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Independence in Isosceles Triangular Chessboard

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Abstract

In this article, we are interested in two classical chessboard problems independence for one and two pieces, namely : rooks, bishops and kings. Our chessboard in this work is isosceles triangular chessboard with square cells. In most cases we determine the possible number of methods of independence (total solution).

1. Introduction

One of the classical chessboard problems is placing a maximum number one kind of pieces such that all unoccupied positions are under attack. This problem is called "independence" and denoted by Ind(-). $\beta(P)$ represents the greatest number of non-attacking pieces "P" which can be placed on a triangular chessboard (for rooks as example by $\beta(R)$). S(Ind(P)) represents the number of methods of placing the pieces for maximum independence .

In n square chessboard (see [4] and [5]) β numbers are studied for rooks "R" , bishops "B" and kings "K" . It had been shown that $\beta(R) = n$, $\beta(B) = 2n - 2$ and $\beta(K) = \left\lfloor \frac{n+2}{2} \right\rfloor^2$.

In [1], JoeMaio and William proved that $\beta(R) = \min\{m, n\}$ for mxn Toroidal chessboard.

Dietrich and Harborth [2] studied the triangular triangle board, the board in the shape of a triangle with triangular cells. They defined the chess pieces, in particular the rook which attacks in straight lines from one side of the triangle to the other, forming rhombuses, the bishop 1 which attacks from vertex to side, side to vertex etc. in straight lines, forming diamonds, and for the bishop 2: the triangular triangle board can be 2-coloured with cells sharing an edge of different colors, and the bishop 2 moves as bishop 1, but attacks only cells of the same color.

Harborth, Kultan, Nyaradyova and Spendelova [3] considered the triangular hexagonal board, in which the cells are hexagons and the board is a triangle. On this board bishops attack in straight lines through the vertices of their cells, rooks attack along straight lines through the centers of the edges of their cells, and queens have both attacks. The only general upper bound they were able to give for the independence number of the queens graph was by the rooks bound, which is $\left\lfloor\frac{2n+1}{2}\right\rfloor$ for all n. For n = 3, 4, 6, 7, 13, 16, 19, 25, 31, they found that $\beta = \left\lfloor\frac{2n+1}{2}\right\rfloor - 1$, and for other $n \leq 31$, $\beta = \left\lfloor\frac{2n+1}{2}\right\rfloor$.

2. Chessboard of two equal sides length

In this work, we consider the isosceles triangular chessboard with square cells and three pieces, rooks, bishops and kings. They move or attack the pieces as usual.We mean by the length of the two equal sides of the board the number of cells (squares) in each side. Let the length of each side is n, consequently the third side (base) is of length $2n$-1.

In matrix form, let r_i denote the i^{th} row measured from above to down, i = 1, 2, …, n. If L_i denotes the length of row r_i, then the first row r_1 which contains one cell will have the length L_1= 1, the second row r_2 which contains 3 cells has the length L_2 = 3, and so on… In general the i^{th} row which contains 2i-1 cells has the length L_i = 2i-1. Let c_j denote the j^{th} column which is numbered from the middle (the middle column has the greatest length of columns), j = 0, ±1, ±2, … , ±(i – 1), i is the row number. Let the middle column c_0 contain n cells, then each of the two columns c_1 and c_{-1} which lie to the right and to the left of c_0 respectively contains n-1cells. In general the j^{th} column contains n – |j| cells. We denote the cell (square) of i^{th} row and j^{th} column by $s_{i,j}$, $i = 1, 2, \ldots n$, and j = 0, ±1, ±2, … , ±(i – 1) and we note that the number of squares in an isosceles triangular chessboard of two equal sides of length n is n^2. We refer to the length of the two equal sides by the size of our chessboard. Figure 1 shows a chessboard of lengths 5, 5, 9.

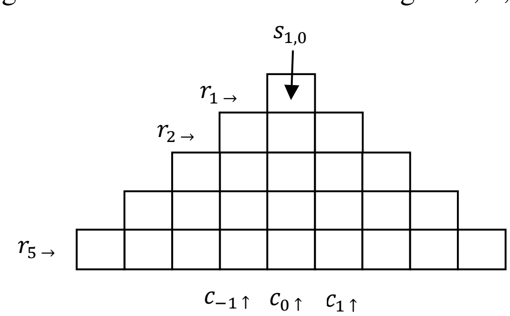


Figure 1

3 Independence of one piece

In this section we will study the independence number β(P) of one piece P (rook ,bishop and king) ,and the number methods of placing the pieces "P" for maximum independence S(Ind (P)) for any piece P .We consider our chessboard of size n .

3.1 Independence of rooks

Theorem 3.1.1.

I) $\beta(R) = n$, II) $S(\text{Ind}(R)) = n$

Proof : I) We place n rooks in n rows such that there is one piece in each row and in each column .Hence $\beta(R) = n$.

II) To determine S(Ind(R)) ,we start with row r_1 which contains one cell , so there exists one possibility to put a rook .For r_2 with $L_2 = 3$, we have two possibilities to put one piece , since there is one piece in row r_1 . By the same manner for each row from r_3 *to* r_n , we get that S(Ind(R)) = n! .

3.2 Independence of bishops

Theorem3.2.1.

I) $\beta(B) = 2n - 1$, II) $S\left(\text{Ind}(B)\right) = 1$

Proof : I) It is obvious if we put one piece in each cell of r_n .

II) The assertion is clear from (I) .

3.3 Independence of kings

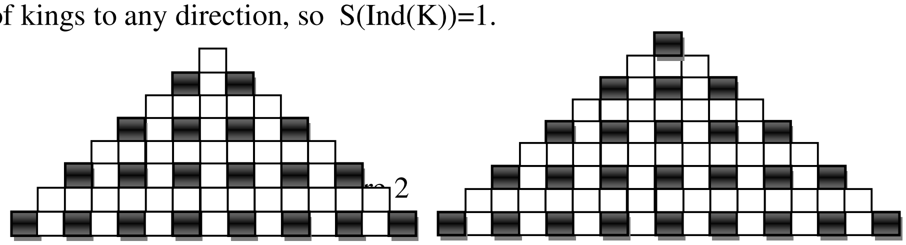
Theorem 3.3.1 .

I) $\beta\,(K) = \begin{cases} \frac{n}{4}(n+2) & , \ if\ n\ is\ even \\ \frac{(n+1)^2}{4} & , \ if\ n\ is\ odd \end{cases}$, II) S(Ind(K))=1

Proof : I) We start to distribute the pieces in the last two rows (r_n , r_{n-1}),since r_n has the greatest "length" which is $2n - 1$ cells .The maximum pieces can be distributed in this row such that no piece is attacked by another is n . In this case we cannot put any piece in each cell of the row r_{n-1}. Again we distribute $n - 2$ pieces in the row r_{n-2} which contains $2n - 5$ cells as shown in Figure 2 (a,n = 8),(b,n = 9). By the same manner , we obtain

$$\beta\,(K) = \sum_{w=0}^{\left\lceil \frac{n-2}{2} \right\rceil} (n - (2w)) = \begin{cases} \frac{n}{4}(n+2) & , \ if\ n\ is\ even \\ \frac{(n+1)^2}{4} & , \ if\ n\ is\ odd \end{cases}$$

II) It is clear that there exists a unique solution because we cannot move any piece of kings to any direction, so S(Ind(K))=1.



a b

The following Lemma of the equilateral triangular chessboard of length n is needed in the next section : independence of two pieces .

Lemma 3.3.2. For a given an equilateral triangular chessboard of length n and square cells as shown in Figure 3 ; $n = 7$, we have

I) $\beta(K) = \sum_{w=0}^{\lceil \frac{n-2}{2} \rceil} (\lceil \frac{n}{2} \rceil - w) = \binom{\lceil \frac{n}{2} \rceil + 1}{\lceil \frac{n}{2} \rceil - 1}$, II) S(Ind(K))=1

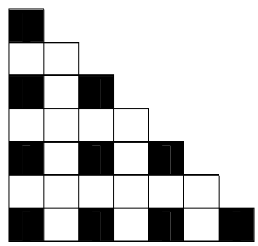


Figure 3

4 Independence of two pieces

In this section, we shall fix a number of one type of pieces P and we determine the independence number of the other type of pieces P^*. We use the symbol n_p to denote the fixed number of the piece P and N_P to denote the number of the cells which are attacked by the piece plus one , this one refers to the cell of this piece .

4.1 Independence of kings with a fixed number of rooks

We denote the independence number of kings with a fixed number of rooks n_r by $\beta(K, n_r)$.

Theorem 4.1.1.

I) **If n is odd** , then we have

i) $\beta(K, n_r) = \binom{\lceil \frac{n-2}{2} \rceil + 1}{\lceil \frac{n-2}{2} \rceil - 1} + \binom{\lceil \frac{n}{2} \rceil + 1}{\lceil \frac{n}{2} \rceil - 1} - (2 n_r + 1)$; $n_r \le \lfloor \frac{n}{2} \rfloor$.

ii) $\beta(K, n_r) = \sum_{w=0}^{\lceil \frac{n-2-2z}{2} \rceil} \left(\lceil \frac{n-1}{2} \rceil - w \right) + \binom{\lceil \frac{n-2z}{2} \rceil + 1}{\lceil \frac{n-2z}{2} \rceil - 1} - (2 \lfloor \frac{n-2z}{2} \rfloor + 1)$; $n > n_r > \lfloor \frac{n}{2} \rfloor$

and $z = n_r - \lfloor \frac{n}{2} \rfloor$.

II) **If n is even** , then we have

i) $\beta(K, n_r) = 2 \binom{\frac{n}{2} + 1}{\frac{n}{2} - 1} - (2 n_r)$; $n_r \le \lfloor \frac{n}{2} \rfloor$.

ii) $\beta(K, n_r) = \sum_{w=0}^{\lceil \frac{n-3-2z}{2} \rceil} \left(\lceil \frac{n-1}{2} \rceil - w \right) + \binom{\lceil \frac{n-2z}{2} \rceil + 1}{\lceil \frac{n-2z}{2} \rceil - 1} - (2 \lceil \frac{n-3z}{2} \rceil + 1$; $n > n_r > \lfloor \frac{n}{2} \rfloor$

and $z = n_r - \lfloor \frac{n}{2} \rfloor$.

proof :The chessboard consists of two equilateral triangles one of them is of size n lying in the right of c_{-1} and the other is of size n-2 lying in the left of c_{-1} if n is odd as shown in Figure 4 ; n=9, and of size n-1 lying in the left of c_0 if n is even as shown in Figure 6 ; $n = 8$. Now we want to put the maximum number of rook pieces in the chessboard such that no piece attacks another , so we determine $\beta(K)$ .To do this we have the following two cases .

I)**If n is odd** , we have the following two steps

i) If $n_r \leq \left\lfloor \frac{n}{2} \right\rfloor$, we distribute the king pieces as in section 3, and we look for a cell to put a rook such that no piece is attacked by another, this cell is one of the cells $s_{n-j,n-j-1}$; $j = 1,3,\ldots,n-2$. Starting with $s_{n-1,n-2}$ there are three kings in cells adjacent to this rook cell, so we must remove these kings. The second piece of rooks should be put in cell $s_{n-2,n-3}$, where this cell is adjacent to two king cells, so we must remove these kings .We continue by the same manner for the other rook pieces until we reach the cell $s_{2,1}$ as shown in Figure 4 , n = 9 .Now by using Lemma 3.3.2. , for the two equilateral triangles ,we have the formula : $\beta(K, n_r) = \sum_{w=0}^{\left\lceil \frac{n-4}{2} \right\rceil} \left(\left\lceil \frac{n-2}{2} \right\rceil - w \right) + \sum_{w=0}^{\left\lceil \frac{n-2}{2} \right\rceil} \left(\left\lceil \frac{n}{2} \right\rceil - w \right) - (2\,n_r + 1) = \binom{\left\lceil \frac{n-2}{2} \right\rceil + 1}{\left\lceil \frac{n-2}{2} \right\rceil - 1} + \binom{\left\lceil \frac{n}{2} \right\rceil + 1}{\left\lceil \frac{n}{2} \right\rceil - 1} - (2\,n_r + 1)$

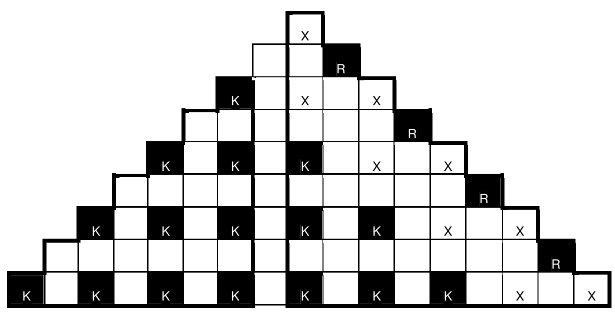


Figure 4

ii) If $n > n_r > \left\lfloor \frac{n}{2} \right\rfloor$ after the distribution of pieces as in (i), it remains $n_r - \left\lfloor \frac{n}{2} \right\rfloor = z$ rooks . The distribution of $\beta(K, n_r)$ pieces is as shown in Figure 4 ; n = 9.

Now , we distribute the remaining pieces of n_r which is z , in the cells $s_{j+1,j}$, j=0,2,…,n-2 . We start with $s_{1,0}$. After we remove the pieces of kings in column c_0, the region of king pieces consists of two equilateral triangles one of them is of size $n - 2z$ lying on the right of $c_1$and the other is of size $n - 2$ lying on the left of cell c_{-1}, see Figure 5; $n = 9$. We put the second piece of z in $s_{3,2}$, and we remove the corresponding king pieces in the column c_2 and in the row r_3, and so on…, until we put the last piece of z in the cell $s_{n-2,n-3}$.We note that the upper bound of the summation of the formula giving $\beta(K, n_r)$ depends on z. Again by using Lemma 3.3.2, the formula of $\beta(K, n_r)$ will be

$$\beta(\mathrm{K}, \mathrm{n_r}) = \sum_{w=0}^{\left\lceil \frac{n-2-2z}{2} \right\rceil} \left(\left\lceil \frac{n-2}{2} \right\rceil - w \right) + \sum_{w=0}^{\left\lceil \frac{n-2-2z}{2} \right\rceil} \left(\left\lceil \frac{n-2z}{2} \right\rceil - w \right) - \left(2 \left\lfloor \frac{n-2z}{2} \right\rfloor + 1 \right)$$

$$= \sum_{w=0}^{\left\lceil \frac{n-2-2z}{2} \right\rceil} \left(\left\lceil \frac{n-2}{2} \right\rceil - w \right) + \binom{\left\lceil \frac{n-2z}{2} \right\rceil + 1}{\left\lceil \frac{n-2z}{2} \right\rceil - 1} - \left(2 \left\lfloor \frac{n-2z}{2} \right\rfloor + 1 \right).$$

The first term gives the number of kings in the left triangle of size n – 2 , while the second term gives the number of kings in the right triangle of size n – 2z , but this is before eliminating the kings due to the existence of rooks , and the third term gives the number of those kings. This is similar to eliminating $2\,\mathrm{n_r} + 1$ in case (i).

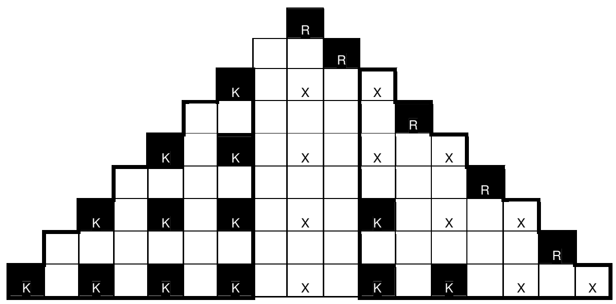


Figure 5

II) **If n is even ,** as in I, there are two cases :

Step 1) If $\mathrm{n_r} \leq \frac{\mathrm{n}}{2}$, we distribute the king pieces as usual and we look for cells to put the rooks such that they attack the minimum number of kings .These cells are some of the cells ($s_{j+1,j}$; $j = 0,2, \dots, n-2$), where no piece of kings is attacked. Start with $s_{1,0}$,there are two "adjacent" king cells to the cell of this rook piece, so we must remove these kings. The second piece of rooks should be put in the cell $s_{3,2}$. This cell is "adjacent "to two king cells , so we must remove these kings .We continue by the same manner for the other rook pieces until we put all rooks in the suitable cells as shown in Figure 6 ; n = 8 .Now by using Lemma 3.3.2. , for the two equilateral triangles, we obtain $\beta(\mathrm{n_r}, \mathrm{K}) = \binom{\left\lceil \frac{n-1}{2} \right\rceil + 1}{\left\lceil \frac{n-1}{2} \right\rceil - 1} +$

$$\binom{\left\lceil \frac{n}{2} \right\rceil + 1}{\left\lceil \frac{n}{2} \right\rceil - 1} - (2\,\mathrm{n_r}) = 2\binom{\frac{n}{2} + 1}{\frac{n}{2} - 1} - (2\,n_r)$$

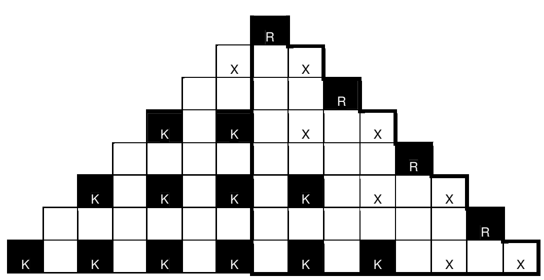


Figure 6

step 2) $n > n_r > \frac{n}{2}$ and $z = n_r - \frac{n}{2}$. After step1,the number of remaining rooks is $n_r - \frac{n}{2} = z$ and the distribution of rook and king pieces is shown in Figure 6; $n = 8$. Now we add to the chessboard after step1 a new piece of n_r which would be put in the cell $s_{2,1}$. Again this piece divides the chessboard into two equilateral triangles as in I(ii), see Figure 7; $n = 8$. Then the formula of $\beta(K, n_r)$ will be

$$\beta(K, n_r) = \sum_{w=0}^{\left\lceil \frac{n-3-2z}{2} \right\rceil} \left(\left\lceil \frac{n-1}{2} \right\rceil - w \right) + \binom{\left\lceil \frac{n-2z}{2} \right\rceil + 1}{\left\lceil \frac{n-2z}{2} \right\rceil - 1} - \left(2 \left\lfloor \frac{n-3z}{2} \right\rfloor + 1 \right)$$

Similar to the case II(ii) , the first term gives the number of kings in the left triangle but of size n – 1, while the second term gives the number of kings in the right triangle of size n – 2z before eliminating the kings due to existence of rooks. The third term gives the number those kings.

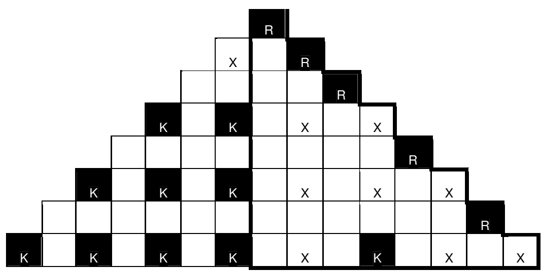


Figure 7

Example 4.1.2 The independence number of a triangular chessboard of size n is given for different values of n_r as follows :

1) $n = 9$, $n_r = 5$, then $z = 1$, then $\beta(K, 5) = 13$, as in Figure 5
2) $n = 8$, $n_r = 5$, then $z = 1$, then $\beta(K, 5) = 10$, as in Figure 7

4.2 Independence of kings with a fixes number of bishops

We denote the independence number of kings with a fixed bishops n_b by $\beta(K, n_b)$.

Theorem 4.2.1.

I) $\beta(K, n_b) = \begin{cases} \frac{n}{4}(n+2) - (n_b + 1) \ , & \textit{if n is even} \\ \frac{(n+1)^2}{4} - (n_b + 1) \ , & \textit{if n is odd} \end{cases}$; $n_b \le n - 1$

II) **If n is odd** , then we have

i) $\beta(K, n_b) = \frac{(n+1)^2}{4} - n - \sum_{i=1}^{z}\left(\left\lfloor \frac{n}{2} \right\rfloor - i - 1 \right)$, $(n-1) < n_b \le (n-1) + \left\lfloor \frac{n}{2} \right\rfloor$

,and $z = n_b - (n-1)$, for $1 \le z \le \left\lfloor \frac{n}{2} \right\rfloor$

ii) $\beta(K, n_b) = \frac{(n+1)^2}{4} - n - \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor}\left(\left\lfloor \frac{n}{2} \right\rfloor - i - 1 \right) - \sum_{i=1}^{z}\left(\left\lfloor \frac{n}{2} \right\rfloor - i \right)$

$(n-1) + \left\lfloor \frac{n}{2} \right\rfloor < n_b < 2n - 1$,and $z = n_b - (n-1) + \left\lfloor \frac{n}{2} \right\rfloor$

III) If **n is even** , then we have $\beta(K, n_b) = \begin{cases} \frac{(n-z)(n-z-2)}{4} \ , & \textit{if z is even} \\ \left(\frac{n-z-1}{2} \right)^2 \ , & \textit{if z is odd} \end{cases}$;

$z = n_b - (n-1)$ and $(n-1) < n_b < 2n - 1$

Proof : I) We put the bishop pieces in our chessboard such that they attack a minimum number of king pieces and no king attacks a bishop .To do this we place these pieces in $s_{n,n-2j}$, $1 \le j \le n-1$.The number of bishops in this step is $(n-1)$ and the kings of maximum number are put as shown in Figure 8 ; $n = 9$.Therefore , by using Theorem 3.3.1, we get

$$\beta(K, n_b) = \begin{cases} \frac{n}{4}(n+2) - (n_b + 1) , & if\ n\ is\ even \\ \frac{(n+1)^2}{4} - (n_b + 1) , & if\ n\ is\ odd \end{cases} ; n_b \le n-1 .$$

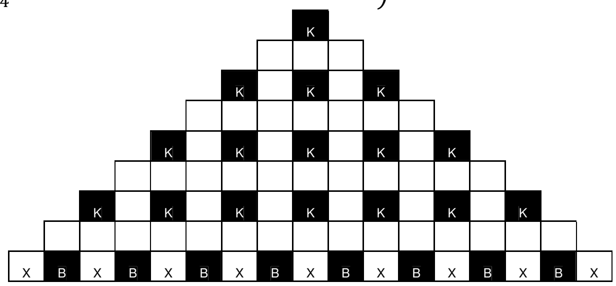


Figure 8

II) **If n is odd** and $(n-1) < n_b < 2n-1$: In the last row r_n, there are two types of empty cells. One type has diagonals containing $\left\lfloor \frac{n}{2} \right\rfloor$ -1 of king pieces and these are the cells $s_{n,3-n+4j}$, $j = 0,1, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1$.The second type of cells their diagonals contain $\left\lfloor \frac{n}{2} \right\rfloor$ of king pieces and these are the cells $s_{n,1-n+4j}$, $j = 0,1, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1$. Therefore we are going to make two successive steps such that we conserve the maximum number of remaining king pieces .

i) $(n-1) < n_b \le (n-1) + \left\lfloor \frac{n}{2} \right\rfloor$, the remaining bishops is $z = n_b - (n-1)$. We start by putting the bishops in the cells $s_{n,3-n+4j}$, $j = 0,1, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1$ as in Figure 9 ; $n = 9$, $z = 1$

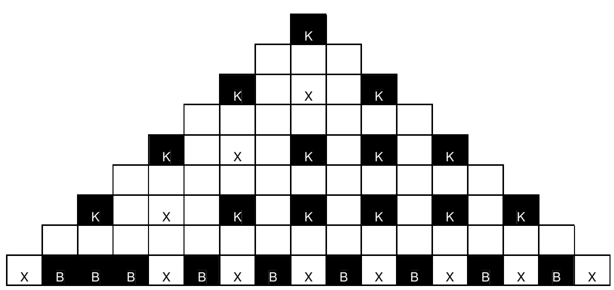


Figure 9

We take the last value of $\beta(K, n_b)$ from (I) i.e. $\beta(K, n-1)$,and removing the king pieces which are attacked by new bishop pieces , we get the formula of $\beta(K, n_b)$ as follow $\beta(K, n_b) = \beta(K, n-1) - \sum_{i=1}^{z}\left(\left\lfloor \frac{n}{2} \right\rfloor - i\right) = \frac{(n+1)^2}{4} - n - \sum_{i=1}^{z}\left(\left\lfloor \frac{n}{2} \right\rfloor - i\right)$

ii) Again we take the last value of $\beta(K, n_b)$ from (i) i.e. $\beta\left(K, (n-1)+\left\lfloor\frac{n}{2}\right\rfloor\right)$. We distribute the new number of bishops. The number of these bishops is $z = n_b - \left((n-1)+\left\lfloor\frac{n}{2}\right\rfloor\right)$, where $(n-1)+\left\lfloor\frac{n}{2}\right\rfloor < n_b < 2n-1$. We distribute the bishop pieces in the cells $s_{n,1-n+4j}$, $j = 0,1,\dots,\left\lfloor\frac{n}{2}\right\rfloor - 1$ whose each diagonal now contains $\left\lfloor\frac{n}{2}\right\rfloor$ of kings . After removing the king pieces which are attacked by the first new piece in this step , we get the new diagonal containing $\left\lfloor\frac{n}{2}\right\rfloor - 1$ king pieces as in Figure 10 ; $n = 9$, and so on … with the other pieces .

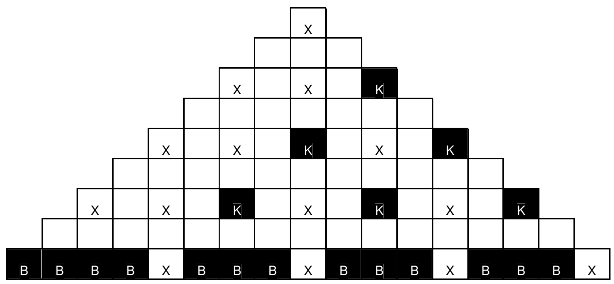


Figure 10

Then we get the formula of $\beta(K, n_b)$ as follows :

$$\beta(K, n_b) = \beta\left(K, (n-1)+\left\lfloor\frac{n}{2}\right\rfloor\right) - \sum_{i=1}^{z}\left(\left\lfloor\frac{n}{2}\right\rfloor + 1 - i\right)$$

$$= \frac{(n+1)^2}{4} - n - \sum_{i=1}^{\left\lfloor\frac{n}{2}\right\rfloor}\left(\left\lfloor\frac{n}{2}\right\rfloor - i\right) - \sum_{i=1}^{z}\left(\left\lfloor\frac{n}{2}\right\rfloor + 1 - i\right)$$

III) **If n is even** and $(n-1) < n_b < 2n-1$: After step (I) we distribute the remaining bishop pieces. The number of these pieces is $z = n_b - (n-1)$. We distribute the bishop pieces in the cells $s_{n,1-n+2j}$, $j = 0,2,\dots,n-3$. After removing the king pieces which are attacked by the new pieces of bishops , the region of pieces of kings become isosceles triangle of side length $n - z$,as a Figure 11 ; $n = 8$, $z = 1$.

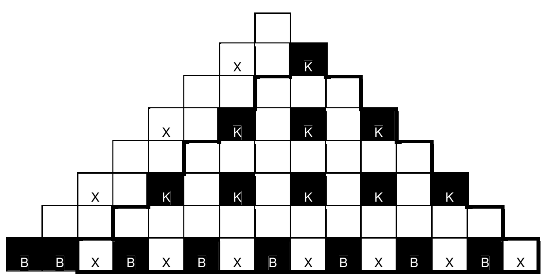


Figure 11

We note that in this triangle, the last row contains $n - z$ of kings which are attacked as in I. So we must remove these kings from $\beta(K)$ .By using Theorem 3.3.1 , we get

$$\beta(K, n_b) = \begin{Bmatrix} \frac{(n-z)(n-z+2)}{4} - (n-z), \textit{if z is even} \\ \left(\frac{n-z+1}{2}\right)^2 - (n-z) \quad , \textit{if z is odd} \end{Bmatrix} = \begin{Bmatrix} \frac{(n-z)(n-z-2)}{4}, \textit{if z is even} \\ \left(\frac{n-z-1}{2}\right)^2 \quad , \textit{if z is odd} \end{Bmatrix}$$

The following example to illustrates the last theorem

Example 4.2.3. The independence number of triangular chessboard where the size n is given for different values of n_b as follows :

1) n = 9 , $n_b = 8$, then $\beta(K, 8) = 16$, as shown in Figure 8

2) n=9 , $n_b = 9$ then z =1 and $\beta(K, 9) = 13$, as shown in Figure 9

3) n = 9 , $n_b = 13$ then z=1 and $\beta(K, 13) = 6$, as shown in Figure 10

4) n=8 , $n_b = 8$ then z =1 and $\beta(K, 8) = 9$, as shown in Figure 11

4.3 Independence of rooks with a fixes number of bishops

We denote the independence number of rooks with a fixed number of bishops n_b by $\beta(R, n_b)$.

Theorem 4.3.1.

$$\beta(R, n_b) = \begin{Bmatrix} n-1 & ; & \begin{Bmatrix} 1 \le n_b \le \left\lfloor \frac{n}{2} \right\rfloor, \textit{if n is odd} \\ 1 \le n_b \le \frac{n}{2} - 1, \textit{if n is even} \end{Bmatrix} \\ n - 1 - \left\lceil \frac{n_b - \left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rceil & ; & \begin{Bmatrix} \left\lfloor \frac{n}{2} \right\rfloor < n_b \le \left\lfloor \frac{n}{2} \right\rfloor + 2\left\lfloor \frac{n}{4} \right\rfloor, \textit{if n is odd} \\ \frac{n}{2} - 1 < n_b \le \frac{n}{2} + 2\left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right), \textit{if n is even} \end{Bmatrix} \end{Bmatrix}$$

Proof : The suitable cells to distribute n rooks in are $s_{i,1-i}$, $i = 1,2, \dots, n$ where $\beta(R) = n$ as in section 3.1.

I) **If n is odd** , we have the following two steps

i) $1 \le n_b \le \left\lfloor \frac{n}{2} \right\rfloor$, we look for the cells to put the fixed number of bishops such that they attack a minimum number of rooks and no rook attacks a bishop. These cells are $s_{n,n-2i}$, $i = 1,2, .., \left\lfloor \frac{n}{2} \right\rfloor$. The rook piece in $s_{n,-(n-1)}$ attacks the bishop pieces, so we must remove this rook piece as shown in Figure 12 ; n=7

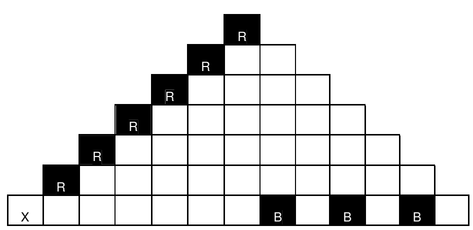


Figure 12

Thus in this case we have $\beta(R, n_b) = \left\{ n - 1 \quad ; \quad 1 \le n_b \le \left\lfloor \frac{n}{2} \right\rfloor \right\}$

ii) If $\left\lfloor \frac{n}{2} \right\rfloor < n_b \le \left\lfloor \frac{n}{2} \right\rfloor + 2\left\lfloor \frac{n}{4} \right\rfloor$ and $n > 3$, we continue to distribute the bishop pieces in $s_{n,n-2i}$, $i = \left\lfloor \frac{n}{2} \right\rfloor + 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor$ .We put the first piece in the cells $s_{n,n-2\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)}$.This piece is attacked by the rook in the cell $s_{2,-1}$ (if we repeat putting bishops several times , then we use the cells $s_{4-2i,1-2i}$, $i = 1 , \dots , \left\lfloor \frac{n}{4} \right\rfloor$), so we remove this rook.This allow us to put another bishop in the cell $s_{n,n-3}$(if we repeat several times , then we use the cells $s_{n,n+1-4j}$, $j = 1 , \dots , \left\lfloor \frac{n}{4} \right\rfloor$), which is not attacked by or attack any rook piece, putting the second (the other) bishop has no influence on the number of $\beta(R , n_b)$ as shown in Figure 13 ; n=7 . We follow this way for putting a new bishop, and so on … until we put the last one in the cell $s_{n,n-2\left(\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor\right)}$, and we get $\beta(R , n_b) = \left\{ n - 1 - \left\lceil \frac{n_b - \left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rceil \;\; ; \left\lfloor \frac{n}{2} \right\rfloor < n_b \le \left\lfloor \frac{n}{2} \right\rfloor + 2 \left\lfloor \frac{n}{4} \right\rfloor \right\}$

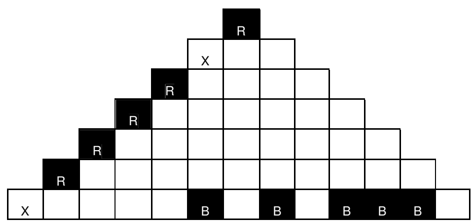


Figure 13

II) If n is even , we have the following two steps

i) If $1 \le n_b \le \frac{n}{2} - 1$, we put the pieces of bishops in the cells $\left(s_{n,n-2i} , i = 1,2,..,\frac{n}{2} - 1 \right)$ as shown in Figure 14 ; n =10. As the same manner in I(i) we get $\beta(R , n_b) = \left\{ n - 1 \;\; ; \;\; 1 \le n_b \le \frac{n}{2} - 1 \right\}$

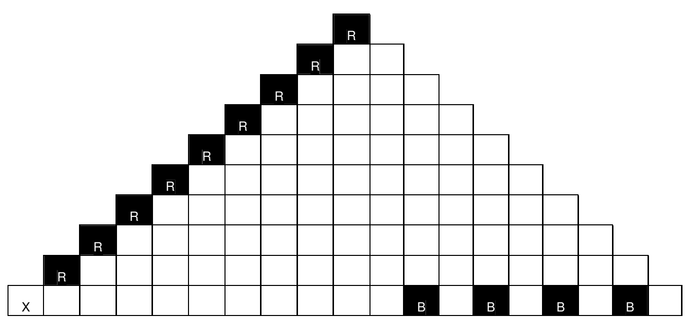


Figure 14

ii) If $\frac{n}{2} - 1 < n_b \le \frac{n}{2} + 2 \left\lfloor \frac{n}{4} \right\rfloor + 1$, we put the pieces of bishops in the cells $s_{n,n-2i}$, $i = \frac{n}{2} , \dots , \frac{n}{2} + \left\lfloor \frac{n}{4} \right\rfloor$, as shown in Figure 15; $n = 10$ (for example).With the same manner in I(ii) we get $\beta(R , n_b) = \left\{ n - \left\lceil \frac{n_b - \frac{n}{2}}{2} \right\rceil - 1 \, ; \;\; \frac{n}{2} - 1 < n_b \le \frac{n}{2} + 2\left(\left\lfloor \frac{n}{4} \right\rfloor + 1 \right) \right\}$

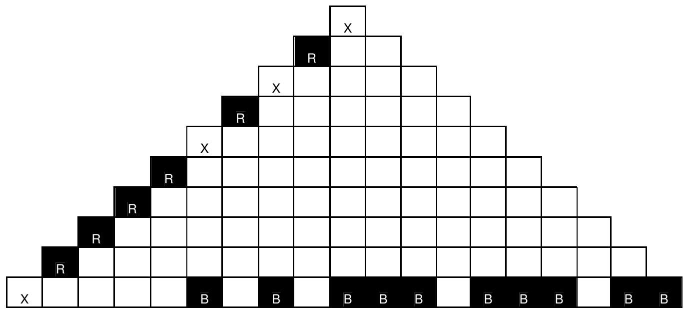


Figure 15

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