
Then $\sigma = \{\{2\},\{1, 2\}, \{2, 3\}, \{3\}, \{1, 3\}, X\}$, and $\tau_\delta = \{X, \emptyset, \{1\}, \{2,3\}\}$.
$\{1, 3\}_{t_\sigma} = \{1\}$, and $\{2, 3\}_{t_\sigma} = X$.

2.2 Follower points

This part sheds light on a complementary definition of the takeoff point in the proximity space, and the characteristics of these points and their relations to the takeoff points are also highlighted.

Definition 2.2.1 Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. A point $x \in X$ is said to be *follower point* of a subset P of X, if and only if for every $\mathcal{U} \in \tau_\delta(x)$, $(\mathcal{U} \cap P)\delta C$ for every $C \in \sigma$.

All the follower point of a set P is denoted by P_{f_σ}. Thus $P_{f_\sigma}(\tau_\delta, \sigma) = \{x \in X; \forall\, \mathcal{U} \in \tau_\delta(x),\ (\mathcal{U} \cap P)\delta C\ \forall\, C \in \sigma\}$.

Example 2.2.2 Let $X = \{1,2,3\}$, δ is a proximity define by: $A\delta B\ \forall A \neq \emptyset, B \neq \emptyset$ except the relation: $\{1\}\overline{\delta}\{2\}, \{2\}\overline{\delta}\{1\}, \{2\}\overline{\delta}\{3\}, \{3\}\overline{\delta}\{2\}, \{2\}\overline{\delta}\{1,3\}, \{1,3\}\overline{\delta}\{2\}$, then $\sigma = \big\{\{1\},\{3\},\{1,2\},\{1,3\},\{2,3\},X\big\}$, and
$\tau_\delta = \{X, \emptyset, \{2\}, \{1,3\}\}$. Then
$\tau_\delta(1) = \{X, \{1,3\}\}$,
$\tau_\delta(2) = \{X, \{2\}\}$,
$\tau_\delta(3) = \{X, \{1,3\}\}$.
If we take $P_1 = \{2,3\}, P_2 = \{2\}, P_3 = X$, then
$\{1,3\} \in \tau_\delta(1), (\{1,3\} \cap \{2,3\})\delta C$ for every $C \in \sigma$, and $X \in \tau_\delta(1), (X \cap \{2,3\})\delta C$ for every $C \in \sigma$. Hence $1 \in P_{1_{f_\sigma}}$.

But there exists $\{2\} \in \tau_\delta(2)$ such that $\{2\} \cap \{2,3\}\overline{\delta}\{1\}$ for some $\{1\} \in \sigma$. Hence $2 \notin P_{1_{f_\sigma}}$.

Also, $X, and\ \{1,3\} \in \tau_\delta(3)$ such that $(\{1,3\} \cap \{2,3\})\delta C$ and $(X \cap \{2,3\})\delta C$ for every $C \in \sigma$. Hence $3 \in P_{1_{f_\sigma}}$.

That is, $P_{1_{f_\sigma}} = \{1,3\}$. In the same way, $P_{2_{f_\sigma}} = \emptyset$, and $P_{3_{f_\sigma}} = \{1,3\}$.

According to the aforementioned discussion about the importance of optimal construction of synonymous cumulative points, the questions arise:

What are the characteristics of these points?

What operations can be performed on them?

Is it possible to create these points, topology or base or sub-base? The following Proposition answers:

Proposition 2.2.3: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. P_1, P_2 nonempty subsets of X. Then

1. If $P_1 \subset P_2$ $then$ ${P_1}_{f_\sigma} \subset {P_2}_{f_\sigma}$;
2. $(P_1 \cup P_2)_{f_\sigma}$ =${P_1}_{f_\sigma} \cup {P_2}_{f_\sigma}$;
3. $(P_1 \cap P_2)_{f_\sigma} \subseteq {P_1}_{f_\sigma} \cap {P_2}_{f_\sigma}$;
4. $P_{f_\sigma} = \tau_\delta - cl(P_{f_\sigma}) \subseteq \tau_\delta - clP$;
5. If $P \notin \sigma$, then $P_{f_\sigma} = \emptyset$;
6. $(P_{f_\sigma})_{f_\sigma} \subseteq P_{f_\sigma}$;
7. $(\emptyset)_{f_\sigma} = \emptyset$;
8. If $G \in \tau_\delta$, then G $\cap P_{f_\sigma} \subseteq (G \cap P)_{f_\sigma}$;
9. If $P\bar{\delta}C$ for some $C \in \sigma$, then $P \notin \sigma$.

Proof.

1) Let $x \in {P_1}_{f_\sigma}$. Then for every $\mathcal{U} \in \tau_\delta(x)$, $(\mathcal{U} \cap P_1)\delta C$ for every $C \in \sigma$. Since $P_1 \subset P_2$ by Theorem 1. 1. 7 part 2, $(\mathcal{U} \cap P_2)\delta C$. Hence $x \in {P_2}_{f_\sigma}$.

2) Evident, ${P_1}_{f_\sigma} \cup {P_2}_{f_\sigma} \subset (P_1 \cup P_2)_{f_\sigma}$. So, let $x \in (P_1 \cup P_2)_{f_\sigma}$, then for every $\mathcal{U} \in \tau_\delta(x)$, and every $C \in \sigma$, $(\mathcal{U} \cap (P_1 \cup P_2))\delta C$, thus $((\mathcal{U} \cap$

$P_1) \cup (\mathcal{U} \cap P_2))\delta C$. By axiom [P4], $(\mathcal{U} \cap P_1)\delta C$ or $(\mathcal{U} \cap P_2)\delta C$ this mean $x \in P_{1_{f_\sigma}}$ or $\in P_{2_{f_\sigma}}$. Hence $x \in (P_{1_{f_\sigma}} \cup P_{2_{f_\sigma}})$.

3) Straight from part 1.

4) Obviously, $P_{f_\sigma} \subset \tau_\delta - cl(P_{f_\sigma})$. Let $x \in \tau_\delta - cl(P_{f_\sigma})$. Then $\{x\}\delta P_{f_\sigma}$. Since $x \in \mathcal{U}$ for every $\mathcal{U} \in \tau_\delta(x)$, $\{x\}\delta\mathcal{U}$. By Theorem 1. 1. 7 part 4, $\mathcal{U}\delta P_{f_\sigma}$. Thus either $(\mathcal{U} \cap P_{f_\sigma}) \neq \emptyset$. Then there exist $y \in \left(\mathcal{U} \cap P_{f_\sigma}\right)$, thus $y \in \mathcal{U}$ and $y \in P_{f_\sigma}$, that is, for every $\mathcal{V} \in \tau_\delta(y)$, $(\mathcal{V} \cap P)\delta C$ for every $C \in \sigma$, but $\mathcal{U}$ is also an τ_δ- open of y this implies that $(\mathcal{U} \cap P)\delta C$ for every $C \in \sigma$ thus $x \in P_{f_\sigma}$.

Or $\left(\mathcal{U} \cap P_{f_\sigma}\right) = \emptyset$ this means $P_{f_\sigma} \subseteq X - \mathcal{U}$. Since $X - \mathcal{U}$ is τ_δ −closed set, $\tau_\delta - cl(P_{f_\sigma}) \subseteq X - \mathcal{U}$. But by hypothesis $x \in \tau_\delta - cl(P_{f_\sigma})$, thus $x \in X - \mathcal{U}$ which is a contradiction. Hence $P_{f_\sigma} = \tau_\delta - cl(P_{f_\sigma})$.

New we show that $P_{f_\sigma} \subseteq \tau_\delta - cl(P)$. Let x $\in P_{f_\sigma}$, Then for every $\mathcal{U} \in \tau_\delta(x)$, $(\mathcal{U} \cap P)\delta C$ for every $C \in \sigma$.By [C3], $(\mathcal{U} \cap P) \in \sigma$. Since $\emptyset \notin \sigma$, we have that $\mathcal{U} \cap P \neq \emptyset$ for every $\mathcal{U} \in \tau_\delta(x)$, thus by axiom [P3], $\mathcal{U}\delta P$ for every $\mathcal{U} \in \tau_\delta(x)$, thus by Proposition 1. 2. 15, we have that $x \in \tau_\delta - cl(P)$.

5) Suppose $P_{f_\sigma} \neq \emptyset$. Then there exist $x \in P_{f_\sigma}$, thus for every $\mathcal{U} \in \tau_\delta(x)$, $(P \cap \mathcal{U})\delta C$ for every $C \in \sigma$. By axiom [C3] and Proposition of 1. 1. 15 part 2, we have that $(P \cap \mathcal{U}) \in \sigma$, and $P \in \sigma$ which is a contradiction with a hypothesis, hence $P_{f_\sigma} = \emptyset$.

6) Let $x \in (P_{f_\sigma})_{f_\sigma}$. Then for every $\mathcal{U} \in \tau_\delta(x)$, $(\mathcal{U} \cap P_{f_\sigma})\delta C$ for every $C \in \sigma$, that means $(\mathcal{U} \cap P_{f_\sigma}) \neq \emptyset$. Then there exists $y \in \left(\mathcal{U} \cap P_{f_\sigma}\right)$ such that $y \in \mathcal{U}$ and $y \in P_{f_\sigma}$ thus for every $\mathcal{V} \in \tau_\delta(y)$, $(\mathcal{V} \cap P)\delta C$ for every $C \in \sigma$. Since $\mathcal{U}$ is also an τ_δ- open of y, $(\mathcal{U} \cap P)\delta C$ for every $C \in \sigma$, hence $x \in P_{f_\sigma}$.

7) It is an immediate consequence of part 5. Because $\emptyset \notin \sigma$.

8) Let $x \in G \cap P_{f_\sigma}$. Then $x \in G$ and $x \in P_{f_\sigma}$, thus for every $\mathcal{U} \in \tau_\delta(x)$, $(\mathcal{U} \cap P)\delta C$ for every $C \in \sigma$. Since $x \in G$, $(G \cap \mathcal{U}) \in \tau_\delta(x)$, hence $\mathcal{U} \cap (G \cap P)\delta C$ for every $C \in \sigma$, that is, $x \in (G \cap P)_{f_\sigma}$.

9) It is an immediate consequence of by axiom [C3] ■

We cane noted that, when studying the follower points on the $\sigma -$ Topological Proximity Space there will be three cases. The following remark explain that.

Remark 2. 2. 4: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space. P nonempty subsets of X. Then

$$P_{f_\sigma} = \begin{cases} X & \text{if } X \text{ is only } \tau_\sigma - open \text{ of } x \text{ and } P \in \sigma \\ \emptyset & othewise \end{cases} .$$

Proof.

1) Let X is only $\tau_\sigma - open$ of x and $P \in \sigma$. Then for every $x \in X$ the only $\tau_\sigma - open$ set is X , thus $(X \cap P)\delta C$ for every $C \in \sigma$, because $P \in \sigma$. Hence $(P)_{f_\sigma} = X$.

2) Suppose$(P)_{f_\sigma} \neq \emptyset$. Then there exist $x \in (P)_{f_\sigma}$such that for every $\mathcal{U} \in \tau_\sigma(x)$, $(\mathcal{U} \cap P)\delta C$ for every $C \in \sigma$. By axiom [C3], $(\mathcal{U} \cap P) \in \sigma$ and by

Proposition 1. 1. 15 part 2 $\mathcal{U} \in \sigma$ which is a contradiction because $\tau_\sigma \cap \sigma = \{X\}$. Hence $(P)_{f_\sigma} = \emptyset$■

For example, in case of indiscrete proximity clear that $\tau_\sigma = \{X, \emptyset\}$ and $\sigma = \{A \subseteq X; A \neq \emptyset\}$, thus follower set is equal X for every nonempty subset of X, because every nonempty subset of X is near all nonempty subset. Hence the follower set of the empty set is only equal empty set.

Note 2. 2. 5: when studying the follower points on the proximity cluster topological space$(X, \tau, \delta, \sigma)$ we have same of these result because τ_δ is topology.

Definition 2. 2. 6: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space, $P \subseteq$ X. Then the closure of P with respect to τ_δ and σ denoted by $cl_{f_\sigma}(P)$ is given by $cl_{f_\sigma}(P) = P \cup P_{f_\sigma}$.

By Example 2. 2. 2 $P_1 = \{2,3\}, P_2 = \{2\}, P_3 = X, P_4 = \emptyset$, then $P_{1_{f_\sigma}} = \{1, 3\}$, $P_{2_{f_\sigma}} = \emptyset$, and $P_{3_{f_\sigma}} = \{1, 3\}$, $P_{4_{f_\sigma}} = \emptyset$. We have that: $cl_{f_\sigma}(P_1) = X$, $cl_{f_\sigma}(P_2) = \{2\}$, $cl_{f_\sigma}(P_3) = X$, and $cl_{f_\sigma}(P_4) = \emptyset$.

Proposition 2. 2. 7: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space, A and B subsets of X, the following statements are hold:

1. $cl_{f_\sigma}(X)$=X and $cl_{f_\sigma}(\emptyset)$=∅.
2. $A \subseteq B \Rightarrow cl_{f_\sigma}(A) \subseteq cl_{f_\sigma}(B)$.
3. $cl_{f_\sigma}(A \cup B) = cl_{f_\sigma}(A) \cup cl_{f_\sigma}(B)$.
4. $cl_{f_\sigma}(A \cap B) \subseteq cl_{f_\sigma}(A) \cap cl_{f_\sigma}(B)$.

5. $cl_{f_\sigma}\left(cl_{f_\sigma}(A)\right) = cl_{f_\sigma}(A)$.
6. $A \subseteq cl_{f_\sigma}(A)$.

Proof.

1) $cl_{f_\sigma}(X) = X \cup X_{f_\sigma} = X$ also, $cl_{f_\sigma}(\emptyset) = \emptyset \cup \emptyset_{f_\sigma} = \emptyset$.

2) Let $A \subseteq$ B. By Proposition 2. 2. 3, $A_{f_\sigma} \subseteq B_{f_\sigma}$, thus $A \cup A_{f_\sigma} \subseteq \mathrm{B} \cup B_{f_\sigma}$, that is, $cl_{f_\sigma}(A) \subseteq cl_{f_\sigma}(B)$.

3) $cl_{f_\sigma}(A \cup B) = (\mathrm{A} \cup \mathrm{B}) \cup (A \cup B)_{f_\sigma}$

$= (A \cup \mathrm{B}) \cup \left(A_{f_\sigma} \cup B_{f_\sigma}\right) = \left(\mathrm{A} \cup A_{f_\sigma}\right) \cup \left(\mathrm{B} \cup B_{f_\sigma}\right)$

$= cl_{f_\sigma}(A) \cup cl_{f_\sigma}(B)$.

4) $cl_{f_\sigma}(A \cap B) = (\mathrm{A} \cap \mathrm{B}) \cup (A \cap B)_{f_\sigma}$

$\subseteq (A \cap \mathrm{B}) \cup \left(A_{f_\sigma} \cap B_{f_\sigma}\right)$

$= ((\mathrm{A} \cap \mathrm{B}) \cup A_{f_\sigma}) \cap ((\mathrm{A} \cap \mathrm{B}) \cup B_{f_\sigma})$

$\subseteq \left(A \cup A_{f_\sigma}\right) \cap \left(\mathrm{B} \cup B_{f_\sigma}\right) = cl_{f_\sigma}(A) \cap cl_{f_\sigma}(B)$.

5) $cl_{f_\sigma}\left(cl_{f_\sigma}(A)\right) = cl_{f_\sigma}(A) \cup (cl_{f_\sigma}(A))_{f_\sigma}$

$= \left(A \cup A_{f_\sigma}\right) \cup (A \cup A_{f_\sigma})_{f_\sigma} = \left(\mathrm{A} \cup A_{f_\sigma}\right) \cup (A_{f_\sigma} \cup (A_{f_\sigma})_{f_\sigma})$

$= \left(A \cup A_{f_\sigma}\right) \cup A_{f_\sigma} = \left(A \cup A_{f_\sigma}\right) = cl_{f_\sigma}(A)$.

6) Clear $cl_{f_\sigma}(A) = A \cup A_{f_\sigma} \supseteq A$, hence $A \subseteq cl_{f_\sigma}(A)$■

Definition 2. 2. 8: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space, A and B subsets of X, the closure operator with respect to τ_δ and σ is defined by:

$cl_{f_\sigma}: \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, satisfying the following four conditions:

1. $cl_{f_\sigma}(\emptyset)=\emptyset$.
2. $A \subseteq cl_{f_\sigma}(A)$.
3. $cl_{f_\sigma}(A \cup B) = cl_{f_\sigma}(A) \cup cl_{f_\sigma}(B)$.
4. $cl_{f_\sigma}\left(cl_{f_\sigma}(A)\right) = cl_{f_\sigma}(A)$.

Remark 2. 2. 9: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. Let cl_{f_σ} be a closure operator such that $cl_{f_\sigma}: \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, satisfying the four conditions in Definition 2. 2. 8, then by Kuratowski closure operator there are unique topology generated by cl_{f_σ}denoted by τ_{f_σ}, that is, $\tau_{f_\sigma} = \{\mathcal{U} \subseteq X; cl_{f_\sigma}(X - \mathcal{U}) = X - \mathcal{U}\}$.

Moreover τ_{f_σ} finer than τ_δ.We can see that by Example 2. 2. 2, $\tau_{f_\sigma} = \tau_\delta$, but Example 2. 2. 10 show that τ_{f_σ} finer than τ_δ.

Example 2. 2. 10: : Let $X = \{1, 2, 3, 4\}$, δ is proximity define by: $A\delta B\ \forall A \neq \emptyset, B \neq \emptyset$ except the relation: $\{1\}\overline{\delta}\{3\}$, $\{1\}\overline{\delta}\{4\}$, $\{2\}\overline{\delta}\{3\}$, $\{2\}\overline{\delta}\{4\}$, $\{1\}\overline{\delta}\{3, 4\}$, $\{2\}\overline{\delta}\{3,4\}$, $\{1,2\}\overline{\delta}\{3,4\}$, $\{3\}\overline{\delta}\{1\}$, $\{4\}\overline{\delta}\{1\}$, $\{3\}\overline{\delta}\{2\}$, $\{4\}\overline{\delta}\{2\}$,$\{3,4\}\overline{\delta}\{1\}$, $\{3,4\}\overline{\delta}\{2\}$, $\{3,4\}\overline{\delta}\{1,2\}$, $\{3\}\overline{\delta}\{1,2\}$, $\{1, 2\}\overline{\delta}\{3\}$, $\{4\}\overline{\delta}\{1,2\}$, $\{1, 2\}\overline{\delta}\{4\}$.

Then $\sigma = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2\}, \{2, 4\}, \{2, 3, 4\}, \{2, 3\}, X\}$, and $\tau_\delta = \{X, \emptyset, \{1, 2\}, \{3, 4\}\}$. Then we can see that:

$\{1\}_{f_\sigma} = \{1, 2\}$ and $cl_{f_\sigma}(\{1\}) = \{1, 2\}$,

$\{2\}_{f_\sigma} = \{1,2\}$ and $cl_{f_\sigma}(\{2\}) = \{1,2\}$,

$\{3\}_{f_\sigma} = \emptyset$ and $cl_{f_\sigma}(\{3\}) = \{3\}$,

$\{4\}_{f_\sigma} = \emptyset$ and $cl_{f_\sigma}(\{4\}) = \{4\}$,

$\{1,2\}_{f_\sigma} = \{1,2\}$ and $cl_{f_\sigma}(\{1,2\}) = \{1,2\}$,

$\{1,3\}_{f_\sigma} = \{1,2\}$ and $cl_{f_\sigma}(\{1,3\}) = \{1,2,3\}$,

$\{2,3\}_{f_\sigma} = \{1,2\}$ and $cl_{f_\sigma}(\{2,3\}) = X$,

$\{1,2,4\}_{f_\sigma} = \{1,2\}$ and $cl_{f_\sigma}(\{1,2,4\}) = \{1,2,4\}$,

$\{1,3,4\}_{f_\sigma} = \{1,2\}$ and $cl_{f_\sigma}(\{1,3,4\}) = X$,

$\{1,4\}_{f_\sigma} = \{1,2\}$ and $cl_{f_\sigma}(\{1,4\}) = \{1,2,4\}$,

$\{2,3,4\}_{f_\sigma} = \{1,2\}$ and $cl_{f_\sigma}(\{2,3,4\}) = X$,

$\{1,2,3\}_{f_\sigma} = \{1,2\}$ and $cl_{f_\sigma}(\{1,2,3\}) = \{1,2,3\}$,

$\{3,4\}_{f_\sigma} = \emptyset$ and $cl_{f_\sigma}(\{3,4\}) = \{3,4\}$,

$\{2,4\}_{f_\sigma} = \{1,2\}$ and $cl_{f_\sigma}(\{2,4\}) = \{1,2,4\}$,

$X_{f_\sigma} = \{1,2\}$ and $cl_{f_\sigma}(X) = X$,

$\emptyset_{f_\sigma} = \emptyset$ and $cl_{f_\sigma}(\emptyset) = \emptyset$,

Then $\tau_{f_\sigma} = \{X, \emptyset, \{3\}, \{4], \{1,2\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}\}$.

Proposition 2. 2. 11: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. Then $\tau_\delta \subseteq \tau_{f_\sigma}$.

Proof.

Let $\mathcal{U} \in \tau_\delta$. Then $\tau_\delta - cl(X - \mathcal{U}) = X - \mathcal{U}$. Therefore $cl_{f_\sigma}(X - \mathcal{U}) = (X - \mathcal{U}) \cup (X - \mathcal{U})_{f_\sigma} \subseteq (X - \mathcal{U}) \cup \tau_\delta - cl(X - \mathcal{U}) = \tau_\delta - cl(X - \mathcal{U}) = (X - \mathcal{U})$, that is, $cl_{f_\sigma}(X - \mathcal{U}) = (X - \mathcal{U})$, by Remark 2. 2. 9, $\mathcal{U} \in \tau_{f_\sigma}$. Hence $\tau_\delta \subseteq \tau_{f_\sigma}$ ■

Definition 2. 2. 12: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space, $P \subseteq$ X. Then the interior operator of P with respect to τ_δ and σ denoted by $int_{f_\sigma}(P)$ is given by $int_{f_\sigma}(P) = P \cap P_{t_\sigma}$.

By Example 2. 2. 2 $P_1 = \{2,3\}, P_2 = \{1, 3\}, P_3 = \emptyset$, then
$P_{1_{t_\sigma}} = \{2\}$, $P_{2_{t_\sigma}} = X$, and $P_{3_{t_\sigma}} = \{ 2\}$. We have that
$int_{f_\sigma}(P_1) = \{2\}$, $int_{f_\sigma}(P_2) = \{1, 3\}$, and $int_{f_\sigma}(P_3) = \emptyset$.

Proposition 2. 2. 13: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space, for any two subsets A and B of X, the following statements are hold:

1. $int_{f_\sigma}(X)$=X and $int_{f_\sigma}(\emptyset)$=$\emptyset$.
2. $A \subseteq B \Longrightarrow int_{f_\sigma}(A) \subseteq int_{f_\sigma}(B)$.
3. $int_{f_\sigma}(A \cap B) = int_{f_\sigma}(A) \cap int_{f_\sigma}(B)$.
4. $int_{f_\sigma}(A) \cup int_{f_\sigma}(B) \subseteq int_{f_\sigma}(A \cup B)$.
5. $int_{f_\sigma}\left(int_{f_\sigma}(A)\right) = int_{f_\sigma}(A)$.
6. $int_{f_\sigma}(A) \subseteq A$.

Proof.

1) $int_{f_\sigma}(X)$ =$X \cap X_{t_\sigma} = X$ also, $int_{f_\sigma}(\emptyset)$=$\emptyset \cap \emptyset_{t_\sigma} = \emptyset$.

2) Let $A \subseteq$ B. By Proposition 2. 1. 3, $A_{t_\sigma} \subseteq B_{t_\sigma}$, so that $A \cap A_{t_\sigma} \subseteq$ B ∩ B_t, hence $int_{f_\sigma}(A) \subseteq int_{f_\sigma}(B)$.

3) $int_{f_\sigma}(A \cap B) = (\mathrm{A} \cap \mathrm{B}) \cap (A \cap B)_{t_\sigma}$

$= (A \cap \mathrm{B}) \cap \left(A_{t_\sigma} \cap B_{t_\sigma}\right) = \left(\mathrm{A} \cap A_{t_\sigma}\right) \cap \left(\mathrm{B} \cap B_{t_\sigma}\right)$
$= int_{f_\sigma}(A) \cap int_{f_\sigma}(B)$.

4) $int_{f_\sigma}(A \cup B) = (A \cup B) \cap (A \cup B)_{t_\sigma}$

$\supseteq (A \cup B) \cap \left(A_{t_\sigma} \cup B_{t_\sigma}\right)$

$= ((A \cup B) \cap A_{t_\sigma}) \cup ((A \cup B) \cap B_{t_\sigma})$

$\supseteq \left(A \cap A_{t_\sigma}\right) \cup \left(B \cap B_{t_\sigma}\right) = int_{f_\sigma}(A) \cup int_{f_\sigma}(B).$

5) $int_{f_\sigma}\left(int_{f_\sigma}(A)\right) = int_{f_\sigma}(A) \cap (int_{f_\sigma}(A))_{t_\sigma}$

$= \left(A \cap A_{t_\sigma}\right) \cap (A \cap A_{t_\sigma})_{t_\sigma} = \left(A \cap A_{t_\sigma}\right) \cap (A_{t_\sigma} \cap (A_{t_\sigma})_{t_\sigma})$

$=\left(A \cap A_{t_\sigma}\right) \cap A_{t_\sigma} =\left(A \cap A_{t_\sigma}\right) = int_{f_\sigma}(A).$

6) Clear $int_{f_\sigma}(A) = A \cap A_{t_\sigma} \subseteq A$■

The following proposition shows the relationship between $follower\ set$ and $takeoff$ set, with respect to τ_δ and σ on X.

Proposition 2. 2. 14: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. P subsets of X. Then

1. $P_{t_\sigma} = X - (X - P)_{f_\sigma}$;
2. $P_{t_\sigma} = (P_{t_\sigma})_{t_\sigma}$ if and only if $(X - P)_{f_\sigma} = ((X - P)_{f_\sigma})_{f_\sigma}$
3. $P_{f_\sigma} = X - (X - P)_{t_\sigma}$;
4. $X - P_{f_\sigma} = (X - P)_{t_\sigma}$;
5. $X - P_{t_\sigma} = (X - P)_{f_\sigma}$;
6. $(\emptyset)_{t_\sigma} = X - X_{f_\sigma}$;
7. $int_{f_\sigma}(P) = X - cl_{f_\sigma}(X - P)$;
8. $cl_{f_\sigma}(P) = X - int_{f_\sigma}(X - P)$;
9. $\tau_\delta - int(P) \subseteq int_{f_\sigma}(P) \subseteq P$.

Proof.

1) $x \in P_{t_\sigma}$ if and only if there exists $\mathcal{U} \in \tau_\delta(x)$such that $(\mathcal{U} \cap (X - P))\overline{\delta}\, C$ for some $C \in \sigma$ if and only if $x \notin (X - P)_{f_\sigma}$if and only if $x \in X - (X - P)_{f_\sigma}$, hence $P_{t_\sigma} = X - (X - P)_{f_\sigma}$.

2) $\left(P_{t_\sigma}\right)_{t_\sigma} = X - \left(X - P_{t_\sigma}\right)_{f_\sigma} = X - (X - X - (X - P)_{f_\sigma})_{f_\sigma} =$
$X - ((X - P)_{f_\sigma})_{f_\sigma} = X - (X - P)_{f_\sigma} = P_{t_\sigma}$.

3) $X - (X - P)_{t_\sigma} = X - (X - (X - (X - P))_{f_\sigma} = P_{f_\sigma}$.

4) $X - P_{f_\sigma} = X - (X - (X - P)_{t_\sigma}) = (X - P)_{t_\sigma}$.

5) $X - P_{t_\sigma} = X - (X - (X - P)_{f_\sigma}) = (X - P)_{f_\sigma}$.

6) $(\emptyset)_{t_\sigma} = X - (X - \emptyset)_{f_\sigma} = X - X_{f_\sigma}$.

7) $X - cl_{f_\sigma}(X - P) = X - \left((X - P) \cup (X - P)_{f_\sigma}\right) =$

$$X - \left((X - P) \cup \left(X - P_{t_\sigma}\right)\right) = P \cap P_{t_\sigma} = int_{f_\sigma}(P).$$

8) $X - int_{f_\sigma}(X - P) = X - \left((X - P) \cap (X - P)_{t_\sigma}\right) =$
$X - \left((X - P) \cap (X - P_{f_\sigma})\right) = P \cup P_{f_\sigma} = cl_{f_\sigma}(P)$.

9) Let $x \in \tau_\delta - int(P)$, there exists $\mathcal{U} \in \tau_\delta(x)$, such that $\mathcal{U} \subseteq P$ this mean $(\mathcal{U} \cap \mathrm{X - P}) = \emptyset$, thus $(\mathcal{U} \cap \mathrm{X - P})\overline{\delta}C$ for every $C \in \sigma$, hence $x \in P_{t_\sigma}$, it follows that $x \in (P \cap P_{t_\sigma})$, that is, $x \in int_{f_\sigma}(P)$■

Note 2. 2. 15: When studying the Proposition 2. 2. 14 in the $\sigma-$ Topological Proximity Space $(X, \delta, \sigma, \tau_\sigma)$, the result is trivial because by Remark 2. 2. 4 $P_{f_\sigma} = \emptyset$ or $P_{f_\sigma} = X$. Thus every parts of Proposition 2. 2. 14 are achieved.

Remark 2. 2. 16: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. P nonempty subsets of X. The following statements are hold:

1. If (X, δ) is a discrete proximity space, then $P_{f_\sigma} \subseteq P_{t_\sigma}$.
2. If (X, δ) be a indiscrete proximity space, then
   i. $P_{t_\sigma} \subseteq P_{f_\sigma}$.
   ii. $\tau_\delta - int\ (P) = P_{t_\sigma} = int_{f_\sigma}(P)$.

Proof.

Let $x \in P_{f_\sigma}$. Then for every $\mathcal{U} \in \tau_\delta(x)$, $(\mathcal{U} \cap P)\delta C$ for every $C \in \sigma$. By axiom [C3], $(\mathcal{U} \cap P) \in \sigma$, by Proposition 1. 1. 15 part 2, $P \in \sigma$. If possible $x \notin P_{t_\sigma}$, then for every $\mathcal{V} \in \tau_\delta(x)$, $(\mathcal{V} \cap (X-P))\delta\ C$ for every $C \in \sigma$. axiom [C3], $(X-P) \in \sigma$, that is following to P and $(X-P) \in \sigma$, but $P \cap (X-P) = \emptyset$ which is a contradiction because δ is a discrete proximity. Hence $P_{f_\sigma} \subseteq P_{t_\sigma}$.

2)*i*. Let $x \in P_{t_\sigma}$. Then there exists $\mathcal{U} \in \tau_\delta(x)$ such that $(\mathcal{U} \cap (X-P))\overline{\delta}\, C$, for some $C \in \sigma$. Since δ is indiscrete proximity, $(\mathcal{U} \cap (X-P)) = \emptyset$ [if not $(\mathcal{U} \cap (X-P))\delta\ C$ for every $C \in \sigma$ because δ is indiscrete proximity]. This implies $\mathcal{U} \subseteq P$, i.e. $x \in P$, that is, $\mathcal{V} \cap P \neq \emptyset$ for every $\mathcal{V} \in \tau_\delta(x)$, but δ is indiscrete proximity, this means $(\mathcal{V} \cap P)\delta C$ for every $C \in \sigma$, thus $x \in P_{f_\sigma}$.

ii. By Proposition 2. 1. 3, $\tau_\delta - int\ (P) \subset P_{t_\sigma}$. Let $x \in P_{t_\sigma}$. Then there exists $\mathcal{U} \in \tau_\delta(x)$ such that $(\mathcal{U} \cap (X - P))\overline{\delta}\ C$, for some $C \in \sigma$. Since δ is indiscrete proximity, $(\mathcal{U} \cap (X - P)) = \emptyset$. This implies to $\mathcal{U} \subseteq P$, thus $x \in \tau_\delta - int\ (P)$. Hence $\tau_\delta - int\ (P) = P_{t_\sigma}$. Moreover, $int_{f_\sigma}(P) = P \cap P_{t_\sigma} = P \cap \tau_\delta - int\ (P) = \tau_\delta - int\ (P) = P_{t_\sigma}$■

Lemma 2. 2. 17: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. P nonempty subsets of X. If P is a closed set, then $P_{f_\sigma} \subseteq P$.

Proof.

Let us suppose that, P is a τ_δ -closed set. Then $(X - P)$ is an τ_δ –open. By Proposition 2. 1. 3, $(X - P) \subseteq (X - P)_{t_\sigma}$, thus $X - (X - P)_{t_\sigma} \subseteq P$. By Proposition 2. 2. 14, $P_{f_\sigma} \subseteq P$■

Proposition 2. 2. 18: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. P nonempty subsets of X. If P is $\tau_\delta - clopen$ set, then $P_{f_\sigma} \subseteq P_{t_\sigma}$.

Proof.

Suppose P is a τ_δ -closed set. By Lemma 2. 2. 17, $P_{f_\sigma} \subseteq P$, but P is τ_δ -open by Proposition 2. 1. 3 $P \subseteq P_{t_\sigma}$. Hence $P_{f_\sigma} \subseteq P_{t_\sigma}$■

Proposition 2. 2. 19: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. $X = X_{f_\sigma}$ if and only if for every $G \in \tau_\delta$, $G_{f_\sigma} = \tau_\delta - cl(G)$.

Proof.

Suppose $X = X_{f_\sigma}$. Then for every $x \in X$, $x \in X_{f_\sigma}$. Thus for every $G \in \tau_\delta(x)$, $(G \cap X)\delta C$ for every $C \in$ σ. If possible there exists $x \in \tau_\delta -$

$cl(G)$, and $x \notin G_{f_\sigma}$, then there exists $\mathcal{U} \in \tau_\delta(x)$, such that $(G \cap \mathcal{U})\overline{\delta}C$ for some $C \in \sigma$. thus $(G \cap \mathcal{U}) \cap X\overline{\delta}C$ for some $C \in \sigma$, this contradicts $X = X_{f_\sigma}$. Hence $G_{f_\sigma} = \tau_\delta - cl(G)$. Conversely is clear ■

We can conclude through the above proposition if $X = X_{f_\sigma}$, then $G \subseteq G_{f_\sigma}$.

Corollary 2. 2. 20: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space, then the following statements are equivalent:

1. If $G \in \tau_\delta$, then $G \subseteq G_{f_\sigma}$.
2. $X = X_{f_\sigma}$.
3. If $G \in \tau_\delta$, then $G_{f_\sigma} = \tau_\delta - cl(G)$.

Proof.

1 ⇒ 2 Suppose $G \in \tau_\delta$ and $G \subseteq G_{f_\sigma}$. Since $X \in \tau_\delta$, $X \subseteq X_{f_\sigma}$, hence $X = X_{f_\sigma}$.

2 ⇒ 3 Suppose $X = X_{f_\sigma}$, by Proposition 2. 2. 19, $G_{f_\sigma} = \tau_\delta - cl(G)$.

3 ⇒ 1 Suppose $\tau_\delta - cl(G) = G_{f_\sigma}$. Then $G \subseteq \tau_\delta - cl(G) = G_{f_\sigma}$, hence $G \subseteq G_{f_\sigma}$■

Proposition 2. 2. 21: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. If A or B is a τ_δ −open set, then $\tau_\delta - int(A_{f_\sigma}) \cap \tau_\delta - int(B_{f_\sigma}) = \tau_\delta - int(A \cap B)_{f_\sigma}$.

Proof.

Let us suppose that $A \in \tau_\delta$. By Proposition 2. 2. 3 part 8, $A \cap B_{f_\sigma} \subseteq (A \cap B)_{f_\sigma}$, thus $\tau_\delta - int\left(A \cap B_{f_\sigma}\right) \subseteq \tau_\delta - int((A \cap B)_{f_\sigma})$, that is follows $\tau_\delta - int(A) \cap \tau_\delta - int(B_{f_\sigma}) \subseteq \tau_\delta - int((A \cap B)_{f_\sigma})$.

Since $\tau_\delta - int(A) = A$, $A \cap \tau_\delta - int(B_{f_\sigma}) \subseteq \tau_\delta - int((A \cap B)_{f_\sigma})$.

By take the follower set and interior of two said we have that, $\tau_\delta - int((A \cap \tau_\delta - int(B_{f_\sigma}))_{f_\sigma}) \subseteq \tau_\delta - int((\tau_\delta - int(A \cap B)_{f_\sigma})_{f_\sigma})$----- (1). Also, $\tau_\delta - int(B_{f_\sigma}) \in \tau_\delta$. Then by Proposition 2. 2. 3 part 8, we have that $\tau_\delta - int(B_{f_\sigma}) \cap A_{f_\sigma} \subseteq (\tau_\delta - int(B_{f_\sigma}) \cap A)_{f_\sigma}$. By take the interior, of two said we have that, $\tau_\delta - int(\tau_\delta - int(B_{f_\sigma})) \cap \tau_\delta - int(A_{f_\sigma}) = \tau_\delta - int(B_{f_\sigma}) \cap \tau_\delta - int(A_{f_\sigma}) \subseteq \tau_\delta - int((\tau_\delta - int(B_{f_\sigma}) \cap A)_{f_\sigma})$. By (1), we get $\tau_\delta - int(B_{f_\sigma}) \cap \tau_\delta - int(A_{f_\sigma}) \subseteq \tau_\delta - int((\tau_\delta - int(A \cap B)_{f_\sigma})_{f_\sigma})$------- (2).

But $(\tau_\delta - int(A \cap B)_{f_\sigma})_{f_\sigma} \subseteq ((A \cap B)_{f_\sigma})_{f_\sigma} \subseteq (A \cap B)_{f_\sigma}$.

Thus $\tau_\delta - int(\tau_\delta - int(A \cap B)_{f_\sigma})_{f_\sigma} \subseteq \tau_\delta - int(A \cap B)_{f_\sigma}$-------- (3).

By (2) and (3) we have $\tau_\delta - int(A_{f_\sigma}) \cap \tau_\delta - int(B_{f_\sigma}) \subset \tau_\delta - int(A \cap B)_{f_\sigma}$. Also, evident $\tau_\delta - int(A_{f_\sigma}) \cap \tau_\delta - int(B_{f_\sigma}) \supset \tau_\delta - int((A \cap B)_{f_\sigma})$. Hence $\tau_\delta - int(A_{f_\sigma}) \cap \tau_\delta - int(B_{f_\sigma}) = \tau_\delta - int(A \cap B)_{f_\sigma}$ ■

Corollary 2. 2. 22: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. If A or B is open set, then

1. $\tau_\delta - int((A_{t_\sigma} \cap B)_{f_\sigma}) = \tau_\delta - int((A_{t_\sigma})_{f_\sigma}) \cap \tau_\delta - int(B_{f_\sigma})$.
2. $\tau_\delta - int((A_{t_\sigma})_{f_\sigma}) \cap \tau_\delta - int((B_{t_\sigma})_{f_\sigma}) = \tau_\delta - int((A \cap B)_{t_\sigma})_{f_\sigma})$.

Proof.

1) By Proposition 2. 2. 21, $\tau_\delta - int((A_{t_\sigma} \cap B)_{f_\sigma}) = \tau_\delta - int(A_{t_\sigma})_{f_\sigma} \cap \tau_\delta - int(B_{f_\sigma})$.

2) Directly by Proposition 2. 1. 3 and Proposition 2. 2. 21, we have that, $\tau_\delta - int((A \cap B)_{t_\sigma})_{f_\sigma}) = \tau_\delta - int(((A_{t_\sigma}) \cap (B_{t_\sigma}))_{f_\sigma})$

$= \tau_\delta - int(A_{t_\sigma})_{f_\sigma} \cap \tau_\delta - int(B_{t_\sigma})_{f_\sigma}$■

Proposition 2. 2. 23: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. P is a nonempty subset of X. Then

1. If $P_1 \subset P_2$, then $(P_{1_{f_\sigma}})_{t_\sigma} \subset (P_{2_{f_\sigma}})_{t_\sigma}$, and $(P_{1_{t_\sigma}})_{f_\sigma} \subset (P_{2_{t_\sigma}})_{f_\sigma}$.
2. $\left(P_{t_\sigma}\right)_{f_\sigma} = X - (X - P)_{f_{\sigma_{t_\sigma}}}$.
3. $\left(P_{f_\sigma}\right)_{t_\sigma} = X - (X - P)_{t_{\sigma_{f_\sigma}}}$.
4. $X - \left(P_{t_\sigma}\right)_{f_\sigma} = (X - P)_{f_{\sigma_{t_\sigma}}}$.
5. $X - \left(P_{f_\sigma}\right)_{t_\sigma} = (X - P)_{t_{\sigma_{f_\sigma}}}$.

Proof.

1) It is an immediate consequence of Proposition 2.1.3 and 2.2.3.

2) $\left(P_{t_\sigma}\right)_{f_\sigma} = X - (X - P_{t_\sigma})_{t_\sigma} = X - (X - (X - (X - P)_{f_\sigma}))_{t_\sigma}$

$= X - ((X - P)_{f_\sigma})_{t_\sigma}$.

3) $\left(P_{f_\sigma}\right)_{t_\sigma} = X - (X - P_{f_\sigma})_{f_\sigma} = X - (X - (X - (X - P)_{t_\sigma}))_{f_\sigma}$

$= X - ((X - P)_{t_\sigma})_{f_\sigma}$.

4) By part 2, $(X - P)_{t_{\sigma_{f_\sigma}}} = X - ((X - X - P)_{f_\sigma})_{t_\sigma} = X - \left(P_{f_\sigma}\right)_{t_\sigma}$.

5) By part 3, $(X - P)_{f_{\sigma_{t_\sigma}}} = X - ((X - X - P)_{t_\sigma})_{f_\sigma}. = X - \left(P_{t_\sigma}\right)_{f_\sigma}$■

Proposition 2. 2. 24: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. Y is a nonempty subset of X, and $P \subset Y$. Then $P_{f_\sigma}(\tau_Y) \subset P_{f_\sigma}(\tau)$.

Proof.

Let $x \in P_{f_\sigma}(\tau_Y)$. Then for every $\mathcal{U} \in \tau_Y(x)$, $(\mathcal{U} \cap P)\delta C$ for every $C \in$ σ. Since $\mathcal{U} \in \tau_Y(x)$, there exists $\mathcal{V} \in \tau(x)$ such that $\mathcal{V} \cap Y = \mathcal{U}$, that is, $\mathcal{U} \subset \mathcal{V}$, by Theorem 1. 1. 7, for every $\mathcal{V} \in \tau(x)$, $(\mathcal{V} \cap P)\delta C$ for every $C \in$ σ. Thus $x \in P_{f_\sigma}(\tau)$.

Definition 2. 3. 1: Let $(X,\delta,\tau_\delta,\sigma)$ be a cluster topological proximity space. P nonempty subsets of X. Then

1. P is called $f_\sigma -$ set if and only if $P = (P_{f_\sigma})_{t_\sigma}$
2. P is called $f_{t\sigma} - set$ if and only if $P \subseteq (P_{f_\sigma})_{t_\sigma}$
3. P is called $t_{f\sigma} -$ set if and only if $P \subseteq (P_{t_\sigma})_{f_\sigma}$.
4. P is called $t_\sigma -$ set if and only if $P = (P_{t_\sigma})_{f_\sigma}$
5. P is called $f_\sigma -$ perfect if and only if $P \subseteq P_{f_\sigma}$.

Example 2. 3. 2: Let $X = \{1,2,3,4,5\}$, δ is proximity define by: $A\delta B$ $\forall A \neq \emptyset, B \neq \emptyset$ except the relation:

$\{1\}\overline{\delta}\{3\}, \{1\}\overline{\delta}\{4\}, \{1\}\overline{\delta}\{5\}, \{2\}\overline{\delta}\{3\}, \{2\}\overline{\delta}\{4\}, \{2\}\overline{\delta}\{5\}, \{1\}\overline{\delta}\{3,4\},$ $\{1\}\overline{\delta}\{4,5\}, \{1\}\overline{\delta}\{3,5\}, \{1\}\overline{\delta}\{3,4,5\}, \{2\}\overline{\delta}\{3,4\}, \{2\}\overline{\delta}\{4,5\}, \{2\}\overline{\delta}\{3,5\},$ $\{2\}\overline{\delta}\{3,4,5\}, \{1,2\}\overline{\delta}\{3,4\}, \{1,2\}\overline{\delta}\{4,5\}, \{1,2\}\overline{\delta}\{3,5\}, \{1,2\}\overline{\delta}\{3,4,5\},$ $\{1,2\}\overline{\delta}\{3\}, \{1,2\}\overline{\delta}\{4\}, \{1,2\}\overline{\delta}\{5\}$, and $\{3\}\overline{\delta}\{1\}, \{4\}\overline{\delta}\{1\}, \{5\}\overline{\delta}\{1\},$ $\{3\}\overline{\delta}\{2\}, \{4\}\overline{\delta}\{2\}, \{5\}\overline{\delta}\{2\}, \{3,4\}\overline{\delta}\{1\}, \{4,5\}\overline{\delta}\{1\}, \{3,5\}\overline{\delta}\{1\},$ $\{3,4,5\}\overline{\delta}\{1\}, \{3,4\}\overline{\delta}\{2\}, \{4,5\}\overline{\delta}\{2\}, \{3,5\}\overline{\delta}\{2\}, \{3,4,5\}\overline{\delta}\{2\}, \{3,4\}\overline{\delta}\{1,2\},$ $\{4,5\}\overline{\delta}\{1,2\}, \{3,5\}\overline{\delta}\{1,2\}, \{3,4,5\}\overline{\delta}\{1,2\}, \{3\}\overline{\delta}\{1,2\}, \{4\}\overline{\delta}\{1,2\}, \{5\}\overline{\delta}\{1,2\}.$

Then $\tau_\delta = \{X,\emptyset,\{1,2\},\{3,4,5\}\}$. And $\sigma = \sigma_1 \cup \sigma_2$ is cluster define on X. Then $\{3,4,5\}$ is $f_\sigma -$ set because $(\{3,4,5\}_{f_\sigma})_{t_\sigma} = \{3,4,5\}$, and $\{1,2\}$ and $\emptyset$ are $t_\sigma -$ set because $(\{1,2\}_{t_\sigma})_{f_\sigma} = \{1,2\}$, and $(\emptyset_{t_\sigma})_{f_\sigma} = \emptyset$. $\{1,3,4\}$ is $f_{t\sigma} - set$ because $(\{1,3,4\}_{f_\sigma})_{t_\sigma} = X$, And $\{2\}$ is a $f_\sigma -$ perfect because $\{2\}_{f_\sigma} = \{1,2\}$.

It is to be noted that, according to earlier results,

- Every $f_\sigma -$ set is $f_{t\sigma} - set$ and every $t_\sigma -$ set is $t_{f\sigma} - set$.
- Every $t_\sigma -$ set is a $\tau_\delta -$closed set. Because P_{f_σ} is $\tau_\delta -$closed set.

- $\emptyset$ and X are not always $t_\sigma -$ set and $f_\sigma -$ set.
- If δ is an indiscrete proximity, then $\emptyset$ and X are $t_\sigma -$ set and $f_\sigma -$ set, because $X_{f_\sigma} = X$ and $\emptyset_{t_\sigma} = \emptyset$.

Definition 2. 3. 3: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. P nonempty subsets of X. Then P is called *scant set* if and only if $(P_{f_\sigma})_{t_\sigma} = \emptyset$.

For example, let $X = \{1,2,3\}$, δ is a discrete proximity, $\sigma = \{\{1\}, \{1,2\}, \{1,3\}, X\}$, and $\tau = \{X, \emptyset, \{1\}, \{1,2\}\}$. If $A = \{2,3\}$, $K = \{3\}$, then A and K are scant sets.

Note 2. 3. 4: Let $(X, \delta, \sigma, \tau_\sigma)$is $\sigma -$ Topological Proximity Space. If δ is non-indiscrete proximity, then we cannot get subset of X is scant set. That is clear by Remarks 2. 1. 5 part 3.

Remark 2. 3. 5: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space. If δ is non-indiscrete proximity, then we cannot get subset of X is a scant set.

Proof.

If possible $(P_{f_\sigma})_{t_\sigma} = \emptyset$, then for every $x \in X$ and every $\mathcal{U} \in \tau_\delta(x)$, $(\mathcal{U} \cap (X - P_{f_\sigma}))\delta C$ for every $C \in \sigma$, that is true for every $\mathcal{U} \in \tau_\delta$. By axiom [C3], it follows that $\mathcal{U} \in \sigma$. By Proposition 1. 2. 12, $(X - \mathcal{U}) \in \tau_\delta$, thus $(X - \mathcal{U}) \in \sigma$. By axiom [C1], $\mathcal{U}\delta(X - \mathcal{U})$, But by axiom [P4], $\{x\}\delta(X - \mathcal{U})$, for some $x \in \mathcal{U}$ which is a contradiction because $\mathcal{U}$ is $\tau_\delta -$open. Hence $(P_{f_\sigma})_{t_\sigma} \neq \emptyset$■

Proposition 2. 3. 6: Let $(X,\delta,\tau_\delta,\sigma)$ be a cluster topological proximity space. If every nonempty open set is a member on cluster, then $(\mathcal{U}_{f_\sigma})_{t_\sigma} = X$.

Proof.

If possible there exist $x \notin (\mathcal{U}_{f_\sigma})_{t_\sigma}$, then there exist $\mathcal{V} \in \tau_\delta(x)$ such that $(\mathcal{V} \cap (\mathrm{X} - \mathcal{U}_{f_\sigma}))\delta C$ for every $C \in \sigma$, that is, $(\mathcal{V} \cap \mathrm{X} - \mathcal{U}_{f_\sigma}) \neq \emptyset$, thus there exist $y \in \mathcal{V}$ and $y \in X - \mathcal{U}_{f_\sigma} \Rightarrow \mathrm{y} \notin \mathcal{U}_{f_\sigma}$this implies, there exists $\mathcal{W} \in \tau_\delta(\mathrm{y})$ such that $(\mathcal{W} \cap \mathcal{U})\overline{\delta}C$ for some $C \in \sigma$ this mean $(\mathcal{W} \cap \mathcal{U}) \notin \sigma$ this contradiction with hypothesis because $(\mathcal{W} \cap \mathcal{U}) \in \tau_\delta$■

Note that, in case of τ_σ, Proposition 2. 3. 6, is true only δ is an indiscrete proximity, because $\tau_\sigma \cap \sigma = \{X\}$.

Proposition 2. 3. 7: Let $(X,\delta,\tau_\delta,\sigma)$ be a cluster topological proximity space. If P is $f_{t\sigma} - set$ and $\tau_\delta - int(P) = P_{t_\sigma}$, then $(P_{f_\sigma})_{t_\sigma} = (((P_{f_\sigma})_{t_\sigma})_{f_\sigma})_{t_\sigma}$.

Proof.

Let us assume that P is $f_{t\sigma} - set$. Then $(P_{f_\sigma})_{t_\sigma} \subseteq (((P_{f_\sigma})_{t_\sigma})_{f_\sigma})_{t_\sigma}$……(1)
Since $P_{t_\sigma} = \tau_\delta - int(P) \subseteq P$, $(P_{f_\sigma})_{t_\sigma} = \tau_\delta - int(P_{f_\sigma}) \subseteq P_{f_\sigma}$. This implies to $((P_{f_\sigma})_{t_\sigma})_{f_\sigma} \subseteq (P_{f_\sigma})_{f_\sigma} \subseteq P_{f_\sigma}$ and so that $(((P_{f_\sigma})_{t_\sigma})_{f_\sigma})_{t_\sigma} \subseteq (P_{f_\sigma})_{t_\sigma}$ …….(2). By (1) and (2) we get $(P_{f_\sigma})_{t_\sigma} = (((P_{f_\sigma})_{t_\sigma})_{f_\sigma})_{t_\sigma}$■

Corollary 2. 3. 8: Let $(X,\delta,\tau_\delta,\sigma)$ be a cluster topological proximity space, and P is a nonempty subset of X. Then

1. $f_{t\sigma} - set$ of $f_{t\sigma} - set$ is also $f_{t\sigma} - set$

2. Every $f_\sigma -$ set is $\tau_\delta -$open set.
3. If P is $f_{t\sigma}$-set and $\tau_\delta - int(P) \cap \left(P_{f_\sigma}\right)_{t_\sigma} = \emptyset$, then $\tau_\delta - int(P) = \emptyset$.
4. Every subset of the scant set is a scant set.

Proof.

1) It is an immediate consequence of Proposition 2. 3. 7.

2) It is an immediate consequence of Proposition 2. 1. 3 and Note 2.1.7, because takeoff set is $\tau_\delta -$open set.

3) Let P is $f_{t\sigma} -\ set$. Then $P \subseteq (P_{f_\sigma})_{t_\sigma}$. Since $\tau_\delta - int(P) \subset P$ and $P \neq \emptyset, \tau_\delta - int(P) \subset \left(P_{f_\sigma}\right)_{t_\sigma}$. Hence $\tau_\delta - int(P) = \emptyset$.

4**)** Let P be a scant set and A subset of P. Then by Proposition 2. 2. 23 part 1, $(A_{f_\sigma})_{t_\sigma} \subseteq (P_{f_\sigma})_{t_\sigma} = \emptyset$. Thus A is a scant set■

Remark 2. 3. 9: P is $f_\sigma -$ set if and only if the complement of P is $t_\sigma -$set.

Proof.

P is $f_\sigma -$set if and only if $P = (P_{f_\sigma})_{t_\sigma}$ if and only if $X - P = X - (P_{f_\sigma})_{t_\sigma}$ if and only if $X - P = (X - P)_{t_{\sigma_{f_\sigma}}}$ if and only if $(X - P)$ is $t_\sigma -$ set■

Remark 2. 3. 10: Let $(X, \delta, \tau_\delta, \sigma)$ be a cluster topological proximity space:

1. The intersection of two $f_\sigma -$ sets is $f_\sigma -$ set.

2. The union of two $t_\sigma -$ sets is $t_\sigma -$ set.

Proof.

1) By Proposition 2. 1. 3 and 2. 2. 3, $(\left((A\cap B)_{f_\sigma}\right)_{t_\sigma} \subset \left(A_{f_\sigma}\right)_{t_\sigma} \cap \left(B_{f_\sigma}\right)_{t_\sigma} = A\cap B$ ……….. (1)

Now, since $\left(A_{f_\sigma}\right)_{t_\sigma}$ is an open, $\left(A_{f_\sigma}\right)_{t_\sigma} \cap B_{f_\sigma} \subset (\left(A_{f_\sigma}\right)_{t_\sigma} \cap B)_{f_\sigma}$, $\left(A_{f_\sigma}\right)_{t_\sigma} \cap \left(B_{f_\sigma}\right)_{t_\sigma} \subset ((\left(A_{f_\sigma}\right)_{t_\sigma})_{t_\sigma} \cap (B_{f_\sigma})_{t_\sigma} \subset ((\left(A_{f_\sigma}\right)_{t_\sigma} \cap B)_{f_\sigma})_{t_\sigma}$, but A and B are $f_\sigma -$ sets , hence $A\cap B \subset ((A\cap B)_{f_\sigma})_{t_\sigma}$………(2). By (1) and (2) the proof is done.

2) Let A and B are $\mathrm{t}_\sigma -$ sets. Then $\mathrm{X}-\mathrm{A}$ and $\mathrm{X}-\mathrm{B}$ are $f_\sigma -$ sets. By part (1) $\mathrm{X}-\mathrm{A}\cap \mathrm{X}-\mathrm{B}$ $is\ f_\sigma -$ set, hence $X - (A\cup B) = (X-(A\cup B))_{f_\sigma t_\sigma}$. So $A\cup B = X - ((X-(A\cup B))_{f_\sigma})_{t_\sigma} = (X - X - ((A\cup B))_{t_\sigma})_{f_\sigma} = ((A\cup B))_{t_\sigma})_{f_\sigma}$this mean $A\cup B\ is\ t_\sigma -$ set■

3.1 Bushy Set

For the role played by resolvable and irresolvable spaces [22, 23, 24, 25] in mathematical programming and solving mathematical systems, many researchers have shown interest in studying these spaces and focused on finding highly connected and inseparable spaces within different spaces. In this section, we will present a concept parallel of resolvable and irresolvable spaces, which we called the dismountable and non-dismountable space, which was built, based on the concept of the bushy set in the proximity cluster topological space, and it was studied on some of the topology concepts. We will recall some of the basic definitions needed in this work.

Definition 3. 1. 1: [22] Let (X, τ) be a topological space. Then the space is an irresolvable space if and only if X cannot contains two disjoint dense subsets. Otherwise X is resolvable space.

Definition 3. 1. 2: [24] Let (X, τ) be a topological space. Then the space is a submaximal space if and only if every dense set is open set.

Definition 3. 1. 3: [24] Let (X, τ) be a topological space. Then the space is hyper connected space if and only if every open set is dense set.

Definition 3. 1. 4: [26] Let (X, τ) be a topological space. The space is called door space if and only if every subset is open or closed.

Definition 3. 1. 5: Let $(\mathrm{X}, \tau, \delta, \sigma)$ be a proximity cluster topological space. A is a nonempty subset of X is called *Bushy set* denoted by $\mathscr{b}(A)$ if and only if every $\mathcal{U} \in \tau$, $(\mathcal{U} \cap A)\delta C$ for every $C \in \sigma$.

In other words A is a bushy set if and only if for every $x \in X$, x is a follower point of A. The family of all bushy set on X denoted by $\mathcal{B}(X, \delta)$.

Example 3. 1. 6: Let $X = \{1,2,3\}$, δ be a discrete proximity , $\sigma = \{\{1\}, \{1,2\}, \{1,3\}, X\}$, and let $\tau = \{X, \emptyset, \{1\}, \{1,2\}\}$. Let $A = \{1,3\}$ *and* $B = \{3\}$. Then A is a bushy set because every $\mathcal{U} \in \tau$, $(\mathcal{U} \cap A)\delta C$ for every $C \in \sigma$. B is not bushy set because $\{1,2\} \in \tau$ but $(\{1,2\} \cap \{3\})\overline{\delta}C$ for every $C \in \sigma$.

Example 3. 1. 7: Let $X = \{1,2,3\}$, δ be a indiscrete proximity , then $\sigma = \{\{1\}, \{1,2\}, \{1,3\}, \{2\}, \{3\}, \{2, 3\}, X\}$, and let $\tau_\delta = \{X, \emptyset\}$. Let $A = \{1,3\}$ *and* $B = \{3\}$. Then A and B are bushy sets because every $\mathcal{U} \in \tau$, $(\mathcal{U} \cap A)\delta C$ and $(\mathcal{U} \cap B)\delta C$ for every $C \in \sigma$. Moreover A is a bushy set for every $A \neq \emptyset$.

Example 3. 1. 8: Let $X = \{1,2,3\}$, δ define by: $A\delta B \Leftrightarrow A \neq \emptyset$ and $B \neq \emptyset$ except: $\{3\}\overline{\delta}\{1,2\}$, $\{3\}\overline{\delta}\{1\}$, $\{3\}\overline{\delta}\{2\}$, $\{1,2\}\overline{\delta}\{3\}$, $\{1\}\overline{\delta}\{3\}$, $\{2\}\overline{\delta}\{3\}$.
Thus $\tau_\delta = \{X, \emptyset, \{3\}, \{1,2\}\}$, and $\sigma = \sigma_1 \cup \sigma_2 =$
$\{\{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$, or $\sigma = \{\{3\}, \{1,3\}, \{2, 3\}, X\}$. Evident $(X, \delta, \tau_\delta, \sigma)$ has not contain bushy set.

Note that by Examples 3. 1. 7 and 3. 1. 8 the space generated from the indiscrete proximity has a bushy set, but if change the proximity to any other proximity, it is difficult to obtain the bushy set in the finite space. The reason is that the topology generated by proximity, each τ_δ −open set is τ_δ −closed set, hence it is difficult to obtain the bushy set in this space.

Due to the difficulty of finding the bushy set in this space, we use the classical topology to obtain the bushy set necessary to achieve most of the topological properties.

Proposition 3. 1. 9: Let$(X, \delta, \tau_\delta, \sigma)$ be a cluster proximity topological space. If δ is non-indiscrete proximity, then the space has no bushy set.

Proof.

If possible there exists subset A of X is a bushy set. Then every $\mathcal{U} \in \tau_\delta, (\mathcal{U} \cap A)\delta C$ for every $C \in \sigma$. By axiom [C3],$\mathcal{U} \in \sigma$ for every $\mathcal{U} \in \tau_\delta$. By Proposition 1. 2. 12, $(X - \mathcal{U}) \in \tau_\delta$, thus $(X - \mathcal{U}) \in \sigma$. By axiom [C1], $\mathcal{U}\delta(X - \mathcal{U})$ which is a contradiction with Proposition 1. 2. 8 because $\mathcal{U}$ is $\tau_\delta - open$. Hence A is not bushy set■

Proposition 3. 1. 10: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. Then:

1. Every super set of the bushy set is a bushy set.
2. Finite union of bushy sets is also a bushy set.
3. Every bushy set of X is a member of cluster. In other word $\mathcal{B}(X, \delta)$ is subset of cluster.

Proof.

1) Suppose that A is a bushy set and $A \subseteq B$. Then every $\mathcal{U} \in \tau, (\mathcal{U} \cap A)\delta C$ for every $C \in \sigma$. Since $A \subseteq B$, by Theorem 1. 1. 7 part 2 we have that, $(\mathcal{U} \cap B)\delta C$ for every $C \in \sigma$. Hence A is bushy set.

2) Suppose that A and B are bushy sets. Then every $\mathcal{U} \in \tau, (\mathcal{U} \cap A)\delta C$ and $(\mathcal{U} \cap B)\delta C$ for every $C \in \sigma$. By axiom [P4] we have that $((\mathcal{U} \cap A) \cup$

$(\mathcal{U} \cap B))\delta C$ for every $C \in \sigma$, that is, $(\mathcal{U} \cap (A \cup B))\delta C$ for every $C \in \sigma$. Hence $A \cup B$ is a bushy set.

3) Suppose that A is a bushy set. Then every $\mathcal{U} \in \tau, (\mathcal{U} \cap A)\delta C$ for every $C \in \sigma$. By axiom [C3] and Proposition 1. 1. 15 part 2, we have that $(\mathcal{U} \cap A) \in \sigma$, and so that $A \in \sigma$■

Remark 3. 1. 11: If δ define of X is a discrete proximity, then $\bigcap \mathcal{B}(X, \delta) \neq \emptyset$.

Proof.

If possible $\bigcap \mathcal{B}(X, \delta) = \emptyset$, then there exists $A \in \mathcal{B}(X, \delta)$ such that $A \cap B = \emptyset$ for some $B \in \mathcal{B}(X, \delta)$. But A and B are bushy set by Proposition 3. 1. 10 part 3 $A, B \in \sigma$ this means $A\delta B$ which is contradiction because δ is a discrete proximity but $A \cap B = \emptyset$. Hence $\bigcap \mathcal{B}(X, \delta) \neq \emptyset$■

Lemma 3. 1. 12: If the space has at least one subset of X is a bushy set, then X is a bushy set.

Proof.

Evident by Proposition 3. 1. 10 part1■

Proposition 3. 1. 13: Let $(X, \delta, \sigma, \tau_\sigma)$ be a σ −Topological Proximity space. Then

$$\mathcal{B}(X, \delta) = \begin{cases} \mathcal{P}(X) / \emptyset & if \delta \text{ is indiscrete} \\ \emptyset & otherwise \end{cases}.$$

Proof.

Case one: If δ is indiscrete proximity, then $\tau_\sigma = \{X, \emptyset\}$, and $\sigma = \{A \subseteq X; A \neq \emptyset\}$. Then only nonempty $\tau_\sigma - open$ set is X for every nonempty subset A of X, that is $(X \cap A)\delta C$ for every $C \in \sigma$. Hence $\mathcal{B}(X, \delta) = \mathcal{P}(X) / \emptyset$.

Case two: If possible there exists nonempty subset A of X is $bushy$ set, then every proper $\mathcal{U} \in \tau_\sigma$, $(\mathcal{U} \cap A)\delta C$ for every $C \in \sigma$, by axiom [C3] $\mathcal{U} \cap A \in \sigma$ and by Proposition 1. 1. 15 part 2, $\mathcal{U} \in \sigma$ which is a contradiction with $\tau_\sigma \cap \sigma = \{X\}$. Hence $\mathcal{B}(X, \delta) = \emptyset$■

Remark 3. 1. 14: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. If A is a bushy set, then A is a dense set.

Proof.

Let A be a bushy set. Then every $\mathcal{U} \in \tau$, $(\mathcal{U} \cap A)\delta C$ for every $C \in \sigma$, that is, $\mathcal{U} \cap A \neq \emptyset$ for every $\mathcal{U} \in \tau$. Hence A is a dense set■

Remark 3. 1. 15: Let $(X, \tau_1, \delta, \sigma)$ and $(X, \tau_2, \delta, \sigma)$be a two proximity cluster topological spaces. If $\tau_1 \subseteq \tau_2$ and A be $\tau_2 - bushy$ set, then A is $\tau_1 - bushy$ set.

Proof.

Let A is $\tau_2 - bushy$ set. Then for every $\mathcal{U} \in \tau_2$, $(\mathcal{U} \cap A)\delta C$ for every $C \in \sigma$. Since $\tau_1 \subseteq \tau_2$, $(\mathcal{V} \cap A)\delta C$ for every $\mathcal{V} \in \tau_1$. Hence A is $\tau_1 - bushy$ set■

Definition 3. 1. 16: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. Then the space is called $Bushy\ space$ if and only if every bushy subset of X is open with respect to τ.

Note 3. 1. 17: The complement of every bushy set in $Bushy\ space$ is a closed set.

Note 3. 1. 18: Every $Bushy\ space$ is a submaximal space.
That is clear because every bushy set is a dense set.

Proposition 3. 1. 19: If $(\mathrm{X}, \tau, \delta, \sigma)$ be a $Bushy\ space$ has at least one subset of X is bushy set, then every member of cluster σ is a *Bushy relative subspace*.

Proof.
Assuming that X is a $Bushy\ space$, $Y \subseteq X$, and $Y \in \sigma$. Let $A \subseteq Y$ be a bushy set in Y. Then for every $\mathcal{V} \in \tau_Y$, $(\mathcal{V} \cap A)\delta C$ for every $C \in \sigma_Y$, that is, $A_{f_\sigma} = Y$. It is sufficient to show that A is τ_Y −open such that $A = \mathcal{U} \cap Y$ for some $\mathcal{U} \in \tau$.
Let $\mathcal{U} = A \cup (X - Y) \Rightarrow \mathcal{U}_{f_\sigma} = (A \cup (X - Y))_{f_\sigma} = A_{f_\sigma} \cup (X - Y)_{f_\sigma}$.
According to the space has at least one bushy set we have that $X = X_{f_\sigma} = (Y \cup (X - Y))_{f_\sigma} = Y_{f_\sigma} \cup (X - Y)_{f_\sigma} =$
$(A_{f_\sigma})_{f_\sigma} \cup (X - Y)_{f_\sigma} \subseteq A_{f_\sigma} \cup (X - Y)_{f_\sigma} = \mathcal{U}_{f_\sigma}$, hence $\mathcal{U}$ is bushy set in X, and from X is $Bushy\ space$ we have that $\mathcal{U}$ is open set in X.
So, $\mathcal{U} \cap Y = \big(A \cup (X - Y)\big) \cap Y = A \cap Y = A$ that is, A is τ_Y −open in Y i.e., Y is *Bushy relative subspace*■

The converse don't always be true, Example 3. 1. 20 explain that.

Example 3. 1. 20: Let $X = \{1,2,3\}$, δ be a discrete proximity , $\sigma = \big\{\{1\}, \{1,2\}, \{1,3\}, X\big\}$, and let $\tau = \{X, \emptyset, \{1\}, \{1,2\}\}$.

Let $A = \{1,3\}$ be a subset of X. Then, $\tau_A = \{A, \emptyset, \{1\}\}$, and $\sigma_A = \{\{1\}, A\}$, we can see that, $\{1\} \subset A$ is τ_A −bushy set and τ_A −open, also $A = \{1,3\}$ is τ_A −bushy and τ_A −open , hence A is *Bushy subspace*. But $\{1,3\}$is subset of X is a bushy set of X but not open, thus X is not *Bushy space*.

Proposition 3. 1. 21: If every member of cluster is open ($\sigma \subset \tau$), then X is a $Bushy\ space$.

Proof.

Let us suppose that A is bushy set in X. Then by Proposition 3. 1. 10 part 3 every bushy set of X is member of cluster, that is, $A \in \sigma$. Since every member of cluster is open it follows that A is open set and so X is $Bushy\ space$■

Proposition 3. 1. 22: The intersection of finite family of bushy sets in $Bushy\ space$ is a bushy set.

Proof.

Let $\{A_i\}_{i=1}^{n}$ be a family of bushy set. Since X is a $Bushy\ space$ then A_i is an open set for every $i \in n$. We prove that by Induction Law. First, to prove the result is true when $i = 2$. Let A_1 and A_2 are bushy sets. Then for every $\mathcal{U} \in \tau$, $(\mathcal{U} \cap A_1)\delta C$ for every $C \in \sigma$. Since X is a $Bushy\ space$, $A_1 \in \tau$, that is $\mathcal{U} \cap A_1$is nonempty open set. Since A_2 is bushy set then for every $(\mathcal{U} \cap A_1) \in \tau$, $(\mathcal{U} \cap (A_1 \cap A_2))\delta C$ for every $C \in \sigma$, hence $A_1 \cap A_2$ is a bushy set. Second, suppose the result is true for $i = n - 1$.

Finally, to prove the result is true for $i = n$. Let A_n is bushy set. Then for every nonempty $\mathcal{U} \in \tau$, $(\mathcal{U} \cap A_n)\delta C$ for every $C \in \sigma$. But $\{\cap A_i\}_{i=1}^{n-1}$ is nonempty open set this implies that $(\mathcal{U} \cap (\{\cap A_i\}_{i=1}^{n-1} \cap A_n))\delta C$ for every $C \in \sigma$. Hence $\{\cap A_i\}_{i=1}^{n}$ is a bushy set■

Corollary 3. 1. 23: The intersection of finite family of dense sets in $Bushy\ space$ is a dense set.

Proof.

By Remark 3. 1. 14 every bushy set is dense set. Then by Proposition 3. 1. 22 the intersection of finite family of dense sets in $Bushy\ space$ is a dense set■

Corollary 3. 1. 24: If A is a bushy set in $Bushy\ space\ X$, then A builds a chain of open and dense subsets of X.

Proof.

This is an immediate consequence of Proposition 3. 1. 10 part 1, Remark 3. 1. 14 and Definition 3. 1. 16■

Proposition 3. 1. 25: If X is a $Bushy\ space$ and $\{A_i \subseteq X, i \in N\}$ is a finite closed cover of X, then $int(A_k) \neq \emptyset$ for some $k \in N$.

Proof.

Suppose $X = \bigcup_{i=1}^{n}(A_i)$. By De-Morgan's Law, $\emptyset = \bigcap_{i=1}^{n}(X - A_i)$. Since X is a $Bushy\ space$ and $X - A_i$ is an open set for every i, clear that by Corollary 3. 1. 23 not every $X - A_i$ is dense set, hence there exist at least one $X - A_k$ is not dense set. Thus there exists nonempty open set $\mathcal{U}$ such that $\mathcal{U} \cap (X - A_k) = \emptyset$ that is $\mathcal{U} \subseteq A_k$ for at least one k, hence $int(A_k) \neq \emptyset$■

Proposition 3. 1. 26: Let f be an open and δ −continuous proximity mapping from $(\mathrm{X}, \tau, \delta_x, \sigma_X)$ into$(\mathrm{Y}, \rho, \delta_Y, \sigma_Y)$. If A is a bushy set in X, then $f(A)$is bushy set in Y.

Proof.

Suppose A is a bushy set in X. Then every $\mathcal{U} \in \tau$, $(\mathcal{U} \cap A)\delta_x C$ for every $C \in \sigma_X$. Since f is δ −continuous, $f(\mathcal{U} \cap A)\delta_y f(C)$. By Theorem 1. 1. 7 part 2 $(f(\mathcal{U}) \cap f(A))\delta_y f(C)$ for every$f(\ C) \in f(\sigma_X)$. Since f is open and δ −continuous, $f(\mathcal{U}) \in \rho$ and by Theorem 1. 1. 25 $f(\sigma_X)$ is cluster in Y. Thus $f(A)$ is bushy set in Y■

Proposition 3. 1. 27: Let f be an injective open δ −continuous mapping from $(\mathrm{X}, \tau, \delta_x, \sigma_X)$ into $\left(\mathrm{Y}, \rho, \delta_y, \sigma_Y\right)$. If Y is $Bushy\ space$, then X is also $Bushy\ space$.

Proof.

Let $A \subseteq X$ is bushy set in X. By Proposition 3. 1. 26, $f(A)$ is also bushy set in Y. Since Y is $Bushy\ space$, $f(A)$ is ρ −open. Further, f is injective continuous this implies $f^{-1}(f(A)) = A$ is τ −open, hence X is $Bushy\ space$■

Definition 3. 1. 28: The space $(\mathrm{X}, \tau, \delta, \sigma)$ is called $Attached\ space$ if and only if every nonempty open subset of X is a bushy set.

Example 3. 1. 29: Let $X = \{1,2,3\}$, δ is a discrete proximity, $\sigma = \{\{3\},\{1,3\},\{2,3\},X\}$. Let $\tau = \{X,\emptyset,\{3\},\{2,3\}\}$. Then $\{2,3\},\{3\}$ are open and bushy sets, hence X is attached space.

It is to be noted that, according to earlier results:

- If (X,τ) is a discrete topology, δ is any proximity relative on X, and σ any cluster define on (X,δ), then X is not attached space. Because every singleton set is an open set but not necessary is a bushy set.
- Also can be note that, attached space cannot be separated by two open nonempty subsets.

Because if possible X can be separated by two nonempty open subsets A and B such that $A \cap B = \emptyset$, $A \cup B = X$, that is, $A = X - B$. Since A is a bushy set, A is a dense set, hence $cl(A) = X$, thus $B = int(B) = X - cl(X - B) = X - cl(A) = X - X = \emptyset$, which is a contradiction.

In addition, it concludes from the above the following proposition:

Proposition 3. 1. 30: Let (X,τ,δ,σ) be a proximity cluster topological space, the following statement are equivalent:

1- X is attached space.

2- $\tau/\{\emptyset\} \subseteq \sigma$.

Proof.

$1 \Rightarrow 2$ Let us assume X is attached space. Then every nonempty open subset is a bushy set. Let us consider G is nonempty open set. So it's a bushy, that is, every $\mathcal{U} \in \tau$, $(\mathcal{U} \cap G)\delta C$ for every $C \in \sigma$, hence $G \in \sigma$.

$2 \Rightarrow 1$ If possible $G \in \tau$ is not bushy set. Then there exists $\mathcal{U} \in \tau$, $(\mathcal{U} \cap G)\overline{\delta}C$ for some $C \in \sigma$ hence $\mathcal{U} \cap G \notin \sigma$ this is a contradiction with hypothesis. Thus G is bushy set■

Corollary 3. 1. 31: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. If $\mathcal{U} \in \sigma$ for every nonempty $\mathcal{U} \in \tau$, then $\mathcal{U}$ is a bushy set.

Proof.

There could exist $x \notin \mathcal{U}_{f_\sigma}$, then there exists $\mathcal{V} \in \tau$ such that $(\mathcal{V} \cap \mathcal{U})\overline{\delta}C$ for some $C \in \sigma$. By axiom [C1], $\mathcal{V} \cap \mathcal{U} \notin \sigma$ but $\mathcal{V} \cap \mathcal{U} \in \tau$ this is a contradiction with hypothesis. Thus $\mathcal{U}$ is a bushy set■

Therefore, it can be concluded from the above corollary, if every non-empty open subset of X is a member on cluster, then $\mathcal{U} \subseteq \mathcal{U}_{f_\sigma}$.

The following proposition shows that the space can have the characteristic of being a *Bushy space* and attached space together.

Proposition 3. 1. 32: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space, τ has F. I. P. (finite intersection property) and every member of cluster is open. Then X is a *Bushy space* and attached space.

Proof.

Let us suppose that A is a nonempty open set. By F. I. P. of τ we have that $\emptyset \neq \mathcal{U} \cap A \in \tau$. Since every $C \in \sigma$ is open, $((\mathcal{U} \cap A) \cap C) \in \tau$ and $(\mathcal{U} \cap A) \cap C \neq \emptyset$ by axioms [P3], $(\mathcal{U} \cap A)\delta C$ for every $C \in \sigma$, hence A is bushy set, hence X is an attached space. Evident X is a *Bushy space* by Proposition 3. 1. 21■

Corollary 3. 1. 33: Let $(\mathrm{X}, \tau, \delta, \sigma)$ be a proximity cluster topological space. If τ has F. I. P, then X is a attached space.

Proof.

Obvious by Proposition 3. 1. 32■

Proposition 3. 1. 34: Let $(\mathrm{X}, \tau, \delta, \sigma)$ be a proximity cluster topological space. Then

1. If $\emptyset_t = \emptyset$, then $\emptyset$ is scant set
2. If A is bushy set then $(X - A)$ is a scant set for every $X - A \notin \sigma$.

Proof.

1) Evident.

2) Since $(X - A) \notin \sigma$, by Proposition 2. 2. 3 part 5 $(X - A)_f = \emptyset$, thus $((X - A)_f)_t = X - (X - (X - A)_f)_f = X - (X - \emptyset)_f = X - X_f =$ $X - X = \emptyset$■

It is also clear that the takeoff set is an open set in the space $(\mathrm{X}, \tau, \delta, \sigma)$, thus this set becomes dense when the space carries an "attached" property.

Note 3. 1. 35: Every *Attached space* is hyper connected space.

Remark 3. 1. 36: Let $(\mathrm{X}, \tau, \delta, \sigma)$ be a proximity cluster topological space. If A is bushy set and $\mathcal{U} \in \tau$, then $\mathcal{U} \subseteq \left((\mathcal{U} \cap A)_{f_\sigma}\right)_{t_\sigma}$.

Proof.

Let $\mathcal{U} \in \tau$. By Proposition 2. 2. 3 part 8 we have that $\mathcal{U} \cap A_{f_\sigma} \subseteq (\mathcal{U} \cap A)_{f_\sigma}$, that is, $(\mathcal{U} \cap A_{f_\sigma})_{t_\sigma} \subseteq ((\mathcal{U} \cap A)_{f_\sigma})_{t_\sigma}$. But $(\mathcal{U} \cap A_{f_\sigma})_{t_\sigma} = \mathcal{U}_{t_\sigma} \cap (A_{f_\sigma})_{t_\sigma} = \mathcal{U}_{t_\sigma} \cap (X)_{t_\sigma} = \mathcal{U}_{t_\sigma}$. Thus $\mathcal{U}_{t_\sigma} = (\mathcal{U} \cap A_{f_\sigma})_{t_\sigma}$. By Proposition 2. 1. 3 part 8 we have that $\mathcal{U} \subseteq \mathcal{U}_{t_\sigma}$. Hence $\mathcal{U} \subseteq \left((\mathcal{U} \cap A)_{f_\sigma}\right)_{t_\sigma}$■

Proposition 3. 1. 37: If A is a bushy set, then $A_{t_\sigma}$is also bushy set.

Proof.

If not, there exists $x \notin \left(A_{t_\sigma}\right)_{f_\sigma}$, then there exist $\mathcal{U} \in \tau$, and $C \in \sigma$ such that $(\mathcal{U} \cap A_{t_\sigma})\overline{\delta}C$. Thus $\mathcal{U} \cap A_{t_\sigma} \notin \sigma$ by Proposition 1. 1. 15 part 2 $\left(\mathcal{U} \cap A_{t_\sigma}\right) \cap A \notin \sigma$. But $A_{t_\sigma} \in \tau$ which means that $\mathcal{U} \cap A_{t_\sigma} \in \tau$ and implies that $\left(\mathcal{U} \cap A_{t_\sigma}\right) \cap A\overline{\delta}C$ for some $C \in \sigma$. Hence $A_{f_\sigma} \neq X$. Hence our assumption leads to a contradiction■

3.2 Co-bushy space.

In this section, we introduce the concept of the co-bushy space, through which it is possible to study the characteristics and properties of the topological proximity concepts. The co-bushy space, presented in this study, means the inclusion of non-empty open sets within the cluster family. This further plays a significant role in the take-off points and the follower points.

Definition 3. 2. 1: $(\mathrm{X}, \tau, \delta, \sigma)$ is called co bushy space if and only if there exist at least one subset of X is a bushy set.

We can see that, By Example 3. 1. 7 $(\mathrm{X}, \delta, \tau_\delta, \sigma)$ is co bushy space if δ is indiscrete proximity but by Example 3. 1. 8 $(\mathrm{X}, \delta, \tau_\delta, \sigma)$ is not co bushy space when δ is non-indiscrete proximity.

Example 3. 2. 2: Let $X = \{1, 2, 3, 4\}$, δ is discrete proximity, let $\sigma = \sigma_4$, and let $\tau = \{\mathrm{X}, \emptyset, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}, \{4\}\}$, then $(\mathrm{X}, \tau, \delta, \sigma)$ is co bushy space because {1, 2, 4}is a bushy set.

Lemma 3. 2. 3: If X is co-bushy space, then $X = X_{f_\sigma}$.

Proof.

It is an immediate consequence of proposition 3. 1. 10 part 1■

Proposition 3. 2. 4: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space, if X is co-bushy, then $P_{t_\sigma} \subseteq P_{f_\sigma}$.

Proof.

Let $x \in P_{t_\sigma}$. Then $x \notin (X-P)_{f_\sigma}$, thus there exists $\mathcal{U} \in \tau(x)$, and $C \in \sigma$ such that $(\mathcal{U} \cap (X-P)\ )\overline{\delta}C$. By Theorem 1. 1. 7, for every $\mathcal{V} \in \tau(x)$ we have that $((\mathcal{U} \cap \mathcal{V}) \cap (X-P))\overline{\delta}C$….(1)

Now, if possible $x \notin P_{f_\sigma}$, then there exist $\mathcal{W} \in \tau(x)$, and $C \in \sigma$ such that $(\mathcal{W} \cap P\ )\overline{\delta}C$. By Theorem 1. 1. 7 part 3, $((\mathcal{U} \cap \mathcal{W}) \cap P)\overline{\delta}C$…..(2)

By (1) and (2) we have that $[(\mathcal{U} \cap \mathcal{V}) \cap (X-P)] \cup [(\mathcal{U} \cap \mathcal{W}) \cap P]\overline{\delta}C$. By Theorem 1. 1. 7 part 3, $[(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) \cap (X-P\ )] \cup [(\mathcal{U} \cap \mathcal{W} \cap \mathcal{V}) \cap P]\overline{\delta}C$, that is, $[(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) \cap (X-P\ \cup P)]\overline{\delta}C$, thus $[(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) \cap X]\overline{\delta}C$ i.e. $((\mathcal{U} \cap \mathcal{V} \cap \mathcal{W})\overline{\delta}C$, that is, $(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) \notin \sigma$. By Proposition 2. 2. 3, part 5, $(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W})_{f_\sigma} = \emptyset$, this is a contradiction with Proposition 2. 2. 19 (If $X = X_{f_\sigma}$, then $G \subseteq G_{f_\sigma}$ for every $G \in \tau$) but $(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W})$ is nonempty open set. Hence $P_{t_\sigma} \subseteq P_{f_\sigma}$■

Corollary 3. 2. 5: Let $(X, \tau, \delta, \sigma)$ be a co-bushy space. Then $\emptyset = \emptyset_{t_\sigma}$.

Proof.

By Proposition 3. 2. 4 $\emptyset_{t_\sigma} \subset \emptyset_{f_\sigma}$ but by proposition 2. 2. 3 $\emptyset_{f_\sigma} = \emptyset$, thus $\emptyset = \emptyset_{t_\sigma}$■

Proposition 3. 2. 6: Let $(X, \tau, \delta, \sigma)$ be a co-bushy space. Then for every closed subset P of X the following statements are hold:

1. $P_{t_\sigma} \subseteq P$.
2. $P_{t_\sigma} = (P_{t_\sigma})_{t_\sigma}$.
3. $f_{t\sigma} -\ set$ is $f_\sigma -$ set.
4. $t_{f\sigma} -\ set$ is $t_\sigma -$ set.

Proof.

1) Since P is a closed set by Lemma 2. 2. 17, $P_{f_\sigma} \subseteq P$ and by Proposition 3. 2. 4 $P_{t_\sigma} \subseteq P_{f_\sigma}$, thus $P_{t_\sigma} \subseteq P$.

2) By Proposition 2. 1. 3 $P_{t_\sigma} \subset (P_{t_\sigma})_{t_\sigma}$. By part 1, $P_{t_\sigma} \subset P$, that is, $(P_{t_\sigma})_{t_\sigma} \subset P_{t_\sigma}$. Hence $P_{t_\sigma} = (P_{t_\sigma})_{t_\sigma}$.

3) Let P is $f_{t\sigma} - set$. Then $P \subseteq (P_{f_\sigma})_{t_\sigma}$. Since P is a closed set, by Lemma 2. 2. 17 we have $P_{f_\sigma} \subseteq P$. By part 1 we ge, $(P_{f_\sigma})_{t_\sigma} \subseteq P_{t_\sigma} \subseteq P$, thus $P = (P_{f_\sigma})_{t_\sigma}$, that is, P is $f_\sigma -$ set.

4) Let P is $t_{f\sigma} - set$. Then $P \subseteq (P_{t_\sigma})_{f_\sigma}$, by Proposition 3. 2. 4 we have $P_{t_\sigma} \subseteq P_{f_\sigma}$. Then $(P_{t_\sigma})_{f_\sigma} \subseteq (P_{f_\sigma})_{f_\sigma} \subseteq P_{f_\sigma}$. Since P is a closed set, $P_{f_\sigma} \subseteq P$, that is, $P = (P_{f_\sigma})_{t_\sigma}$, thus P is $t_\sigma -$ set■

Proposition 3. 2. 7: Let $(X, \tau, \delta, \sigma)$ be a co-bushy space. Then for every nonempty open subset P of X, the following statements are hold:

1. $P_{f_\sigma} = (P_{f_\sigma})_{f_\sigma}$.
2. P is $f_{t\sigma} - set$.
3. $P \in \sigma$.
4. $P \subseteq P_{t_\sigma} \subseteq P_{f_\sigma} = cl(P)$.
5. Every nonempty open set is a bushy set.
6. Every nonempty open set is a dense set.

Proof.

1) By Proposition 2. 2. 3 part 6, $(P_{f_\sigma})_{f_\sigma} \subset P_{f_\sigma}$. By Proposition 3. 2. 4, $P_{t_\sigma} \subseteq P_{f_\sigma}$. Since P is open by Proposition 2. 1. 3 part 5, $P \subset P_{t_\sigma}$, thus $P_{f_\sigma} \subset (P_{f_\sigma})_{f_\sigma}$. Hence $P_{f_\sigma} = (P_{f_\sigma})_{f_\sigma}$.

2) By Proposition 3. 2. 4, $P_{t_\sigma} \subseteq P_{f_\sigma}$, then $P_{t_\sigma} \subseteq (P_{t_\sigma})_{t_\sigma} \subseteq (P_{f_\sigma})_{t_\sigma}$ but P is an open, by Proposition 2. 1. 3 part 5 we have that $P \subseteq (P_{f_\sigma})_{t_\sigma}$.

3) If not, $P \notin \sigma$ then by Proposition 2. 2. 3 $P_{f_\sigma} = \emptyset$ which is a contradiction with Proposition 2. 2. 19 because P is open.

4) It is an immediate consequence of propositions 2. 1. 3, 3. 2. 4 and 2. 2. 19.

5) Let P is open set. Then by part 3 $P \in \sigma$ so that by Corollary 3. 1. 31, P is bushy set**.**

6) Clear because every bushy set is a dense set■

Corollary 3. 2. 8: Let $(X, \tau, \delta, \sigma)$ be a co-bushy space. The following statements are hold:

1. $(\emptyset_{f_\sigma})_{t_\sigma} = \emptyset$.
2. $(X_{f_\sigma})_{t_\sigma} = X$.
3. $P_{t_\sigma} \subseteq (P_{f_\sigma})_{t_\sigma}$.

Proof.

1) By Corollary 3. 2. 5.

2) By Lemma 3. 2. 3.

3) By Proposition 3. 2. 4, $P_{t_\sigma} \subseteq P_{f_\sigma}$ but $P_{t_\sigma} \subseteq (P_{t_\sigma})_{t_\sigma}$, thus $P_{t_\sigma} \subseteq (P_{f_\sigma})_{t_\sigma}$■

Proposition 3. 2. 9: $(X, \tau, \delta, \sigma)$ is co-bushy space if and only if X is attached space.

Proof.

Let $P \in \tau$ by Proposition 3. 2. 7 part 3 $P \in \sigma$ that is $\tau/\{\emptyset\} \subseteq \sigma$. By Proposition 3. 1. 30, X is attached space. Conversely, suppose X is attached space. Then every nonempty open set is a bushy set, that is, X is has at least one nonempty open set is a bushy set. Hence X is a co bushy space■

Note .3. 2. 10: Let $(X, \tau, \delta, \sigma)$ be a co-bushy space. Then

1. If P is a bushy set, then P_{f_σ} and $(P_{f_\sigma})_{t_\sigma}$ are bushy sets.

   That is clear because $P_{f_\sigma} = X$ thus $(P_{f_\sigma})_{f_\sigma} = X_{f_\sigma} = X$. And also$((P_{f_\sigma})_{t_\sigma})_{f_\sigma} = ((X)_{t_\sigma})_{f_\sigma} = X_{f_\sigma} = X$.

2. If P_{f_σ} is a bushy set, then P is a bushy set and P is $f_{t\sigma} - set$.

   Because $X = (P_{f_\sigma})_{f_\sigma} \subset P_{f_\sigma}$ thus $P_{f_\sigma} = X$. Also $P \subset P_{f_\sigma}$ but $(P_{f_\sigma})_{t_\sigma} = X$, thus $P \subset (P_{f_\sigma})_{t_\sigma}$.

Proposition 3. 2. 11: If $\mathcal{U}$ and $\mathcal{V}$ are bushy sets in proximity cluster topological space and $\mathcal{U}$ or $\mathcal{V}$ is an open set, then $\mathcal{U} \cap \mathcal{V}$ is also bushy set.

Proof.

Let $\mathcal{U}$ and $\mathcal{V}$ are bushy sets. Then $\mathcal{U}_{f_\sigma} = \mathcal{V}_{f_\sigma} = X$. Let us suppose that $\mathcal{U}$ is an open set. By Proposition 2. 2. 3 it follows that $\mathcal{U} \cap \mathcal{V}_{f_\sigma} \subset (\mathcal{U} \cap \mathcal{V})_{f_\sigma}$, thus $\mathcal{U} \subset (\mathcal{U} \cap \mathcal{V})_{f_\sigma}$ and so that $\mathcal{U}_{f_\sigma} \subset \left((\mathcal{U} \cap \mathcal{V})_{f_\sigma}\right)_{f_\sigma} \subset (\mathcal{U} \cap \mathcal{V})_{f_\sigma}$ but $\mathcal{U}$ is a bushy set, thus $X \subset (\mathcal{U} \cap \mathcal{V})_{f_\sigma}$, that is, $(\mathcal{U} \cap \mathcal{V})_{f_\sigma} = X$, hence $\mathcal{U} \cap \mathcal{V}$ is a bushy set■

Remark 3. 2. 12: Let $(X, \tau, \delta, \sigma)$ be a co-bushy space. Then every nonempty open subset of X is a dense set if and only if every nonempty open subset of X is a bushy set.

Proof.

Let $\mathcal{U}$ is a bushy set. Then for every $\mathcal{V} \in \tau, (\mathcal{V} \cap \mathcal{U})\delta C$ for every $C \in \sigma$ this implies $\mathcal{V} \cap \mathcal{U} \neq \emptyset$ for every $\mathcal{V} \in \tau$ thus $\mathcal{U}$ is a dense set.

Conversely, since X is co-bushy by Proposition 2. 2. 19, $\mathcal{U}_{f_\sigma} = cl(\mathcal{U}) = X$, hence $\mathcal{U}$ is bushy set■

It is to be noted that, according to earlier results:

1. Every bushy set is $f_\sigma -$ perfect
2. Every open set in co bushy space is $f_\sigma -$ perfect.
3. If X is co bushy space, then $\emptyset$ and X are $t_\sigma -$ set and $f_\sigma -$ set.

The proof is evident

Lemma 3. 2. 13: Let $(X, \tau, \delta, \sigma)$ be a co-bushy space, then

$$\left(P_{f_\sigma}\right)_{t_\sigma} = \left(\left(\left(P_{f_\sigma}\right)_{t_\sigma}\right)_{f_\sigma}\right)_{t_\sigma}.$$

Proof.

By Corollary 3. 2. 8 part (3) we have that $((P_{f_\sigma})_{t_\sigma})_{t_\sigma} \subset (((P_{f_\sigma})_{t_\sigma})_{f_\sigma})_{t_\sigma}$. Since $(P_{f_\sigma})_{t_\sigma} \subset ((P_{f_\sigma})_{t_\sigma})_{t_\sigma}$, $(P_{f_\sigma})_{t_\sigma} \subset \left(((P_{f_\sigma})_{t_\sigma})_{f_\sigma}\right)_{t_\sigma}$ ---------- (1)

Also, P_{f_σ} is a closed set, By Proposition 3. 2. 6 part 1, we have that $(P_{f_\sigma})_{t_\sigma} \subset P_{f_\sigma}$, hence $((P_{f_\sigma})_{t_\sigma})_{f_\sigma} \subset (P_{f_\sigma})_{f_\sigma} \subset P_{f_\sigma}$, that is, $\left(((P_{f_\sigma})_{t_\sigma})_{f_\sigma}\right)_{t_\sigma} \subset (P_{f_\sigma})_{t_\sigma}$------------(2).

By (1) and (2) it is following that $(P_{f_\sigma})_{t_\sigma} = \left(((P_{f_\sigma})_{t_\sigma})_{f_\sigma}\right)_{t_\sigma}$ ∎

Proposition 3. 2. 14: Let $(X, \delta, \tau, \sigma)$ be a cluster topological proximity space, X is a co-bushy. Then the following statements are equivalent:

1. P is $f_{t\sigma} - set$.
2. There exist G is $f_\sigma -$ set such that $P \subseteq G$ and $P_{f_\sigma} = G_{f_\sigma}$.
3. P is the intersection of $f_\sigma -$ set and bushy set .

Proof.

$1 \Rightarrow 2$ Suppose $P \subset (P_{f_\sigma})_{t_\sigma}$. By Lemma 3. 2. 13 $(P_{f_\sigma})_{t_\sigma} = \left(((P_{f_\sigma})_{t_\sigma})_{f_\sigma}\right)_{t_\sigma}$ but $(P_{f_\sigma})_{t_\sigma} \in \tau$, if put $G = (P_{f_\sigma})_{t_\sigma}$, then G is $f_\sigma -$ set and $P \subset G$ this implies to $P_{f_\sigma} \subset G_{f_\sigma}$. So by Proposition 3. 2. 4 $P_{t_\sigma} \subset P_{f_\sigma}$, thus $(P_{f_\sigma})_{t_\sigma} \subset (P_{f_\sigma})_{f_\sigma} \subset P_{f_\sigma}$ but $G = (P_{f_\sigma})_{t_\sigma}$, that is, $G \subset P_{f_\sigma}$ and $G_{f_\sigma} \subset (P_{f_\sigma})_{f_\sigma} \subset P_{f_\sigma}$. Hence $P_{f_\sigma} = G_{f_\sigma}$.

$2 \Rightarrow 3$ let $D = P \cup (X - G)$ this implies $D_{f_\sigma} = P_{f_\sigma} \cup (X - G)_{f_\sigma}$ by 2 $P_{f_\sigma} = G_{f_\sigma}$ thus $D_{f_\sigma} = G_{f_\sigma} \cup (X - G)_{f_\sigma} = X_{f_\sigma} = X$ hence D is a bushy set . Hence $P = G \cap D$ where G is $f_\sigma -$ set and D is a bushy set.

$3 \Rightarrow 1$ Suppose $P = G \cap D$ where G is $f_\sigma -$ set and D is a bushy set. Since G is $f_\sigma -$ set, G is open set. By Proposition 2. 2. 3, $G \cap D_{f_\sigma} \subseteq (G \cap D)_{f_\sigma} = P_{f_\sigma}$. So that $P \subseteq G \subseteq (G \cap D)_{f_\sigma} = P_{f_\sigma}$. But by Proposition 2. 1. 3, $G \subseteq G_{t_\sigma}$, it follows that $P \subseteq G \subseteq G_{t_\sigma} \subseteq \left(P_{f_\sigma}\right)_{t_\sigma}$. Hence P is $f_{t\sigma} -$ set■

Proposition 3. 2. 15: Let $(\mathrm{X}, \tau, \delta, \sigma)$ be a proximity cluster topological space, X is co-bushy, P is $f_{t\sigma} -$ set if and only if P is the intersection of open set and bushy set.

Proof.

Suppose P is $f_{t\sigma} -$ set and P= $(P \cup X - P_{f_\sigma}) \cap \left(P_{f_\sigma}\right)_{t_\sigma}$, by Note 2. 1. 7, $\left(P_{f_\sigma}\right)_{t_\sigma} \in \tau$. To prove $(P \cup X - P_{f_\sigma})$ is a bushy set. If not, there exist $x \notin (P \cup X - P_{f_\sigma})_{f_\sigma}$. Then there exists $\mathcal{U} \in \tau(x)$, $(\mathcal{U} \cap (P \cup X - P_{f_\sigma}))\overline{\delta}C$ for some $C \in \sigma$ thus $((\mathcal{U} \cap P) \cup (\mathcal{U} \cap X - P_{f_\sigma}))\overline{\delta}C$ this implies to $(\mathcal{U} \cap P) \notin \sigma$ by Proposition 2. 2. 3, we have that $(\mathcal{U} \cap P)_{f_\sigma} = \emptyset$. But $\mathcal{U} \cap P_{f_\sigma} \subseteq (\mathcal{U} \cap P)_{f_\sigma} = \emptyset$ this implies to $\mathcal{U} \cap P_{f_\sigma} = \emptyset$, thus $\mathcal{U} \subseteq X - P_{f_\sigma}$. Also $(\mathcal{U} \cap X - P_{f_\sigma}) \notin \sigma$, hence $(\mathcal{U} \cap X - P_{f_\sigma})_{f_\sigma} = \emptyset$ this means $\mathcal{U}_{f_\sigma} = \emptyset$ but $\mathcal{U} \neq \emptyset$ hence $\mathcal{U}_{f_\sigma} \subseteq \mathcal{U}$ which is a contradiction with Proposition 3. 2. 7 part 4, thus $(P \cup X - P_{f_\sigma})$ is a bushy set.

Conversely, Suppose $P = \mathcal{U} \cap D$ where $\mathcal{U}$ is open set and D is a bushy set. Since $\mathcal{U}$ is an open set then by Proposition 2. 2. 3, $\mathcal{U} \cap D_{f_\sigma} \subseteq (\mathcal{U} \cap D)_{f_\sigma} = P_{f_\sigma}$. So that $P \subseteq \mathcal{U} \subseteq (\mathcal{U} \cap D)_{f_\sigma} = P_{f_\sigma}$. But by Proposition 2. 1. 3, $\mathcal{U} \subseteq \mathcal{U}_{t_\sigma}$, it follows that $P \subseteq \mathcal{U} \subseteq \mathcal{U}_{t_\sigma} \subseteq \left(P_{f_\sigma}\right)_{t_\sigma}$. Hence P is $f_{t\sigma} -$ set■

We can note that by Proposition 3. 2. 15, each open set in the co-bushy space is an intersection of $f_\sigma -\ set$ and bushy set.

Proposition 3. 2. 16: Let $(\mathrm{X}, \tau, \delta, \sigma)$ be a co-bushy space, $f_{\boldsymbol{t\sigma}}(\{X\}) = \tau$ if and only if X is a $Bushy\ space$. where $f_{\boldsymbol{t\sigma}}(\{X\})$ the family of all $f_{t\sigma} -$ set.

Proof.

Suppose $f_{\boldsymbol{t\sigma}}(\{X\}) = \tau$, and P is a bushy set. Then $\left(P_{f_\sigma}\right)_{t_\sigma} = X$ hence $P \subseteq \left(P_{f_\sigma}\right)_{t_\sigma}$ this mean P is $f_{t\sigma} -\ set,$ thus $P \in f_{t\sigma}(\{X\})$, thus $P \in \tau$, that is, X is a $Bushy\ space$. Conversely, since X is a co bushy then $X = X_{f_\sigma}$this means X is bushy set, and for every $\mathcal{U} \in \tau$, then $\mathcal{U} = \mathcal{U} \cap \mathrm{X}$ by Proposition 3. 2 15, $\mathcal{U}$ is $f_{t\sigma} -\ set$ this mean $\tau \subseteq f_{\boldsymbol{t\sigma}}(\{X\})$.

Let $P \in f_{t\sigma}(\{X\})$ by Proposition 3. 2 15 $P = \mathcal{U} \cap D$ where D is a bushy set. Since X is a $Bushy\ space$, D is an open set, thus $\mathcal{U} \cap \mathrm{D} \in \tau$, so that $P \in \tau$. Hence $f_{t\sigma}(\{X\}) = \tau$■

3.3 Dismountable Space

In this section, we find new concepts parallel to resolvable, and irresolvable space and study them within this space. This section is concerned with studying the ability of space to be dismountable or non-dismountable and the effect of the co-bushy space on topological proximity spaces.

Definition 3. 3. 1:$(\mathrm{X}, \tau, \delta, \sigma)$ is a non-dismountable space if and only if X cannot contains two disjoint bushy subsets. Otherwise X is a dismountable space.

It is easy to see that:

- $(\mathrm{X}, \delta, \sigma, \tau_{\sigma})$ is a non-dismountable space when δ is non-indiscrete proximity because it doesn't have any bushy subset in this space.

- $(\mathrm{X}, \delta, \tau_{\delta}, \sigma)$ is a non-dismountable space when δ is non-indiscrete proximity because it doesn't have any bushy subset in this space.

Proposition 3. 3. 2: Let $(\mathrm{X}, \tau, \delta, \sigma)$ be a dismountable space. Then X is a resolvable space.

Proof.

Suppose that X is a dismountable space. Then there exist two disjoint bushy sets A, B such that $A \cap B = \emptyset, A \cup B = X$. Since every bushy set is a dense set, $A\ and\ B$ are dense sets, hence X is a resolvable■

Example 3. 3. 3: Let $X = \{1, 2, 3, 4\}$, δ is proximity define by: $A\delta B\ \forall A \neq \emptyset, B \neq \emptyset$ except the relation: $\{1\}\overline{\delta}\{2\}$, $\{1\}\overline{\delta}\{3\}$, $\{1\}\overline{\delta}\{4\}$, $\{2\}\overline{\delta}\{3\}$,

$\{2\}\overline{\delta}\{4\}$, $\{1\}\overline{\delta}\{2,3\}$, $\{1\}\overline{\delta}\{2,4\}$, $\{1\}\overline{\delta}\{3,4\}$, $\{1\}\overline{\delta}\{2,3,4\}$, $\{2\}\overline{\delta}\{3,4\}$, $\{2\}\overline{\delta}\{1,3\}$, $\{2\}\overline{\delta}\{1,4\}$, $\{2\}\overline{\delta}\{1,3,4\}$, $\{1,2\}\overline{\delta}\{3,4\}$, $\{1,2\}\overline{\delta}\{3\}$, $\{1,2\}\overline{\delta}\{4\}$ $\{2\}\overline{\delta}\{1\}$, $\{3\}\overline{\delta}\{1\}$, $\{4\}\overline{\delta}\{1\}$, $\{3\}\overline{\delta}\{2\}$, $\{4\}\overline{\delta}\{2\}$, $\{2,3\}\overline{\delta}\{1\}$, $\{2,4\}\overline{\delta}\{1\}$, $\{3,4\}\overline{\delta}\{1\}$, $\{2,3,4\}\overline{\delta}\{1\}$, $\{3,4\}\overline{\delta}\{2\}$, $\{1,3\}\overline{\delta}\{2\}$, $\{1,4\}\overline{\delta}\{2\}$, $\{1,3,4\}\overline{\delta}\{2\}$, $\{3,4\}\overline{\delta}\{1,2\}$. $\{3\}\overline{\delta}\{1,2\}$, $\{4\}\overline{\delta}\{1,2\}$.

Then $\sigma = \{\{3\}, \{1,3\}, \{2,3,4\}, \{1,2,3\}, \{2,3\}, \{3,4\}, \{1,3,4\}, \{4\}, \{1,4\}, \{1,2,4\}, \{2,4\}, X\}$. Let $\tau = \{X, \emptyset, \{3,4\}\}$, we can see that $\{1,3\}$ and $\{2,4\}$ are bushy disjoint sets, thus X is dismountable space.

Proposition 3. 3. 4: Let $(X, \tau, \delta, \sigma)$ be an attached space and submaximal. Then X is a dismountable space if and only if X is an resolvable space.

Proof.

Assume that X is a resolvable space. Then there exist two disjoint dense sets A, B such that $A \cap B = \emptyset, A \cup B = X$. Since every dense set is open and X is attached space, A *and* B are bushy sets, that is, X is dismountable space. Conversely, clear by Proposition 3. 3. 2 ■

Proposition 3. 3. 5: Let τ_1 *and* τ_2 be two topological defined on X, such that $\tau_1 \subseteq \tau_2$. If $(X, \tau_1, \delta, \sigma)$ is non-dismountable space, then $(X, \tau_2, \delta, \sigma)$ is non-dismountable space.

Proof.

Let us suppose that τ_2 is dismountable space. Then there exist two disjoint τ_2 −bushy sets A, B such that $A \cap B = \emptyset, A \cup B = X$. By Remark 3. 1. 15 every τ_2 −bushy set is τ_1 −bushy set, that is τ_1 is dismountable space, this is a contradiction, hence $(X, \delta, \tau_2, \sigma)$ is non-dismountable space■

The above proposition can be generalized to any topology that is finer than the non-dismountable topology defined on the same cluster.

Definition 3. 3. 6: $(X, \tau, \delta, \sigma)$is called:

1- Hereditarily non-dismountable space if and only if every subspace is non-dismountable.
2- Strongly non-dismountable space if and only if each open subspace is non-dismountable.

For example, $(X, \delta, \tau_\delta, \sigma)$ or $(X, \delta, \sigma, \tau_\sigma)$ if δ is non-indiscrete proximity, then the spaces:

- Is non-dismountable space.
- The subspace of X is also non-dismountable space.
- Is hereditarily non-dismountable space.
- Is strongly non-dismountable Space.

Because has not have two disjoint bushy sets.

Proposition 3. 3. 7: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. Then every $Bushy\ space$ is non-dismountable space.

Proof.

If possible that X is a dismountable space. Then there exist two nonempty disjoint bushy sets A, B such that $A \cap B = \emptyset, A \cup B = X$, that is, $A = X - B$ and $B = X - A$. Since A is bushy set, A is dense set, thus $cl(A) = X$, but $B = int(B) = X - cl(X - B) = X - cl(A) = X - X = \emptyset$, which is a contradiction with hypothesis. Thus X is a non-dismountable space■

(Hint:$B = int(B)$ because X is $Bushy\ space$ and B is bushy set)

Proposition 3. 3. 8: Let $(X, \tau, \delta, \sigma)$ be a co bushy space. The following statements are equivalent:

1- X is Strongly non-dismountable space.
2- Every takeoff set of bushy set is also bushy set.
3- Every complement bushy subset of X is scant set.
4- Every closed subset of X is the union of open set and scant set.

Proof.

$1 \Rightarrow 2$ Suppose every open is non-dismountable. Let A is bushy subset of X, that means $A_{f_\sigma} = X$.

If possible $(A_{t_\sigma})_{f\sigma} \neq X$, then $G = X - (A_{t_\sigma})_{f\sigma} \neq \emptyset$. Since $(A_{t_\sigma})_{f\sigma}$ is closed set, G is open set, and so $G \cap A_{f\sigma} \subseteq (G \cap A)_{f\sigma}$, hence $G = G \cap (G \cap A)_{f\sigma}$, that is, $(A \cap G)$ is bushy in G.

Since A_{t_σ} is open set, by Proposition 3. 2. 7 part 4 $A_{t_\sigma} \subseteq \left(A_{t_\sigma}\right)_{f\sigma}$, it follows that $G = X - \left(A_{t_\sigma}\right)_{f\sigma} \subseteq X - A_{t_\sigma} = (X - A)_{f\sigma}$.

Since G is open set, $G = G \cap (X - A)_{f\sigma} \subseteq (G \cap X - A)_{f\sigma}$ this implies to $G = G \cap (G - A)_{f\sigma}$ this means $(G - A)$ is bushy set in G. Further, $G = (G \cap A) \cup (G - A)$ and $(G \cap A) \cap (G - A) = \emptyset$, that is, $(G \cap A), (G - A)$ are bushy set in G hence G is dismountable subspace, which is a contradiction with hypothesis. Hence $(A_{t_\sigma})_{f\sigma} = X$.

$2 \Rightarrow 3$ Suppose $X - A$ is bushy set. By part 2, $\left((X - A)_{t_\sigma}\right)_{f\sigma} = X$. By Proposition 2. 2. 23 part 3, $\emptyset = X - \left((X - A)_{t_\sigma}\right)_{f\sigma} = \left(A_{f_\sigma}\right)_{t\sigma}$, hence A is scant set.

$3 \Rightarrow 4$ Let A closed subset of X. Then $A_{t_\sigma} = \emptyset$ or $A_{t_\sigma} \neq \emptyset$. In case of $A_{t_\sigma} = \emptyset$ this implies $\emptyset = A_{t_\sigma} = X - (X - A)_{f_\sigma} \Rightarrow (X - A)_{f_\sigma} = X$, that is, $(X - A)$ is bushy set. By part 3, A is scant set, hence $A = \emptyset \cup A$ where $\emptyset \in \tau$ and A is scant set.

In case of $A_{t_\sigma} \neq \emptyset$. $(A - A_{t_\sigma})_{t_\sigma} = (A \cap (X - A_{t_\sigma}))_{t_\sigma} = A_{t_\sigma} \cap \big((X - A_{t_\sigma})\big)_{t_\sigma} = (A_{t_\sigma})_{t_\sigma} \cap (X - A_{t_\sigma})_{t_\sigma} = (A_{t_\sigma} \cap X - A_{t_\sigma})_{t_\sigma} = \emptyset_{t_\sigma} = \emptyset$ .Thus $(A - A_{t_\sigma})_{t_\sigma}=\emptyset$, and so that by Proposition 2. 2. 14 part 1, we have that $\emptyset = X - (X - A - A_{t_\sigma})_{f_\sigma} \Rightarrow (X - A - A_{t_\sigma})_{f_\sigma} = X$ this means $(X - A - A_{t_\sigma})$ is bushy set, that is, $(A - A_{t_\sigma})$ is scant set, and so $A = A_{t_\sigma} \cup (A - A_{t_\sigma})$where A_{t_σ} is open and $(A - A_{t_\sigma})$ is scant set. (Hint: Since A is closed set, $A_{t_\sigma} \subseteq A$).

$4 \Rightarrow 1$ Suppose that G is nonempty open dismountable. Then there exist A, B subsets of G such that $G = A \cup B$, $A \cap B = \emptyset$ and $G = A_{f_\sigma}$, $G = B_{f_\sigma}$. Since $A_{f_\sigma}$is closed set, $A_{f_\sigma} = \mathcal{U} \cup C$ where $\mathcal{U}$ is open set and C is scant set, so that $\mathcal{U} \neq \emptyset$ otherwise $A_{f_\sigma} = C$, this mean A_{f_σ} is scant set thus $((A_{f_\sigma})_{f_\sigma})_{t_\sigma} = \emptyset$ but $G = A_{f_\sigma}$ by Proposition 3. 2. 4, $G \subset G_{f_\sigma} = (A_{f_\sigma})_{f_\sigma}$this implies to $G \subseteq (A_{f_\sigma})_{f_\sigma}$and so that $G_{t_\sigma} \subseteq ((A_{f_\sigma})_{f_\sigma})_{t_\sigma}$ but G is open set we get by Proposition 2. 1. 3 $G \subseteq G_{t_\sigma} \subseteq ((A_{f_\sigma})_{f_\sigma})_{t_\sigma} = \emptyset \Rightarrow G = \emptyset$ this is a contradiction, hence $\mathcal{U} \neq \emptyset$. Clear that $int(\mathcal{U}) = \mathcal{U}$ and $int(\mathcal{U}) \subseteq int(A)$. Since B is bushy set in G, $int(A) \cap B \neq \emptyset \Rightarrow A \cap B \neq \emptyset$ which is a contradiction. Thus G is non-dismountable■

Proposition 3. 3. 9: Let $(X, \tau, \delta, \sigma)$ be a co bushy and $Bushy\ space$. Then X is hereditary non-dismountable.

Proof.

Let Y is nonempty dismountable subspace of X. Then there exist nonempty A, B are subsets of Y such that $Y = A \cup B$, $A \cap B = \emptyset$ and $Y = A_{f_\sigma}$, $Y = B_{f_\sigma}$, that is, $X - A_{f_\sigma} = X - Y$and also we have that $(X - A_{f_\sigma})_{f_\sigma} = (X - Y)_{f_\sigma}$.

Since X is co bushy and by Proposition 2. 2. 24, $X = X_{f_\sigma} = ((X - A_{f_\sigma}) \cup A_{f_\sigma})_{f_\sigma} = \left(X - A_{f_\sigma}\right)_{f_\sigma} \cup (A_{f_\sigma})_{f_\sigma} \subseteq \left(X - A_{f_\sigma}\right)_{f_\sigma} \cup A_{f_\sigma} = (X - Y)_{f_\sigma} \cup A_{f_\sigma}$=$((X - Y) \cup A)_{f_\sigma} = (X - B)_{f_\sigma}$, this mean $(X - B)_{f_\sigma} = X$, hence $X - B$ is bushy set in X. Since B is closed and by Proposition 2. 2. 24, $Y = B_{f_\sigma}(\tau_Y, \sigma) \subseteq B_{f_\sigma}(\tau, \sigma) \subseteq B$ that is, $Y = B$ this implies that $A = \emptyset$ which is a contradiction with hypothesis. Thus Y is nonempty non-dismountable subspace■

Proposition 3. 3. 10: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. X is non-dismountable space if and only if there is no bushy set A for which $X - A$ is also bushy set.

Proof.

If possible there exist $A_{f_\sigma} = (X - A)_{f_\sigma} = X$. Then $X = A \cup (X - A)$, and $A \cap (X - A) = \emptyset$, thus X is dismountable space, which is a contradiction with hypothesis.

Conversely, if X is dismountable space, then there exist nonempty $A, B$are bushy subsets of X such that $A_{f_\sigma} = B_{f_\sigma} = X$, but $X = A \cup B$, $A \cap B = \emptyset$, thus $B = X - A$, and $B_{f_\sigma} = (X - A)_{f_\sigma}$, that is, $A_{f_\sigma} = (X - A)_{f_\sigma} = X$, this is a contradiction with hypothesis. Hence X is non-dismountable space■

Corollary 3. 3. 11: The space is dismountable if and only if there exists a set and complement of these set are bushy sets.

The proof clear by Proposition 3. 3. 10.

Proposition 3. 3. 12: Let $(X, \tau, \delta, \sigma)$ be a cluster topological proximity space, and $X = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \emptyset$ and Y_1 is closed set. If $(Y_1, \delta_{Y_1}, \tau_{Y_1}, \sigma_{Y_1})$ and $(Y_2, \delta_{Y_2}, \tau_{Y_2}, \sigma_{Y_2})$ are hereditarily non-dismountable subspaces, then $(X, \tau, \delta, \sigma)$ is hereditarily non -dismountable space.

Proof.

Suppose P is nonempty subset of X and $P \in \sigma$, and $(P, \delta_P, \tau_P, \sigma_P)$ is dismountable. Then there exist nonempty $A, B$are bushy subsets in P such that $A_{f_\sigma} = B_{f_\sigma} = P$, $P = A \cup B$, $A \cap B = \emptyset$. since Y_1 is closed set this implies Y_2 is open set . If possible $Y_1 \cap A \neq \emptyset$ and $Y_2 \cap B \neq \emptyset$. Since $A, B$are bushy subsets in P, then for every $G \in \tau$, $(G \cap P) \cap A \delta_P C$ for every $C \in \sigma_P$. But$(Y_2 \cap G) \in \tau$ this implies to $((Y_2 \cap G \cap P) \cap A)\delta_P C$ for every $C \in \sigma_P \Rightarrow [G \cap (Y_2 \cap P) \cap (Y_2 \cap A)]\delta_{(Y_2 \cap P)} C$, hence $(Y_2 \cap A)_{f_\sigma}$=$Y_2 \cap P$, similarly $(Y_2 \cap B)_{f_\sigma}$=$Y_2 \cap P$ this mean $(Y_2 \cap A)$ and $(Y_2 \cap B)$ are nonempty disjoint bushy subset of $(Y_2 \cap P)$ this mean $(Y_2 \cap P)$ is dismountable subspace of $(Y_2, \delta_{Y_2}, \tau_{Y_2}, \sigma_{Y_2})$, this is a contradiction.

If possible $Y_2 \cap A = \emptyset$ or $Y_1 \cap B = \emptyset$. Let us suppose that $Y_2 \cap A = \emptyset$ and $Y_1 \cap B \neq \emptyset$. Then $A \subset Y_1$. By hypothesis $A_{f_\sigma} = P$, that is, for every $\mathcal{U} \in \tau(x)$, $(\mathcal{U} \cap P) \cap A \delta_P C$ for every $C \in \sigma$. By Theorem 1. 1. 7, $\mathcal{U} \cap (P \cap Y_1)\delta_P C$ for every $C \in \sigma$.Thus $P = (P \cap Y_1)_{f_\sigma} \subset P_{f_\sigma} \cap Y_{1_{f_\sigma}} \subset Y_{1_{f_\sigma}} \subset Y_1$. That means P is a dismountable subspace of Y_1 which is a contradiction. Similarly for $Y_1 \cap B = \emptyset$ and $Y_2 \cap A \neq \emptyset$. Hence $(X, \tau, \delta, \sigma)$ is hereditarily non -dismountable space■

Proposition 3. 3. 13: Let (X,τ,δ,σ) be a proximity cluster topological space. For every $x \notin P$ and P is a bushy set, then $\{x\}$ is not open set.

Proof.

Suppose P is bushy subset of X. Then $P_{f_\sigma} = X$, thus for every $\mathcal{U} \in \tau, (\mathcal{U} \cap P)\delta C$ for every $C \in \sigma$ this implies $\mathcal{U} \cap P \neq \emptyset$ for every $\mathcal{U} \in \tau$. If possible $\{x\}$ is open set . Then $\{x\} \cap P \neq \emptyset$, thus $x \in P$ which is a contradiction, thus $\{x\}$ is not open■

Corollary 3. 3. 14: If the space is a door space the singleton set is a closed set for every $\{x\}$ not belong in the bushy set.

Proof.

Clear by Proposition 3. 3. 13, $\{x\}$ is not open set. But X is a door space that must $\{x\}$ is closed set■

Corollary 3. 3. 15: If the space has a singleton open set, then the space is non-dismountable.

Proof.

If X is dismountable space, then there exist nonempty A, B are bushy subsets of X such that $A_{f_\sigma} = B_{f_\sigma} = X$, $X = A \cup B, A \cap B = \emptyset$. Then for every $\mathcal{U} \in \tau, (\mathcal{U} \cap A)\delta C$ and $(\mathcal{U} \cap B)\delta C$ for every $C \in \sigma$. Since $\{x\} \in \tau$ this implies $\{x\} \cap A \neq \emptyset$ and $\{x\} \cap B \neq \emptyset$ thus $A \cap B \neq \emptyset$ which is a contradiction■

Remark 3. 3. 16: If the space is dismountable and the door, then the space is T_1-space.

Proof.

Suppose X is a dismountable and the door space. Then there exist nonempty $A, B$are bushy subsets of X such that $A_{f_\sigma} = B_{f_\sigma} = X$, $X = A \cup B$, $A \cap B = \emptyset$. Then for every $x \in X$ either $x \in A$ and $x \notin B$. By Proposition 3. 3. 13 $\{x\}$ is not open set. Since X is a door space, $\{x\}$ is closed set. Or $x \in B$ and x $\notin A$. Similarly $\{x\}$ is closed set. Thus the space is T_1-space■

Definition 3. 3. 17: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. P subset of X is called *dismountable set* if and only if there exists subset two bushy sets P_1 and P_2 such that $P = P_1 \cup P_2$ and $P \subseteq {P_1}_{f_\sigma}, P \subseteq {P_2}_{f_\sigma}$.

Proposition 3. 3. 18: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. The union of any family of dismountable set is dismountable set.

Proof.

Let $\{P_\lambda, \lambda \in \Delta\}$ the family of all dismountable subsets of X , for every $\lambda \in \Delta$ there exist nonempty $A_\lambda, B_\lambda$are bushy subsets in P such that $P_\lambda \subseteq {A_\lambda}_{f_\sigma}$, $P_\lambda \subseteq {B_\lambda}_{f_\sigma}$, $P_\lambda = A_\lambda \cup B_\lambda$. Since $P_\lambda \subseteq {A_\lambda}_{f_\sigma}$we get $\bigcup_{\lambda\in\Delta}(P_\lambda) \subseteq \bigcup_{\lambda\in\Delta}({A_\lambda}_{f_\sigma}) \subseteq (\bigcup_{\lambda\in\Delta}A_\lambda)_{f_\sigma}$. Similarly $\bigcup_{\lambda\in\Delta}(P_\lambda) \subseteq (\bigcup_{\lambda\in\Delta}B_\lambda)_{f_\sigma}$. Since there exist $\lambda \in \Delta$ such that $P_\lambda = A_\lambda \cup B_\lambda$, $\bigcup_{\lambda\in\Delta}(P_\lambda) = \bigcup_{\lambda\in\Delta}(A_\lambda \cup B_\lambda) = \bigcup_{\lambda\in\Delta}(A_\lambda) \cup \bigcup_{\lambda\in\Delta}(B_\lambda)$. Thus $\bigcup_{\lambda\in\Delta}(P_\lambda)$ is dismountable set■

Remark 3. 3. 19: If $(X, \tau, \delta, \sigma)$ is dismountable space, then there exist disjoint takeoff sets is empty sets.

Proof.

Suppose X is dismountable, there exist two sets $A, B \subseteq X$, such that $A_{f_\sigma} = B_{f_\sigma} = X, A \cup B = X,\ A \cap B = \emptyset$ this implies to $X - A_{f_\sigma} = X - B_{f_\sigma} = \emptyset$. By Proposition 2. 2. 14 we have that $(X - A)_{t_\sigma} = (X - B)_{t_\sigma} = \emptyset$ but $(X - A) = B$ and $(X - B) = A$, Hence $(A)_{t_\sigma} = (B)_{t_\sigma} = \emptyset$■

4.1 Cluster outer, Cluster disputed, and Cluster brim sets

This part introduces the definition of three disjoint sets and studies the relationship between them, as well as their relationship to the non-dismountable space.

Definition 4. 1. 1: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. A point $x \in X$ is said to be *cluster outer point* of subset P of X, if and only if there exists $\mathcal{U} \in \tau(x)$, such that $(\mathcal{U} \cap P)\overline{\delta}C$ for some $C \in \sigma$.

All the cluster outer points of set P is denoted by P_{O_σ}. Hence $P_{O_\sigma}(\tau, \sigma) = \{x \in X; \exists\, \mathcal{U} \in \tau(x)\ s.t\ (\mathcal{U} \cap P)\overline{\delta}C$ for some $C \in \sigma\}$. The collection of all cluster outer set denoted by $O_\sigma(X)$.

Example 4. 1. 2: Let $X = \{1,2,3\}, \tau = \{X, \emptyset, \{3\}, \{1\}, \{1,3\}\}, \delta$ is discrete proximity, hence $\sigma = \big\{\{2\}, \{1,2\}, \{2,3\}, X\big\}$. If $P_1 = \{2,3\}, P_2 = \{3\}, P_3 = X$, $then$ $P_{1_{O_\sigma}} = \{1,3\}, P_{2_{O_\sigma}} = X, P_{3_{O_\sigma}} = \{1,3\}$.

Proposition 4. 1. 3: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. P is a non-empty subsets of X, the following statements are equivalent:

1. $x \in P_{O_\sigma}$.
2. $x \in (X - P)_{t_\sigma}$.
3. $x \in X - P_{f_\sigma}$.

Proof.

Let $x \in P_{O_\sigma}$. Then there exists $\mathcal{U} \in \tau(x)$, such that $(\mathcal{U} \cap P)\overline{\delta}C$ for some $C \in \sigma$. By Definition 2. 1. 6 if and only if $x \in (X-P)_{t_\sigma}$. By Proposition 2. 2. 14 part 4, if and only if $x \in X - P_{f_\sigma}$. Thus $x \notin P_{f_\sigma}$ if and only if by Definition 2.2.1 there exists $\mathcal{U} \in \tau(x)$, such that $(\mathcal{U} \cap P)\overline{\delta}C$ for some $C \in \sigma$. Hence $x \in P_{O_\sigma}$■

It easy to see that, $(X-P)_{t_\sigma}$ and $(X - P_{f_\sigma})$ are equivalent definitions of the cluster outer set.

Remark 4. 1. 4: The cluster outer set is an open set.

Proof.

That is clear because the cluster outer set is complement of the follower set. By Proposition 2. 2. 3 part 4, P_{f_σ} is closed set. Thus cluster outer set is an open set■

Proposition 4. 1. 5: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. P_1, P_2 are subsets of X. Then

1. If $P_1 \subseteq P_2$, *then* $P_{2_{O_\sigma}} \subseteq P_{1_{O_\sigma}}$.
2. If $P_1 \subseteq P_2$, *then* $(P_{1_{O_\sigma}})_{O_\sigma} \subseteq (P_{2_{O_\sigma}})_{O_\sigma}$.
3. $(P_1 \cup P_2)_{O_\sigma} = P_{1_{O_\sigma}} \cap P_{2_{O_\sigma}}$.
4. $P_{1_{O_\sigma}} \cap P_{2_{O_\sigma}} \subseteq (P_1 \cap P_2)_{O_\sigma}$.
5. $P_{O_\sigma} = int(P_{O_\sigma}) \supseteq int(X-P)$.
6. If $P \notin \sigma$, then $P_{O_\sigma} = X$.
7. $P_{O_\sigma} \subseteq \left(X - P_{O_\sigma}\right)_{O_\sigma} = (P_{f_\sigma})_{O_\sigma}$.
8. $(\emptyset)_{O_\sigma} = X$.

9. $X_{O_\sigma} = X - X_{f_\sigma}$.

10. $(P_{O_\sigma})_{O_\sigma} = (P_{f_\sigma})_{t_\sigma}$.

Proof.

1) Let $x \in P_{2_{O_\sigma}}$. Then there exists $\mathcal{U} \in \tau(x)$, such that $(\mathcal{U} \cap P_2)\overline{\delta}C$ for some $C \in \sigma$, but $P_1 \subset P_2$ then by property of proximity space $(\mathcal{U} \cap P_1)\overline{\delta}C$ for some $C \in \sigma$, hence $x \in P_{1_{O_\sigma}}$.

2) Evident through part 1.

3) $(P_1 \cup P_2)_{O_\sigma} = X - (P_1 \cup P_2)_{f_\sigma} = X - (P_{1_{f_\sigma}} \cup P_{2_{f_\sigma}}) = X - (P_{1_{f_\sigma}}) \cap X - (P_{2_{f_\sigma}}) = P_{1_{O_\sigma}} \cap P_{2_{O_\sigma}}$.

4) By part 1.

5) $P_{O_\sigma} = X - P_{f_\sigma} = X - cl(P_{f_\sigma}) = int(X - P_{f_\sigma}) = int(P_{O_\sigma})$. Also, $X - cl(P_{f_\sigma}) \supseteq X - cl(P) = int(X - P)$.

6) Let $P \notin \sigma$. Then $P\overline{\delta}C$ for some $C \in \sigma$. By Theorem 1. 1. 7 part 3, for every $x \in X$ there exists $\mathcal{U} \in \tau(x)$ such that $(\mathcal{U} \cap P)\overline{\delta}C$ for some $C \in \sigma$, hence $x \in P_{O_\sigma}$, its complete result.

7) $P_{O_\sigma} = X - P_{f_\sigma} \subseteq X - \left(X - (X - P_{f_\sigma})\right)_{f_\sigma} = X - \left(X - P_{O_\sigma}\right)_{f_\sigma} = \left(X - P_{O_\sigma}\right)_{O_\sigma} = \left(P_{f_\sigma}\right)_{O_\sigma}$.

8) $(\emptyset)_{O_\sigma} = X - \emptyset_{f_\sigma} = X - \emptyset = X$.

9) Evident.

10) $(P_{O_\sigma})_{O_\sigma} = (X - P_{f_\sigma})_{O_\sigma} = X - (X - P_{f_\sigma})_{f_\sigma} = \left(X - X - P_{f_\sigma}\right)_{t_\sigma} = \left(P_{f_\sigma}\right)_{t_\sigma}$ ■

Corollary 4. 1. 6: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. P_1, P_2 are subsets of X, then:

1. If $(P_{O_\sigma})_{O_\sigma} = \emptyset$, then P is a scant set, and the opposite is true.
2. If $P_{O_\sigma} = \emptyset$, then P is a member of the cluster family.
3. If X is co bushy space, then $(P_{O_\sigma})_{O_\sigma} \subset P_{f_\sigma}$.
4. If P is closed set, then $(P_{O_\sigma})_{O_\sigma} \subset P_{t_\sigma}$.
5. $(X - P)_{O_\sigma} = P_{t_\sigma}$.

Proof.

1) By Proposition 4. 1. 5 part 10, $(P_{O_\sigma})_{O_\sigma} = (P_{f_\sigma})_{t_\sigma}$. Thus P is scant set.

2) By Proposition 4. 1. 5 part 6, if $P_{O_\sigma} = \emptyset$, then $P \in \sigma$.

3) Since X is co bushy space, by Proposition 3. 2. 4 $(P)_{t_\sigma} \subseteq (P)_{f_\sigma}$. Then $(P_{f_\sigma})_{t_\sigma} \subseteq (P_{f_\sigma})_{f_\sigma} \subseteq P_{f_\sigma}$, thus by Proposition 4. 1. 5 part 10, we get that $(P_{O_\sigma})_{O_\sigma} \subseteq P_{f_\sigma}$.

4) Since P is closed by Lemma 2. 2. 17, $(P)_{f_\sigma} \subseteq P$, thus by Proposition 4. 1. 5 part 10, $(P_{O_\sigma})_{O_\sigma} = (P_{f_\sigma})_{t_\sigma} \subseteq (P)_{t_\sigma}$.

5) By Proposition 4. 1. 3, $P_{O_\sigma} = (X - P)_{t_\sigma}$, thus $(X - P)_{O_\sigma} = (X - X - P)_{t_\sigma} = P_{t_\sigma}$ ■

Proposition 4. 1. 7: If (X,τ,δ,σ) is a dismountable space, then there exists A$\subseteq X$, such that $A_{O_\sigma} = \emptyset$.

Proof.

Suppose X is dismountable. Then there exist two sets $A, B \subseteq X$, such that $A_{f_\sigma} = B_{f_\sigma} = X$, $A \cup B = X$, and $A \cap B = \emptyset$. Thus $X - A_{f_\sigma} = \emptyset$, and by Proposition 4. 1. 3, $A_{O_\sigma} = \emptyset$■

It can be conclude by Proposition 4. 1. 7 that a space is dismountable if and only if there are at least two empty disjoint cluster outer sets.

Definition 4. 1. 8: Let (X,τ,δ,σ) be a proximity cluster topological space. P is said to be *cluster disputed set* denoted by $D_\delta(P)$ if and only if $D_\delta(P) = P_{f_\sigma} \cap P_{t_\sigma}$.

Example 4. 1. 9: Let $X = \{1,2,3,4,5\}$, δ is discrete proximity , and $\sigma = \sigma_5$, and $\tau = \{X, \emptyset, \{1,3,5\}, \{3\}\}$. If we take $\{2,5\}, \{1,2\}, X$ and $\emptyset$, then:

$\{2,5\}_{f_\sigma} = \{1,2,4,5\}$ and $\{2,5\}_{t_\sigma} = X$, thus $D_\delta(\{2,5\}) = \{1,2,4,5\}$,

$\{1,2\}_{f_\sigma} = \emptyset$, and $\{1,2\}_{t_\sigma} = \{3\}$, thus $D_\delta(\{1,2\}) = \emptyset$,

$X_{f_\sigma} = \{1,2,4,5\}$, and $X_{t_\sigma} = X$, thus $D_\delta(X) = \{1,2,4,5\}$,

$\emptyset_{f_\sigma} = \emptyset$, and $\emptyset_{t_\sigma} = \{3\}$, thus $D_\delta(\emptyset) = \emptyset$.

Proposition 4. 1. 10: Let (X,τ,δ,σ) be a proximity cluster topological space. P_1, P_2 are subsets of X. Then

1. $D_\delta(\emptyset) = \emptyset$, and $D_\delta(X) = X_{f_\sigma}$.
2. If $P_1 \subset P_2$ *then* $D_\delta(P_1) \subset D_\delta(P_2)$.
3. $D_\delta(P_1 \cup P_2) \supseteq D_\delta(P_1) \cup D_\delta(P_2)$.

4. $D_\delta(P_1 \cap P_2) \subseteq D_\delta(P_1) \cap D_\delta(P_2)$.
5. If $C \notin \sigma$, then $D_\delta(C) = \emptyset$,and $D_\delta(X - C) = (X - C)_{f_\sigma}$.
6. If P is closed set , then $D_\delta(P) \subseteq int_{f_\sigma}(P) \subseteq P$.
7. If P is a clopen set, then $D_\delta(P) = P_{f_\sigma}$.
8. $D_\delta(P) = P_{f_\sigma} - (X - P)_{f_\sigma}$.
9. $D_\delta(P) = \emptyset$ if and only if $P_{f_\sigma} \subseteq (X - P)_{f_\sigma}$or $P_{t_\sigma} \subseteq (X - P)_{t_\sigma}$.
10. $D_\delta\big(D_\delta(P)\big) \subseteq D_\delta\big(P_{f_\sigma}\big) \cap D_\delta\big(P_{t_\sigma}\big)$.
11. $D_\delta\Big(D_\delta\big(D_\delta(P)\big)\Big) \subseteq D_\delta\big((P_{f_\sigma})_{f_\sigma}\big) \cap D_\delta\big((P_{t_\sigma})_{t_\sigma}\big) \cap D_\delta\big((P_{f_\sigma})_{t_\sigma}\big) \cap D_\delta\big((P_{t_\sigma})_{f_\sigma}\big)$.

Proof.

1) Evident, because $\emptyset_{f_\sigma} = \emptyset$, and X_{f_σ} is not necessary equal X.

2) $D_\delta(P_1) = {P_1}_{f_\sigma} \cap {P_1}_{t_\sigma} \subseteq {P_2}_{f_\sigma} \cap {P_2}_{t_\sigma} = D_\delta(P_2)$.

3) Clear by part 2.

4) Clear by part 2.

5) Suppose $C \notin \sigma$ by Proposition 2. 2. 3 $C_{f_\sigma} = \emptyset$, thus $D_\delta(C) = \emptyset$.
By Proposition 1. 1. 15, $(X - C) \in \sigma$. Also, by Proposition 2. 1. 3 part 8, $(X - C)_{t_\sigma} = X$, thus $D_\delta(X - C) = (X - C)_{f_\sigma}$.

6) $D_\delta(P) = P_{f_\sigma} \cap P_{t_\sigma}$ but P is closed so by Lemma 2. 2. 17 we have $P_{f_\sigma} \subseteq P$, hence $P_{f_\sigma} \cap P_{t_\sigma} \subseteq P \cap P_{t_\sigma} = int_{f_\sigma}(P) \subseteq P$.

7) By fact, if P is a both open and closed, then $P_{f_\sigma} \subseteq P_{t_\sigma}$.

8) $D_\delta(P) = P_{f_\sigma} \cap P_{t_\sigma} = P_{f_\sigma} \cap (X - (X - P)_{f_\sigma}) = P_{f_\sigma} - (X - P)_{f_\sigma}$.

9) Suppose $D_\delta(P) = \emptyset$. Then $P_{f_\sigma} \cap P_{t_\sigma} = \emptyset$, by Proposition 2. 2. 23 either $P_{f_\sigma} \subseteq (X - P_{t_\sigma}) = (X - P)_{f_\sigma}$, or $P_{t_\sigma} \subseteq (X - P_{f_\sigma}) = (X - P)_{t_\sigma}$. Thus $P_{f_\sigma} \subseteq (X - P)_{f_\sigma}$or $P_{t_\sigma} \subseteq (X - P)_{t_\sigma}$. Conversely, suppose$P_{f_\sigma} \subseteq X - P_{t_\sigma}$. Then $P_{f_\sigma} \cap P_{t_\sigma} = \emptyset$, thus $D_\delta(P) = \emptyset$.

10) $D_\delta(D_\delta(P)) = D_\delta(P_{f_\sigma} \cap P_{t_\sigma}) = (P_{f_\sigma} \cap P_{t_\sigma})_{f_\sigma} \cap (P_{f_\sigma} \cap P_{t_\sigma})_{t_\sigma} \subseteq (P_{f_\sigma})_{f_\sigma} \cap (P_{t_\sigma})_{f_\sigma} \cap (P_{f_\sigma})_{t_\sigma} \cap (P_{t_\sigma})_{t_\sigma} = ((P_{f_\sigma})_{f_\sigma} \cap (P_{f_\sigma})_{t_\sigma}) \cap ((P_{t_\sigma})_{f_\sigma} \cap (P_{t_\sigma})_{t_\sigma}) = D_\delta(P_{f_\sigma}) \cap D_\delta(P_{t_\sigma})$.

11) By part 10, $D_\delta\left(D_\delta(D_\delta(P))\right) \subseteq D_\delta\left(D_\delta(P_{f_\sigma}) \cap D_\delta(P_{t_\sigma})\right)$

$= D_\delta\left((P_{f_\sigma})_{f_\sigma} \cap (P_{f_\sigma})_{t_\sigma} \cap (P_{t_\sigma})_{f_\sigma} \cap (P_{t_\sigma})_{t_\sigma}\right)$

$= \left((P_{f_\sigma})_{f_\sigma} \cap (P_{f_\sigma})_{t_\sigma} \cap (P_{t_\sigma})_{f_\sigma} \cap (P_{t_\sigma})_{t_\sigma}\right)_{f_\sigma}$

$\cap \left((P_{f_\sigma})_{f_\sigma} \cap (P_{f_\sigma})_{t_\sigma} \cap (P_{t_\sigma})_{f_\sigma} \cap (P_{t_\sigma})_{t_\sigma}\right)_{t_\sigma}$

$\subseteq [((P_{f_\sigma})_{f_\sigma})_{f_\sigma} \cap ((P_{f_\sigma})_{t_\sigma})_{f_\sigma} \cap ((P_{t_\sigma})_{f_\sigma})_{f_\sigma} \cap ((P_{t_\sigma})_{t_\sigma})_{f_\sigma}] \cap$
$[((P_{f_\sigma})_{f_\sigma})_{t_\sigma} \cap ((P_{f_\sigma})_{t_\sigma})_{t_\sigma} \cap ((P_{t_\sigma})_{f_\sigma})_{t_\sigma} \cap ((P_{t_\sigma})_{t_\sigma})_{t_\sigma}]$

$= ((P_{f_\sigma})_{f_\sigma})_{f_\sigma} \cap ((P_{f_\sigma})_{f_\sigma})_{t_\sigma} \cap ((P_{t_\sigma})_{t_\sigma})_{f_\sigma}) \cap ((P_{t_\sigma})_{t_\sigma})_{t_\sigma} \cap$
$((P_{f_\sigma})_{t_\sigma})_{f_\sigma} \cap ((P_{f_\sigma})_{t_\sigma})_{t_\sigma} \cap ((P_{t_\sigma})_{f_\sigma})_{f_\sigma} \cap ((P_{t_\sigma})_{f_\sigma})_{t_\sigma}$

$= D_\delta\left((P_{f_\sigma})_{f_\sigma}\right) \cap D_\delta\left((P_{t_\sigma})_{t_\sigma}\right) \cap D_\delta\left((P_{f_\sigma})_{t_\sigma}\right) \cap D_\delta\left((P_{t_\sigma})_{f_\sigma}\right)$■

Definition 4. 1. 11: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. P is said to be *cluster brim set* denoted by $B_\delta(P)$ if and only if $B_\delta(P) = P_{f_\sigma} \cap (X - P)_{f_\sigma}$.

By Example 4. 1. 9, $B_\delta(P) = \emptyset$ for every subset of X, we can conclusion that $B_\delta(P) \neq \emptyset$ if and only if P and $X - P$ belong to cluster family, because if P or $X - P$ not belong to cluster by Proposition 2. 2. 3 part 5, P or $X - P$ equal empty set**.**

It is noted that the cluster brim set depends on the proximity relationship defined on the cluster. So if we assume that the proximity relationship is indiscrete proximity on any topology, then the cluster brim set is non-empty, but in the case of the discrete proximity relationship defined on the cluster, the cluster brim set is always equal to the empty set. This does not mean that the topology has no effect, but the effect of the proximity relationship is stronger than the effect of the topology.

By Examples 3. 1. 8**,** $\tau_\delta = \{X, \emptyset, \{3\}, \{1,2\}\}$, and $\sigma = \{\{1\},\{1,2\},\{1,3\},\{2\},\{2,3\}, X\}$,. If $P = \{2\}$, then $X - P = \{1,3\}$, thus $\{2\}_{f_\sigma} = \{1,2\}$, and $\{1,3\}_{f_\sigma} = \{1,2\}$, that is, $B_\delta(\{2\}) = \{1,2\}$.

Examples 4. 1. 12: Let $X = \{1,2,3\}$, δ is discrete proximity. Then and $\tau_\delta = \{X, \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$. And let $\sigma = \{\{2\},\{1,2\},\{2,3\}, X\}$. If $P = \{2\}$, then $X - P = \{1,3\}$, thus $\{2\}_{f_\sigma} = \{2\}$, and $\{1,3\}_{f_\sigma} = \emptyset$, that is, $B_\delta(\{2\}) = \emptyset$.

Proposition 4. 1. 13: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space, $x \in B_\delta(P)$ if and only if $x \in (P_{f_\sigma} - P_{t_\sigma})$.

Proof.

Let $x \in B_\delta(P)$. Then $x \in P_{f_\sigma}$ and $x \in (X-P)_{f_\sigma}$, and so that $x \notin X-(X-P)_{f_\sigma}$, by Proposition 2. 2. 14, $x \notin P_{t_\sigma}$ hence $x \in (P_{f_\sigma} - P_{t_\sigma})$.

Conversely, let $x \in P_{f_\sigma} - P_{t_\sigma}$. This means $x \in P_{f_\sigma}$ and $x \notin P_{t_\sigma}$ so that $x \in X - P_{t_\sigma}$ but $X - P_{t_\sigma} = (X-P)_{f_\sigma}$, hence $x \in B_\delta(P)$■

The above proposition is an equivalent definition to the cluster brim. This could be used to prove some elements of the following proposition, where the most important properties of cluster brim are mentioned.

Proposition 4. 1. 14: Let (X,τ,δ,σ) be a proximity cluster topological space. Then

1. $B_\delta(\emptyset) = \emptyset$, and $B_\delta(X) = \emptyset$.
2. $B_\delta(X) \subseteq B_\delta(P)$, for every nonempty subset of X.
3. $B_\delta(C) = \emptyset$, for every $C \notin \sigma$.
4. $B_\delta(P)$ is closed set.
5. $B_\delta(P_1 \cup P_2) \subseteq B_\delta(P_1) \cup B_\delta(P_2)$.
6. $B_\delta\big(B_\delta(P)\big) \subseteq B_\delta(P)$.
7. $B_\delta(X-P) = B_\delta(P)$.
8. $cl_{f_\sigma}(P) = D_\delta(P) \cup B_\delta(P) \cup P$.

Proof.

1) $B_\delta(X) = X_{f_\sigma} \cap (X-X)_{f_\sigma} = \emptyset$.

2) Evident by 1.

3) That is clear because by Preposition 2. 2. 3 part5, $C_{f_\sigma} = \emptyset$.

4) Evident, because P_{f_σ} is closed set.

5) $B_\delta(P_1 \cup P_2) = (P_1 \cup P_2)_{f_\sigma} \cap \big(X - (P_1 \cup P_2)\big)_{f_\sigma} \subseteq (P_{1_{f_\sigma}} \cup P_{2_{f_\sigma}}) \cap ((X - P_1)_{f_\sigma} \cap (X - P_2)_{f_\sigma}) = (P_{1_{f_\sigma}} \cap (X - P_1)_{f_\sigma} \cap (X - P_2)_{f_\sigma}) \cup (P_{2_{f_\sigma}} \cap (X - P_1)_{f_\sigma} \cap (X - P_2)_{f_\sigma})) \subseteq ((P_{1_{f_\sigma}} \cap ((X - P_1)_{f_\sigma}) \cup \Big(P_{2_{f_\sigma}} \cap ((X - P_2)_{f_\sigma}\Big) = B_\delta(P_1) \cup B_\delta(P_2).$

6) $B_\delta\big(B_\delta(P)\big) = B_\delta\big(P_{f_\sigma} \cap (X - P)_{f_\sigma}\big)$

$= \big(P_{f_\sigma} \cap (X - P)_{f_\sigma}\big)_{f_\sigma} \cap \big(X - (P_{f_\sigma} \cap (X - P)_{f_\sigma})\big)_{f_\sigma}$

$= \big(P_{f_\sigma} \cap (X - P)_{f_\sigma}\big)_{f_\sigma} \cap \big((X - P_{f_\sigma}) \cup (P)_{f_\sigma})\big)_{f_\sigma}$

$\subseteq ((P_{f_\sigma})_{f_\sigma} \cap ((X - P)_{f_\sigma})_{f_\sigma}) \cap (\big(X - P_{f_\sigma}\big)_{f_\sigma} \cup ((P)_{f_\sigma})_{f_\sigma})$

$\subseteq P_{f_\sigma} \cap (X - P)_{f_\sigma} = B_\delta(P).$

7) $B_\delta(X - P) = (X - P)_{f_\sigma} \cap (X - X - P)_{f_\sigma} = P_{f_\sigma} \cap (X - P)_{f_\sigma} = B_\delta(P).$

8) $D_\delta(P) \cup B_\delta(P) \cup P = ((P_{f_\sigma} \cap P_{t_\sigma}) \cup (P_{f_\sigma} \cap (X - P)_{f_\sigma}) \cup P$

$= (P_{f_\sigma} \cap (P_{t_\sigma} \cup (X - P)_{f_\sigma}) \cup P = (P_{f_\sigma} \cap (P_{t_\sigma} \cup X - P_{t_\sigma})\,) \cup P$

$= (P_{f_\sigma} \cap X) \cup P = P_{f_\sigma} \cup P = cl_{f_\sigma}(P)$■

Proposition 4. 1. 15: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space, $B_\delta(P) = \emptyset$ if and only if $P_{f_\sigma} \subseteq P_{t_\sigma}$.

Proof.

If $P_{f_\sigma} = \emptyset$ the proof is done.

Suppose $B_\delta(P) = \emptyset$ and $P_{f_\sigma} \neq \emptyset$. Then for every $x \in P_{f_\sigma}$, $x \notin (X-P)_{f_\sigma}$, that is, $x \in X-(X-P)_{f_\sigma}$, hence $x \in P_{t_\sigma}$.

Conversely, suppose $P_{f_\sigma} \subseteq P_{t_\sigma}$, that is, $P_{f_\sigma} \cap (X - P_{t_\sigma}) = \emptyset$ but by Proposition 2. 2. 14, $X - P_{t_\sigma} = (X-P)_{f_\sigma}$, hence $B_\delta(P) = \emptyset$■

By Propositions 4. 1. 13 and 4.1. 15, as well as by relying on some properties of the follower and takeoff sets, we have some results mentioned in the following proposition:

Proposition 4. 1. 16: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space the following statements are hold:

1. If P is open set, then $B_\delta(P) \subseteq P_{f_\sigma} - P$.
2. If P is closed set, then $B_\delta(P) \subseteq P - P_{t_\sigma}$.
3. If P is clopen set, then $B_\delta(P) = \emptyset$.

Proof.

1) Let P be an open set. Then by Proposition 2. 1. 3 part 5, we have $P \subseteq P_{t_\sigma}$. So that $B_\delta(P) = P_{f_\sigma} - P_{t_\sigma} \subseteq P_{f_\sigma} - P$.

2) Let P be a closed set. Then by Lemma 2. 2. 17 we have $P_{f_\sigma} \subseteq P$. So that $B_\delta(P) = P_{f_\sigma} - P_{t_\sigma} \subseteq P - P_{t_\sigma}$.

3) Let P be a clopen set. Then by Lemma 2. 2. 18, we have $P_{f_\sigma} \subseteq P_{t_\sigma}$. So that $B_\delta(P) = P_{f_\sigma} - P_{t_\sigma} = \emptyset$■

After examining the properties of the disjoint sets, it is important to study the relationship among them and their effect on the proximity space. The following proposition explains the most important relationships between these sets.

Proposition 4. 1. 17: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space, then the following statements are hold:

1. $B_\delta(P) = P_{f_\sigma} \cap (X - P_{t_\sigma})$.
2. $B_\delta(P) \cap D_\delta(P) = \emptyset$.
3. $P_{f_\sigma} = B_\delta(P) \cup D_\delta(P)$.

Proof.

1) Clear.

2) $B_\delta(P) \cap D_\delta(P) = (P_{f_\sigma} \cap \left(X - P\right)_{f_\sigma}) \cap (P_{f_\sigma} \cap P_{t_\sigma}) = (P_{f_\sigma} \cap X - P_{t_\sigma}) \cap (P_{f_\sigma} \cap P_{t_\sigma}) = P_{f_\sigma} \cap (P_{t_\sigma} \cap X - P_{t_\sigma}) = \emptyset$.

3) $B_\delta(P) \cup D_\delta(P) = (P_{f_\sigma} \cap (X - P)_{f_\sigma}) \cup (P_{f_\sigma} \cap P_{t_\sigma})$
$= P_{f_\sigma} \cap ((X - P_{t_\sigma}) \cup P_{t_\sigma})) = P_{f_\sigma} \cap X = P_{f_\sigma}$■

Proposition 4. 1. 18: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space, then the cluster outer, cluster disputed and cluster brim divided space into three pairwise disjoint sets.

Proof.

By Proposition 4. 1. 17 part 2, $B_\delta(P) \cap D_\delta(P) = \emptyset$, to prove $B_\delta(P) \cap P_{O_\sigma} = \emptyset$ and $D_\delta(P) \cap P_{O_\sigma} = \emptyset$.

$B_\delta(P) \cap P_{O_\sigma} = (P_{f_\sigma} \cap (X-P)_{f_\sigma}) \cap X - P_{f_\sigma} = (P_{f_\sigma} \cap X - P_{f_\sigma}) \cap (X - P)_{f_\sigma} = \emptyset$. Also, $D_\delta(P) \cap P_{O_\sigma} = (P_{f_\sigma} \cap P_{t_\sigma}) \cap X - P_{f_\sigma} = P_{t_\sigma} \cap (P_{f_\sigma} \cap X - P_{f_\sigma}) = \emptyset$. Also, by Proposition 4. 1. 17 part 3, $B_\delta(P) \cup D_\delta(P) = P_{f_\sigma}$, thus $(B_\delta(P) \cup D_\delta(P)) \cup P_{O_\sigma} = P_{f_\sigma} \cup (X - P_{f_\sigma}) = X$■

Note 4. 1. 19: If P is a bushy set, then P divides the space into two disjoint sets $B_\delta(P)$ and $D_\delta(P)$. Because if P is bushy set, then:

- $P_{O_\sigma} = \emptyset$.
- $B_\delta(P) = X - P_{t_\sigma}$.
- $D_\delta(P) = P_{t_\sigma}$.

Proposition 4. 1. 20: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space, and A, B are subsets of X. If X is a dismountable space, then the following holds:

1. $B_\delta(A) = X$.
2. $D_\delta(A) = \emptyset$
3. $A_{O_\sigma} = \emptyset$.

Proof.

1) Suppose X is dismountable space, then there exist nonempty A, B disjoint bushy subsets of X, such that $A_{f_\sigma} = B_{f_\sigma} = X$, and $B = X - A$. $B_\delta(A) = A_{f_\sigma} \cap (X - A)_{f_\sigma} = (X - A)_{f_\sigma} = B_{f_\sigma} = X$.

2) $D_\delta(A) = A_{f_\sigma} \cap A_{t_\sigma} = A_{t_\sigma} = X - (X - A)_{f_\sigma} = X - X = \emptyset$.

3) Evident.

4. 2 Cluster Too Intense and Cluster Semi Intense sets

This section presents new concepts for four of the sets that were built by depend on the concept of follower set and takeoff set with the study of the most important properties and characteristics of these sets and their relationship to open and closed sets in proximity cluster topological space as well as their relationship with each other. Moreover, it is building other sets that depend on those sets and studying their most important characteristics within the co bushy spaces.

Definition 4. 2. 1: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space, A is subset of X is called:

1. Cluster too intense set denoted by $C_{TI}(A)$ if and only if $A_{t_\sigma} = (A_{f_\sigma})_{t_\sigma}$.
2. Cluster semi intense set denoted by $C_{SI}(A)$ if and only if $A_{f_\sigma} = (A_{t_\sigma})_{f_\sigma}$.
3. Cluster intense set denoted by $C_I(A)$ if and only if $A_{t_\sigma} = (A_{f_\sigma})_{t_\sigma} and\ A_{f_\sigma} = (A_{t_\sigma})_{f_\sigma}$.

Hence A is cluster intense if and only if A is $C_{TI}(A)$ and $C_{SI}(A)$.
So that $C_{TI}(X), C_{SI}(X), C_I(X)$ denoted the collection of all cluster too intense, cluster semi intense, cluster intense, sequentially.

Note 4. 2. 2: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. Then

1. X is a cluster semi intense. Because by Proposition 2. 1.3, $X_{t_\sigma} = \mathrm{X}$.
2. $\emptyset$ is a cluster too intense. Because by Proposition 2. 2. 3, $\emptyset_{f_\sigma} = \emptyset$.

3. If the space is co bushy, then X and $\emptyset$ are cluster intense sets. That is clear by Lemma 3. 2. 3, and Corollary 3. 2. 5.
4. If A is an open set, then every cluster too intense is f_{t_σ} −set.
5. If A is open in co bushy space, then every cluster semi intense is t_{f_σ} −set. because by Proposition 3. 2. 7 part 4, $A \subseteq A_{f_\sigma}$.

Examples 4. 2. 3: Let $X = \{1,2,3,4\}$, if δ is indiscrete proximity, then $\sigma = \{A \subseteq X;\ A \neq \emptyset\}$. Let $\tau = \{X, \emptyset, \{3\}, \{1,3\}\}$,
$P_1 = \{3\}, P_2 = \{1,2\}, P_3 = \{1,3,4\}, P_4 = \{1,2,4\} then$
P_1 is cluster semi intense, P_2 is not.
P_4 is cluster too intense, P_3 is not.

It is possible to use the previous results on the cluster proximity topological space$(X, \delta, \tau_\delta, \sigma)$, for Example 2. 2. 2 , $X = \{1,2,3\}$, $\tau_\delta = \{X, \emptyset, \{2\}, \{1,3\}\}$, and let $\sigma = \{\{2\},\{1,2\},\{2,3\}, X\}$. Then $\{2\}_{f_\sigma} = \{2\}$, and $\{2\}_{t_\sigma} = X$, that is, $(\{2\}_{f_\sigma})_{t_\sigma} = X$. But $(\{2\}_{t_\sigma})_{f_\sigma} = \{2\}$. Thus $\{2\}$ is cluster intense set.

Proposition 4. 2. 4: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space, A is a cluster too intense if and only if the complement of A is a cluster semi intense.

Proof.
A is a cluster too intense if and only if $A_{t_\sigma} = (A_{f_\sigma})_{t_\sigma}$ if and only if $X - A_{t_\sigma} = X - (A_{f_\sigma})_{t_\sigma}$ if and only if $(X - A)_{f_\sigma} = ((X - A)_{t_\sigma})_{f_\sigma}$ if and only if the complement of A is a cluster semi intense■

Proposition 4. 2. 5: Let (X,τ,δ,σ) be a co bushy space. Then:

1. If A is closed set, then A is cluster too intense if and only if $(A_{f_\sigma})_{t_\sigma} \subseteq A$.
2. If A is open set, then A is cluster semi intense if and only if $A \subseteq (A_{t_\sigma})_{f_\sigma}$.
3. If A is clopen set, then A is cluster intense if and only if $(A_{f_\sigma})_{t_\sigma} = (A_{t_\sigma})_{f_\sigma}$.

Proof.

1) Suppose A is $C_{TI}(A)$. Then $A_{t_\sigma} = (A_{f_\sigma})_{t_\sigma}$. By Proposition 3. 2. 4, and Lemma 2. 2. 17 we get $(A_{f_\sigma})_{t_\sigma} = A_{t_\sigma} \subseteq A_{f_\sigma} \subseteq A$. Conversely, suppose $(A_{f_\sigma})_{t_\sigma} \subseteq A$. Since A is a closed set in co bushy space by Proposition 3. 2. 6 part 2, $(A_{f_\sigma})_{t_\sigma} = ((A_{f_\sigma})_{t_\sigma})_{t_\sigma} \subseteq A_{t_\sigma}$, that is, $(A_{f_\sigma})_{t_\sigma} \subseteq A_{t_\sigma}$. By Proposition 3. 2. 4, $P_{t_\sigma} \subseteq P_{f_\sigma}$for any $P \subseteq X$, that is follows $A_{t_\sigma} = (A_{t_\sigma})_{t_\sigma} \subseteq (A_{f_\sigma})_{t_\sigma}$, thus $A_{t_\sigma} \subseteq (A_{f_\sigma})_{t_\sigma}$, that is, $(A_{f_\sigma})_{t_\sigma} = A_{t_\sigma}$. Hence A is cluster too intense.

2) Let us asumme that A is a cluster semi intense. Since A is open set by Proposition 3. 2. 7 part 4, we have that $A \subseteq A_{f_\sigma} = (A_{t_\sigma})_{f_\sigma}$.

Conversely, let $A \subseteq (A_{t_\sigma})_{f_\sigma}$, that is, $A_{f_\sigma} \subseteq (\left(A_{t_\sigma}\right)_{f_\sigma})_{f_\sigma} = (A_{t_\sigma})_{f_\sigma}$. By Proposition 3. 2. 4, $P_{t_\sigma} \subseteq P_{f_\sigma}$, for any $P \subseteq X$, that is, $(A_{t_\sigma})_{f_\sigma} \subseteq \left(A_{f_\sigma}\right)_{f_\sigma} = A_{f_\sigma}$. Hence A is a cluster semi intense set.

3) The proof is easily because if A is open and close by Proposition 3. 2. 6, and Proposition 3. 2. 7 we get $A_{t_\sigma} = A = A_{f_\sigma}$. Since A is a cluster intense, $(A_{t_\sigma})_{f_\sigma} = A_{f_\sigma} = A$ and $(A_{f_\sigma})_{t_\sigma} = A_{t_\sigma} = A$ hence $(A_{f_\sigma})_{t_\sigma} =$

$(A_{t_\sigma})_{f_\sigma}$. Conversely, since A is clopen we have that $A_{t_\sigma} = A = A_{f_\sigma}$, thus $(A_{t_\sigma})_{f_\sigma} = A_{f_\sigma}$ and so that $(A_{f_\sigma})_{t_\sigma} = A_{t_\sigma}$, thus A is a cluster intense set■

Proposition 4. 2. 6: Let $(X, \tau, \delta, \sigma)$ be a co bushy space. If A or B is an open set, then $(A_{f_\sigma})_{t_\sigma} \cap (B_{f_\sigma})_{t_\sigma} = (A \cap B)_{f_\sigma})_{t_\sigma}$.

Proof.

Evedint by propositios 2. 1. 3, and 2. 2. 3, $((A \cap B)_{f_\sigma})_{t_\sigma} \subset (A_{f_\sigma})_{t_\sigma} \cap (B_{f_\sigma})_{t_\sigma}$. It suffice to show that $(A_{f_\sigma})_{t_\sigma} \cap (B_{f_\sigma})_{t_\sigma} \subset ((A \cap B)_{f_\sigma})_{t_\sigma}$.

Let us assume that $A \in \tau$. By Proposition 2. 2 .3 part 8, $A \cap B_{f_\sigma} \subset (A \cap B)_{f_\sigma}$, so that $\left(A \cap B_{f_\sigma}\right)_{t_\sigma} \subset \left((A \cap B)_{f_\sigma}\right)_{t_\sigma}$, that is, $A_{t_\sigma} \cap (B_{f_\sigma})_{t_\sigma} \subset \left((A \cap B)_{f_\sigma}\right)_{t_\sigma}$. Since A is an open set, and by Proposition 2. 1. 3 part 5, we have $A \subset A_{t_\sigma}$ thus $A \cap (B_{f_\sigma})_{t_\sigma} \subset A_{t_\sigma} \cap (B_{f_\sigma})_{t_\sigma} \subset \left((A \cap B)_{f_\sigma}\right)_{t_\sigma}$, and by Lemma 3. 2. 13, we have that $\left([A \cap ((B_{f_\sigma})_{t_\sigma}]_{f_\sigma}\right)_{t_\sigma} \subset \left(((A \cap B)_{f_\sigma})_{t_\sigma})_{f_\sigma}\right)_{t_\sigma} = \left((A \cap B)_{f_\sigma}\right)_{t_\sigma}$. Thus $\left([A \cap ((B_{f_\sigma})_{t_\sigma}]_{f_\sigma}\right)_{t_\sigma} \subset \left((A \cap B)_{f_\sigma}\right)_{t_\sigma}$--------- (1).

Since $\left(B_{f_\sigma}\right)_{t_\sigma}$ is an open set , $A_{f_\sigma} \cap (B_{f_\sigma})_{t_\sigma} \subset [A \cap ((B_{f_\sigma})_{t_\sigma}]_{f_\sigma}$ by take the takeoff we have that, $(A_{f_\sigma})_{t_\sigma} \cap ((B_{f_\sigma})_{t_\sigma})_{t_\sigma} \subset \left([A \cap ((B_{f_\sigma})_{t_\sigma}]_{f_\sigma}\right)_{t_\sigma}$,that is, $(A_{f_\sigma})_{t_\sigma} \cap (B_{f_\sigma})_{t_\sigma} \subset (A_{f_\sigma})_{t_\sigma} \cap ((B_{f_\sigma})_{t_\sigma})_{t_\sigma} \subset \left([A \cap ((B_{f_\sigma})_{t_\sigma}]_{f_\sigma}\right)_{t_\sigma}$--------(2). By (1) and (2) it is following that $(A_{f_\sigma})_{t_\sigma} \cap (B_{f_\sigma})_{t_\sigma} \subset ((A \cap B)_{f_\sigma})_{t_\sigma}$. Thus $(A_{f_\sigma})_{t_\sigma} \cap (B_{f_\sigma})_{t_\sigma} \subset ((A \cap B)_{f_\sigma})_{t_\sigma}$■

We can see that in case of τ_σ , the above proposition is not true because takeoff set is not $\tau_\sigma - open\ set$.

Corollary 4. 2. 7: Let (X,τ,δ,σ) be a co bushy space. If A or B is open, then

1. The intersection of cluster too intense sets is a cluster intense set.
2. The union of complement too intense sets is a cluster sime intense set.

Proof.

1) Suppose A, B are cluster too intense. By Propositions 2. 1. 3, and 4. 2. 6, we get, $(\mathrm{A}\cap\mathrm{B})_{t_\sigma} = (A)_{t_\sigma} \cap (\mathrm{B})_{t_\sigma} = (A_{f_\sigma})_{t_\sigma} \cap (B_{f_\sigma})_{t_\sigma} = ((A\cap B)_{f_\sigma})_{t_\sigma}$. Hence $(\mathrm{A}\cap\mathrm{B})$ is a cluster too intense set.

2) Let A, B be a cluster too intense sets. Then $A_{t_\sigma} = (A_{f_\sigma})_{t_\sigma}$, and $B_{t_\sigma} = (B_{f_\sigma})_{t_\sigma}$. By Propositions 2. 2. 3, 2. 2. 14, and 4. 2. 6, its follows that:

$((\mathrm{X}-A)\cup(\mathrm{X}-B))_{f_\sigma} = (X-A)_{f_\sigma} \cup (X-B)_{f_\sigma} = (\mathrm{X}-A_{t_\sigma} \cup \mathrm{X}-B_{t_\sigma})$

$= \mathrm{X}-(A_{f_\sigma})_{t_\sigma} \cup \mathrm{X}-(B)_{f_\sigma})_{t_\sigma} = X-\big((A_{f_\sigma})_{t_\sigma} \cap (B_{f_\sigma})_{t_\sigma}\big)$

$= X-((A\cap B)_{f_\sigma})_{t_\sigma} = ((\mathrm{X}-(A\cap B))_{t_\sigma})_{f_\sigma} = ((\mathrm{X}-A\cup\mathrm{X}-B)_{t_\sigma})_{f_\sigma}$,

thus $(\mathrm{X}-A)\cup(\mathrm{X}-B)$ is a cluster sime intense■

Proposition 4. 2. 8: Let $(\mathrm{X},\tau,\delta,\sigma)$ be a co bushy space. Then

1. H is $f_\sigma -$ set if and only if $H = H_{t_\sigma}$ and H is cluster too intense.
2. H is $t_\sigma -$ set if and only if $\mathrm{H} = H_{f_\sigma}$ and H is cluster semi intense.

Proof.

1) Let H is $f_\sigma -$ set . Then $H = (H_{f_\sigma})_{t_\sigma}$. Since X is co bushy, $H_{t_\sigma} \subseteq H_{f_\sigma}$. Then $H_{t_\sigma} \subset (H_{t_\sigma})_{t_\sigma} \subset (H_{f_\sigma})_{t_\sigma} = H$.

Also, $H = (H_{f_\sigma})_{t_\sigma} \subset ((H_{f_\sigma})_{t_\sigma})_{t_\sigma} = H_{t_\sigma}$. Hence $H = H_{t_\sigma}$.

Now, to prove H is cluster too intense. Since X is co bushy, $H_{t_\sigma} \subset (H_{t_\sigma})_{t_\sigma} \subset (H_{f_\sigma})_{t_\sigma}$ and so that $(H_{f_\sigma})_{t_\sigma} \subset ((H_{f_\sigma})_{t_\sigma})_{t_\sigma} = H_{t_\sigma}$. This means H is cluster too intense. Conversely, $H = H_{t_\sigma} = (H_{f_\sigma})_{t_\sigma}$, hence H is $f_\sigma -$ set.

2) Let us suppose that H is $t_\sigma -$ set. By Remark 2. 3. 9, $X - H$ is $f_\sigma -$ set . By part 1, $X - H = (X - H)_{t_\sigma}$ and $X - H$ is a cluster too intense. So, $X - H = (X - H)_{t_\sigma} = X - H_{f_\sigma}$. Hence $H = H_{f_\sigma}$. Since $X - H$ is a cluster too intense, $(X - H)_{t_\sigma} = ((X - H)_{f_\sigma})_{t_\sigma}$, by Proposition 2. 2. 23 $X - H_{f_\sigma} = X - (H_{t_\sigma})_{f_\sigma}$. Hence $H_{f_\sigma} = (H_{t_\sigma})_{f_\sigma}$.

Conversely, suppose H is a cluster semi intense and $H = H_{f_\sigma}$, $X - H = X - H_{f_\sigma} = (X - H)_{t_\sigma}$. But H is a cluster semi intense, by Proposition 4. 2. 4, $X - H$ is cluster too intense and $(X - H)_{t_\sigma} = ((X - H)_{f_\sigma})_{t_\sigma}$, hence $X - H$ is $f_\sigma -$ set, that is by Remark 2. 3. 9, H is $t_\sigma -$ set■

Proposition 4. 2. 9: Let $(X, \tau, \delta, \sigma)$ be a co bushy space. H is a cluster intense set if and only if cluster brim of H consider by: $B_\delta(H) = (H_{t_\sigma})_{f_\sigma} \cap ((X - H)_{t_\sigma})_{f_\sigma}$.

Proof.

Suppose H is a cluster intense. Then $H_{f_\sigma} = (H_{t_\sigma})_{f_\sigma}$ and $H_{t_\sigma} = (H_{f_\sigma})_{t_\sigma}$. So that $X - H_{t_\sigma} = X - (H_{f_\sigma})_{t_\sigma}$ by Proposition 2. 2. 23 part 5, $(X - H)_{f_\sigma} = ((X - H)_{t_\sigma})_{f_\sigma}$, hence $B_\delta(H) = H_{f_\sigma} \cap (X - H)_{f_\sigma}) = (H_{t_\sigma})_{f_\sigma} \cap ((X - H)_{t_\sigma})_{f_\sigma}$.

Conversely, $B_\delta(H) = H_{f_\sigma} - H_{t_\sigma}$. Then $H_{f_\sigma} = B_\delta(H) \cup (H_{f_\sigma} \cap H_{t_\sigma}) = (B_\delta(H) \cup H_{t_\sigma}) \cap (B_\delta(H) \cup H_{f_\sigma}) \subseteq B_\delta(H) \cup H_{t_\sigma} = ((H_{t_\sigma})_{f_\sigma} \cap ((X - H)_{t_\sigma})_{f_\sigma}) \cup H_{t_\sigma} \subseteq (H_{t_\sigma})_{f_\sigma} \cup H_{t_\sigma}$. But by Proposition 3. 2. 7 part 4, $H_{t_\sigma} \subseteq (H_{t_\sigma})_{f_\sigma}$. Thus$(H_{t_\sigma})_{f_\sigma} \cup H_{t_\sigma} = (H_{t_\sigma})_{f_\sigma}$, that is, $H_{f_\sigma} \subseteq (H_{t_\sigma})_{f_\sigma}$. Again, since X is a co bushy space, $H_{t_\sigma} \subset H_{f_\sigma}$, and so that $(H_{t_\sigma})_{f_\sigma} \subset (H_{f_\sigma})_{f_\sigma} \subset H_{f_\sigma}$. Hence H is cluster semi intense set.

Now,by Proposition 4. 1. 14 part 7, $B_\delta(H) = B_\delta(X - H) = (X - H)_{f_\sigma} - (X - H)_{t_\sigma} = \left((X - H)_{t_\sigma}\right)_{f_\sigma} - ((\mathrm{X} - H)_{f_\sigma})_{t_\sigma}$. Then $(X - H)_{f_\sigma} = ((X - H)_{t_\sigma})_{f_\sigma}$ if and only if $X - H_{t_\sigma} = X - (H_{f_\sigma})_{t_\sigma}$if and only if $H_{t_\sigma} = (H_{f_\sigma})_{t_\sigma}$. Hence H is cluster too intense so H is cluster intense set■

Definition 4. 2. 10: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. H is called $f_\sigma -$ brim if and only if $H = (H_{f_\sigma})_{t_\sigma} \cap ((X - H)_{f_\sigma})_{t_\sigma} = (B_\delta(H))_{t_\sigma}$ denoted by $B_{t_\sigma}(H)$.

Note 4. 2. 11: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. Then $f_\sigma -$ brim is open set.

That is clear because $B_{t_\sigma}(H) = (B_\delta(H))_{t_\sigma}$, but by Note 2. 1. 7 part 2, takeoff set is an $\tau -$open set. Thus $B_{t_\sigma}(H)$ is open set.

Remark 4. 2. 12: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space.

1. If $H \notin \sigma$ then $B_{t_\sigma}(H) = \emptyset_{t_\sigma}$.
2. If X is co bushy space and $H \notin \sigma$, then $B_{t_\sigma}(H) = \emptyset$.
3. If H is scant set, then $B_{t_\sigma}(H) = \emptyset$.

Proof.

1) $(B_\delta(H))_{t_\sigma} = (H_{f_\sigma})_{t_\sigma} \cap ((X - H)_{f_\sigma})_{t_\sigma} = (H_{f_\sigma} \cap (X - H)_{f_\sigma})_{t_\sigma}$ but by proposition 2. 2. 3 $H_{f_\sigma} = \emptyset$ that is, $(B_\delta(H))_{t_\sigma} = (\emptyset)_{t_\sigma}$.

2) Clear by part 1 and Corollary 3. 2. 5.

3) Clear because if H is scant set, then$((\mathrm{H})_{f_\sigma})_{t_\sigma} = \emptyset$■

Proposition 4. 2. 13 : Let $(X, \tau, \delta, \sigma)$ be a co bushy space. Then $B_{t_\sigma}(H)$ is $f_\sigma -$ set.

Proof.

Since X is a co bushy space and $B_\delta(H)$ is a closed set, by Prposition 3. 2. 6, $(B_\delta(H))_{t_\sigma} \subseteq B_\delta(H)$. So that $B_{t_\sigma}(H) = (B_\delta(H))_{t_\sigma} \subseteq B_\delta(H)$.

By Lemma 2. 2. 17, we have $(B_{t_\sigma}(H))_{f_\sigma} \subseteq (B_\delta(H))_{f_\sigma} \subseteq B_\delta(H)$, and so that $((B_{t_\sigma}(\mathrm{H}))_{f_\sigma})_{t_\sigma} \subseteq (B_\delta(H))_{t_\sigma} = B_{t_\sigma}(H)$.

Since X is a co bushy space, and $B_{t_\sigma}(H)$ is open by Proposition 3. 2. 7 part 2, $B_{t_\sigma}(H) \subseteq ((B_{t_\sigma}(\mathrm{H}))_{f_\sigma})_{t_\sigma}$. Hence $B_{t_\sigma}(H)$ is $f_\sigma -$ set■

Proposition 4. 2. 14: Let $(X, \tau, \delta, \sigma)$ be a co bushy space and . Then

1- H is a cluster too intense if and only if H_{t_σ} is $f_\sigma -$ set and $B_{t_\sigma}(H) = \emptyset$.

2- H is a cluster semi intense if and only if H_{f_σ} is $t_\sigma -$ set and $B_{t_\sigma}(H) = \emptyset$.

Proof.

1) Suppose H is cluster too intense. Then $H_{t_\sigma} = (H_{f_\sigma})_{t_\sigma}$, so that $((H_{t_\sigma})_{f_\sigma})_{t_\sigma} = \left(((H_{f_\sigma})_{t_\sigma})_{f_\sigma}\right)_{t_\sigma} = (H_{f_\sigma})_{t_\sigma} = H_{t_\sigma}$. Thus H_{t_σ} is f $-$set.

Now, H_{t_σ} is an open set by Proposition 3. 2. 7 part 4, $H_{t_\sigma} \subseteq \left(H_{t_\sigma}\right)_{f_\sigma}$. Since H is a cluster too intense, $(H_{f_\sigma})_{t_\sigma} = H_{t_\sigma} \subseteq \left(H_{t_\sigma}\right)_{f_\sigma}$ that is $X - \left(H_{t_\sigma}\right)_{f_\sigma} \subseteq X - \left(H_{f_\sigma}\right)_{t_\sigma}$, but $B_{t_\sigma}(H) = (H_{f_\sigma})_{t_\sigma} \cap ((X - H)_{f_\sigma})_{t_\sigma} = (H_{f_\sigma})_{t_\sigma} \cap X - \left(H_{t_\sigma}\right)_{f_\sigma} \subseteq (H_{f_\sigma})_{t_\sigma} \cap X - \left(H_{f_\sigma}\right)_{t_\sigma} = \emptyset$. Hence $B_{t_\sigma}(H) = \emptyset$.

Conversely, let $B_{t_\sigma}(H)=\emptyset$ we have that $(H_{f_\sigma})_{t_\sigma}\subseteq \left(H_{t_\sigma}\right)_{f_\sigma}$. So that $(H_{f_\sigma})_{t_\sigma}=(\left(H_{f_\sigma}\right)_{t_\sigma})_{t_\sigma}\subseteq(\left(H_{t_\sigma}\right)_{f_\sigma})_{t_\sigma}=H_{t_\sigma}$. Also by Propositon 3. 2. 4, $H_{t_\sigma}\subseteq\left(H_{t_\sigma}\right)_{t_\sigma}\subseteq\left(H_{f_\sigma}\right)_{t_\sigma}$. Hence $H_{t_\sigma}=\left(H_{f_\sigma}\right)_{t_\sigma}$.

2) Let H be a cluster semi intense. Then $H_{f_\sigma}=\left(H_{t_\sigma}\right)_{f_\sigma}$. To show that $H_{f_\sigma}=(\left(H_{f_\sigma}\right)_{t_\sigma})_{f_\sigma}$. Since X is a co bushy, $H_{t_\sigma}\subseteq H_{f_\sigma}$, that follows it $(\left(H_{f_\sigma}\right)_{t_\sigma})_{f_\sigma}\subseteq(\left(H_{f_\sigma}\right)_{f_\sigma})_{f_\sigma}\subseteq\left(H_{f_\sigma}\right)_{f_\sigma}\subseteq H_{f_\sigma}$……..(1).

Also, $H_{f_\sigma}=\left(H_{t_\sigma}\right)_{f_\sigma}\subseteq(\left(H_{t_\sigma}\right)_{t_\sigma})_{f_\sigma}\subseteq(\left(H_{f_\sigma}\right)_{t_\sigma})_{f_\sigma}$…….(2).

By (1) and (2), $H_{f_\sigma}=(\left(H_{f_\sigma}\right)_{t_\sigma})_{f_\sigma}$. Hence H_{f_σ} is $t-$set.

So that $B_{t_\sigma}(H)=(H_{f_\sigma})_{t_\sigma}\cap((X-H)_{f_\sigma})_{t_\sigma}=(H_{f_\sigma})_{t_\sigma}\cap X-\left(H_{t_\sigma}\right)_{f_\sigma}=(H_{f_\sigma})_{t_\sigma}\cap(X-H_{f_\sigma})\subseteq(H_{f_\sigma})_{f_\sigma}\cap(X-H_{f_\sigma})\subseteq H_{f_\sigma}\cap(X-H_{f_\sigma})=\emptyset$.

Hence $B_{t_\sigma}(H)=\emptyset$.

Conversely, let $B_{t_\sigma}(H)=\emptyset$, $(H_{f_\sigma})_{t_\sigma}\subseteq\left(H_{t_\sigma}\right)_{f_\sigma}$. Then $H_{f_\sigma}=(\left(H_{f_\sigma}\right)_{t_\sigma})_{f_\sigma}\subseteq(\left(H_{t_\sigma}\right)_{f_\sigma})_{f_\sigma}\subseteq\left(H_{t_\sigma}\right)_{f_\sigma}$. But $\left(H_{t_\sigma}\right)_{f_\sigma}\subseteq\left(H_{f_\sigma}\right)_{f_\sigma}\subseteq H_{f_\sigma}$. Thus H is a cluster semi intense■

Corollary 4. 2. 15: Let (X,τ,δ,σ) be a co bushy space. H is a cluster intense set if and only if

1- H_{t_σ} is $f_\sigma-$ set.

2- H_{f_σ} is $t_\sigma-$ set.

3- $B_{t_\sigma}(H)=\emptyset$.

Proof.

Let H be a cluster intense. Then H is a cluster too intense by Proposition 4. 2. 14, H_{t_σ} is $f_\sigma-$set. Since H is also cluster sime intense by Proposition

4. 2. 14, H_{f_σ} is $t_\sigma -$ set, and also $B_{t_\sigma}(H) = \emptyset$. Conversly, since H_{t_σ} is $f_\sigma -$ set and $B_{t_\sigma}(H) = \emptyset$ by Proposition 4. 2. 14, H is a cluster too intense and also, since H_{f_σ} is $t_\sigma -$ set and $B_{t_\sigma}(H) = \emptyset$ by Proposition 4. 2. 14, H is a cluster sime intense, thus H be a cluster intense■

Remark 4. 2. 16: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. H is a subset of X, then the following statements are hold:

1- If H satisfy $(H_{f_\sigma})_{t_\sigma} \subseteq \left(H_{t_\sigma}\right)_{f_\sigma}$, then $(X - H)$ also satisfy $((X - H)_{f_\sigma})_{t_\sigma} \subseteq \left((X - H)_{t_\sigma}\right)_{f_\sigma}$.

2- If H satisfy $(H_{t_\sigma})_{f_\sigma} \subseteq \left(H_{f_\sigma}\right)_{t_\sigma}$, then $(X - H)$ also satisfy $((X - H)_{t_\sigma})_{f_\sigma} \subseteq \left((X - H)_{f_\sigma}\right)_{t_\sigma}$.

Proof.

1) Let H satisfy $(H_{f_\sigma})_{t_\sigma} \subset \left(H_{t_\sigma}\right)_{f_\sigma}$. Then $X - (H_{t_\sigma})_{f_\sigma} \subset \mathrm{X} - \left(H_{f_\sigma}\right)_{t_\sigma}$by Proposition 2. 2. 23, $((\, X - H)_{f_\sigma})_{t_\sigma} \subset ((\mathrm{X} - H)_{t_\sigma})_{f_\sigma}$.

2) Let H satisfy $(H_{t_\sigma})_{f_\sigma} \subset \left(H_{f_\sigma}\right)_{t_\sigma}$. Then $X - (H_{f_\sigma})_{t_\sigma} \subset \mathrm{X} - \left(H_{t_\sigma}\right)_{f_\sigma}$by Proposition 2. 2. 23, $((\, X - H)_{t_\sigma})_{f_\sigma} \subset ((\mathrm{X} - H)_{f_\sigma})_{t_\sigma}$■

Proposition 4. 2. 17: Let $(X, \tau, \delta, \sigma)$ be a co bushy space, and $\left(H_{f_\sigma}\right)_{t_\sigma} \subseteq \left(H_{t_\sigma}\right)_{f_\sigma}$. Then

1- $\left(H_{f_\sigma}\right)_{t_\sigma} = (\left(H_{t_\sigma}\right)_{f_\sigma})_{t_\sigma}$.

2- $\left(H_{t_\sigma}\right)_{f_\sigma} = (\left(H_{f_\sigma}\right)_{t_\sigma})_{f_\sigma}$.

3- If 1 and 2 are holds, then $\left(H_{f_\sigma}\right)_{t_\sigma} = \left(H_{t_\sigma}\right)_{f_\sigma}$.

Proof.

1) Let $(H_{f_\sigma})_{t_\sigma} \subseteq (H_{t_\sigma})_{f_\sigma}$. Then $(H_{f_\sigma})_{t_\sigma} \subset ((H_{f_\sigma})_{t_\sigma})_{t_\sigma} \subset ((H_{t_\sigma})_{f_\sigma})_{t_\sigma}$. Since X is a co bushy, $H_{t_\sigma} \subset H_{f_\sigma}$, that is, $(H_{t_\sigma})_{f_\sigma} \subset (H_{f_\sigma})_{f_\sigma} \subset H_{f_\sigma}$ and so that $((H_{t_\sigma})_{f_\sigma})_{t_\sigma} \subset (H_{f_\sigma})_{t_\sigma}$. Hence$(H_{f_\sigma})_{t_\sigma} = ((H_{t_\sigma})_{f_\sigma})_{t_\sigma}$.

2) By Remark 4. 2. 16 part 1, $((X-H)_{f_\sigma})_{t_\sigma} \subseteq ((X-H)_{t_\sigma})_{f_\sigma}$by 1, $((X-H)_{f_\sigma})_{t_\sigma} = (((X-H)_{t_\sigma})_{f_\sigma})_{t_\sigma}$ by Proposition 2. 2. 23, we have that $X-(H_{t_\sigma})_{f_\sigma} = X-((H_{f_\sigma})_{t_\sigma})_{f_\sigma}$, then $(H_{t_\sigma})_{f_\sigma} = ((H_{f_\sigma})_{t_\sigma})_{f_\sigma}$.

3) If $(H_{t_\sigma})_{f_\sigma} = ((H_{f_\sigma})_{t_\sigma})_{f_\sigma}$ hold. Since X is a co bushy, by Proposition 3. 2. 4, $((H_{f_\sigma})_{t_\sigma})_{t_\sigma} \subset ((H_{f_\sigma})_{t_\sigma})_{f_\sigma} = (H_{t_\sigma})_{f_\sigma}$but H_{f_σ} is closed set by proposition 3. 2. 6, $(H_{f_\sigma})_{t_\sigma} = ((H_{f_\sigma})_{t_\sigma})_{t_\sigma}$ thus $(H_{f_\sigma})_{t_\sigma} \subset (H_{t_\sigma})_{f_\sigma}$.
Also, $(H_{f_\sigma})_{t_\sigma} = (((H)_{t_\sigma})_{f_\sigma})_{t_\sigma}$ is hold. Since X is a co bushy, by Proposition 3. 2. 4, $(((H)_{t_\sigma})_{f_\sigma})_{t_\sigma} \subset (((H)_{t_\sigma})_{f_\sigma})_{f_\sigma} \subset (H_{t_\sigma})_{f_\sigma}$, hence$(H_{f_\sigma})_{t_\sigma} \subset (H_{t_\sigma})_{f_\sigma}$ ■

5.1 Focal Cluster $\oint_\sigma$

After formulating the major relation of construction, we will introduce a class of sets whose construction is based on open sets in the $\sigma -$ proximity topological space τ_σ with the cluster σ. The properties of this family are studied.

Definition 5. 1. 1: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space, then *focal cluster* denoted by $\oint_\sigma$ is family of all subsets of X satisfies the condition:

$\oint_\sigma = \{ A \subset X; there\ exist\ \mathcal{U} \in \tau_\sigma\ such\ that\ \mathcal{U} \propto_\sigma A\}$.

Example 5. 1. 2: Let $X = \{1,2,3\}$ and δ is a discrete proximity.

$\sigma = \big\{\{3\}, \{1,3\}, \{2,3\}, X\big\}$, and let $\tau_\sigma = \{X, \emptyset, \{2\}, \{1\}, \}$. Then

$\oint_\sigma = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

It is noticed that, if we replace $there\ exist\ \mathcal{U} \in \tau_\sigma$ in above definition by $every\ \mathcal{U} \in \tau_\sigma$ the result will not be the same, because $\oint_\sigma = \{\emptyset\}$.

The following is a review of the most important properties of the focal cluster, which have a direct and indirect effect on building density space, resolvability, as well as their transferability between functions.

Theorem 5. 1. 3: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space, and A, B nonempty subsets of X, and $\oint_\sigma$ focal cluster. Then

1. $A \in \oint_\sigma$ if and only if $X \propto_\sigma A$;
2. If $A \notin \sigma, then\ A \in \oint_\sigma$;
3. If $B \subset A\ and\ A \in \oint_\sigma, then\ B \in \oint_\sigma$;
4. A or $B \in \oint_\sigma$ if and only if $A \cap B \in \oint_\sigma$;

5. If A is a proper τ_σ −open set, then $A \in \oint_\sigma$;
6. $X \notin \oint_\sigma$ and $\emptyset \in \oint_\sigma$;
7. If $(A \cup B) \in \oint_\sigma$, then A and $B \in \oint_\sigma$.

Proof.

1) Let A $\in \oint_\sigma$. Then there exists $\mathcal{U} \in \tau_\sigma$ $such\ that\ \mathcal{U} \propto_\sigma A$, thus $(\mathcal{U} \cap X - A) \in \sigma$. But $(\mathcal{U} \cap X - A) \subset (X \cap X - A)$, thus $(X \cap X - A) \in \sigma$, this means $X \propto_\sigma A$. Conversely, obviously by Definition 5.1.1.

2) Let us suppose $A \notin \sigma$. By Proposition 1. 1. 15 part 1, $(X - A) \in \sigma$. Then there exists $X \in \tau_\sigma$ such that $(X \cap X - A) = (X - A) \in \sigma$, hence $X \propto_\sigma A$, that is, $A \in \oint_\sigma$.

3) Let $A \in \oint_\sigma$ and $B \subset A$. Then there exists $\mathcal{U} \in \tau_\sigma$ such that $\mathcal{U} \propto_\sigma A$, thus $(\mathcal{U} \cap X - A) \in \sigma$ but $(\mathcal{U} \cap X - A) \subset (\mathcal{U} \cap X - B)$, this implies to $\mathcal{U} \propto_\sigma B$, thus $B \in \oint_\sigma$.

4) Suppose A or $B \in \oint_\sigma$. Then by part 3, $(A \cap B) \in \oint_\sigma$. Conversely, let (A$\cap B) \in \oint_\sigma$. Then there exists $\mathcal{U} \in \tau_\sigma$ such that $\mathcal{U} \propto_\sigma (A \cap B)$, this means $((\mathcal{U} - A) \cup (\mathcal{U} - B)) \in \sigma$. By [C2], $(\mathcal{U} - A) \in \sigma$ or $(\mathcal{U} - B) \in \sigma$, that is, $A \in \oint_\sigma or\ B \in \oint_\sigma$.

5) Let A $\in \tau_\sigma$. Then A $\notin \sigma$, by part 2 we have that $A \in \oint_\sigma$.

6) Clear by 3.

7) Clear by 3■

Remark 5. 1. 4: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space and δ be a discrete proximity. Then

1. $A \notin \sigma$ if and only if $A \in \oint_\sigma$;
2. $A, B \in \oint_\sigma$ if and only if $A \cup B \in \oint_\sigma$.

Proof.

1) Let $A \in \oint_\sigma$. Then there exists $\mathcal{U} \in \tau_\sigma$ such that $\mathcal{U} \propto_\sigma A$. But $\mathcal{U} - A \subset X - A$, thus $X - A \in \sigma$. If possible $A \in \sigma$. This implies that $A \cap X - A = \emptyset$ which is a contradiction because δ is a discrete proximity, hence $A \notin \sigma$.

2) $A, B \in \oint_\sigma$ by part 1 if and only if $A \notin \sigma$ *and* $B \notin \sigma$ if and only if $(A \cup B) \notin \sigma$ if and only if $(A \cup B) \in \oint_\sigma$■

Remark 5. 1. 5: Let $(X, \delta, \sigma, \tau_\sigma)$ be a $\sigma -$ Topological Proximity Space. Then

1. Focal cluster in discrete proximity space is ideal.
2. $\oint_{\sigma_x} = \{A \subset X \,; x\bar{\delta}A\}$ is a focal cluster.

Proof.

1) By Theorem 5. 1. 3 part 6, $\emptyset \in \oint_\sigma$, and by part 3, we have that, for every $B \subset A$ *and* $A \in \oint_\sigma$, *then* $B \in \oint_\sigma$. Also by Remark 5. 1. 4 part 2, $A, B \in \oint_\sigma$ if and only if $(A \cup B) \in \oint_\sigma$. Thus $\oint_\sigma$ is ideal.

2) Let $x\bar{\delta}A$. Then $A \notin \sigma_x$. By Proposition 1. 1 15 part 1, $X - A \in \sigma_x$, hence $X \propto_\sigma A$, that is, $\oint_{\sigma_x}$ is focal cluster■

Lemma 5. 1. 6: If $\oint_{\sigma_x}$ and $\oint_{\sigma_y}$ are two focal clusters in (X, δ) and $\oint_{\sigma_x} \subset \oint_{\sigma_y}$ then $\oint_{\sigma_x} = \oint_{\sigma_y}$.

Proof.

Let $A \notin \oint_{\sigma_x}$. Then for every $\mathcal{U} \in \tau_\sigma$, $\mathcal{U} \overline{\propto_\sigma} A$ i.e. $X \overline{\propto_\sigma} A$ this implies $X - A \notin \sigma_x$, thus $X - A \notin \sigma_y$. If not, $X - A \in \sigma_y$ this implies to $\sigma_x \subset \sigma_y$ which is a contradiction, hence $A \notin \oint_{\sigma_y}$, that is, $\oint_{\sigma_x} = \oint_{\sigma_y}$ ■

Proposition 5. 1. 7: Let $f: (X, \tau_\sigma) \longrightarrow (Y, \tau_{\breve{\sigma}})$ be a proximally open and $f^{-1}$is continuous function, then $f\left(\oint_\sigma\right) = \{A \subseteq Y; f^{-1}(A) \in \oint_\sigma\}$ is focal cluster in Y.

Proof.

Let $A \in f\left(\oint_\sigma\right) \Rightarrow f^{-1}(A) \in \oint_\sigma$. Since $\oint_\sigma$ is a focal cluster in X, then there exists $\mathcal{U} \in \tau_\sigma$ such that $\mathcal{U} - f^{-1}(A) \in \sigma$ by Theorem 1. 1. 7, $X - f^{-1}(A) \in \sigma$. Since $X - f^{-1}(A) = f^{-1}(Y - A) \in \sigma$, then $f(f^{-1}(Y - A)) \in f(\sigma)$, and $f(f^{-1}(Y - A)) \subset Y - A$. By Theorem 1. 1. 24, $f(\sigma)$ is a cluster relation on Y, and $Y - A \in f(\sigma)$ by Definition 5. 1. 1, we have that $f\left(\oint_\sigma\right)$ is focal cluster set in Y■

5.2 Sporadic

In this section, we introduce the definition of a Sporadic using the concept of cluster and thus obtain a family negative to the cluster family.

Definition 5. 2. 1: Sporadic denoted by $\mathcal{S}$ is a family of all subsets of X satisfies the condition:
$\mathcal{S} = \{ H \subset X;\ such\ that\ (X - H) \in \sigma\}$.

Examples 5. 2. 2:
1) Let $X = \{1,2,3\}$ and δ is a discrete proximity.
$\sigma = \{\{3\},\{1,3\},\{2,3\},X\}$. Then $\mathcal{S} = \{\emptyset,\{1\},\{2\},\{1,2\}\}$.

2) If δ is an indiscrete proximity, then $\mathcal{S} = \{A \subset X; A \neq X\}$ is sporadic.

That is clear because if $A \in \mathcal{S}$, then A is proper subset of X, thus $(X - A) \neq \emptyset$, but δ is indiscrete proximity, thus $(X - A) \in \sigma$, by Definition 5. 2. 1, $\mathcal{S}$ is sporadic.

It is to be noted that, according to earlier results, every Sporadic is a focal cluster.

Theorem 5. 2. 3: Let (X, δ) be a proximity space. H, D subsets of X. Then

1. If $H \notin \sigma, then\ H \in \mathcal{S}$.
2. If $D \subset H\ and\ H \in \mathcal{S}, then\ D \in \mathcal{S}$.
3. H or $D \in \mathcal{S}$ if and only if $H \cap D \in \mathcal{S}$.
4. $X \notin \mathcal{S}\ and\ \emptyset \in \mathcal{S}$.
5. If $(H \cup D) \in \mathcal{S}$, then H and $D \in \mathcal{S}$.

Proof.

1) Let us suppose $H \notin \sigma$. By Proposition 1. 1. 15 part1, X $-H \in \sigma$, that is, $H \in \mathcal{S}$.

2) Let $H \in \mathcal{S}$ *and* $D \subset H$. Then $X - H \in \sigma$. By Proposition 1. 1. 15 part 2, we have that $X - H \subset X - D \in \sigma$, thus $D \in \mathcal{S}$.

3) Let H or $D \in \mathcal{S}$. By part 2, $(H \cap D) \in \mathcal{S}$. Conversely, let H $\cap D \in \mathcal{S}$. Then $X - (H \cap D) \in \sigma$, and $(X - H) \cup (X - D) \in \sigma$, thus $X - H \in \sigma$ or $X - D \in \sigma$, that is, $H \in \mathcal{S}$ *or* $D \in \mathcal{S}$.

4) Clear.

5) Let $(H \cup D) \in \mathcal{S}$. Then $((X - H) \cap (X - D)) \in \sigma$. By Proposition 1. 1. 15 part 2,$(X - H) \in \sigma$ and $(X - D) \in \sigma$, thus H and $D \in \mathcal{S}$■

Proposition 5. 2. 4: Let δ_1 *and* δ_2 be two proximity defined on X such that $\delta_2 > \delta_1$. Then

1. $\sigma(\delta_2) \subset \sigma(\delta_1)$.
2. $\mathcal{S}(\delta_2) \subset \mathcal{S}(\delta_1)$.

Proof.

1) Let H $\in \sigma(\delta_2)$. Then H$\delta_2 D$ for every $D \in \sigma(\delta_2)$. Since $\delta_2 > \delta_1$, by Definition 1. 1.5, $H\delta_2 D$ implies $H\delta_1 D$ for every $D \in \sigma(\delta_1)$. Thus H $\in \sigma(\delta_1)$.

2) Let $H \in \mathcal{S}(\delta_2)$. Then $(X - H) \in \sigma(\delta_2)$. By part 1, $(X - H) \in \sigma(\delta_1)$. Thus $H \in \mathcal{S}(\delta_1)$■

Definition 5. 2. 5: The quadruple $(X, \delta, \tau_\delta, \mathcal{S})$ is called a sporadic topological proximity space, where (X, τ_δ) is a topological proximity space and (X, δ) is a proximity space.

That means the topology proximity and sporadic depend upon the proximity.

Definition 5. 2. 6: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space. A point $x \in X$ is said to be *sporadic follower point* of a subset H of topological space (X, τ_δ) , if for every $\mathcal{U} \in \tau_\delta(x)$, there exist $O \in \mathcal{S}$ such that $(\mathcal{U} \cap H)\delta O$. All the sporadic follower points of a set H are denoted by $H_{\mathcal{S}_f}$ as $\mathcal{S}$ *−follower set*.

Proposition 5. 2. 7: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space. H, D are subsets of X. Then

1. If $H \subset D, then\ H_{\mathcal{S}_f} \subset D_{\mathcal{S}_f}$.
2. $(H \cup D)_{\mathcal{S}_f} = H_{\mathcal{S}_f} \cup D_{\mathcal{S}_f}$.
3. $(H \cap D)_{\mathcal{S}_f} \subseteq H_{\mathcal{S}_f} \cap D_{\mathcal{S}_f}$.
4. $(H_{\mathcal{S}_f})_{\mathcal{S}_f} \subseteq H_{\mathcal{S}_f}$.
5. $(\emptyset)_{\mathcal{S}_f} = \emptyset$.
6. If $G \in \tau_\delta$, then G $\cap H_{\mathcal{S}_f} \subseteq (G \cap H)_{\mathcal{S}_f}$.
7. $(H - D_{\mathcal{S}_f}) \cap (H - D_{\mathcal{S}_f})_{\mathcal{S}_f} = \emptyset$.
8. $H_{\mathcal{S}_f} = \tau_\delta - cl\left(H_{\mathcal{S}_f}\right) \subseteq \tau_\delta - cl(H)$.
9. If $H\bar{\delta}O$ for every $O \in \mathcal{S}$, then $H = \emptyset$ or $H \notin \mathcal{S}$.

Proof.

1) Let $x \in H_{\mathcal{S}_f}$. Then for every $\mathcal{U} \in \tau_\delta(x)$ there exists $O \in \mathcal{S}$ such that $(\mathcal{U} \cap H)\delta O$. Since $H \subset D$, by Theorem 1. 1. 7 part 2, $(\mathcal{U} \cap D)\delta O$ for every $\mathcal{U} \in \tau_\delta(x)$, thus $x \in D_{\mathcal{S}_f}$.

2) Evident $H_{\mathcal{S}_f} \cup D_{\mathcal{S}_f} \subset (H \cup D)_{\mathcal{S}_f}$. We show that $(H \cup D)_{\mathcal{S}_f} \subset H_{\mathcal{S}_f} \cup D_{\mathcal{S}_f}$. Let $x \in (H \cup D)_{\mathcal{S}_f}$. Then for every $\mathcal{U} \in \tau_\delta(x)$ there exists $O \in \mathcal{S}$ such that$\big(\mathcal{U} \cap (H \cup D)\big)\delta O$, that is, $((H \cap \mathcal{U}) \cup (D \cap \mathcal{U}))\delta O$ thus $(H \cap \mathcal{U})\delta C$ or $(D \cap \mathcal{U})\delta C$ i.e., $x \in H_{\mathcal{S}_f}$ or $x \in D_{\mathcal{S}_f}$ hence $x \in (H_{\mathcal{S}_f} \cup D_{\mathcal{S}_f})$.

3) Straight from the part 1.

4) Let $x \in (H_{\mathcal{S}_f})_{\mathcal{S}_f}$. Then for every $\mathcal{U} \in \tau_\delta(x)$, $(\mathcal{U} \cap H_{\mathcal{S}_f})\delta O$ for some $O \in \mathcal{S}$, that is, $\mathcal{U} \cap H_{\mathcal{S}_f} \neq \emptyset$ thus there exists $y \in (\mathcal{U} \cap H_{\mathcal{S}_f})$ such that $y \in \mathcal{U}$ and $y \in H_{\mathcal{S}_f}$ and so that for every $\mathcal{V} \in \tau_\delta(y)$, $(\mathcal{V} \cap H)\delta O$ for some $O \in \mathcal{S}$. Since $\mathcal{U}$ is also an τ_δ −open of y, $(\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}$ this is true for every $\mathcal{U} \in \tau_\delta(x)$, thus $x \in H_{\mathcal{S}_f}$.

5) Suppose$(\emptyset)_{\mathcal{S}_f} \neq \emptyset$, Then there exist $x \in (\emptyset)_{\mathcal{S}_f}$such that $(\mathcal{U} \cap \emptyset)\delta O$ for every $\mathcal{U} \in \tau_\delta(x)$, and some $O \in \mathcal{S}$ which is a contradiction because $\emptyset\overline{\delta}A$ for every $A \subseteq X$, thus $(\emptyset)_{\mathcal{S}_f} = \emptyset$.

6) Let $x \in (G \cap H_{\mathcal{S}_f})$. Then $x \in G$ and $x \in H_{\mathcal{S}_f}$, thus for every $\mathcal{U} \in \tau_\delta(x)$, $(\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}$. Since $x \in G$, $G \cap \mathcal{U} \in \tau_\delta(x)$, hence $\mathcal{U} \cap (G \cap H)\delta O$ for some $O \in \mathcal{S}$, that is, $x \in (G \cap H)_{\mathcal{S}_f}$.

7) Suppose$(H - H_{\mathcal{S}_f}) \cap (H - H_{\mathcal{S}_f})_{\mathcal{S}_f} \neq \emptyset$. Then there exists $x \in (H - H_{\mathcal{S}_f})$ and $x \in (H - H_{\mathcal{S}_f})_{\mathcal{S}_f}$. This is follows $x \in H$ and $x \notin H_{\mathcal{S}_f}$. Then there exists $\mathcal{U} \in \tau_\delta(x)$ such that $(\mathcal{U} \cap H)\overline{\delta} O$ for every $O \in \mathcal{S}$……. (1). and $x \in (H - H_{\mathcal{S}_f})_{\mathcal{S}_f}$ i.e., for every $\mathcal{V} \in \tau_\delta(x)$, $(\mathcal{V} \cap (H - H_{\mathcal{S}_f}))\delta O$ for some $O \in \mathcal{S}$this implies $(\mathcal{V} \cap H)\delta O$ for some $O \in \mathcal{S}$, but this is a contradiction with (1), thus $(H - H_{\mathcal{S}_f}) \cap (H - H_{\mathcal{S}_f})_{\mathcal{S}_f} = \emptyset$.

8) Let $x \in \tau_\delta - cl(H_{\mathcal{S}_f})$. Then $\mathcal{U}\delta H_{\mathcal{S}_f}$ for every $\mathcal{U} \in \tau_\delta(x)$, thus $\mathcal{U} \cap H_{\mathcal{S}_f} \neq \emptyset$ if not we have $H_{\mathcal{S}_f} \subseteq X - \mathcal{U}$, thus $\tau_\delta - cl(H_{\mathcal{S}_f}) \subseteq X - \mathcal{U}$ that means $x \in X - \mathcal{U}$ which is a contradiction, thus $\mathcal{U} \cap H_{\mathcal{S}_f} \neq \emptyset$ for every $\mathcal{U} \in \tau_\delta(x)$. Then there exists $y \in \mathcal{U} \cap H_{\mathcal{S}_f} \Rightarrow y \in \mathcal{U}$ and $y \in H_{\mathcal{S}_f}$. That is, for every $\mathcal{V} \in \tau_\delta(y)$, $(\mathcal{V} \cap H)\delta O$ for some $O \in \mathcal{S}$, but $\mathcal{U}$ is also an τ_δ −open of y this implies $(\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}$ this is true for every $\mathcal{U} \in \tau_\delta(x)$ hence $x \in H_{\mathcal{S}_f}$.

Now we show that $H_{\mathcal{S}_f} \subseteq \tau_\delta - cl(H)$. Let $x \in H_{\mathcal{S}_f}$. Then for every $\mathcal{U} \in \tau_\delta(x)$, $(\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}$, that is, $\mathcal{U} \cap H \neq \emptyset$ for every $\mathcal{U} \in \tau_\delta(x)$, by [P3] we have $\mathcal{U}\delta H$ for every $\mathcal{U} \in \tau_\delta(x)$by Proposition 1. 2. 15, $x \in cl(H)$.

9) Suppose $H \neq \emptyset$ and $H\overline{\delta}O$ for every $O \in \mathcal{S}$, by Theorem 1. 1. 7 part 9, $H \cap O = \emptyset$ for every $O \in \mathcal{S}$. If possible $H \in \mathcal{S}$, that is, $H \cap H = \emptyset$ which is a contradiction with hypothesis, hence $H \notin \mathcal{S}$■

Proposition 5. 2. 8: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space. H is nonempty subsets of X. Then the following statements are equivalent:

1. If $H \cap H_{\mathcal{S}_f} = \emptyset$, then $H_{\mathcal{S}_f} = \emptyset$.
2. $(H - H_{\mathcal{S}_f})_{\mathcal{S}_f} = \emptyset$.
3. $(H \cap H_{\mathcal{S}_f})_{\mathcal{S}_f} = H_{\mathcal{S}_f}$.

Proof.

$1 \Rightarrow 2$ By Proposition 5. 2. 7 part 7, $(H - H_{\mathcal{S}_f}) \cap (H - H_{\mathcal{S}_f})_{\mathcal{S}_f} = \emptyset$ and by (1) we get $(H - H_{\mathcal{S}_f})_{\mathcal{S}_f} = \emptyset$.

$2 \Rightarrow 3$ Since $H = (H - H_{\mathcal{S}_f}) \cup (H \cap H_{\mathcal{S}_f})$, $H_{\mathcal{S}_f} = (H - H_{\mathcal{S}_f})_{\mathcal{S}_f} \cup (H \cap H_{\mathcal{S}_f})_{\mathcal{S}_f}$. By (2) we get $H_{\mathcal{S}_f} = (H \cap H_{\mathcal{S}_f})_{\mathcal{S}_f}$.

$3 \Rightarrow 1$ Let $(H \cap H_{\mathcal{S}_f}) = \emptyset$. Then $(H \cap H_{\mathcal{S}_f})_{\mathcal{S}_f} = (\emptyset)_{\mathcal{S}_f} = \emptyset$. By (3) $(H \cap H_{\mathcal{S}_f})_{\mathcal{S}_f} = H_{\mathcal{S}_f}$, hence $H_{\mathcal{S}_f} = \emptyset$■

Note that from what has been mentioned that $X_{\mathcal{S}_f} \subseteq X$, but it is possible to obtain equality, that is, $X = X_{\mathcal{S}_f}$ in two cases:

First case: If the topological defined on X is indiscrete topology. That is clear because X is only τ_δ −open of each point.

Second case: If the topological defined on X is τ_σ. The following proposition explains that:

Proposition 5. 2. 9: Let $(\mathrm{X}, \delta, \tau_\sigma, \mathcal{S})$ be a $\sigma-$ Topological Proximity space. Then

1. $X = X_{\mathcal{S}_f}$
2. $H \subseteq H_{\mathcal{S}_f}$.
3. $H_{\mathcal{S}_f} = \tau_\sigma - cl(H)$.

Proof.

1) Evidently $X_{\mathcal{S}_f} \subset X$, we show that $X \subset X_{\mathcal{S}_f}$. Let $x \in X$. Then for every $\mathcal{U} \in \tau_\sigma(x)$, $(\mathcal{U} \cap X)\delta\, \mathcal{U}$. Since $\mathcal{U} \in \tau_\sigma$, $\mathcal{U} \notin \sigma$. By Theorem 5. 2. 3 part 1, $\mathcal{U} \in \mathcal{S}$, thus $x \in X_{\mathcal{S}_f}$. In case of $\mathcal{U} = X$ this is also true because $X\, \delta O$ for every $O \neq \emptyset$.

2) Let $x \in H$. If possible $x \notin H_{f_\emptyset}$, then there exists $\mathcal{U} \in \tau_\sigma(x)$ such that $(\mathcal{U} \cap H)\, \overline{\delta} O$ for every $O \in \mathcal{S}$, but $\mathcal{U} \cap H \neq \emptyset$ because $x \in (\mathcal{U} \cap H)$ that is follows for Proposition 5. 2. 7 part 9, $(\mathcal{U} \cap H) \notin \mathcal{S}$ by Theorem 5. 2. 3 part 1, $\mathcal{U} \cap H \in \sigma$ thus $\mathcal{U} \in \sigma$ this is in contradiction with the definition of τ_σ, thus $x \in H_{\mathcal{S}_f}$, that is, $H \subseteq H_{\mathcal{S}_f}$.

3) Let $x \in \tau_\sigma - cl(H)$. Then $\mathcal{U}\delta H$ for every $\mathcal{U} \in \tau_\delta(x)$, thus $\mathcal{U} \cap H \neq \emptyset$ if not we have $H \subseteq X - \mathcal{U}$, thus $\tau_\delta - cl(H) \subseteq X - \mathcal{U}$ this means $x \in X - \mathcal{U}$ which is a contradiction, thus $\mathcal{U} \cap H \neq \emptyset$ for every $\mathcal{U} \in \tau_\sigma(x)$. If possible $x \notin H_{\mathcal{S}_f}$ there exists $\mathcal{U} \in \tau_\sigma(x)$ such that $(\mathcal{U} \cap H)\, \overline{\delta} O$ for every $O \in \mathcal{S}$. By Proposition 5. 2. 7 part 9, implies that $\mathcal{U} \cap H \notin \mathcal{S}$. Thus by Proposition 5. 2. 3 part 1 $\mathcal{U} \cap H \in \sigma$, thus $\mathcal{U} \in \sigma$ this is a contradiction with the definition of τ_σ thus $x \in H_{\mathcal{S}_f}$. Hence $H_{\mathcal{S}_f} = \tau_\sigma - cl(H)$■

Proposition 5. 2. 10: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space, and $H \in \mathcal{S}$. Then the following statements are hold:

1. H $\subseteq H_{\mathcal{S}_f}$.
2. $\tau_\delta - cl(H) = H_{\mathcal{S}_f}$.
3. $X = X_{\mathcal{S}_f}$.

Proof.

1) Let $x \in H$. Then for every $\mathcal{U} \in \tau_\delta(x)$, $(\mathcal{U} \cap H) \cap H \neq \emptyset$, that is, $(\mathcal{U} \cap H)\delta H$, but $H \in \mathcal{S}$, thus $x \in H_{\mathcal{S}_f}$.

2) By Proposition 5. 2. 7 part 8, $\tau_\delta - cl(H) \supset H_{\mathcal{S}_f}$. We show that $\tau_\delta - cl(H) \subset H_{\mathcal{S}_f}$. Let $x \in \tau_\delta - cl(H)$. Then for every $\mathcal{U} \in \tau_\delta(x)$, $\mathcal{U}\delta H$, and $(\mathcal{U} \cap H)\delta(\mathcal{U} \cap H)$, by Theorem 1. 1.7 $(\mathcal{U} \cap H)\delta H$ for some $H \in \mathcal{S}$, thus $x \in H_{\mathcal{S}_f}$.

3) Evident by 2■

Note that by definition of sporadic follower set, we cannot replace (there exists $O \in \mathcal{S}$)by (for every $O \in \mathcal{S}$) because $\emptyset \in \mathcal{S}$, thus $H\overline{\delta}\emptyset$ for every $H \subseteq X$. But if replace (for every $\mathcal{U} \in \tau_\delta(x)$) by (there exists $\mathcal{U} \in \tau_\delta(x)$) the result $H_{\mathcal{S}_f} = \emptyset \; or \; H_{\mathcal{S}_f} = X$. In case of (there exists $\mathcal{U} \in \tau_\delta(x)$, for every $O \in \mathcal{S}$ such that $(\mathcal{U} \cap H)\overline{\delta}O$) the result , sporadic follower set equal $X - H_{\mathcal{S}_f}$.

Proposition 5. 2. 11: Let $(X, \delta, \mathcal{S})$ be sporadic proximity space. Let $\tau_{\delta_1}, \tau_{\delta_2}$ are two topologies defied on X. If $\tau_{\delta_1} \subseteq \tau_{\delta_2}$, then $H_{\mathcal{S}_f}(\tau_{\delta_2}) \subseteq H_{\mathcal{S}_f}(\tau_{\delta_1})$.

Proof.

Let $x \in H_{\mathcal{S}_f}(\tau_{\delta_2})$. Then every $\mathcal{U} \in \tau_{\delta_2}(x)$, $(\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}$. Since $\tau_{\delta_1} \subseteq \tau_{\delta_2}$, for every $\mathcal{V} \in \tau_{\delta_1}(x)$, $(\mathcal{V} \cap H)\delta O$ for some $O \in \mathcal{S}$ thus $x \in H_{\mathcal{S}_f}(\tau_{\delta_1})$ ■

Definition 5. 2. 12: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space. H nonempty subsets of X, then H is called $\mathcal{S}_f - set$ if and only if $H \subseteq \tau_\delta - int(H_{\mathcal{S}_f})$. Therefor $\mathcal{S}_f(X) = \{H \subseteq X, H \subseteq \tau_\delta - int(H_{\mathcal{S}_f})\}$.

Definition 5. 2. 13: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be sporadic topological proximity space. A point $x \in X$ is said to be *sporadic takeoff point* of a subset H of topological space (X, τ_δ) , if there exists $\mathcal{U} \in \tau_\delta(x)$, such that $(\mathcal{U} \cap (X - H))\overline{\delta}O$ for every $O \in \mathcal{S}$.

All the sporadic takeoff points of a set H are denoted by $H_{\mathcal{S}_t}$. Thus $H_{\mathcal{S}_t}(\tau_\delta, \mathcal{S}) = \{x \in X;\ \exists\ \mathcal{U} \in \tau_\delta(x)\ s.t\ (\mathcal{U} \cap\ X - H)\overline{\delta}O$ for every $O \in \mathcal{S}\}$. By Example 5. 2. 2 part 1, $(\{2,3\})_{\mathcal{S}_t} = \{2,3\}$, $(\{1\})_{\mathcal{S}_t} = \{1,3\}$, $(X)_{\mathcal{S}_t} = X$.

The following Proposition shows the most important characteristics of this set.

Proposition 5. 2. 14: Let $(X, \tau_\delta, \delta, \mathcal{S})$ be sporadic topological proximity space. H, D subsets of X. Then

1. If $H \subset D, then\ H_{\mathcal{S}_t} \subset D_{\mathcal{S}_t}$;
2. $(H \cup D)_{\mathcal{S}_t} \supseteq H_{\mathcal{S}_t} \cup D_{\mathcal{S}_t}$;
3. $(H \cap D)_{\mathcal{S}_t} = H_{\mathcal{S}_t} \cap D_{\mathcal{S}_t}$;
4. $H_{\mathcal{S}_t} = \cup\{\mathcal{U} \in \tau_\delta\ ;\ \mathcal{U} \cap (X - H)\overline{\delta}\ O$, for every $O \in \mathcal{S}\}$;
5. $H_{\mathcal{S}_t}$ is τ_δ –open set;
6. If $G \in \tau_\delta$, then G $\subseteq G_{\mathcal{S}_t}$;
7. $H_{\mathcal{S}_t} \subseteq (H_{\mathcal{S}_t})_{\mathcal{S}_t}$;
8. $X_{\mathcal{S}_t} = X$;
9. $\tau_\delta - int(H) \subseteq H_{\mathcal{S}_t}$;

10. $(\emptyset)_{\mathcal{S}_t} \subseteq H_{\mathcal{S}_t}$ for every $H \subseteq X$.

Proof.

1) Let $x \in H_{\mathcal{S}_t}$. Then there exists $\mathcal{U} \in \tau_\delta(x)$,such that $\mathcal{U} \cap (X - H))\overline{\delta}\, O$ for every $O \in \mathcal{S}$. Since $H \subset D$, $(X - D) \subseteq (X - H)$, that is, $(\mathcal{U} \cap (X - D))\overline{\delta}\, O$ for every $O \in \mathcal{S}$. Hence $H_{\mathcal{S}_t} \subset D_{\mathcal{S}_t}$.

2) By part 1.

3) By part 1, we get $(H \cap D)_{\mathcal{S}_t} \subset H_{\mathcal{S}_t} \cap D_{\mathcal{S}_t}$, we show that $H_{\mathcal{S}_t} \cap D_{\mathcal{S}_t} \subset (H \cap D)_{\mathcal{S}_t}$. Let $x \in (H_{\mathcal{S}_t} \cap D_{\mathcal{S}_t})$. Then $x \in H_{\mathcal{S}_t}$ and $x \in D_{\mathcal{S}_t}$, and so that there exists $\mathcal{U}$ *and* $\mathcal{V} \in \tau_\delta(x)$ such that $\mathcal{U} \cap (X - H)\overline{\delta}\, O$ and $\mathcal{V} \cap (X - D)\overline{\delta}\, O$, for every $O \in \mathcal{S}$.......(1)

If possible $x \notin (H \cap D)_{\mathcal{S}_t}$, then for every $\mathcal{W} \in \tau_\delta(x)$, $(\mathcal{W} \cap (X - (H \cap D)))\delta\, O$ for some $O \in \mathcal{S}$ this implies to $\mathcal{W} \cap [(X - H) \cup (X - D)]\delta\, O$ for some $O \in \mathcal{S}$, that is $(\mathcal{W} \cap (X - H))\delta\, O$ for every $\mathcal{W} \in \tau_\delta(x)$ or $(\mathcal{W} \cap (X - D))\delta\, O$ for every $\mathcal{W} \in \tau_\delta(x)$ which is a contradiction with (1), thus $x \in (H \cap D)_{\mathcal{S}_t}$ and so, $H_{\mathcal{S}_t} \cap D_{\mathcal{S}_t} = (H \cap D)_{\mathcal{S}_t}$.

4) $x \in H_{\mathcal{S}_t}$ if and only if there exists $\mathcal{U} \in \tau_\delta(x)$ such that $(\mathcal{U} \cap (X - H))\overline{\delta} O$ for every $O \in \mathcal{S}$ if and only if $x \in \{\mathcal{U} \in \tau_\delta(x) ; (\mathcal{U} \cap (X - H))\overline{\delta}\, O$ for every $O \in \mathcal{S}\}$ if and only if $x \in \bigcup\{\mathcal{U} \in \tau_\delta(x) ; (\mathcal{U} \cap (X - H))\overline{\delta}\, O$ for every $O \in \mathcal{S}\}$.

5) It is obvious from part 4 and a fact the union of τ_δ −open sets is an τ_δ −open set.

6) Let $x \in G$ *and* $G \in \tau_\delta$,this implies $(G \cap (X - G))\overline{\delta}\, O$ for every $O \in \mathcal{S}$, thus $x \in G_{\mathcal{S}_t}$.

7) Let $x \notin (H_{\mathcal{S}_t})_{\mathcal{S}_t}$. Then for every $\mathcal{U} \in \tau_\delta(x), (\mathcal{U} \cap (X - H_{\mathcal{S}_t}))\ \delta O$ for some $O \in \mathcal{S}$, thus $(\mathcal{U} \cap (X - H_{\mathcal{S}_t})) \neq \emptyset$. Then there exists $y \in (\mathcal{U} \cap (X - H_{\mathcal{S}_t}))$ such that $y \in \mathcal{U}$ *and* $y \in (X - H_{\mathcal{S}_t})$.

Thus $y \notin H_{\mathcal{S}_t}$, and so that for every $\mathcal{V} \in \tau_\delta(y)$, $(\mathcal{V} \cap (X - H))\delta O$ for some $O \in \mathcal{S}$, but $\mathcal{U} \in \tau_\delta(y)$, that is, $(\mathcal{U} \cap (X - H))\delta O$ for some $O \in \mathcal{S}$ this means $x \notin H_{\mathcal{S}_t}$, thus $H_{\mathcal{S}_t} \subseteq (H_{\mathcal{S}_t})_{\mathcal{S}_t}$.

Moreover, we can also prove part 7 by parts 5 and 6. That is, by part 5, $H_{\mathcal{S}_t}$ is $\tau_\delta -$open set, and by part 6, $H_{\mathcal{S}_t} \subseteq (H_{\mathcal{S}_t})_{\mathcal{S}_t}$.

8) For every $x \in X$ there exists $X \in \tau_\delta(x)$, such that $(X \cap (X - X)\overline{\delta} O$ for every $O \in \mathcal{S}$, thus $x \in X_{\mathcal{S}_t}$.

9) Let $x \in \tau_\delta - int(H)$. Then there exists $\mathcal{U} \in \tau_\delta(x)$, such that $\mathcal{U} \subseteq H$ this mean $\mathcal{U} \cap (X - H) = \emptyset$, thus $(\mathcal{U} \cap (X - H))\overline{\delta} O$ for every $O \in \mathcal{S}$, thus $x \in H_{\mathcal{S}_t}$.

10) Let $x \in (\emptyset)_{\mathcal{S}_t}$. Then there exists $\mathcal{U} \in \tau_\delta(x)$, such that $(\mathcal{U} \cap (X - \emptyset))\overline{\delta} O$ for every $O \in \mathcal{S}$ this mean $\mathcal{U}\overline{\delta} O$ for every $O \in \mathcal{S}$. Since $\mathcal{U} \cap (X - H) \subseteq \mathcal{U}$, $(\mathcal{U} \cap (X - H))\overline{\delta} O$ for every $O \in \mathcal{S}$ and every $H \subseteq X$, hence $x \in H_{\mathcal{S}_t}$■

We can use Definition 5. 2. 13 on $(X, \delta, \tau, \mathcal{S})$ and have same result because τ_δ is topology. But if use $(X, \delta, \tau_\sigma, \mathcal{S})$ Proposition 5. 2. 13 part 5, is not true because the union of τ_σ −open sets is not necessary is τ_σ −open set.

The following proposition shows the relationship between $\mathcal{S}-follower\ set$ and $\mathcal{S}-takeoff\ set$.

Proposition 5. 2. 15: Let $(X, \tau_\delta, \delta, \mathcal{S})$ be sporadic topological proximity space. H is a subsets of X. Then

1. $H_{\mathcal{S}_t} = X - (X-H)_{\mathcal{S}_f}$;
2. $H_{\mathcal{S}_t} = (H_{\mathcal{S}_t})_{\mathcal{S}_t}$ if and only if $(X-H)_{\mathcal{S}_f} = (X-H)_{\mathcal{S}_{f_{\mathcal{S}_f}}}$;
3. $H_{\mathcal{S}_f} = X - (X-H)_{\mathcal{S}_t}$;
4. $X - H_{\mathcal{S}_f} = (X-H)_{\mathcal{S}_t}$;
5. $X - H_{\mathcal{S}_t} = (X-H)_{\mathcal{S}_f}$;
6. $(\emptyset)_{\mathcal{S}_t} = X - X_{\mathcal{S}_f}$.

Proof.

1) $x \in (H)_{\mathcal{S}_t}$ if and only if there exists $\mathcal{U} \in \tau_\delta(x)$, such that $(\mathcal{U} \cap (X-H)))\overline{\delta}O$ for every $O \in \mathcal{S}$ if and only if $x \notin (X-H)_{\mathcal{S}_f}$ if and only if $x \in X - (X-H)_{\mathcal{S}_f}$.

2) $(H_{\mathcal{S}_t})_{\mathcal{S}_t} = X - (X - H_{\mathcal{S}_t})_{\mathcal{S}_f} = X - (X - X - (X-H)_{\mathcal{S}_f})_{\mathcal{S}_f} = X - ((X-H)_{\mathcal{S}_f})_{\mathcal{S}_f} = X - (X-H)_{\mathcal{S}_f} = H_{\mathcal{S}_t}$, so part 2 is true.

3) $X - (X-H)_{\mathcal{S}_t} = X - (X - (X - (X-H))_{\mathcal{S}_f} = H_{\mathcal{S}_f}$.

4) $X - H_{\mathcal{S}_f} = X - (X - (X-H)_{\mathcal{S}_t}) = (X-H)_{\mathcal{S}_t}$.

5) $X - H_{\mathcal{S}_t} = X - \left(X - (X - H)_{\mathcal{S}_f}\right) = (X - H)_{\mathcal{S}_f}$.

6) $(\emptyset)_{\mathcal{S}_t} = X - (X - \emptyset)_{\mathcal{S}_f} = X - X_{\mathcal{S}_f}$■

Remark 5. 2. 16: Let $(\mathrm{X}, \tau, \delta_I, \mathcal{S})$ be sporadic topological proximity space. H is nonempty subsets of X. Then

1. $H_{\mathcal{S}_t} \subseteq H_{\mathcal{S}_f}$;
2. $int\ (H) = H_{\mathcal{S}_t}$;
3. $(\emptyset)_{\mathcal{S}_t} = \emptyset$.

Proof.

1) Let $x \in H_{\mathcal{S}_t}$. Then there exists $\mathcal{U} \in \tau(x)$,such that $(\mathcal{U} \cap (X - H))\overline{\delta}\, O$ for every $O \in \mathcal{S}$. By Proposition 5. 2. 7 part 9, either $\mathcal{U} \cap (X - H) \notin \mathcal{S}$, thus by Definition of Sporadic $X - (\mathcal{U} \cap (X - H)) \notin \sigma$, thus $(X - \mathcal{U}) \cup H \notin \sigma$ By axiom [P4] $H \notin \sigma$ which is a contradiction because δ is indiscrete proximity and $H \neq \emptyset$. Or $\mathcal{U} \cap (X - H) = \emptyset$, that is, $\mathcal{U} \subseteq H$, thus $x \in H$, that mean $(\mathcal{V} \cap H)\delta O$ for some $O \in \mathcal{S}$,this is true for every $\mathcal{V} \in \tau(x)$, hence $x \in H_{\mathcal{S}_f}$.

2) By Proposition 5. 2. 14 part 9, $int\ (H) \subseteq H_{\mathcal{S}_t}$, we show that $H_{\mathcal{S}_t} \subseteq int\ (H)$. By the same method of proof above we get $\mathcal{U} \subseteq H$ thus $x \in int\ (H)$ for every $x \in H_{\mathcal{S}_t}$.

3) By part 1 and Proposition 5. 2. 7 part 5, $(\emptyset)_{\mathcal{S}_t} \subseteq (\emptyset)_{\mathcal{S}_f} = \emptyset$■

The Remark is also true when using τ_{δ}. Because $(P)_{\mathcal{S}_t} = \emptyset$, for every proper P subset of X.

5. 3. $\mathcal{S} -$ *Bushy set*

Now, we construct a special kind of density through the concept of sporadic. Also, the properties of this concept are examined.

Definition 5. 3. 1: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space. A nonempty subsets of X, then H is called $\mathcal{S} -$ *bushy set* if and only if for every $x \in X$ *and* every $\mathcal{U} \in \tau_\delta(x), (\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}$.

In other words H is $\mathcal{S} -$ *bushy set* if and only if for every $x \in X$, x is $\mathcal{S} -$ *follower point* of H. The family of all $\mathcal{S} -$ *bushy set* on X denoted by $\mathcal{S}(X, \delta)$.

Example 5. 3. 2: $(\mathrm{X}, \delta_D, \tau_\delta, \mathcal{S})$ is not contain $\mathcal{S} -$ bushy set.

Example 5. 3. 3: In space $(\mathrm{X}, \delta_I, \tau_\delta, \mathcal{S})$ every nonempty set is $\mathcal{S} -$ bushy set.
Because X is only nonempty τ_δ −open set, thus every $H \subseteq X, (X \cap H)\delta O$ for some $O \in \mathcal{S}$.

Proposition 5. 3. 4: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space. Then every singleton set is not $\mathcal{S} -$ bushy set.

Proof.
If possible $\{x\}$ is $\mathcal{S} -$ bushy set, then every $\mathcal{U} \in \tau_\delta$ such that $(\mathcal{U} \cap \{x\})\delta O$ for some $O \in \mathcal{S}$ but by Proposition 1. 2. 12, $X - \mathcal{U}$ is τ_δ −open and $((X - \mathcal{U}) \cap \{x\}) = \emptyset$ which is a contradiction. Hence $\{x\}$ is not $\mathcal{S} -$ bushy set■

Proposition 5. 3. 5: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space. Then

1. Every super set of the $\mathcal{S} -$ bushy set is a $\mathcal{S} -$ bushy set.
2. If H or D is $\mathcal{S}$ −bushy set, then $(H \cup D)$ is also $\mathcal{S}$ −bushy set.

Proof.

1) Suppose that H is a $\mathcal{S}$ −bushy set and H ⊆ D . Then every $\mathcal{U} \in \tau_{\delta}, (\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}$. Since H ⊆ D, by Theorem 1. 1. 7 part 2 we have that , $(\mathcal{U} \cap D)\delta O$ for some $O \in \mathcal{S}$. Hence H is $\mathcal{S}$ −bushy set.

2) Suppose that H or D is a $\mathcal{S}$ −bushy set. Then every $\mathcal{U} \in \tau_{\delta}, (\mathcal{U} \cap H)\delta O$ or $(\mathcal{U} \cap D)\delta O$ for some $O \in \mathcal{S}$. By axiom [P4] we have that $((\mathcal{U} \cap H) \cup (\mathcal{U} \cap D))\delta O$ for some $O \in \mathcal{S}$, that is, $(\mathcal{U} \cap (H \cup D))\delta O$ for some $O \in \mathcal{S}$. Hence $(H \cup D)$ is $\mathcal{S}$ −bushy set■

It is possible to rely on part 1 in the above proof.

Proposition 5. 3. 6: If H is a $\mathcal{S}$ −bushy set, then $\mathcal{U} \cap H \neq \emptyset$ for every nonempty $\mathcal{U} \in \tau_{\delta}$.

The proof is clear because $\emptyset\bar{\delta}O$ for every $O \in \mathcal{S}$. The convers is not always is true, Example 5. 3. 7 explain that.

Example 5. 3. 7: Let $X = \{1, 2, 3\}$, δ_i is proximity define by:

$A\delta_i B$ $\forall H \neq \emptyset, D \neq \emptyset$ except the

relation: $\{2\}\bar{\delta}\{1\}, \{2\}\bar{\delta}\{1\}, \{\ 3\}\bar{\delta}\{1\}, \{1\}\bar{\delta}\{3\}, \{1\}\bar{\delta}\{2, 3\}, \{2, 3\}\bar{\delta}\{1\}$.

Then $\tau_{\delta_i} = \{X, \emptyset, \{1\}, \{2,3\}\}$.

And $\mathcal{S}_i = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$. We can see that $\{1, 2\} \cap \mathcal{U} \neq \emptyset$ for every nonempty $\mathcal{U} \in \tau_{\delta}$ but $\{1, 2\}$ is not $\mathcal{S}$ −bushy set.

Remark 5. 3. 8: Let $(X, \delta, \tau_{\delta}, \mathcal{S})$ be a sporadic topological proximity space. If H is a $\mathcal{S}$ −bushy set, then H is a τ_{δ} −dense set.

Proof.

Let H be a $\mathcal{S}$ −bushy set. Then by Proposition 5. 3. 6, $\mathcal{U} \cap H \neq \emptyset$ for every $\mathcal{U} \in \tau_{\delta}$. by [P3] $\mathcal{U}\delta H$ for every $\mathcal{U} \in \tau_{\delta}$, by Proposition 1. 2. 15, $\tau_{\delta} - cl(H) = X$, that is, H is a τ_{δ} −dense set■

The convers is not always is true. For Example 5. 3. 7, $\{1, 3\}$ is a τ_{δ} −dense set but not $\mathcal{S}$ −bushy set because $\{1, 3\} \cap [1] = \{1\} but\ \{1\}\bar{\delta}O$ for every $O \in \mathcal{S}$ that is $(\{1, 3\})_{\mathcal{S}_f} \neq X$.

Proposition 5. 3. 9: Let $(X, \delta, \tau_{\delta}, \mathcal{S})$ be a sporadic topological proximity space. If H is a proper $\mathcal{S}$ −bushy subset of X, then H is not τ_{δ} −open subset of X.

Proof.

Suppose H is a proper $\mathcal{S}$ −bushy subset of X. Then for every $\mathcal{U} \in \tau_{\delta}$, $(\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}$. By Proposition 1. 2. 12, $\mathcal{U}$ is also τ_{δ} −closed set, thus $(X - \mathcal{U})$ is τ_{δ} −open set. Thus $\mathcal{U} \cap H \neq \emptyset$ and $(X - \mathcal{U}) \cap H \neq \emptyset$, thus $\tau_{\delta} - int(H) \neq H$. That mean H is not τ_{δ} −open subset of X■

The converse is not always true, by Example 5. 3. 7, $\{1, 2\}$ is not τ_{δ} −open, and also it is not $\mathcal{S}$ −bushy set.

Remark 5. 3. 10: Let $(X, \delta, \tau_{\delta}, \mathcal{S})$ be a sporadic topological proximity space. If H is a $\mathcal{S}$ −bushy set and $\mathcal{U} \in \tau_{\delta}$, then $\mathcal{U} \subseteq \left((\mathcal{U} \cap H)_{\mathcal{S}_f}\right)_{\mathcal{S}_t}$.

Proof.

Let $x \in \mathcal{U}$. By Proposition 5. 2. 7 part 6, we have that $\mathcal{U} \cap H_{\mathcal{S}_f} \subseteq (\mathcal{U} \cap H)_{\mathcal{S}_f}$, that is, $(\mathcal{U} \cap H_{\mathcal{S}_f})_{\mathcal{S}_t} \subseteq ((\mathcal{U} \cap H)_{\mathcal{S}_f})_{\mathcal{S}_t}$. But $(\mathcal{U} \cap H_{\mathcal{S}_f})_{\mathcal{S}_t} =$

$\mathcal{U}_{\mathcal{S}_t} \cap (H_{\mathcal{S}_f})_{\mathcal{S}_t} = \mathcal{U}_{\mathcal{S}_t} \cap (X)_{\mathcal{S}_t} = \mathcal{U}_{\mathcal{S}_t}$. Thus $\mathcal{U}_{\mathcal{S}_t} = (\mathcal{U} \cap H_{\mathcal{S}_f})_{\mathcal{S}_t}$. Since $\mathcal{U} \in \tau_\delta$ By Proposition 5. 2. 14 part 6, we have that $\mathcal{U} \subseteq \mathcal{U}_{\mathcal{S}_t}$.
Hence $\mathcal{U} \subseteq (\mathcal{U} \cap H_{\mathcal{S}_f})_{\mathcal{S}_t} \subseteq ((\mathcal{U} \cap H)_{\mathcal{S}_f})_{\mathcal{S}_t}$■

Proposition 5. 3. 11: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space. If $\mathcal{S} \cap \sigma \neq \emptyset$, then every bushy set is $\mathcal{S}$ −bushy set.

Proof.
Let H is a bushy set. Then every $\mathcal{U} \in \tau_\delta, (\mathcal{U} \cap H)\delta C$ for every $C \in \sigma$. Since $\mathcal{S} \cap \sigma \neq \emptyset$, there exists $O_i \in (\mathcal{S} \cap \sigma)$, that is, $O_i \in \mathcal{S}$, thus H is $\mathcal{S}$ −bushy set■

Proposition 5. 3. 12: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space. If H is a $\mathcal{S}$ − bushy set. Then

1. Only τ_δ −closed set contained H is X.
2. Only τ_δ −open set disjoint from H is $\emptyset$.

Proof.
1) Let H is $\mathcal{S}$ − bushy set. Then by Proposition 5. 3. 6 $(\mathcal{U} \cap H) \neq \emptyset$ for every $\mathcal{U} \in \tau_\delta$. Since $\mathcal{U}, \ and \ (X - \mathcal{U}) \in \tau_\delta$, X is only τ_δ −closed set contained H.

2) That is clear because $(\mathcal{U} \cap H) \neq \emptyset$ for every $\mathcal{U} \in \tau_\delta$■

Proposition 5. 3. 13: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space. Then $H_{\mathcal{S}_t} = \emptyset$ if and only if $(X - H)$ is $\mathcal{S}$ − bushy set.

Proof.

$H_{\mathcal{S}_t} = \emptyset$ if and only if $X - (X - H)_{\mathcal{S}_f} = \emptyset$ if and only if $(X - H)_{\mathcal{S}_f} = X$ if and only if $(X - H)$ is $\mathcal{S} -$ bushy set■

Proposition 5. 3. 14: Let $(X, \delta_1, \tau_{\delta 1}, \mathcal{S}_1)$ *and* $(X, \delta_2, \tau_{\delta 2}, \mathcal{S}_2)$ be a two sporadic proximity topological spaces. If $\delta_2 > \delta_1$ and H be $\mathcal{S}_2 - bushy$ set, then H is $\mathcal{S}_1 - bushy$ set.

Proof.

Let H is $\mathcal{S}_2 - bushy$ set. Then for every $\mathcal{U} \in \tau_{\delta 2}$, $(\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}_2$. Since $\delta_2 > \delta_1$, by Proposition 1. 2. 4, $\tau_{\delta 1} \subseteq \tau_{\delta 2}$ and by Proposition 5. 2. 4, $\mathcal{S}_2 \subseteq \mathcal{S}_1$ thus for every $\mathcal{V} \in \tau_{\delta 1}$, $(\mathcal{V} \cap H)\delta O$ for some $O \in \mathcal{S}_1$. Hence H is $\mathcal{S}_1 - bushy$ set■

Convers of this proposition not always is true Example 5. 3. 17 explain that.

Example 5. 3. 15: Let $X = \{1, 2, 3, 4, 5\}$, δ_1 is a proximity define by: $H\delta_1 D$ for every $H \neq \emptyset$, and $D \neq \emptyset$ except the relation:

$\{2\}\overline{\delta}\{1\}, \{2\}\overline{\delta}\{4\}, \{2\}\overline{\delta}\{1, 4\}, \{2\}\overline{\delta}\{5\}, \{2\}\overline{\delta}\{1, 5\}, \{2\}\overline{\delta}\{4, 5\},$

$\{2\}\overline{\delta}\{1, 4, 5\}, \{3\}\overline{\delta}\{4\}, \{3\}\overline{\delta}\{1\}, \{3\}\overline{\delta}\{1, 4\}, \{3\}\overline{\delta}\{5\}, \{3\}\overline{\delta}\{1, 5\},$

$\{3\}\overline{\delta}\{4, 5\}, \{3\}\overline{\delta}\{1, 4, 5\}, \{4\}\overline{\delta}\{2,3\}, \{5\}\overline{\delta}\{2,3\}, \{1\}\overline{\delta}\{2,3\},$

$\{2,3\}\overline{\delta}\{1, 4\}, \{2,3\}\overline{\delta}\{1,5\}, \{2,3\}\overline{\delta}\{4,5\}, \{2,3\}\overline{\delta}\{1,4,5\}$, In addition to the commutative relationships. Then $\tau_{\delta_1} = \{X, \emptyset, \{2, 3\}, \{1, 4, 5\}\}$.

And $\mathcal{S}_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 2, 4\}, \{1, 2, 5\}, \{2, 4, 5\},$ $\{1, 2, 4, 5\}, \{1, 4\}, \{1, 5\}, \{1, 3, 4\}, , \{1, 3, 5\}, \{1, 4, 5\}, \{3, 4, 5\}, \{1, 3, 4, 5\}\}$.

And let δ_2 is proximity define by: $H\delta_2 D$ $\forall H \neq \emptyset, D \neq \emptyset$ except the relation: $\{4\}\overline{\delta}\{1\}, \{4\}\overline{\delta}\{2\}, \{4\}\overline{\delta}\{1, 2\}, \{4\}\overline{\delta}\{3\}, \{4\}\overline{\delta}\{1, 3\}, \{4\}\overline{\delta}\{2, 3\},$

$\{4\}\overline{\delta}\{1,2,3\}, \{4\}\overline{\delta}\{5\}, \{4\}\overline{\delta}\{1,5\}, \{4\}\overline{\delta}\{2,5\}, \{4\}\overline{\delta}\{3,5\}, \{4\}\overline{\delta}\{1,2,5\}$,

$\{5\}\overline{\delta}\{2,3,4\}, \{4\}\overline{\delta}\{1,3,5\}, \{4\}\overline{\delta}\{2,3,5\}, \{4\}\overline{\delta}\{1,2,3,5\}, \{2,3\}\overline{\delta}\{1,4,5\}$,

$\{2\}\overline{\delta}\{1\}, \{2\}\overline{\delta}\{4\}$, and$\{2\}\overline{\delta}\{1,4\}, \{2\}\overline{\delta}\{5\}, \{2\}\overline{\delta}\{1,5\}, \{2\}\overline{\delta}\{4,5\}$,

$\{2\}\overline{\delta}\{1,4,5\}, \{3\}\overline{\delta}\{1\}, \{3\}\overline{\delta}\{4\}, \{3\}\overline{\delta}\{1,4\}, \{3\}\overline{\delta}\{5\}, \{3\}\overline{\delta}\{1,5\}$,

$\{3\}\overline{\delta}\{4,5\}, \{3\}\overline{\delta}\{1,4,5\}, \{5\}\overline{\delta}\{4\}, \{2,3\}\overline{\delta}\{1,4\}, \{2,3\}\overline{\delta}\{1,5\}, \{1\}\overline{\delta}\{2,4\}$

$\{2,3\}\overline{\delta}\{4,5\}, \{1,5\}\overline{\delta}\{2,3,4\}, \{1\}\overline{\delta}\{2\}, \{1\}\overline{\delta}\{3\}.\{1\}\overline{\delta}\{2,3\}, \{1\}\overline{\delta}\{4\}, \{1\}\overline{\delta}\{3,4\}$,

$\{1\}\overline{\delta}\{2,3,4\}, \{5\}\overline{\delta}\{2\}, \{5\}\overline{\delta}\{3\}, \{5\}\overline{\delta}\{2,3\}$,

$\{5\}\overline{\delta}\{2,4\}, \{5\}\overline{\delta}\{3,4\}, \{1,5\}\overline{\delta}\{2,3\}, \{1,5\}\overline{\delta}\{2,4\}, \{1,5\}\overline{\delta}\{3,4\}$. In addition to the commutative relationships.

Then $\tau_{\delta_2} = \{X, \emptyset, \{4\}, \{2,3\}, \{1,5\}, \{1,4,5\}, \{2,3,4\}, \{1,2,3,5\}\}$.

And $\mathcal{S}_2 = \mathcal{S}_1$ is

Let $H = \{1,2\}$. Then H is a $\mathcal{S}_1$ −bushy set with respect τ_{δ_1}but H is not $\mathcal{S}_2$ −bushy set with respect τ_{δ_2}.

Proposition 5. 3. 16: Let $(X, \delta, \tau_{\sigma W}, \mathcal{S})$ be a $\sigma - weekly$ Topological proximity space with sporadic and $\tau_{\sigma W}$ is a topology. H is $\mathcal{S}_f - set$ if and only if $H = \mathcal{U} \cap D$ where $\mathcal{U} \in \tau_{\sigma W}$ and D is $\mathcal{S}$ − bushy set.

Proof.

Suppose $H = \mathcal{U} \cap D$ where $\mathcal{U} \in \tau_{\sigma W}$ *and* $D_{\mathcal{S}_f} = X$. Since $\mathcal{U} \in \tau_{\sigma W}$, $\mathcal{U} \cap D_{\mathcal{S}_f} \subseteq (\mathcal{U} \cap D)_{\mathcal{S}_f} = H_{\mathcal{S}_f}$, thus $\mathcal{U} \subseteq H_{\mathcal{S}_f}$, and so that $\mathcal{U} \subseteq \tau_{\sigma W} - int(H_{\mathcal{S}_f})$. But $H = \mathcal{U} \cap D \subseteq \mathcal{U} \subseteq \tau_{\sigma W} - int(H_{\mathcal{S}_f})$, that is, H is $\mathcal{S}_f - set$. Conversely, suppose H is $\mathcal{S}_f - set$. Then $H \subseteq \tau_{\sigma W} - int(H_{\mathcal{S}_f})$ so that $(H \cup (X - H_{\mathcal{S}_f})) \cap \tau_{\sigma W} - int(H_{\mathcal{S}_f}) = (H \cap \tau_{\sigma W} - int(H_{\mathcal{S}_f}) \cup ((X - H_{\mathcal{S}_f}) \cap \tau_{\sigma W} - int(H_{\mathcal{S}_f})) = H \cup \emptyset = H$, evident $\tau_{\sigma W} - int(H_{\mathcal{S}_f}) \in \tau_{\sigma W}$. It remains to show that $(H \cup (X - H_{\mathcal{S}_f})$ is a $\mathcal{S}$ −bushy set. Let $x \in X$ if possible $x \notin (H \cup (X - H_{\mathcal{S}_f}))_{\mathcal{S}_f}$, then there exists $\mathcal{U} \in \tau_{\sigma W}$, such that $\mathcal{U} \cap (H \cup (X - H_{\mathcal{S}_f}))\overline{\delta}\, O$ for every $O \in \mathcal{S}$, thus $(\mathcal{U} \cap H)\overline{\delta}\, O$ and $(\mathcal{U} \cap (X - H_{\mathcal{S}_f}))\overline{\delta}\, O$ for every $O \in \mathcal{S}$ ……..(1).

This implies $(\mathcal{U} \cap (X - H_{\mathcal{S}_f})) \neq \emptyset$ otherwise for every $x \in \mathcal{U}$, $x \notin (X - H_{\mathcal{S}_f})$ this implies $x \in H_{\mathcal{S}_f}$, thus for every $\mathcal{V} \in \tau_{\sigma W}(x)$, there exists $O \in \mathcal{S}$ such that $(\mathcal{V} \cap H)\delta O$ but $x \in \mathcal{U}$ this implies $(\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}$ this is in contradiction with (1), thus $(\mathcal{U} \cap (X - H_{\mathcal{S}_f}) \neq \emptyset$ this mean $(\mathcal{U} \cap (X - H_{\mathcal{S}_f})) \notin \mathcal{S}$, thus $(\mathcal{U} \cap (X - H_{\mathcal{S}_f})) \in \sigma$ that implies $\mathcal{U} \in \sigma$ this is in contradiction with definition of $\tau_{\sigma W}$. Thus $x \in (H \cup (X - H_{\mathcal{S}_f}))_{\mathcal{S}_f}$, and so that $H \cup (X - H_{\mathcal{S}_f})$ is a $\mathcal{S}$ − bushy set■

The following proposition shows the most important possible results if a set of the intersections between the topology and the sporadic is an empty set.

Proposition 5. 3. 17: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a not trivial sporadic topological proximity space. If $\tau_\delta \cap \mathcal{S} = \emptyset$. Then

1. $\tau_\delta - int(O) = \emptyset$ for every $O \in \mathcal{S}$;
2. The τ_δ −interior set of the $\mathcal{S}$ −follower set is empty, for every non $\mathcal{S}$ −bushy set;
3. If $H \in \mathcal{S}$ and H is non $\mathcal{S}$ −bushy set, then $H_{\mathcal{S}_f} \in \mathcal{S}$;
4. If $H - H_{\mathcal{S}_f} \in \mathcal{S}$, $H$is non $\mathcal{S}$ −bushy set, then $H_{\mathcal{S}_f} \in \mathcal{S}$;
5. If H has a cover of τ_δ −open sets each of whose intersection with H is in $\mathcal{S}$, then $H \in \mathcal{S}$.

Proof.

1) Suppose $\tau_\delta - int(O) \neq \emptyset$. Then there exist $\mathcal{U} \in \tau_\delta(x)$ such that $\mathcal{U} \subseteq O$. Since $O \in \mathcal{S}$, $\mathcal{U} \in \mathcal{S}$, which is a contradiction with hypothesis, thus $\tau_\delta - int(O) = \emptyset$.

2) Since $H_{\mathcal{S}_f}$ is τ_δ −closed, $X - H_{\mathcal{S}_f}$is τ_δ −open. But $\tau_\delta \cap \mathcal{S} = \emptyset$, we have that $X - H_{\mathcal{S}_f} \notin \mathcal{S}$, but $\emptyset \in \mathcal{S}$ and $\emptyset = (H_{\mathcal{S}_f} \cap (X - H_{\mathcal{S}_f})\ )$ and by Theorem 5. 2. 3 part 3, it is must $H_{\mathcal{S}_f} \in \mathcal{S}$, so by part(1), $\tau_\delta - int(H_{\mathcal{S}_f}) = \emptyset$.

3) If $H = \emptyset$,then by Proposition 5. 2. 7 part 5, and Proposition 5. 2. 3 part 4, $H_{\mathcal{S}_f} = \emptyset \in \mathcal{S}$. Otherwise $H \neq \emptyset$, if possible $H_{\mathcal{S}_f} \notin \mathcal{S}$, we have two cases: First $(X - H_{\mathcal{S}_f}) \in \mathcal{S}$ which is a contradiction because $\left(X - H_{\mathcal{S}_f}\right)$is τ_δ −open set. Second case $(X - H_{\mathcal{S}_f}) \notin \mathcal{S}$this also is in contradiction because $(X - H_{\mathcal{S}_f}) \notin \mathcal{S}$and $H_{\mathcal{S}_f} \notin \mathcal{S}$ this mean($(X - H_{\mathcal{S}_f}) \cap H_{\mathcal{S}_f}) = \emptyset \notin \mathcal{S}$, hence $H_{\mathcal{S}_f} \in \mathcal{S}$.

4) Suppose $\mathrm{H} \cap (\mathrm{X} - H_{\mathcal{S}_f}) \in \mathcal{S}$ by Theorem 5. 2. 3 part 3, either $X - H_{\mathcal{S}_f} \in \mathcal{S}$ which is a contradiction because $(\mathrm{X} - H_{\mathcal{S}_f})$ is τ_δ −open or $H \in \mathcal{S}$, thus $H \in \mathcal{S}$. By part 3, $H_{\mathcal{S}_f} \in \mathcal{S}$.

5) Let $H \subseteq \mathrm{X}$ such that $H \subseteq \cup_{\lambda \in \Lambda} (G)_\lambda$. Then for every $\lambda \in \Lambda, G_\lambda \in \tau_\delta$, and by hypothesis $H \cap G_\lambda \in \mathcal{S}$ by Theorem 5. 2. 3 part 3, either $H \in \mathcal{S}$ or $G_\lambda \in \mathcal{S}$. Since $\tau_\delta \cap \mathcal{S} = \emptyset$ we have that $H \in \mathcal{S}$■

The $cl_{\mathcal{S}}: \mathcal{P}(X) \to \mathcal{P}(X)$, defined by $cl_{\mathcal{S}}(H) = H \cup (H_{\mathcal{S}_f})$ is a Kuratowski closure operator. It is denoted by $\tau_{\mathcal{S}}$ of the topology generated by $cl_{\mathcal{S}}$, that is, $\tau_{\mathcal{S}} = \{\mathcal{U} \subseteq X; cl_{\mathcal{S}}(X - \mathcal{U}) = X - \mathcal{U}\}$. So that $(X - H)$ is $\tau_{\mathcal{S}}$- closed if and only if H is $\tau_{\mathcal{S}}$- open if and only if $H \in \tau_{\mathcal{S}}$ .Already observed that if $\mathcal{S} = \{H \subset X; H \text{ is proper subset }\}$, then $cl_{\mathcal{S}}(H) = cl(H)$ and $\tau_{\mathcal{S}} = \tau$ (where τ is any topology on X). In general case, $\tau \subseteq \tau_{\mathcal{S}}$, because by Proposition 5. 3. 8, $H_{\mathcal{S}_f} = cl(H_{\mathcal{S}_f}) \subseteq cl(H)$, thus $H \cup H_{\mathcal{S}_f} \subseteq H \cup cl(H)$, that is, $cl_{\mathcal{S}}(H) \subseteq cl(H)$.

Example 5. 3. 18: Let $X = \{1,2,3\}$, $\mathcal{S} = \{\emptyset, \{1\}, \{3\}, \{1,3\}\}$, and $\tau = \{\emptyset, X, \{2\}, \{2,3\}\}$. Then

$cl_{\mathcal{S}}(\{1\}) = \{1\}_{\mathcal{S}_f} \cup \{1\} = \{1\} \cup \{1\} = \{1\} \Rightarrow \{2,3\} \in \tau_{\mathcal{S}}$.

$cl_{\mathcal{S}}(\{2\}) = \{2\}_{\mathcal{S}_f} \cup \{2\} = \emptyset \cup \{2\} = \{2\} \Rightarrow \{1,3\} \in \tau_{\mathcal{S}}$.

$cl_{\mathcal{S}}(\{3\}) = \{3\}_{\mathcal{S}_f} \cup \{3\} = \{1,3\} \cup \{3\} = \{1,3\} \Rightarrow \{2\} \notin \tau_{\mathcal{S}}$.

$cl_{\mathcal{S}}(\{1,3\}) = \{1,3\}_{\mathcal{S}_f} \cup \{1,3\} = \{1,3\} \cup \{1,3\} = \{1,3\} \Rightarrow \{2\} \in \tau_{\mathcal{S}}$.

$cl_{\mathcal{S}}(\{1,2\}) = \{1,2\}_{\mathcal{S}_f} \cup \{1,2\} = \{1\} \cup \{1,2\} = \{1,2\} \Rightarrow \{3\} \in \tau_{\mathcal{S}}$.

$cl_{\mathcal{S}}(\{2,3\}) = \{2,3\}_{\mathcal{S}_f} \cup \{2,3\} = \{1,3\} \cup \{2,3\} = X \Rightarrow \{1\} \notin \tau_{\mathcal{S}}$.

$cl_{\mathcal{S}}(\emptyset) = \emptyset_{\mathcal{S}_f} \cup \emptyset = \emptyset \cup \emptyset = \emptyset \Rightarrow X \in \tau_{\mathcal{S}}$.

$cl_{\mathcal{S}}(X) = X_{\mathcal{S}_f} \cup X = \{1,3\} \cup X = X \Rightarrow \emptyset \in \tau_{\mathcal{S}}$.

Thus $\tau_{\mathcal{S}} = \{\emptyset, X, \{2\}, \{3\}, \{1,3\}, \{2,3\}\}$.

Note 5. 3. 19:

I. If $\mathcal{U} \in \tau$, then $\mathcal{U} \cap cl_{\mathcal{S}}(H) \subseteq cl_{\mathcal{S}}(\mathcal{U} \cap H)$. Because $\mathcal{U} \cap cl_{\mathcal{S}}(H) = \mathcal{U} \cap (H_{\mathcal{S}_f} \cup H) = (\mathcal{U} \cap H_{\mathcal{S}_f}) \cup (\mathcal{U} \cap H)$. By Proposition 5. 2. 7 part 6, $(\mathcal{U} \cap H_{\mathcal{S}_f}) \subseteq (\mathcal{U} \cap H)_{\mathcal{S}_f}$. Thus $\mathcal{U} \cap cl_{\mathcal{S}}(H) \subseteq cl_{\mathcal{S}}(\mathcal{U} \cap H)$.

II. $(cl_{\mathcal{S}}(H))_{\mathcal{S}_f} = H_{\mathcal{S}_f}$. Because$(cl_{\mathcal{S}}(H))_{\mathcal{S}_f} = (H \cup H_{\mathcal{S}_f})_{\mathcal{S}_f} = H_{\mathcal{S}_f} \cup (H_{\mathcal{S}_f})_{\mathcal{S}_f} = H_{\mathcal{S}_f}$.

III. $H\ is\ \tau_{\mathcal{S}}$- closed if and only if $H_{\mathcal{S}_f} \subseteq H$. Because $H\ is\ \tau_{\mathcal{S}}$- closed if and only if$(X - H)\ is\ \tau_{\mathcal{S}}$- open if and only if $(X - H) \in \tau_{\mathcal{S}}$if and only if $cl_{\mathcal{S}}(H) = H$ if and only if $H_{\mathcal{S}_f} \cup H = H$ if and only if $H_{\mathcal{S}_f} \subseteq H$.

IV. If $H \subseteq H_{\mathcal{S}_f}$, then $H_{\mathcal{S}_f} = cl(H_{\mathcal{S}_f}) = cl(H) = cl_{\mathcal{S}}(H)$. Because by Proposition 5. 2. 7 part 8, $H_{\mathcal{S}_f} = cl(H_{\mathcal{S}_f}) \subset cl(H)$. But $H \subset H_{\mathcal{S}_f}$, that is $cl(H) \subset cl(H_{\mathcal{S}_f})$. And so that $H_{\mathcal{S}_f} = H_{\mathcal{S}_f} \cup H = cl_{\mathcal{S}}(H)$.

V. $H \in \tau_{\mathcal{S}}$ if and only if $H = int_{\mathcal{S}}(H)$ (where $int_{\mathcal{S}}(H) = X - cl_{\mathcal{S}}(X - H)$). Because $H \in \tau_{\mathcal{S}}$ if and only if $H\ is\ \tau_{\mathcal{S}}$- open if and only if $X - H\ is\ \tau_{\mathcal{S}}$- closed if and only if $cl_{\mathcal{S}}(X - H) = X - H$ if and only if $X - cl_{\mathcal{S}}(X - H) = X - X - H = H$ if and only if $int_{\mathcal{S}}(H) = H$.

VI. Every $\mathcal{S}$ –bushy set is $\tau_{\mathcal{S}}$- bushy set (H is called $\tau_{\mathcal{S}}$- bushy set if and only if $cl_{\mathcal{S}}(H) = X$). Because $cl_{\mathcal{S}}(H) = H_{\mathcal{S}_f} \cup H = X \cup H = X$.

VII. Every $\tau_{\mathcal{S}}$-bushy set is dense. Because $X = cl_{\mathcal{S}}(H) = H_{\mathcal{S}_f} \cup H \subseteq cl(H) \cup H = cl(H)$.

Proposition 5. 3. 20: Let $(X, \delta, \mathcal{S})$ be proximity sporadic space. Then

1. τ_1, τ_2 are two topologies. If $\tau_1 \subseteq \tau_2$ and H is $\tau_{\mathcal{S}2}$- bushy set, then H is $\tau_{\mathcal{S}1}$- bushy set.
2. If $H \subseteq H_{\mathcal{S}_f}$, then H is $\tau_{\mathcal{S}}$- bushy if and only if H is τ-dense.
3. If H is $\tau_{\mathcal{S}}$-bushy set and $\mathcal{U} \in \tau$, then $\mathcal{U} \subseteq cl_{\mathcal{S}}(\mathcal{U} \cap H)$.
4. If H is $\mathcal{S}$ –bushy set and $\mathcal{U} \in \tau$, then $\mathcal{U} \subseteq (\mathcal{U} \cap H)_{\mathcal{S}_f}$.

Proof.

1) For every $x \in X$, $x \in cl_{\tau_{\mathcal{S}2}}(H)$thus $x \in (H_{\mathcal{S}_{f2}} \cup H)$, that is either $x \in H$, thus $x \in (H_{\mathcal{S}_{f1}} \cup H)$, thus $cl_{\tau_{\mathcal{S}1}}(H) = X$, that is, H is $\tau_{\mathcal{S}1}$- bushy set. Or $x \in H_{\mathcal{S}_{f2}}$, this implies for every $\mathcal{U} \in \tau_2(x)$, $(\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}$ since $\tau_1 \subseteq \tau_2$, that is for every $\mathcal{V} \in \tau_1(x)$ such that $(\mathcal{V} \cap H)\delta O$ for some $O \in \mathcal{S}$ thus $x \in H_{\mathcal{S}_{f1}}$, that is $cl_{\tau_{\mathcal{S}1}}(H) = X$.

2) Let $H \subseteq H_{\mathcal{S}_f}$. By Note 5. 3. 21, $cl(H) = cl_{\mathcal{S}}(H)$. Thus H is $\tau_{\mathcal{S}}$- bushy.

3) Clear.

4) Clear■

Definition 5. 3. 21: $(X, \delta, \tau_{\delta}, \mathcal{S})$ is called *Co* $\mathcal{S}$ − *bushy space* if and only if there exist at least one subset of X is a $\mathcal{S}$ − bushy set.

Lemma 5. 3. 22: If X is *Co* $\mathcal{S}$ − *bushy space*, then $X = X_{\mathcal{S}_f}$.

Proof.

It is an immediate consequence of Proposition 5. 3. 5 part 1■

Proposition 5. 3. 23: If X is *Co* $\mathcal{S} -$ *bushy space*. Then

1. $H_{\mathcal{S}_t} \subseteq H_{\mathcal{S}_f}$;
2. $\emptyset = \emptyset_{\mathcal{S}_t}$;
3. $(\emptyset_{\mathcal{S}_f})_{\mathcal{S}_t} = \emptyset$;
4. $(X_{\mathcal{S}_f})_{\mathcal{S}_t} = X$;
5. $H_{\mathcal{S}_t} \subseteq (H_{\mathcal{S}_f})_{\mathcal{S}_t}$.

Proof.

1) Let $x \in H_{\mathcal{S}_t}$. Then $x \notin (X - H)_{\mathcal{S}_f}$. By Definition 5. 2. 1, there exists $\mathcal{U} \in \tau_\delta(x)$, such that $(\mathcal{U} \cap (X - H))\overline{\delta}O$for every $O \in \mathcal{S}$. Now if possible $x \notin H_{\mathcal{S}_f}$, then there exist $\mathcal{V} \in \tau_\delta(x)$ such that $(\mathcal{V} \cap H)\overline{\delta}O$for every $O \in \mathcal{S}$. By axiom [P4] we get $((\mathcal{U} \cap X - H) \cup (\mathcal{V} \cap H)\overline{\delta}O$ for every $O \in \mathcal{S}$. By Theorem1. 1. 7 part 3, we have that $(\big((\mathcal{U} \cap \mathcal{V}) \cap X - H\big) \cup \big((\mathcal{U} \cap \mathcal{V}) \cap H\big))\overline{\delta}O$.And so that $((\mathcal{U} \cap \mathcal{V}) \cap (X - H \cup H))\overline{\delta}O$ that is, $(\mathcal{U} \cap \mathcal{V})\overline{\delta}O$ for every $O \in \mathcal{S}$...... (1).

But the space is co $\mathcal{S} -$ bushy that implies $X = X_{\mathcal{S}_f}$thus for every $G \in \tau_\delta$, $(G \cap X)\delta O$ for some $O \in \mathcal{S}$. But $(\mathcal{U} \cap \mathcal{V}) \in \tau_\delta$ that is $((\mathcal{U} \cap \mathcal{V}) \cap X)\delta O$ for some $O \in \mathcal{S}$ which is a contradiction with (1). Hence $H_{\mathcal{S}_t} \subseteq H_{\mathcal{S}_f}$.

2) It is an immediate consequence of Proposition 5. 2. 15 part 6, and Lemma 5. 3. 22.

3) By Proposition 5. 2. 7 part 5 and by part 2.

4) By Lemma 5. 3 22 and Proposition 5. 2. 14 part 8.

5) By part 1, we get $H_{\mathcal{S}_t} \subseteq H_{\mathcal{S}_f}$, that is, $H_{\mathcal{S}_t} \subseteq (H_{\mathcal{S}_t})_{\mathcal{S}_t} \subseteq (H_{\mathcal{S}_f})_{\mathcal{S}_t}$■

Proposition 5. 3. 24: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a *Co* $\mathcal{S} -$ *bushy space*. Then H is τ_δ −closed subset of X the following statements are hold:

1. $H_{\mathcal{S}_f} \subseteq H$;
2. $H_{\mathcal{S}_t} \subseteq H$;
3. $H_{\mathcal{S}_t} = (H_{\mathcal{S}_t})_{\mathcal{S}_t}$.

Proof.

1) Let H is a τ_δ −closed set. Then $X - H$ is a τ_δ −open set. By Proposition 5. 2. 14 part 6, $(X - H) \subseteq (X - H)_{\mathcal{S}_t}$, thus $X - (X - H)_{\mathcal{S}_t} \subseteq H$, that is, $H_{\mathcal{S}_f} \subseteq H$.

2) By Proposition 5. 3. 26, $H_{\mathcal{S}_t} \subseteq H_{\mathcal{S}_f}$ **,** and by part 1, $H_{\mathcal{S}_f} \subseteq H$.

3) By Proposition 5. 2. 14, $H_{\mathcal{S}_t} \subset (H_{\mathcal{S}_t})_{\mathcal{S}_t}$. By part 2, $H_{\mathcal{S}_t} \subset H$, that is, $(H_{\mathcal{S}_t})_{\mathcal{S}_t} \subset H_{\mathcal{S}_t}$. Hence $H_{\mathcal{S}_t} = (H_{\mathcal{S}_t})_{\mathcal{S}_t}$■

Proposition 5. 3. 25: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a *Co* $\mathcal{S} -$ *bushy space*. Then H is nonempty τ_δ −open subset of X the statements holds:

1. $H_{\mathcal{S}_f} = (H_{\mathcal{S}_f})_{\mathcal{S}_f}$;
2. $H \subseteq (H_{\mathcal{S}_f})_{\mathcal{S}_t}$;
3. $H \subseteq H_{\mathcal{S}_t} \subseteq H_{\mathcal{S}_f} = \tau_\delta - cl(H)$.

Proof.

1) By Proposition 5. 2. 7, $(H_{\mathcal{S}_f})_{\mathcal{S}_f} \subset H_{\mathcal{S}_f}$. By Proposition 5. 3. 26, $H_{\mathcal{S}_t} \subseteq H_{\mathcal{S}_f}$. Since H is τ_δ −open, $H \subset H_{\mathcal{S}_t}$, thus $H_{\mathcal{S}_f} \subset (H_{\mathcal{S}_f})_{\mathcal{S}_f}$. Hence $H_{\mathcal{S}_f} = (H_{\mathcal{S}_f})_{\mathcal{S}_f}$.

2) By Proposition 5. 3. 26, $H_{\mathcal{S}_t} \subseteq H_{\mathcal{S}_f}$, then $H_{\mathcal{S}_t} \subset (H_{\mathcal{S}_t})_{\mathcal{S}_t} \subseteq (H_{\mathcal{S}_f})_{\mathcal{S}_t}$ but H is τ_δ −open, thus $H \subseteq (H_{\mathcal{S}_f})_{\mathcal{S}_t}$.

3) By Proposition 5. 2. 7, $H_{\mathcal{S}_f} \subseteq \tau_\delta - cl(H)$. Now let $x \in \tau_\delta - cl(H)$. Then for every $G \in \tau_\delta(x)$, $\mathcal{U}\delta$H, that is, $(G \cap H) \neq \emptyset$. If possible $x \notin H_{\mathcal{S}_f}$. Then there exists $\mathcal{U} \in \tau_\delta(x)$ that is $(\mathcal{U} \cap H)\overline{\delta}O$ for every $O \in \mathcal{S}$…… (1).

Since$(\mathcal{U} \cap H) \in \tau_\delta$ and X is $\mathcal{S}$ − *bushy set*, that is, $((\mathcal{U} \cap H) \cap X)\delta O$ for some $O \in \mathcal{S}$ which is a contradiction with (1). Thus $H_{\mathcal{S}_f} = cl(H)$■

Note 5. 3. 26: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a *Co* $\mathcal{S}$ − *bushy space*. Then

1. If H is a $\mathcal{S}$ −bushy set, then $H_{\mathcal{S}_f}$ and $(H_{\mathcal{S}_f})_{\mathcal{S}_t}$ are $\mathcal{S}$ −bushy sets.

   That is clear because $H_{\mathcal{S}_f} = X$ thus $(H_{\mathcal{S}_f})_{\mathcal{S}_f} = X_{\mathcal{S}_f} = X$. And also$((H_{\mathcal{S}_f})_{\mathcal{S}_t})_{\mathcal{S}_f} = ((X)_{\mathcal{S}_t})_{\mathcal{S}_f} = X_{\mathcal{S}_f} = X$.

2. If $H_{\mathcal{S}_f}$ is $\mathcal{S}$ −bushy set, then H is $\mathcal{S}$ −bushy set.

   Because $X = (H_{\mathcal{S}_f})_{\mathcal{S}_f} \subset H_{\mathcal{S}_f}$, that is, $H_{\mathcal{S}_f} = X$.

Lemma 5. 3. 27: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a *Co* $\mathcal{S}$ − *bushy space*, then

$$(H_{\mathcal{S}_f})_{\mathcal{S}_t} = (((H_{\mathcal{S}_f})_{\mathcal{S}_t})_{\mathcal{S}_f})_{\mathcal{S}_t}.$$

Proof.

By Propositions 5. 2. 7 and 5. 2. 14, we get $(H_{\mathcal{S}_f})_{\mathcal{S}_t} \subset (((H_{\mathcal{S}_f})_{\mathcal{S}_t})_{\mathcal{S}_f})_{\mathcal{S}_t}$. Agian, by Proposition 5. 3. 26, $H_{\mathcal{S}_t} \subseteq H_{\mathcal{S}_f}$, thus $(H_{\mathcal{S}_f})_{\mathcal{S}_t} \subset (H_{\mathcal{S}_f})_{\mathcal{S}_f} \subset H_{\mathcal{S}_f}$, and so that $((H_{\mathcal{S}_f})_{\mathcal{S}_t})_{\mathcal{S}_f} \subset (H_{\mathcal{S}_f})_{\mathcal{S}_f} \subset H_{\mathcal{S}_f}$, hence$(((H_{\mathcal{S}_f})_{\mathcal{S}_t})_{\mathcal{S}_f})_{\mathcal{S}_t} \subset (H_{\mathcal{S}_f})_{\mathcal{S}_t}$, that is, $(H_{\mathcal{S}_f})_{\mathcal{S}_t} = (((H_{\mathcal{S}_f})_{\mathcal{S}_t})_{\mathcal{S}_f})_{\mathcal{S}_t}$■

Definition 5. 3. 28: $(X, \delta, \tau_\delta, \mathcal{S})$ is a dismountable space with respect to sporadic if and only if X contains two disjoint $\mathcal{S} -$ bushy sets. Otherwise X is non-dismountable space with respect to sporadic.

Example 5. 3. 29: Let $X = \{1, 2, 3, 4, 5\}$, δ is proximity define by: $A\delta B\ \forall A \neq \emptyset, B \neq \emptyset$ except the relation:

$\{1\}\overline{\delta}\{3\}, \{1\}\overline{\delta}\{4\}, \{1\}\overline{\delta}\{2\}, \{5\}\overline{\delta}\{3\}, \{5\}\overline{\delta}\{4\}, \{5\}\overline{\delta}\{2\}, \{1\}\overline{\delta}\{3,4\},$ $\{1\}\overline{\delta}\{4,2\}, \{1\}\overline{\delta}\{3,2\}, \{1\}\overline{\delta}\{3,4,2\}, \{5\}\overline{\delta}\{3,4\}, \{5\}\overline{\delta}\{2,4\}, \{5\}\overline{\delta}\{3,2\},$ $\{5\}\overline{\delta}\{3,4,2\}, \{1,5\}\overline{\delta}\{3,4\}, \{1,5\}\overline{\delta}\{4,2\}, \{1,5\}\overline{\delta}\{3,2\}, \{1,5\}\overline{\delta}\{2,3,4\},$ $\{1,5\}\overline{\delta}\{3\}, \{1,5\}\overline{\delta}\{4\}, \{1,5\}\overline{\delta}\{2\}$, and $\{3\}\overline{\delta}\{1\}, \{4\}\overline{\delta}\{1\}, \{2\}\overline{\delta}\{1\},$ $\{3\}\overline{\delta}\{5\}, \{4\}\overline{\delta}\{5\}, \{5\}\overline{\delta}\{2\}, \{3,4\}\overline{\delta}\{1\}, \{4,2\}\overline{\delta}\{1\}, \{3,2\}\overline{\delta}\{1\},$ $\{2,3,4\}\overline{\delta}\{1\}, \{3,4\}\overline{\delta}\{5\}, \{4,2\}\overline{\delta}\{5\}, \{3,2\}\overline{\delta}\{5\}, \{3,4,2\}\overline{\delta}\{5\}, \{3,4\}\overline{\delta}\{1,5\},$ $\{4,2\}\overline{\delta}\{1,5\}, \{3,2\}\overline{\delta}\{1,5\}, \{3,4,2\}\overline{\delta}\{1,5\}, \{3\}\overline{\delta}\{1,5\}, \{4\}\overline{\delta}\{1,5\}, \{2\}\overline{\delta}\{1,5\}$.

Then $\tau_\delta = \{X, \emptyset, \{1,5\}, \{2,3,4\}\}$, and $\mathcal{S} =$ $\{\emptyset, \{4\}, \{1\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{1,2,3\},$ $\{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}, \{2,3,4,5\}, \{1,2,3,4\}\}$

Then X is dismountable with respect to sporadic because there exists $\{1,4\}$ and $\{2,3,5\}$ such that $\{1,4\}_{\mathcal{S}_f} = \{2,3,5\}_{\mathcal{S}_f} = X$, $\{1,4\} \cap \{2,3,5\} = \emptyset$, and $\{1,4\} \cup \{2,3,5\} = X$.

Proposition 5. 3. 30: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a dismountable space with respect to sporadic. Then X is a resolvable space.

Proof.

Suppose that X is dismountable with respect to sporadic. Then there exist two disjoint $\mathcal{S} -$ bushy sets H, D such that $H \cap D = \emptyset, H \cup D = X$. Since every $\mathcal{S} -$ bushy set is dense set, $H \; and \; D$ are dense sets, hence X is resolvable■

Proposition 5. 3. 31: If $(X, \delta, \tau_\delta, \mathcal{S})$ is a dismountable space with respect to sporadic, then $\bigcap \mathcal{S}(X, \delta) = \emptyset$.

Proof.

Let X is a dismountable space with respect to sporadic. Then there exists at least two disjoint $\mathcal{S} -$ bushy set H, D such that $H \cap D = \emptyset$, thus $\bigcap \mathcal{S}(X, \delta) = \emptyset$■

Corollary 5. 3. 32: If $\bigcap \mathcal{S}(X, \delta) \neq \emptyset$, then X is a non-dismountable space with respect to sporadic.

Proposition 5. 3. 33: Let $(X, \tau_{\delta 1}, \delta_1, \; \mathcal{S}_1) \; and \; (X, \tau_{\delta 2}, \delta_2, \; \mathcal{S}_2)$ be two topological defined on X, such that $\delta_1 \; > \delta_2$. If $(X, \tau_{\delta 1}, \delta_1, \; \mathcal{S}_1)$ is non-dismountable space with respect to sporadic, then $(X, \tau_{\delta 2}, \delta_2, \; \mathcal{S}_2)$ is non-dismountable space with respect to sporadic.

Proof.

Let us suppose that$(X, \tau_{\delta 2}, \delta_2, \; \mathcal{S}_2)$ is a dismountable space with respect to sporadic. Then there exist two disjoint τ_2 −bushy sets H, D such that $H \cap D = \emptyset, H \cup D = X$. By Proposition 5. 3. 14**,** every τ_2 −bushy set is τ_1 −bushy set, that is $(X, \tau_{\delta 1}, \delta_1, \; \mathcal{S}_1)$ is a dismountable space which is a contradiction, hence $(X, \tau_{\delta 2}, \delta_2, \mathcal{S}_2)$ is a non-dismountable space with respect to sporadic■

Proposition 5. 3. 34: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space. X is a non-dismountable space with respect to sporadic if and only if there is no $\mathcal{S} -$ bushy set H for which $(X - H)$ is also $\mathcal{S} -$ bushy set.

Proof.

If possible there exist $H_{\mathcal{S}_f} = (X - H)_{\mathcal{S}_f} = X$, then $X = H \cup (X - H)$, and so that $H \cap (X - H) = \emptyset$, that is, X is a dismountable space which is a contradiction with hypothesis.

Conversely, if X is a dismountable space with respect to sporadic, then there exist nonempty $H, D$are $\mathcal{S} -$ bushy sets of X such that $H_{\mathcal{S}_f} = D_{\mathcal{S}_f} = X$, but $X = H \cup D$, $H \cap D = \emptyset$ thus $D = X - H$, and so that $D_{\mathcal{S}_f} = (X - H)_{\mathcal{S}_f}$, that is, $H_{\mathcal{S}_f} = (X - H)_{\mathcal{S}_f} = X$, this is a contradiction with hypothesis. Hence X is non-dismountable space with respect to sporadic■

Corollary 5. 3. 35: Space is a dismountable with respect to sporadic if and only if there exists a set and complement of these set are sporadic bushy sets.

Proposition 5. 3. 36: Let $(X, \delta, \tau_\delta, \mathcal{S})$ be a sporadic topological proximity space. For every $x \notin H$ and H is a $\mathcal{S} -$ bushy set, then $\{x\}$ is not $\tau_\delta -$open set.

Proof.

Suppose H is $\mathcal{S} -$ bushy subset of X. Then $H_{\mathcal{S}_f} = X$, thus for every $\mathcal{U} \in \tau_\delta$, $(\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}$ this implies $\mathcal{U} \cap H \neq \emptyset$ for every $\mathcal{U} \in \tau_\delta$.

If possible $\{x\}$ is τ_δ −open set . Then $\{x\} \cap H \neq \emptyset$, that is, $x \in H$ which is a contradiction, thus $\{x\}$ is not τ_δ −open■

Corollary 5. 3. 37: If the space has a singleton τ_δ −open set, then the space is a non-dismountable with respect to sporadic.

Proof.

If X is dismountable space with respect to sporadic, then there exist nonempty $H, D$are $\mathcal{S} -$ bushy sets of X such that $H_{\mathcal{S}_f} = D_{\mathcal{S}_f} = X$, $X = H \cup D, H \cap D = \emptyset$. Then for every $\mathcal{U} \in \tau_\delta, (\mathcal{U} \cap H)\delta O$ for some $O \in \mathcal{S}$ but $\{x\} \in \tau_\delta$ this implies $\{x\} \cap H \neq \emptyset$, and so that for every $\mathcal{V} \in \tau_\delta, (\mathcal{V} \cap D)\delta O$ for some $O \in \mathcal{S}$ but $\{x\} \in \tau_\delta$ this implies $\{x\} \cap D \neq \emptyset$ thus $H \cap D \neq \emptyset$ which is a contradiction■

Remark 5. 3. 38: If $(X, \delta, \tau_\delta, \mathcal{S})$ is a dismountable space with respect to sporadic, then there exist disjoint $\mathcal{S} -$ takeoff sets is empty sets.

Proof.

Suppose X is dismountable with respect to sporadic. Then there exist two sets $H, D \subseteq X$, such that $H_{\mathcal{S}_f} = D_{\mathcal{S}_f} = X, H \cup D = X, H \cap D = \emptyset$ this implies to $X - H_{\mathcal{S}_f} = X - D_{\mathcal{S}_f} = \emptyset$. By Proposition 5. 2. 15 part 4, we have that $(X - H)_{\mathcal{S}_t} = (X - D)_{\mathcal{S}_t} = \emptyset$ but $(X - H) = D$ and $(X - D) = H$, Hence $H_{\mathcal{S}_t} = D_{\mathcal{S}_t} = \emptyset$■

6. 1 Applications of proximity space

One of the most important problems facing mathematicians is how to link mathematical concepts to life.

The space of proximity is to solve this problem, as most mathematical concepts can be linked to our lives. Proximity plays an important role in solving a lot of problems that depend on human perception that emerged in the fields of image analysis, image processing, facial recognition, and Information Systems, as well as science problems, forgery, medicine, engineering, chemistry, physics, and various other sciences, due to the ease of linking this space to various sciences [27, 28, 29, 30, 31, 32, 33, 34].

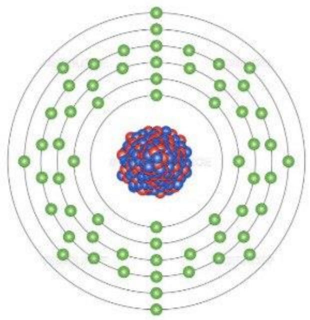
- For example, we can use proximity in what is known as visual promotion that the owners of the shops use to display their marketing products, where similar products are displayed on vertical or horizontal shelves according to the type of category, size, or color that helps in promoting these goods.

In the picture below A it represents the Makki Mouse Games collection. C is the Mashha Games set, but B also represents Masha games of lower sizes. Efromovic proximity can be used to display these products.

Where: $A\bar{\delta}B, then\ there\ exists\ C\ s.t C\bar{\delta}A\ and\ (X-C)\bar{\delta}B$

1. Figure Sample Efromovic Displays

- The atom represents a good example of the proximity space, where we notice that if the electrons are near to the nucleus, the more the core attractiveness strength of it. This requires a large energy to free these electrons from the atom. However, if electrons are far from the nucleus the easier it is to free these electrons from the atom.



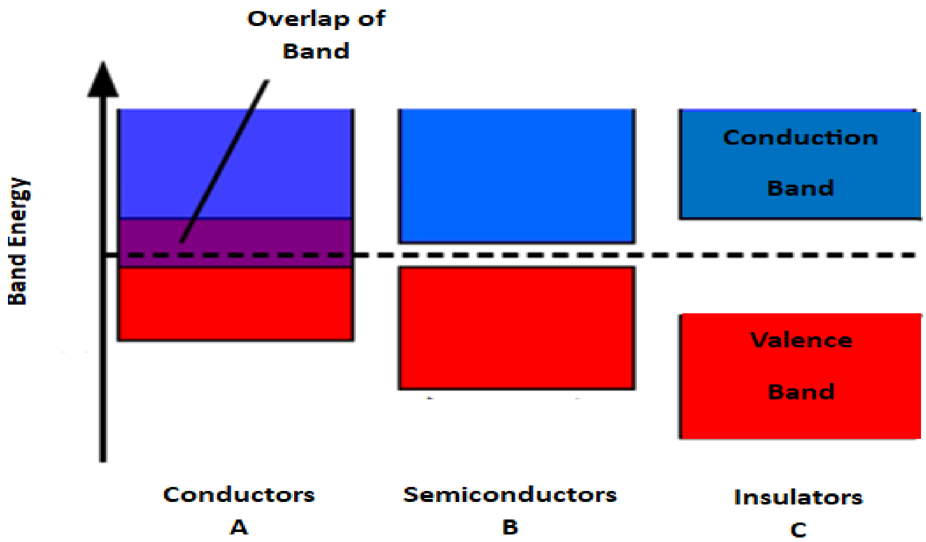
2. Figure Atoms

Electrons energy also differs in the atoms isolated from each other, according to their different locations in the external covers of the atom, and it is classified according to its capacity to two bands:

1- Valence band: It is the band in which the electrons are not ready for movement from one atom to another.

2- Conduction band: It is the band in which the electrons can move freely and move from one atom to another.

We notice through the drawing:



3. Figure Energy Bands in the Atoms

A- There is a large gap between the valence band and conduction band. This means there is a far between the two bands, called the materials in this case with insulating materials.

B - There are a few gaps between the two bands, so these materials are semiconductors.

C - We notice that there is a common area between the two bands, meaning that the two bands are near each other, which facilitates the process of electronic transmission, called the materials in this case conductive materials.

- Let X be a collection of all people in the hospital. A, B, and C subset of X. Let δ be a binary relationship between two people with diabetes, meaning:
  $A\delta B$ if and only if A and B are infected with diabetes, that is A near B, in the relationship of proximity if and only if A and B are infected with diabetes. Then δ fulfills Cech's axioms, and therefore, (X, δ) represents proximity space.

1- If A is near B, then A and B are infected with diabetes, thus B is also near A, that is, $A\delta B \Rightarrow B\delta A$.

2- If A is near $(B \cup C)$, that mean B or C is infected with diabetes. Thus A is near B or A is near C. Also, if A is near B or A is near C we have that A is near $(B \cup C)$, thus $A\delta(B \cup C)$if and only if $A\delta B$ or $A\delta C$.

3- If A is near B, this means that both of them carry diabetes, meaning that A and B people are not without disease, thus if $A\delta B \Rightarrow A \neq \emptyset$, and $B \neq \emptyset$.

4- If A intersects with B in carry diabetes, this leads to A near B, that is, $A \cap B \neq \emptyset \Rightarrow A\delta B$.

Accordingly, Cech's proximity axioms are achieved, meaning that (X, δ) represents a proximity space.

Let σ represent the family of all people whose blood insulin level is more than 200. We note that the axioms of Definition 1. 1. 14 are achieved.

1- If A and B are subsets of σ, then A and B are infected with diabetes, thus $A\delta B$.

2- We note that A union B is a max of the infected with diabetes between A and B, thus either A is infected with diabetes or B is infected with diabetes, meaning A is an element of the σ or B is an element of the σ.

3- If A is near each element of B, and $B \in \sigma$. This means A is infected with diabetes, thus A belongs to σ, that is, if $A\delta\{b\}$ for each $b \in B$, then $A \in \sigma$.

Example 6. 1. 1: Let $X = \{a, b, c\}$. Where a represent the people with diabetes whose diabetes ranges between (80-140), b represent the people with diabetes whose diabetes ranges between (140-250), and c represent the people with diabetes whose diabetes ranges between (250-350). Let δ be a binary relationship between two people with diabetes. Let σ represent the family of all people whose blood insulin level is more than 200.

Let $\tau = \{X, \emptyset, \{a\}, \{a, c\}\}$, $\sigma = \{X, \{c\}, \{a, c\}, \{b, c\}\}$,

$\mathcal{S} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then

sets	follower set	Takeoff set	$\mathcal{S}$ -follower set	$\mathcal{S}$ -Takeoff set
$\{a\}$	$\emptyset$	$\{a\}$	X	$\{a, c\}$
$\{b\}$	$\emptyset$	$\{a\}$	$\{b\}$	$\emptyset$
$\{c\}$	$\{b, c\}$	X	$\emptyset$	$\emptyset$
$\{a, b\}$	$\emptyset$	$\{a\}$	X	X
$\{a, c\}$	$\{b, c\}$	X	X	$\{a, c\}$
$\{b, c\}$	$\{b, c\}$	X	$\{b\}$	$\emptyset$
X	$\{b, c\}$	X	X	X
$\emptyset$	$\emptyset$	$\{a\}$	$\emptyset$	$\emptyset$

1. Table Diabetes Levels

We note through the table that the follower set gives people with diabetes and people at risk of developing diabetes. As for the takeoff set, it gives people who do not have diabetes. As for the $\mathcal{S}-$ follower set, it gives the people at risk of diabetes. As for the $\mathcal{S}-$takeoff set, it was given to people with and without diabetes. As for the X, it appeared with people with no diabetes and the empty set appeared with people with diabetes.

6. 2 Discussion and Conclusions

In this study, the topology generated within the proximity space was discussed. We concluded that each τ_δ −open set is τ_δ −closed within this topology. Due to the method of generating of the τ_δ −open andτ_δ − closed sets within proximity space. Proposition 1. 2. 12 explains that.

The symmetry relationship was also employed between two sets $A\Delta B$ and the relationship of the difference $A - B$ and linking it to the concept of a cluster to create two new binary relationships $\approx_\sigma$ and $\propto_\sigma$. The first had a great role in building a σ −Topological Proximity space, while the second relationship was building a focal cluster family.

Family (τ_σ) does not represent a topology because the conditions of intersections and union are not achieved, as well as the converse is not achieved. Nevertheless, under certain conditions, each one can lead to the other. Therefore, classical topology is σ −Topological Proximity if satisfy condition four of Definition 1. 4. 1. Converse, we need to satisfy the intersection and union conditions. In other words, the families are independent in general cases. Example 1. 4. 4 explains that.

In addition, we can see that $\tau_\sigma - int(A)$ and $\tau_\sigma - cl(A)$ are not $\tau_\sigma - open$ and $\tau_\sigma - closed$ sequentially. We say σ− weekly topological Proximity denoted by $\tau_{w\sigma}$ if every $\tau_\sigma - int(A)$ is $\tau_\sigma - open$ and every $\tau_\sigma - cl(A)$ is $\tau_\sigma - closed$. Hence, $\tau_{w\sigma}$ is a special case of τ_σ, that is, $\tau_{w\sigma}$ is τ_σ, but the converse is not always true.

The question here is whether $\tau_{w\sigma}$ represents topology? The answer is no because the conditions of intersect and the union are still not achieved. This is explained by Proposition 1. 4. 22, Definition 1. 4. 23, and Proposition 1. 4. 25.

In this dissertation, we used topology as well as the topology generated within the proximity space and the family (τ_σ) to build three types of spaces. The question that arises here is why we didn't settle for one space?

The answer to this question is that the basis of work and the construction of concepts mathematical include topology were on the space (X, τ_σ) that was built within this dissertation. However, these concepts were studied in the spaces (X, τ) and (X, τ_δ) to clarify the difference if these concepts were built within these spaces. In addition, the results of some concepts are unsatisfactory in this space, so we try to study them wider in another space, for example:

Follower points when studying it in space (X, τ_σ), then the results either X or empty set. While inside the space (X, τ) or (X, τ_δ), the results were satisfactory. Additionally, takeoff points, are effective within the three spaces where we get almost equal results within the three spaces except that P_{t_σ} is not τ_σ −open. Proposition 2. 1.3 and Remark 2. 2.4 shows that.

Another concept of density was presented in this dissertation called Bushy set. The findings of this concept are ineffective within the space of τ_σ or τ_δ. This mean cannot be get a bushy set within τ_σ or τ_δ when τ_σ or τ_δ are nontrivial, because the space has a bushy set if and only if every open is a member of cluster, but that is impossible within τ_σ due to the fourth condition of Definition 1. 4. 1. Also in case of τ_δ, because A is bushy if and only if $\forall A \in \tau_\delta \Rightarrow A\delta(X-A)$ but that difficult because $\forall x \in A \Rightarrow \{x\}\overline{\delta}(X-A)$ and by [P3] we get $\cup \{x\}\overline{\delta}(X-A)$, thus $A\overline{\delta}(X-A)$ but $A \ and \ (X-A) \in \tau_\delta$ and by [C1] $A \ or \ (X-A) \notin \sigma$. Remark 3. 1. 9, shows that.

Hence, we use a space τ in order to obtain better results, but this does not mean that τ_{σ} or τ_{δ} are ineffective spaces. However, we concluded that these spaces are connected spaces which cannot be separated by two disjoint bushy sets. When using the dense set yes it can be separated but by bushy it's totally different.

The reader may have the following question: How to build mathematical concepts within this work?

The answer is that first concept follower set: Definition 2. 2. 1 each open is corresponding to each element of the cluster. If the definition is changed to $\exists\ \mathcal{U} \in \tau(x), \forall\ C \in \sigma$ s.t $(\mathcal{U} \cap P)\delta C$, then $P_{f_{\sigma}} = \emptyset$ if $P \notin \sigma$ and $P_{f_{\sigma}} = X$ if $P \in \sigma$, thus the most characteristics are achieved, compact, regular, normal, connected etc

Also if the definition is changed to $\forall\ \mathcal{U} \in \tau(\mathrm{x}), \exists\ C \in \sigma$ s.t $(\mathcal{U} \cap P)\delta C$, then $P_{f_{\sigma}} = cl(P)$, thus most of the results will be equivalent to the results achieved in the closure set.

In addition , if the definition is changed to $\exists \mathcal{U} \in \tau(x), \exists\ C \in \sigma$ s.t $(\mathcal{U} \cap P)\delta C$, then $P_{f_{\sigma}} = X$ for every nonempty subset of X.

Second concept Takeoff set: Definition 2. 1. 1 there exists open is corresponding to there exists element of the cluster. If the definition is changed to $\forall\ \mathcal{U} \in \tau(x), \forall\ C \in \sigma$ s.t $(\mathcal{U} \cap (\mathrm{X} - P))\overline{\delta} C$, then $P_{t_{\sigma}} = \emptyset$ for every $P \subseteq X$. Because $\mathrm{X} \in \tau(x)$ and X is near every nonempty set.

Also if the definition is changed to $\forall\ \mathcal{U} \in \tau(\mathrm{x}), \exists\ C \in \sigma$ s.t $(\mathcal{U} \cap (\mathrm{X} - P))\overline{\delta} C$, then $P_{t_{\sigma}} = \mathrm{X}$ if $(X - P) \notin \sigma$ and $P_{t_{\sigma}} = \emptyset$ if $(X - P) \in \sigma$.

Also, if the definition is changed to $\exists \mathcal{U} \in \tau(x), \forall\ C \in \sigma$ s.t $(\mathcal{U} \cap (\mathrm{X} - P))\overline{\delta} C$, then we have two cases. Case one $P_{t_{\sigma}} = \emptyset$ if $\mathcal{U} \cap (\mathrm{X} - P) \neq \emptyset$. Two case $x \in P_{t_{\sigma}}$ if $\mathcal{U} \cap (\mathrm{X} - P) = \emptyset$.

Three concept cluster outer set: The same results as those of the second case. In summary, the construction of mathematical concepts was not arbitrary or by chance.

The last question remains, is it possible to build the concept of a follower set on the τ_σ? Answer. Yes, through the building of a new mathematical concept, which we called Sporadic, during which all the problems that appeared previously were solved. Whereas, Follower points are effective in a space $(X, \delta, \tau_\sigma, \mathcal{S})$ as well as the bushy set concept that can be studied in the space $(X, \delta, \tau_\delta, \mathcal{S})$ as well as space $(X, \delta, \tau_\sigma, \mathcal{S})$. In addition to that X inside a space$(X, \delta, \tau_\sigma, \mathcal{S})$ always is a bushy set. Therefore a space $(X, \delta, \tau_\sigma, \mathcal{S})$ is always Co $\mathcal{S} - sporadic$. But we found that when using the concept of sporadic, the space could not be Bushy Space, and the attached property cannot be achieved in this space.

6. 3 Future works

There are some issues that we did not address inside the dissertation due to the time factor, so we leave them as future works:

1- Most of the mathematical concepts presented in this dissertation did not touch on the possibility of transferring these concepts by functions. For example, follower set and takeoff set, we did not touch on the transfer of these sets from one space to another by functions. Additionally, do these sets have a topology property that can preserve their results when moving from one space to another?

2- The sets: $f_\sigma -$ set, $f_{t\sigma} - set$, $t_{f\sigma} -$ set , $t_\sigma -$ set, and $f_\sigma -$ perfect.

In the future, they can be studied more extensively. For instance, if we take the intersection of all complements of $t_{f\sigma} -$ set and denoted by $cl_{f_\sigma}(X)$ as well as the unions of all sets of $t_{f\sigma} -$ set and denoted by$int_{t_\sigma}(X)$. What are the properties of these two sets? Are they open or closed sets or not. And is it that $cl_{f_\sigma}(P) = P \cup \left(P_{f_\sigma}\right)_{t_\sigma}$, and $int_{t_\sigma}(P) = P \cap (P_{t_\sigma})_{f_\sigma}$ or the relationship is not true?

3- In addition, the concept of Sporadic, we hope in the future to study it in more details, where all the results obtained by researchers when studying the concept of cluster can be re-studied on the concept of sporadic. For example, Smirnov taking the family of all the clusters, and build a topology accordingly, and expanding the space of real numbers to a compact space. It could be re-studied and Smirnov, but on Sporadic families and find out the results.

4- Although there is an abundance of research on fuzzy topology, further study is still needed. The mathematical concepts presented by this dissertation can be re-studied on important spaces such as fuzzy, soft or Neutrosophic spaces, etc.

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المستخلص

نظرا لسهولة ربط الرياضيات بالواقع الحياتي داخل فضاء القرب تهدف هذه الدراسة الى ايجاد مفاهيم رياضية جديدة داخل فضاء القرب ودراسة خصائصها وعلاقتها بالمفاهيم الرياضية المتفق عليها وبالتالي الحصول على نتائج نأمل في ان تكون مفيدة في حل بعض المشاكل الرياضية المتزايدة بتزايد متطلبات الحياة اليومية.

حيث تتركز الاطروحة على خمسة محاور وهي كما يلي:

المحور الاول تم تقديم مفهوم $\sigma-$Topological Proximity بالاعتماد على علاقة ثنائية تم بناءها داخل هذا العمل بمساعدة مفهوم cluster . هذه العائلة تحقق شروط عائلة التبلوجي ماعدا خاصية الاتحاد والتقاطع وهذا بدوره اتاح لنا فرصة دراسة مفاهيم النقاط الداخلية والخارجية والحدودية وكذلك نقاط الغاية داخل هذه العائلة وتوضيح الفرق بينها وبين دراسة هذه المفاهيم داخل التبلوجي . وبالتالي اصبح لدينا داخل هذا العمل ثلاث فضاءات هي: $(X, \tau , \delta, \sigma)$ ، $(X, \delta, \tau_\delta , \sigma)$ ، $(X, \delta, \sigma, \tau_\sigma)$ حيث X مجموعة غير خالية . جميع المفاهيم الرياضية التي تم تقديمها داخل هذه الاطروحة تم دراستها على هذه الفضاءا ت الثلاث.

المحور الثاني تم تقديم نوعين من النقاط Follower و Takeoff ومجموعة كل هذه النقاط اطلقنا عليها Follower set و Takeoff set والتي من خلالهما استطعنا من توليد تبلوجيات بالاعتماد على نظرية كورتوفسكي . كذلك من خلال مجموعة Follower اصبح بالامكان تقديم تعريف موازي للكثافة اطلقنا عليه المجموعة الكثة (Bushy set) والفضاء الذي يحوي على الاقل مجموعة واحده كثة اطلقنا عليه Co Bushy Space حيث الكثير من الخواص والنتائج لايمكن تحقيقها الا داخل هذا الفضاء .

المحور الثالث تم توظيف مفهوم المجموعة الكثة في دراسة قدرة الفضاء على التفكك (Dismountable) من خلال مجموعتين كثتين منفصلتين والذي يمثل تقليص لمفهوم قابلية الفضاء للحل (resolvable space) حيث كل فضاء قابل للتفكك يكون قابل للحل .
كذلك تم تقديم مفهوم Bushy Space, Attached Space ودراسة العلاقة بينهم وبين Submaximal, and Hyper connected Space

المحور الرابع تم دراسة ثلاث انواع من المجاميع المنفصلة التي تولد الفضاء اطلقنا عليها: (Cluster outer, cluster brim, and cluster disputed sets) التي تم بناءها بالاعتماد على نقاط Follower ونقاط Takeoff مع دراسة اهم الخصائص التي نحصل عليها بالاعتماد على هذه المجاميع وعلاقتها بقدرة الفضاء على التفكك (dismountable space) وبالتالي قدرة الفضاء على الحل(resolvable space). كذلك تم تقديم نوع خاص من المجاميع المفتوحة والمغلقة داخل فضاء القرب وبالتالي يمكن استغلالها في بناء فضاءات تبلوجية انعم او اخشن من التبلوجي المتولد داخل فضاء القرب.

المحور الخامس من الاطروحة تناول تقديم مفهوم مرادف لعائلة (Cluster) حيث تم اخذ المتممه لكل مجاميع عائلة Cluster وبالتالي تولدت لدينا عائلة اطلقنا عليها Sporadic واصبح بالامكان دراسة كل المفاهيم السابقة مع هذه العائلة .على الرغم من ان دراسة هذه العائلة كان بشكل مختصر الا انه استطعنا من الحصول على بعض النتائج المهمه التي ترتبط بها هذه العائلة مع Cluster ونتائج اخرى تتميز بها عن عائلة Cluster.

المحور السادس يقدم فيه الباحثين بعض التطبيقات حول فضاء القرب وربط النتائج التي توصل اليها الباحثين بالواقع الصحي.

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تعميم المفاهيم التبولوجية عبر الفضاءات العنقودية

أطروحة

مقدمة إلى مجلس كلية التربية للعلوم الصرفة / جامعة بابل

كجزء من متطلبات نيل درجة الدكتوراه فلسفة في التربية / الرياضيات

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