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DOMINATION IN RHOMBUS CHESSBOARD

Essam EL-Seidy

Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Cairo, Egypt

Ahmed A. Omran

Department of Mathematics, College of Education for Pure Science, Babylon University, Babylon, Iraq

ABSTRACT

In this article, we are interested in the problem of dominance in the chessboard with square cells. Beginning we will examine this problem on one piece of the chess pieces. In certain technique, we will continue to study the problem of domination in two different pieces. In the case of one piece, chess pieces which we will deal with in this article are rooks, bishops, and kings. The pieces of the two different types in our study are: kings with rooks, kings with bishops, and rooks with bishops. If possible, we will identify the number of different ways to place the minimum number of domination (total solution).

Keywords: Chessboard, Kings, Bishops, Rooks, and domination number

1. INTRODUCTION

One of the interesting problems in chessboard is the problem of placing a minimum number of chess pieces P to dominates the chessboard. The domination means that all vacant positions are under attack. This problem is called the domination number problem of P and this number is denoted by $\gamma(\mathrm{P})$. The number of different ways for placing pieces of P type to obtain the minimum domination number of P in each time denoted by $S(\gamma(P))$. Also we are interested in the domination number of two different types of pieces by fixing a number n_p of one type P of pieces and determine the domination number $\gamma(P^*, n_p)$ of another type P^* of pieces. Finally we will compute the number of different ways $S(\gamma(P^*, n_p))$ to place the minimum number of P^* pieces with a fixed number n_p of P pieces to dominate the chessboard.

In n square chessboard the γ numbers are studied in (Hon-Chan and Ting-Yem [1], Joe and William [2], John and Christopher [3] and Odile [4]) for rook "R", bishop "B" and king "K". They proved that, $\gamma(R) = n, \gamma(B) = n$ and $\gamma(K) = \left\lfloor \frac{n+2}{3} \right\rfloor^2$.

In Dietrich and Harborth [5], JoeMaio and William proved that $\gamma(R) = \min(\mathrm{m}, n)$ for

$m \times n$ Toroidal chessboard.

In EL-Seidy and Omran [6] and Seoud, et al. [7], we studied the isosceles triangle chessboard with square cells and one type of the pieces of rooks, bishops and kings. In [7], we proved each of the following:

$$\gamma(R) = n - \left\lfloor \frac{n+1}{3} \right\rfloor , \left(n - \left\lfloor \frac{n+1}{3} \right\rfloor\right)! \le S(\gamma(R)) \le \left(2\left\lfloor \frac{n+1}{3} \right\rfloor + 1\right)!,$$

$$\gamma(B) = n , \; S(\gamma(B)) = \begin{Bmatrix} n & , if\ n\ is\ odd \\ n-1 & , if\ n\ is\ even \end{Bmatrix},$$

$$\gamma(K) = \begin{Bmatrix} \frac{\left\lceil \frac{2n-1}{3} \right\rceil}{4}\left(\left\lceil \frac{2n-1}{3} \right\rceil + 2\right) , if \left\lceil \frac{2n-1}{3} \right\rceil\ is\ even \\ \frac{\left(\left\lceil \frac{2n-1}{3} \right\rceil + 1\right)^2}{4} \quad , if \left\lceil \frac{2n-1}{3} \right\rceil\ is\ odd \end{Bmatrix}, \text{ and} \tag{1}$$

$$S^*(\gamma(K)) = \begin{Bmatrix} \prod_{w=0}^{\left\lceil \frac{n}{3} \right\rceil - 1} \left\lceil \frac{2n-(3+6w)}{3} \right\rceil & if\ n \equiv 0(mod\ 3) \\ \prod_{w=1}^{\left\lceil \frac{n}{3} \right\rceil - 1} (2w^2 + w) & , if\ n \equiv 1(mod\ 3) \\ 1 \quad , & if\ n \equiv 2(mod\ 3) \end{Bmatrix} \tag{2}$$

Where S^* is the number of the different ways of distribution of K pieces in row movement only. For the equilateral triangle chessboard of length n, we proved that

$$\gamma(K) = \binom{\left\lceil \frac{n}{3} \right\rceil + 1}{\left\lceil \frac{n}{3} \right\rceil - 1} = \binom{\left\lceil \frac{n}{3} \right\rceil + 1}{2} \; \forall n \ge 2 \tag{3}$$

$$S^*(\gamma(K)) = \begin{Bmatrix} 1 \quad , & if\ n \equiv 0(mod\ 3) \\ \prod_{w=0}^{\left\lceil \frac{n}{3} \right\rceil - 2} \frac{1}{2}\left(\left\lfloor \frac{n-3w}{3} \right\rfloor^2 + 3\left\lfloor \frac{n-3w}{3} \right\rfloor\right) & , if\ n \equiv 1(mod\ 3) \\ \prod_{w=0}^{\left\lceil \frac{n}{3} \right\rceil - 1} \left(\left\lceil \frac{n-3w}{3} \right\rceil\right) & , if\ n \equiv 2(mod\ 3) \end{Bmatrix} \tag{4}$$

For two different types of pieces, we have got new results for domination of two types of pieces by fixing a number of pieces P of one type, and determine the domination number of the other type of pieces P^*.

2. RHOMBUS CHESSBOARDS

We consider the rhombus chessboard with square cells and one type of pieces of rooks R, bishops B and kings K, where the pieces move or attack as usual.

We mean by the length of a side the number of cells (squares) in that side. The length of each side in the rhombus chessboard is n.

In matrix form, let r_i denote the i^{th} row which is numbered from the middle (the middle row has the greatest length); $i = 0, \pm 1, \pm 2, \ldots, \pm(n-1)$ see Figure 1. Then the middle row r_0 contains $2n-1$ cells, and each of the two rows r_1 and r_{-1} which are lying above and below r_0

respectively contains $2n-3$ cells. In general the i^{th} row contains $2n-1-2|i|$ cells and this length is denoted by L_i. By the same manner the j^{th} column is denote by c_j ; $j = 0, \pm1, \pm2, \dots, \pm(n-1)$: and $Lj = 2n-1-2|j|$ .We denote the cell (square) of i^{th} row and j^{th} column by $s_{i,j}$, $i,j = 0, \pm1, \pm2, \dots, \pm(n-1)$ and we note that the total number of cells in rhombus chessboard of length n is $2n^2-2n+1$. Figure 1 shows chessboard of length 4.

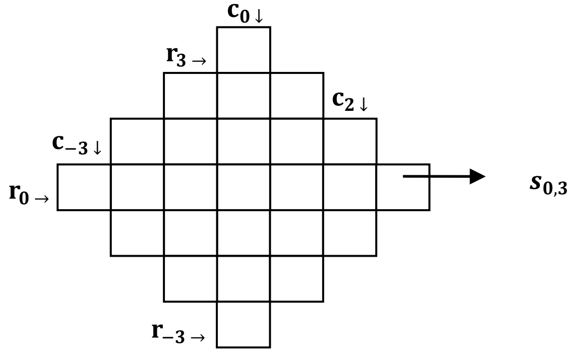

Figure-1.

3. DOMINATION OF ONE TYPE OF PIECE

In this section we compute the domination number $\gamma(P)$ of one type of pieces and its total solution $S(\gamma(P))$.

Theorem-3.1. For bishop B pieces, we have

(I) $\gamma(B) = 2n-1$

(II) $S(\gamma(B)) = (2n^n - n!)(2(n-1)^{n-1} - (n-1)!)$

Proof. (I) It is clear, if we put one B piece in each cell of the column c_0 in the chessboard that

$$\gamma(B) = 2n-1.$$

(II) Let us consider the two sets $D_w = \{s_{i,j}; i+j = n-w, 1 \le w \le 2n-1\}$ and

$d_w = \{s_{i,j}; i-j = n-w, 1 \le w \le 2n-1\}$, each set of D_w and d_w contains n cells when w is an odd number (the cells of black color) and intersect in $s_{n-w,0}$, and $n-1$ when w is an even number (the cells of white color) and intersect in $s_{n-w,0}$ as shown in Figure 2 ; $n = 4$. It is clear that these sets are independent (black and white) with respect to B pieces. We distribute n pieces of bishop in black cells and $n-1$ as in white cells, and then we distribute these pieces in two successive steps as follows:

Step 1: "The distribution of n pieces of bishop (domination of black cells)". In this step, we distribute the B pieces in two different sets of D_i and d_j , where i and j are odd . In this step we have two cases as follows:

(i) If we place n B pieces in D_1 as shown in Figure 2(a), then these pieces dominate all black cells. Each one of these pieces can move down in n cells to the left diagonally corresponding to its d_i^{th} diagonal. Thus there exist n^n methods to distribute n B pieces, to dominate the black cells.
(ii) If we put n B pieces in d_1 as shown in Figure 2(b), then as (i) there are n^n methods, to dominate the black cells.

From (i) and (ii), we have $2n^n$ ways for distribution of the black cells, but not all are different. The common ways for distribution in (i) and (ii) occur when the black diagonal contains no more than one piece. Then the number of ways together is $n!$. Thus the total solution of domination of n B pieces to the black cells in (i) and (ii) is $2n^n - n!$.

Step 2: "The distribution of n B pieces (domination of white cells)"

Similarly as in step 1 after we interchange n by $n-1$ and use the two sets D_i and d_j, where i and j are even, then the total solution of domination of $n-1$ B pieces to the white cells is $2(n-1)^{n-1} - (n-1)!$. Thus from the two above steps we have

$$S(\gamma(B)) = (2n^n - n!)(2(n-1)^{n-1} - (n-1)!).$$

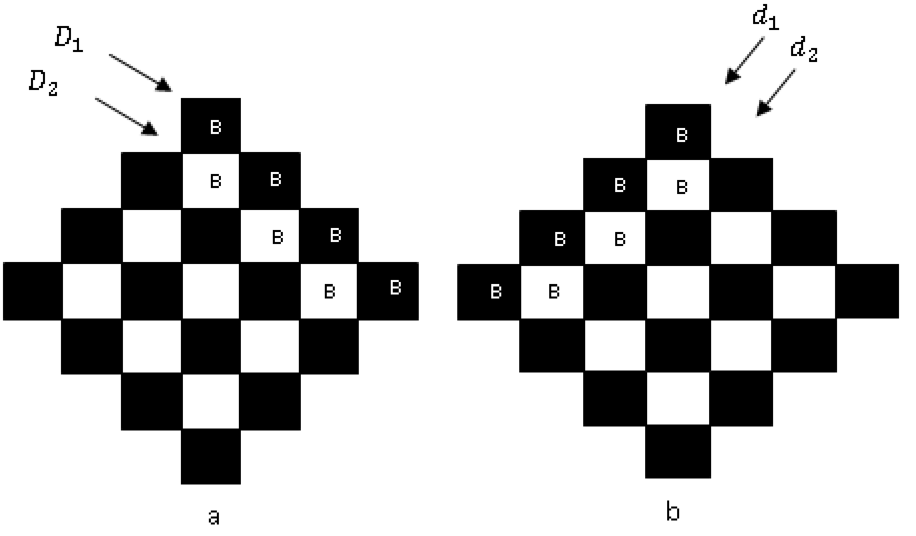

Figure-2.

Theorem-3.2. The domination number and the total solution for the R pieces are given by the following:

(I) $\gamma(R) = \begin{cases} n & \text{, if } n \text{ is odd} \\ n-1 & \text{, if } n \text{ is even} \end{cases}$

(II) $S(\gamma(R)) = \begin{cases} n! & , if\ n\ is\ odd \\ (n-1)! & , if\ n\ is\ even \end{cases}$

Proof. There are two cases that depend on n *as follows*:

(I) *If* n *is odd*, place the R pieces in the unique square of dimension $2\left\lfloor\frac{n}{2}\right\rfloor + 1 = n$, starting from above r_i ; $i = \left\lfloor\frac{n}{2}\right\rfloor$, such that there exist one piece only in any row or column of this square. Clearly these R pieces dominate all cells in chessboard and the number of these pieces is the minimum number of domination, that is $\gamma(R) = n$ as shown in Figure 3(a); $n = 5$. Since every row or column contains only one R piece then $S\big(\gamma(R)\big) = n!$.

(II) *If* n *is even*, similarly as in (i), but we use the middle square of dimension $n-1$ and the corners $s_{\pm(\frac{n}{2}-1),\pm(\frac{n}{2}-1)}$ as shown in Figure 3(b) ; $n = 6$,we get $\gamma(R) = n-1$ and $S(\gamma(R)) = (n-1\ \ )!$.

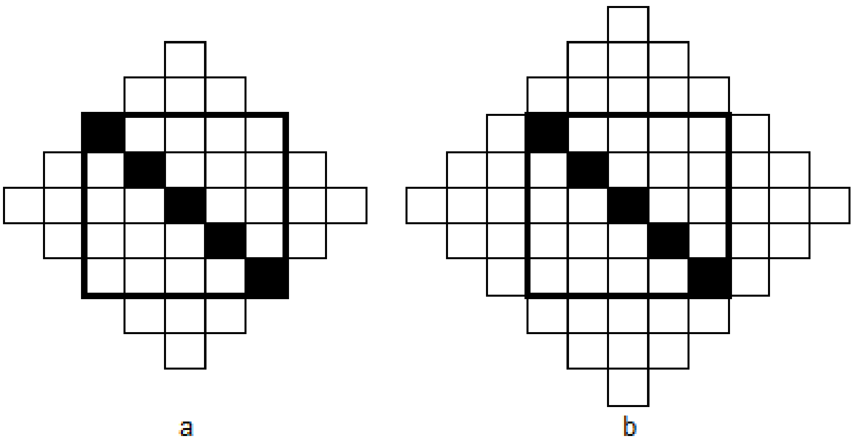

Figure-3.

In some cases for the total solution S of K pieces, we will be interested only in the row movement for the distribution of pieces . The reason is that the column movement is much more complicated. We will denote this special type of total solution of K pieces in row movement by $S^*(\gamma(K))$.

Theorem-3.3. Consider $m = \left\lceil\frac{2n-1}{3}\right\rceil$,and $l = \left\lceil\frac{2n-3}{3}\right\rceil$, then the domination number and special type of the total solution for the K pieces are given by the following:

$$\text{(I) } \gamma(K) = \begin{cases} \frac{m}{4}(m+2) + \frac{(l+1)^2}{4} & , if\ n \equiv 0 (mod\ 3\ ) \\ \frac{(m+1)^2}{4} + \frac{l}{4}(l+2) & , if\ n \equiv 1 (mod\ 3\ ) \\ \frac{(m+1)^2}{2} & , if\ n \equiv 2 (mod\ 3\ ) \end{cases}$$

(II) $S^*\big(\gamma\ (K\ )\big) =$

$$\begin{cases} \prod_{w=0}^{\left\lceil\frac{n}{3}\right\rceil-1} \left\lceil\frac{2n-(3+6w)}{3}\right\rceil & if\ n \equiv 0(mod\ 3\ ) \\ \prod_{w=1}^{\left\lceil\frac{n}{3}\right\rceil-1} (2w^2+w) * \prod_{w=0}^{\left\lceil\frac{n-1}{3}\right\rceil-1} \left\lceil\frac{2(n-1)-(3+6w)}{3}\right\rceil & ,if\ n \equiv 1(mod\ 3\ ) \\ \prod_{w=1}^{\left\lceil\frac{n-1}{3}\right\rceil-1} (2w^2+w) & ,if\ n \equiv 2(mod\ 3\ ) \end{cases}$$

Proof. (I) To simplify the distribution, we divide the rhombus chessboard into two isosceles triangles. One of these triangles with size n lying above r_{-1} and the other with size $n-1$ lying bellow r_0 as shown in Figure 4; (a) $n = 5$, (b) $n = 6$. If m is even, then $n \equiv 0(mod\ 3\ )$, if m is odd and l is even, then $n \equiv 1(mod\ 3\ )$, and if both m and l are odd then $n \equiv 2(mod\ 3\ )$. By using equation (1) (see the introduction), we get the result.
(II) From (I), and by using equation (2) (see the introduction), the assertion is clear.

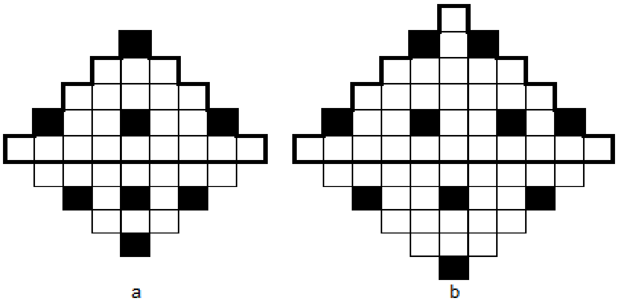

Figure-4.

The following example illustrates the above theorem for some different values of n.

Example-3.4.

(1) If $n = 5$, then $\gamma(K) = \frac{(m+1)^2}{2} = 8$ and

$$S^*\ (\gamma\ (K)) = \prod_{w=1}^{\left\lceil\frac{n-1}{3}\right\rceil-1} (2w^2+w) = 3,$$ (see Figure 4 (a)).

(2) If $n = 6$, then $\gamma(K) = \frac{m}{4}(m+2) + \frac{(l+1)^2}{4} = 10$ and

$$S^*(\gamma\ (K)) = \prod_{w=0}^{\left\lceil\frac{n}{3}\right\rceil-1} \left\lceil\frac{2n-(3+6w)}{3}\right\rceil = 3,$$ (see Figure 4 (b).

4. DOMINATION OF TWO DIFFERENT TYPES OF PIECES

In this section, we fix a number of one type P pieces and determine the domination number of another type of P^* pieces. By n_P we mean the fixed number of P pieces and by $\mathrm{N_P}$ the number of the cells which are attacked by the P piece and the cell of that piece.

4.1. Domination of K Pieces together with a Fixed Number of R Pieces

We denote the domination number of K pieces with a fixed number n_r of R pieces by $\gamma(K\,, n_r)$.

Theorem-4.1.1. If $n - n_r - 1 \equiv 0(mod\ 3)$, and n_r is odd less than n, then

(I) $\gamma(K\,, n_r) = 4\left(\frac{\left\lceil\frac{n-n_r-1}{3}\right\rceil + 1}{2}\right)$

(II) $S\big(\gamma\ (K, n_r)\big) = n_r!$

Proof. (I) If n_r is odd and $n - n_r - 1 \equiv 0(mod\ 3)$, then the suitable cells to place the R pieces are in the cells $s_{i,i}; i = 0, \pm1, \ldots, \pm\left\lfloor\frac{n_r}{2}\right\rfloor$. The position of these pieces divides the chessboard into four equilateral triangles with same size $n - n_r - 1$ as shown in Figure 5; $n = 10$ and $n_r = 3$. This division is the best, since these triangles are congruent to zero. Using equation (3) (see the introduction), we get the result.

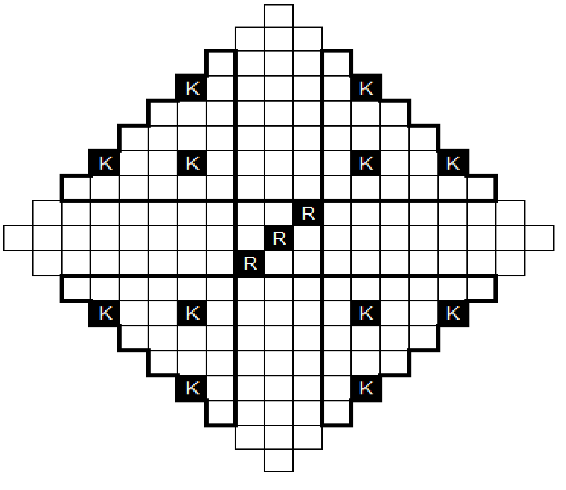

Figure-5.

(II) We place the R pieces in a square region of n_r dimension located in the center of the rhombus chessboard as shown in Figure 5. In this square, the number of ways to distribute n_r pieces of rook such that only one piece exist in any row or column is $n_r!$. For this distribution of n_r pieces, we get the maximum attacked neighborhood. The cells which are not attacked by these pieces constitute four equilateral triangles with same size $n-n_r-1$, where $n-n_r-1\equiv 0(mod\ 3)$ as shown in Figure 5. Using equation (4) (see the introduction), we get the result.

The following example illustrates the above theorem for some different values of n and n_r.

Example-4.1.2.

(1) If $n=10, n_r=3$, then $\gamma(K,3)=4\left(\frac{\left\lceil\frac{n-n_r-1}{3}\right\rceil}{2}+1\right)=12$ and

$S(\gamma(K,3))=n_r!=6$, (see Figure 5).

4.2. Domination of K Pieces together with a Fixed Number of B Pieces

We denote the domination number of K pieces with a fixed number n_b of B pieces by $\gamma(K,n_b)$.

Theorem-4.2.1. Consider $= n-\left\lceil\frac{n_b}{2}\right\rceil$, $m_1=\left\lceil\frac{2n-n_b-2}{3}\right\rceil$, $m_2=\left\lceil\frac{2n-n_b-1}{3}\right\rceil$, and $m_3=\left\lceil\frac{2n-n_b-3}{3}\right\rceil$,then

(I) If n_b is odd, we have $\gamma(K,n_b)=2\left\{\begin{array}{ll}\frac{m_1}{4}(m_1+2), & if m_1\ is\ even\\ \frac{(m_1+1)^2}{4}, & if\ m_1\ is\ odd\end{array}\right\} n\geq 3.$

(II) If n_b is even, we have

$$\gamma(K,n_b)=\left\{\begin{array}{ll}\frac{m_2}{4}(m_2+2)+\frac{(m_3+1)^2}{4} & ,if\ m\equiv 0(mod\ 3)\\ \frac{(m_2+1)^2}{4}+\frac{m_3}{4}(m_3+2) & ,\ if\ m\equiv 1(mod\ 3)\\ \frac{(m_2+1)^2}{2} & ,if\ m\equiv 2(mod\ 3)\end{array}\right\}.$$

(III) If $\mu_1=\left\{\begin{array}{ll}\prod_{w=0}^{\left\lceil\frac{m}{3}\right\rceil-1}\left\lceil\frac{2m-(3+6w)}{3}\right\rceil & ,if\ m\equiv 0(mod\ 3)\\ \prod_{w=1}^{\left\lceil\frac{m}{3}\right\rceil-1}(2w^2+w) & ,if\ m\equiv 1(mod\ 3)\\ 1 & ,if\ m\equiv 2(mod\ 3)\end{array}\right\}$, and

$$\mu_2=\left\{\begin{array}{ll}\prod_{w=0}^{\left\lceil\frac{m}{3}\right\rceil-1}\left\lceil\frac{2m-(3+6w)}{3}\right\rceil & ,if\ m\equiv 0(mod\ 3)\\ \prod_{w=1}^{\left\lceil\frac{m}{3}\right\rceil-1}(2w^2+w)*\prod_{w=0}^{\left\lceil\frac{m-1}{3}\right\rceil-1}\left\lceil\frac{2(m-1)-(3+6w)}{3}\right\rceil & ,if\ m\equiv 1(mod\ 3)\\ \prod_{w=1}^{\left\lceil\frac{m-1}{3}\right\rceil-1}(2w^2+w) & ,if\ m\equiv 2(mod\ 3)\end{array}\right\}$$

,we have

$$S^*(\gamma\,(K,n_b)) = 2(n_b + 1)\begin{cases} {\mu_1}^2 & if\ n_b\ is\ odd \\ \mu_2 & ,if\ n_b\ is\ even \end{cases}.$$

Proof. We place the B pieces in one of either ways vertical diagonal or horizontal diagonal. Let us start the distribution with the vertical diagonal; there are two cases that depend on n_b as follows:

(I)*If* n_b *is odd*, then we place the first B piece in a cell such that we obtain the maximum number of N_b for this piece. This cell is either $s_{n-1,0}$ or $s_{-(n-1),0}$. Starting with the cell $s_{n-1,0}$, this B piece attacks the two sides adjacent to it, we see that, the cells which are not attacked by this piece constitute two isosceles triangles chessboard with size $n-1$. In the same manner if n_b is odd, the cells which are not attacked by these B pieces constitute two isosceles triangles chessboard form of size $n - \frac{n_b+1}{2} = m$. By interchange n by m, and by using equation (1) (see the introduction), we get the result.

(II) *If* n_b *is even*, the cells which are not attacked by these B pieces constitute a rhombus chessboard form with size m. Using Theorem 3.3, we obtain the result.

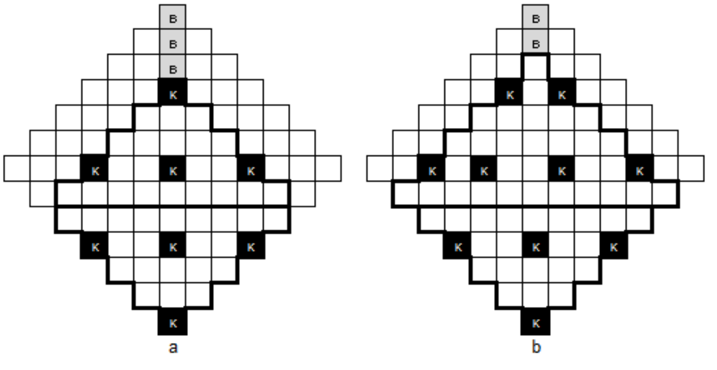

Figure-6.

(III) Again we start with vertical diagonal. We place the first B piece in either $s_{n-1,0}$ or $s_{-(n-1),0}$ cell. So there are two ways to place this piece. If $n_b = 2$, then there are three distributions of n_b B pieces , two above diagonal and nothing below, one above diagonal and one below and nothing above diagonal and two below. In general there are $(n_b + 1)$ ways to distribute the n_b B pieces. From (I) if n_b is odd, the cells which are not attacked by these B pieces takes the form as two isosceles triangles chessboard of size m. Thus By using equation (2) (see the introduction), we obtain the result. Using Theorem 3.3, where n_b is even, then, we obtain the result.

The following example illustrates the above theorem for some different values of n and n_b.

Examples-4.2.2.

(1) If $n = 7, n_b = 3$, then, $\gamma(K\,,3) = 2\left(\frac{(m_1+1)^2}{4}\right) = 8$ and

$S^*(\gamma\,(K\,,3\,)) = 2(n_b + 1\,)\mu_1{}^2 = 8$ (see Figure 6(a)).

(2) If $n = 7, n_b = 2$, then, $\gamma(K\,,2) = \frac{m_2}{4}(m_2+2) + \frac{(m_3+1)^2}{4} = 10$

and $S^*(\gamma\,(K\,,2\,)) = 2(n_b + 1\,)\mu_2 = 18$ (see Figure 6(b)).

4.3. Domination of *R* Pieces together with a Fixed Number of *B* Pieces

We denote the domination number of R pieces with a fixed number n_b of B pieces by $\gamma(R\,,n_b)$.

Theorem-4.3.1. Consider $m = n - \left\lceil \frac{n_b}{2} \right\rceil$, then

(I) $\gamma(R\,,n_b) = \begin{cases} m-1 & ,\ if\ \ m\ is\ even\ with\ \ n_b\ is\ even \\ m & ,\qquad otherwise \end{cases}$

(II) $S(\,\gamma\,(R\,,n_b\,)) = 2(n_b + 1\,)\begin{cases} 2(m)! & ,\text{if}\ \ n_b\ is\ odd \\ (m)! & ,\text{if}\ m \text{ is odd } with\ n_b\ is\ even \\ (m-1)! & ,\text{if}\ m \text{ is even } with\ n_b\ is\ even \end{cases}$.

Proof. As the last theorem there are two ways to distribute B pieces in cells of vertical diagonal or horizontal diagonal, we start with the first method.

(I) In this way there are two cases that depend on n_b as follows:

(i) *If* n_b *is odd*, place the first B piece in a cell such that we obtain the maximum number of N_b for this piece. This cell is either $s_{n-1,0}$ or $s_{-(n-1),0}$, let us start with the cell $s_{n-1,0}$. The non-attacked cells by these pieces constitute a rhombus contains two largest middle rows r_{0a} and r_{0d} , for above and below respectively with size m . Consecutively, we have the following two steps.

(a) *If* m *is odd*, we place the R pieces in the square chessboard of dimension m. We start to distribute the R pieces from the above row r_m with corners $s_{\lfloor\frac{m}{2}\rfloor,\pm\lfloor\frac{m}{2}\rfloor}$ and $s_{-(\lfloor\frac{m}{2}\rfloor-1),\pm\lfloor\frac{m}{2}\rfloor}$ as shown in Figure 7(a); $n = 6$, such that every column or row in that square chessboard contains only one of these pieces, then we get the result.

(b) *If* m *is even***,** we place the R pieces in a square chessboard of dimension m with corners $s_{\pm(\frac{m}{2}-1),-(\frac{m}{2})}$ and $s_{\pm(\frac{m}{2}-1),(\frac{m}{2}-1)}$, such that every column or row in that square chessboard contains only one of these R pieces, so $\gamma(R\,,n_b) = m$.

(ii) *If* n_b *is even*, the cells non-attacked by the B pieces constitute a rhombus chessboard form with length m, then by using Theorem 3.3.1, we get the result.

(II) As in Theorem 4.2.1.(III), there are $2(n_b + 1)$ ways to distribute n_b B pieces, and to compute the ways to distribute R pieces, there are two cases that depend on n_b as follows:

(i) *If n_b is odd***,** there are two squares of length m to distribute the R pieces. One of them is in I(i(a)) and I(i(b)), and the others are in the corners

$\left(s_{\left\lfloor\frac{m}{2}\right\rfloor-1,\pm\left\lfloor\frac{m}{2}\right\rfloor} \text{ and } s_{-\left\lfloor\frac{m}{2}\right\rfloor,\pm\left\lfloor\frac{m}{2}\right\rfloor}\right)$ and $\left(s_{\pm\left(\frac{m}{2}-1\right),-\left(\frac{m}{2}-1\right)} \text{ and } s_{\pm\left(\frac{m}{2}-1\right),\left(\frac{m}{2}\right)}\right)$ respectively.

Thus the total solution is $2(m)!$.

(ii) *If n_b is even***,** from I(ii) and by Theorem 3.3, we get the result.

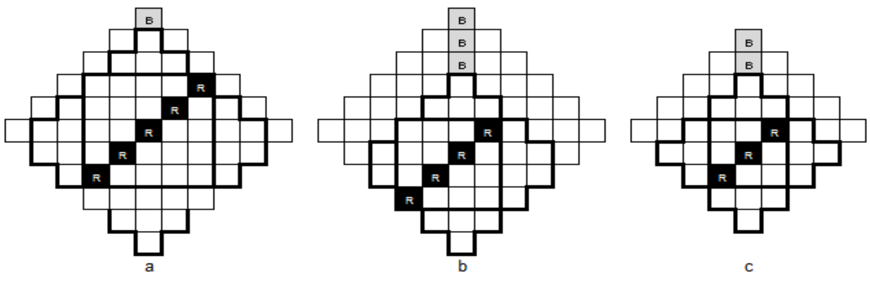

Figure-7.

The following example illustrates the above theorem for some different values of n and n_b.

Examples-4.3.2.

(1) If $n = 6, n_b = 1,$ then $\gamma(R, 1) = m = 5$ and

$S(\gamma(R, 1)) = 4(n_b + 1)(m)! = 960$ (see Figure 7(a)).

(2) If $n = 6, n_b = 3,$ then $\gamma(R, 3) = m = 4$ and

$S(\gamma(R, 3)) = 4(n_b + 1)(m)! = 384$ (see Figure 7(b)).

(3) If $n = 5, n_b = 2,$ then $\gamma(R, 2) = m - 1 = 3$ and

$S(\gamma(R, 2)) = 2(n_b + 1)(m - 1)! = 36$ (see Figure 7(c)).

4.4. Open Problems for two Pieces

If it is possible, find a general form a number for each one of (R, n_k) , $\gamma(B, n_k)$ and $\gamma(B, n_r)$.

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