NATURAL FREQUENCIES AND PARAMETRIC INSTABILITY OF TAPERED CURVED BEAMS SUBJECTED TO PERIODICAL LOAD AND ELASTIC END SUPPORT

Ahmed A.Al-Rajihy
University Of Babylon, College of Engineering, Mechanical Engineering

Abstract

The present paper describes dynamical behaviour of a uniformly curved beam of tapered cross section under the action of periodical load. The investigated dynamics divided into two main parts ; the first deals with the effect of the slope of taper on the natural frequency. The second part studies the effect of the periodical load on the stability of the beam. The effects of the value of the axial force and the total angle of curvature are discussed. Also the effects of excitation parameter and the periodical force are shown through graphs.

Nomeclature

E	Young's modulus (N/m^2)
F	shear force (N)
I	second moment of area (m^4)
I_o	second moment of inertia at the clamped end (m^4)
I_ϕ	second moment of inertia at angle of curvature, ϕ (m^4)
k_s	stiffness of elastic end support (N/m)
k_e	dimensionless end stiffness, $=k_s R^3/EI_o$
M	bending moment (N.m)
m	beam mass per unit length (kg/m)
P	axial force (N)
R	radius of curvature of beam (m)
t	time (s)
u	displacement along radius of beam (m)
w	displacement along the beam centerline (m)
α	total angle of beam
γ	tapered ratio
Δ	excitation parameter
η	dimensionless transverse displacement
ξ	dimensionless displacement along beam centerline.
Ω	dimensionless frequency
ω	circular frequency (Hz)
μ	dimensionless periodical force.
τ	dimensionless time

1- Introduction

The free vibration of curved beams has been studied by many investigators. Culver [1]* used the Rayliegh-Ritz procedure together with the Lagrange multiplier concept to find the natural frequencies of a two – span curved beam with non – yielding supports. Lee [2] studied the natural frequencies of an intermediately supported U-bend tube using the conventional methods. Chen [3] developed the dynamic three-moment equation for determining the natural frequencies of multispan curved beams on rigid, non –twisting supports.

The out – of – plane vibrations of continuous curved beams neglecting the effect of damping and rotary inertia was studied by Wang [4]. Natural frequencies for a two – span curved beam were determined. It was found that the natural frequency increases with increasing the central angle of the arc. The same authors has another paper [5] which deals with the effect of an elastic foundation on the out – of –plane vibration of a circular curved beams, It was concluded that the natural frequencies decrease with increasing the central angle.

The free –out –of plane vibration of circular rings on identical equi-spaced elastic supports was studied by Mallik and Murty [6]. They calculated the natural frequencies and mode shape using a wave approach. Some numerical results were presented for six numbers of supports.

The dynamic behaviour of non-uniform cantilever straight beams was presented by Al-Rajihy and Al-Daami [7]. They were concluded that stiffening the end support shifts the resonance limits to a higher values.

From the literature available and to our knowledge there is no reference deals with the problem of curved beams under effect of periodical load.

All the aforemensioned and many other works for the dynamics of curved beams were limited to beams of uniform cross sections and no external periodical load. The purpose of this paper is to present the analysis of tapered beam of uniform curvature and the effects of the external periodical loads on the beam under consideration.

2- Equation Of Motion

The equation of motion is derived using Newton's second law by considering the element of the beam as shown in Fig.(1). The differential equation for the transitory motions in radial and tangential directions of the element are:

$$\frac{\partial M}{\partial \theta} + RF = 0 \qquad \ldots(1)$$

$$\frac{\partial F}{\partial \theta} + P = mR\frac{\partial^2 u}{\partial t^2} \qquad \ldots(2)$$

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*Square brackets refer to reference number.

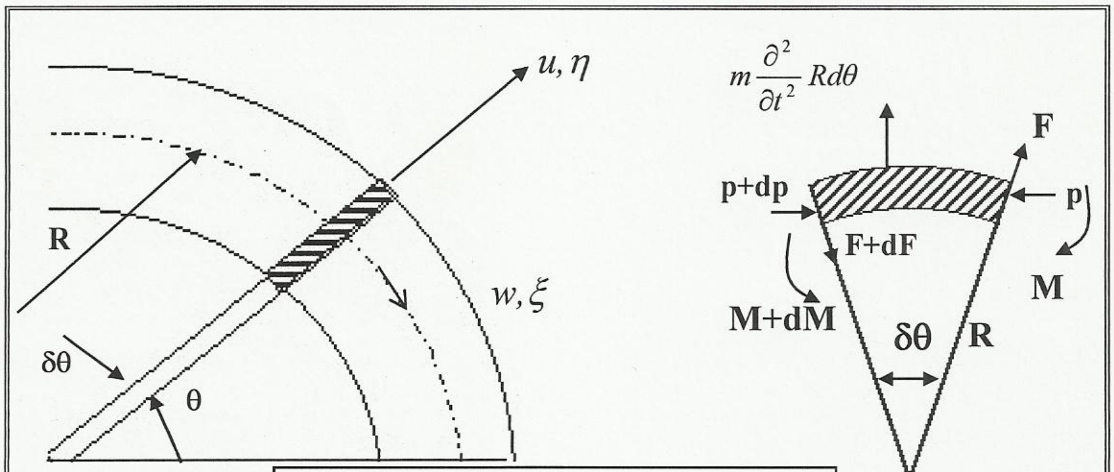

Fig.(1): Definition of coordinates and loads on an element of the beam

$$\frac{\partial P}{\partial \theta}+F=mR\frac{\partial^2 w}{\partial t^2} \qquad ...(3)$$

The bending moment is given by :

$$M=\frac{EI}{R^2}\left(\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial w}{\partial \theta}\right) \qquad ...(4)$$

The radius of curvature after deformation, R' , is given by:

$$\frac{1}{R'}=\frac{1}{R}\left(1+\frac{u}{R}+\frac{1}{R}\frac{\partial^2 u}{\partial \theta^2}\right) \qquad ...(5)$$

For in –plane motion, the condition of no extension in the centerline of the tube requires that the displacements u and w related to each other by :

$$u=\frac{\partial w}{\partial \theta} \qquad ...(6)$$

Utilizing Eqs. (4), (5)& (6) and neglecting the higher order terms, Eqs.(1), (2) and (3) reduced to a single differential equation as:

$$\frac{\partial^6 w}{\partial \theta^6}+\left(2+\frac{R^2 P}{EI}\right)\frac{\partial^4 w}{\partial \theta^4}+\left(1+\frac{R^2 P}{EI}\right)\frac{\partial^2 w}{\partial \theta^2}+\frac{mR^4}{EI}\frac{\partial^4 w}{\partial t^2 \partial \theta^2}-\frac{mR^4}{EI}\frac{\partial^2 w}{\partial t^2}=0 \qquad ...(7)$$

Upon introducing the following dimensionless parameters

$$\eta=\frac{u}{R},\xi=\frac{w}{R},\tau=\left(\frac{EI}{m}\right)^{\frac{1}{2}}\frac{t}{R^2},\mu=\frac{PR^2}{EI} \qquad ...(8)$$

the equation of motion becomes:

$$\frac{\partial^6\xi}{\partial\theta^6}+(2+\mu)\frac{\partial^4\xi}{\partial\theta^4}+(1+\mu)\frac{\partial^2\xi}{\partial\theta^2}+\frac{\partial^4\xi}{\partial\theta^2}+\frac{\partial^4\xi}{\partial\tau^2\partial\theta^2}-\frac{\partial^2\xi}{\partial\tau^2}=0 \quad ...(9)$$

The second moment of area, I and mass distribution, m are written as:

$$I(\theta)=I_\Phi[1+\gamma(1-\alpha)] \quad ...(10)$$

where: $$m(\theta)=m_\Phi[1+\gamma(1-\alpha)] \quad ...(11)$$

$$\gamma=\frac{I_0-I_\Phi}{I_\Phi} \quad ...(12)$$

$$\alpha=\frac{\theta}{\Phi} \quad ...(13)$$

Φ: Total angle of curvature

The dimensionless force μ(t) is periodical which is written as [8]

$$\mu=\mu_0(1+\Delta\cos\Omega\tau) \quad ...(14)$$

where Δ is the excitation parameter of the force, μ_o is the steady state force, and Ω is the dimensionless frequency which related to the circular frequency ω as:

$$\Omega=(\frac{m}{EI})^{\frac{1}{2}}R^2\omega \quad ...(15)$$

substituting Eq. (14) into (9) yields :

$$\frac{\partial^6\xi}{\partial\theta^6}+[2+\mu_o(1+\Delta\cos\Omega\tau)]\frac{\partial^4\zeta}{\partial\theta^4}+[1+\mu_0(1+\Delta\cos\Omega\tau)]\frac{\partial^2\xi}{\partial\theta^2}+\frac{\partial^4\xi}{\partial\theta^2\tau^2}+\frac{\partial^2\xi}{\partial\tau^2}=0 \quad ...(16)$$

3-Boundary Conditions:

The beam is clamped at one end and supported by elastic support at the other, which can be written as:

a – at the clamped support :

$$\xi(0,\tau)=\eta(0,\tau)=\eta'(0,\tau)=0 \quad ...(17)$$

b – at the elastic end:

$$\left.\begin{array}{l}\eta(\Phi,\tau)k_e=\eta'''(\Phi,\tau)\\ \xi'''(\Phi,\tau)=0\\ EA\eta'''(\Phi,\tau)-\xi'''(\Phi,\tau)=0\end{array}\right\} \quad ...(18)$$

where(′) denotes differentiation with respect to θ and $k_e=k_s R^3/EI_o$.

4- Bands Of Parametric Instabilities

For systems under parametric forces, Bolotin [9] has shown that the instability occur in two separated bands, which are the primary and secondary bands. The bands of unbounded solution are separated from the bands of bounded solution by two periodic periods T and 2T which are corresponding to the secondary and primary instability bands respectively. Applying Bolotin's concept directly, the displacement ξ is expressed as:

$$\xi(\theta,\tau)=\sum_{k=1,3,5,..} X_k(\theta)\sin(\frac{1}{2}k\Omega\tau)+Y_k(\theta)\cos(\frac{1}{2}k\Omega\tau)] \qquad ...(19)$$

Substituting Eq. (19) into Eq. (9) yields:

$$\sum_{k=1,3,5}[[\frac{\partial^6 X_k}{\partial\theta^6}+(2+\mu_0)\frac{\partial^4 X_k}{d\theta^4}+(1+\mu_0-\frac{k^2\Omega^2}{4})\frac{\partial^2 X_k}{\partial\theta^2}+(\mu^0+\frac{k^2\Omega^2}{4})X_k]\sin(\frac{k}{2}\Omega\tau)+$$

$$[\frac{\partial^6 Y_k}{\partial\theta^6}+(2+\mu_0)\frac{\partial^2 Y_k}{\partial\theta^{26}}+((\mu_0+\frac{K^2\Omega^2}{4})Y_k]\cos(\frac{k}{2}\Omega\tau)+[\mu_0^2\Delta\frac{\partial^4 X_k}{\partial\theta^4}+(\mu_0\Delta)\frac{\partial^2 Y_k}{\partial\theta^2}+$$

$$\frac{\mu_0\Delta}{2}X_k]\sin(\frac{k-2}{2}\Omega\tau)+[\mu_0^2\Delta\frac{\partial^4 Y_k}{\partial\theta^4}+\mu_0\Delta\frac{\partial^2 Y_k}{\partial\theta^2}Y_k]\cos(\frac{k-2}{2}\Omega\tau)+[\mu_0^2\Delta\frac{\partial^4 Y_k}{\partial\theta^4}+(\mu_0\Delta)$$

$$\frac{\partial^2 Y_k}{\partial\theta^2}+\frac{\mu_0\Delta}{2}Y_k]\cos(\frac{k-2}{2}\Omega\tau)+[\mu_0^2\Delta\frac{\partial^4 X_k}{\partial\theta^4}(\mu_0\Delta)\frac{\partial^2 X_k}{\partial\theta^2}+\frac{\mu_0\Delta}{2}X_k]\sin(\frac{k+2}{2}\Omega\tau)+$$

$$[\mu_0^2\Delta\frac{\partial^4 Y_k}{\partial\theta^4}+\mu_0\Delta\frac{\partial^2 Y_k}{\partial\theta^2}+\frac{\mu_0\Delta}{2}Y_k]\cos(\frac{k+2}{2}\Omega\tau)+\frac{\mu_0^2\Delta^2}{4}\frac{\partial^4 X_k}{\partial\theta^4}\sin(\frac{k-4}{4}\Omega\tau)+$$

$$\frac{\mu_0^2\Delta^2}{4}\frac{\partial^4 Y_k}{\partial\theta^4}\cos(\frac{k-4}{4}\Omega\tau)+\frac{\mu_0^2\Delta^2}{4}\frac{\partial^4 X_k}{\partial\theta^4}\sin(\frac{k+4}{4}\Omega\tau)+\frac{\mu_0^2\Delta^2}{4}\frac{\partial^4 Y_k}{\partial\theta^4}\cos(\frac{k+4}{4}\Omega\tau)]=0$$

...(20)

Truncating the series in Eq. (20) at k=1 and equating the coefficients of $\sin(\frac{1}{2}\Omega\tau)$ and $\cos(\frac{1}{2}\Omega\tau)$ results in the following two equations :

$$\frac{\partial^6 X_1}{\partial\theta^6}+(2+\mu_0+\mu_0\Delta)\frac{\partial^4 X_1}{\partial\theta^4}+(2+\mu_0-\frac{\Omega^2}{4}-\mu_0\Delta)\frac{\partial^2 X_1}{\partial\theta^2}+(\mu_0+\frac{\Omega^2}{4}+\frac{\mu_0\Delta}{2})X_1=0 \qquad ...(21)$$

and

$$\frac{\partial^6 Y_1}{\partial\theta^6}+(2+\mu_0+\mu_0\Delta)\frac{\partial^4 Y_1}{\partial\theta^4}+(1+\mu_0-\frac{\Omega^2}{4}+\mu_0\Delta)\frac{\partial^2 Y_1}{\partial\theta^2}+(\mu_0+\frac{\Omega^2}{4}+\frac{\mu_0\Delta}{2})Y_1=0 \qquad ...(22)$$

Equations (21) and (22) determines the upper and lower limits of the primary instability regions respectively. The solutions of Eqs.(21) and (22) are respectively :

$$X_1 = \sum_{i=1}^{6} A_i e^{r_{1i}\theta} \quad \text{... (23)}$$

$$Y_1 = \sum_{j=1}^{6} B_j e^{r_{1j}\theta} \quad \text{...(24)}$$

Substituting each of Eqs. (23) and (24) into the boundary conditions, Eqs. (17) & (18), results in 6 equations in 6 unknowns for each band. Searching for frequencies which vanish the determinants give the corresponding limit of the primary band.

The principal secondary instability regions are determined by expressing the displacement, ξ, as:

$$\xi(\theta,\tau) = \sum_{k=0,2,4} [X_k(\theta)\sin(\frac{1}{2}k\Omega\tau) + Y_k(\theta)\cos(\frac{1}{2}k\Omega\tau)] \quad \text{...(27)}$$

Substituting Eq. (27) into (16) results in Eq. (20) with k=0,2,4. The principal secondary instability bands can be estimated with a good accuracy by truncating the series at k=2. Equating the constant terms and coefficients of $\sin(\Omega\tau)$ and $\cos(\Omega\tau)$ gives:

$$\frac{\partial^6 X_2}{\partial\theta^6} + (2+\mu_0)\frac{\partial^4 X_2}{\partial\theta^4} + (1+\mu_0)\frac{\partial^2 X_2}{\partial\theta^2} + \Omega^2 X_2 = 0 \quad \text{...(28)}$$

$$\frac{\partial^6 Y_0}{\partial\theta^6} + (2+\mu_0)\frac{\partial^4 Y_0}{\partial\theta^4} + \mu_0\Delta\frac{\partial^4 Y_2}{\partial\theta^4} + (1+\mu_0)\frac{\partial^2 Y_0}{\partial\theta^2} + \mu_0 Y_0 + \frac{\mu_0\Delta}{2}Y_2 = 0 \quad \text{...(29)}$$

and

$$\frac{\partial^6 Y_2}{\partial\theta^6} + (2+\mu_0)\frac{\partial^4 Y_2}{\partial\theta^4} + (\mu_0\Delta + \frac{\mu_0^2\Delta^2}{2})\frac{\partial^4 Y_0}{\partial\theta^4} + (1+\mu_0-\Omega^2)\frac{\partial^2 Y_2}{\partial\theta^2} + 2\mu_0\Delta\frac{\partial^2 Y_0}{\partial\theta^2}(\mu_0+\Omega^2)Y_2 + \mu_0\Delta Y_1 = 0 \quad \text{... (30)}$$

Equation (28) is uncoupled 6-order differential equation can be solved to give the upper limit as:

$$X_2 = \sum_{i=1}^{6} a_i e^{r_{2i}\theta} \quad \text{...(31)}$$

where a_i's are arbitrary constants, and r_{2i}'s are the roots of the polynomial:

$$r_2^6 + (2+\mu_0)r_2^4 + (1+\mu_0)r_2^2 + \Omega^2 = 0 \quad \text{...(32)}$$

Equations (29) and (30) are two coupled differential equations gives the lower limit of the secondary instability regions. They are solved as:

$$Y_0 = \sum_{j=1}^{12} b_0 e^{s_{0j}\theta} \qquad ...(33)$$

$$Y_2 = \sum_{n=1}^{12} b_2 e^{s_{0j}\theta} \qquad ...(34)$$

Where s_{0j}'s are the roots of:

$$\left[S_0^6 + (2+\mu_0^2)S_0^4 + (1+\mu_0)S_0^2 + \mu_0\right]\left[S_0^6 + (2+\mu_0^2)S_0^4 + (1+\mu_0-\Omega^2)S_0^2 + \mu_0 + \Omega^2\right] -$$

$$\left[\mu_0^2 DS_0^4 + \mu_0 DS_0^4 + \frac{\mu_0\Delta}{2}S_0^6 + (2+\mu_0^2)S_0^4\right]\left[(2+\mu_0^2\Delta + \frac{\mu_0^2\Delta^2}{2})S_0^4 + 2\mu_0\Delta S_0^2 + \mu_0\Delta\right] = 0 \qquad ...(35)$$

Substituting Eqs.(31) , (33) and (34) into the boundary conditions results in 12 equations in 12 unknowns for each band.

5- Disscussions

The effect of the exciting periodical force (μ) on the bands of parametric instability regions is indicated in Fig. (2) while Fig.(3) shows the variation of the regions of instability with the excitation parameter (Δ). The size of an unstable region increases with increasing the value of the exciting force for both the primary and secondary instabilities. This phenomenon can be elaborated as follows:

* The time duration through the reduction in force value from (μ_{Max}) to the mean value (μ_O) increases with increasing the exciting force.
* The inertia force delivered from the fluctuations of the force value increases with increasing the value of the periodical force (μ_o). Hence this force will be an auxiliary reason in resonance appearance.
* Increasing the value of exciting periodical force increasing the energy transferred to the excited structure; i.e. the kinetic energy of the structure will be increased which cause an increase in the size of unstable regions.

The excitation parameter has an identical effect on regions of parametric instabilities as that of the exciting periodical force (μ_o). This can be explained as; the time duration δt (δt=t at μ_{max} – t at μ_o) is proportional to (μ_{max} -μ_o) and μ_{max} depend on amplitude of excitation, therefore δt is proportional to Δ.

The effect of axial force (nonperiodical) on the natural frequency of the beam is shown in Figs. (4) and (5). In Fig. (4) the axial force is compressive while in Fig.(5) the force is tensile. It is shown that increasing the compressive force decreases the natural frequency. When the compressive force reach a certain value, the beam loses its stability by buckling. The value of force at which the loses its stability is known as "the critical force". In the case of tensile force, the natural frequency increases with increasing the force, then it decreases when the stresses developed in the beam reaches the elastic limit.

Figure (6) shows the variation of the natural frequency with the total angle of curvature of the beam. The natural frequency decreases with increasing the angle of curvature. This is due to the fact that the stiffness of curved beams decrease with increasing the angle of curvature as the case of helical springs.

The effect of the elastic end support on the natural frequency is shown in Fig. (7). It shown that the natural frequency increases with increasing the stiffness of the end support.

Through out the computational process, it is found that the taper ratio has no effect on the dynamical characteristics of the beam. Therefore, the graph corresponding to this parameter is not presented here.

6- Conclutions

Throughout the obtained results, the following conclusions are obtained:

1- Increasing both of the exciting periodical force and the excitation parameter cause an increase in the size of the parametric instability regions. This indicate that care should be taken into account during the design stage to prevent failure.

2- The taper ratio (slope taper of beam) has a non effective effect on the dynamical characteristics of the beam.

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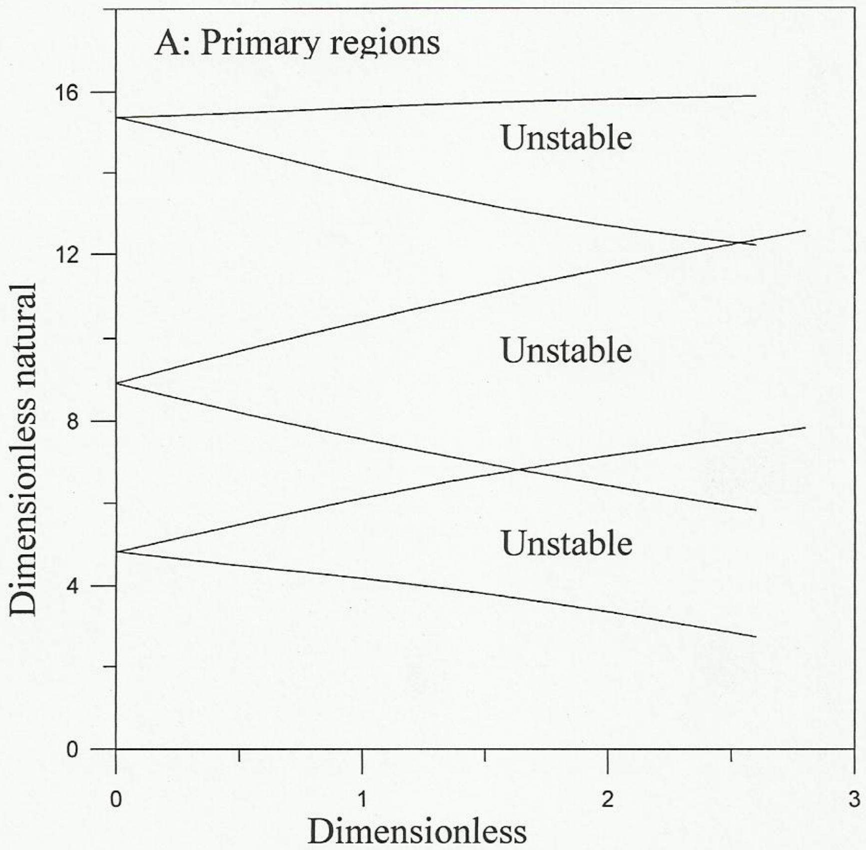
ألذبذبات الطبيعية والاأستقرارية لعتبة منحنية مسلوبة تقع تحت تأثير حمل دوري مسنده بمسند طرفي مرن

الخلاصة

يتناول هذا البحث السلوك الديناميكي لعتبة منحنية ومسلوبة تقع تحت حمل دوري . ويقع البحث تحت جزأين رئيسيين؛ الأول يتناول تأثير مقدار السلبة على الاهتزاز الحر للعتبة ' بينما يتناول الجزء الثاني تأثير القوة المحورية الدورية على أستقرارية العتبة . لقد تم وصف تأثير كل من مقدار الحمل الدوري وزاوية الانحناء الكلية على الخصائص الديناميكية للعتبة ' وقد عرضت النتائج على شكل منحنيات .



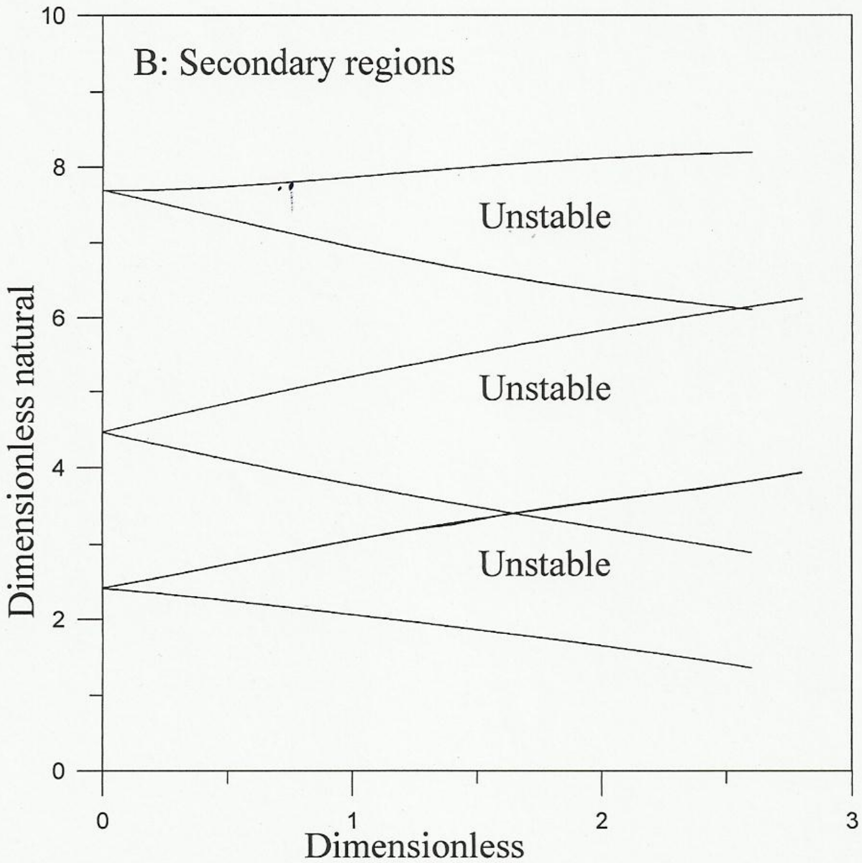


Figure (2): Effect of periodical force on regions of parametric instabilities, α=180^0, Δ = 0.5, end stiffens=1.

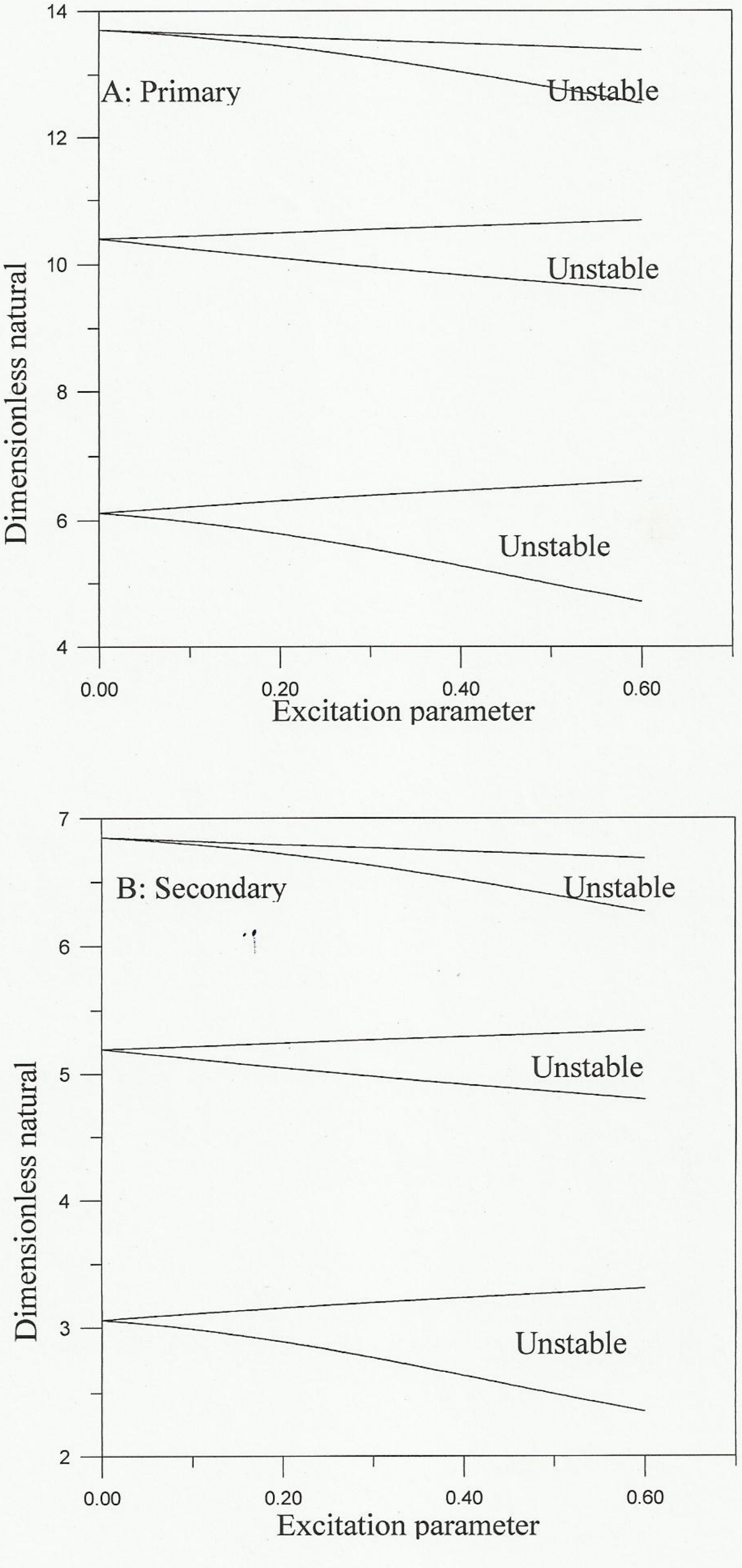


Figure (3): Effect of excitation parameter on regions of parametric instabilities, α=180^0, $\mu=1$, end stiffens=1.

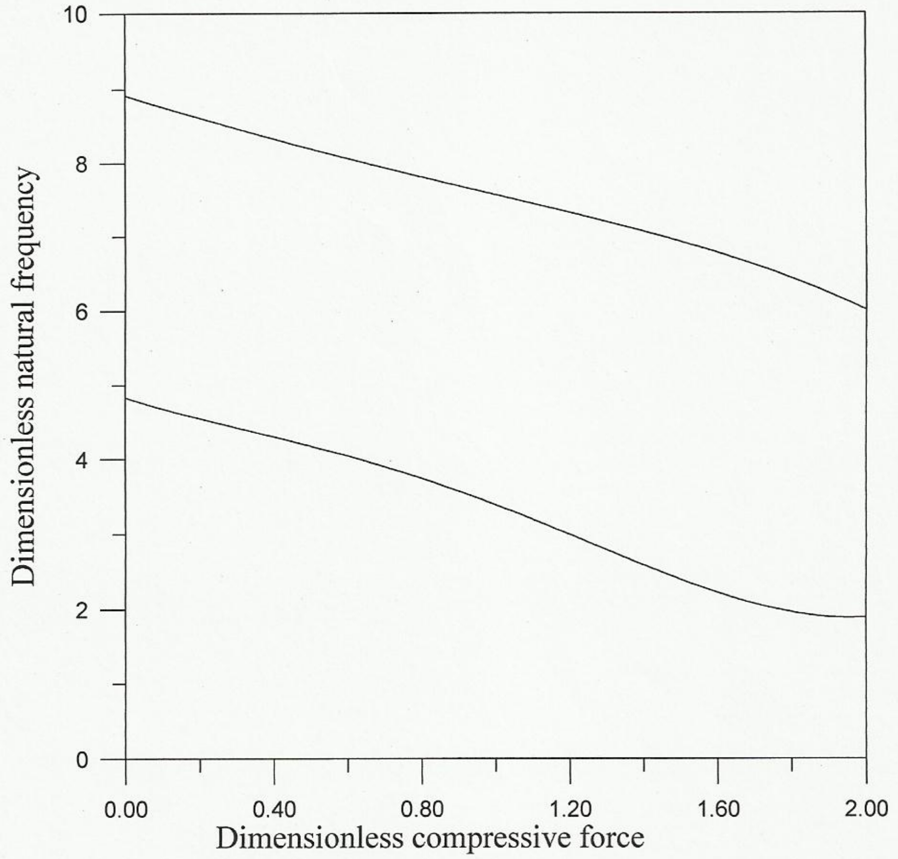


Figure (4): Effect of compressive dimensionless axial force on natural frequency, $\Delta = 0$, end stiffness =1.

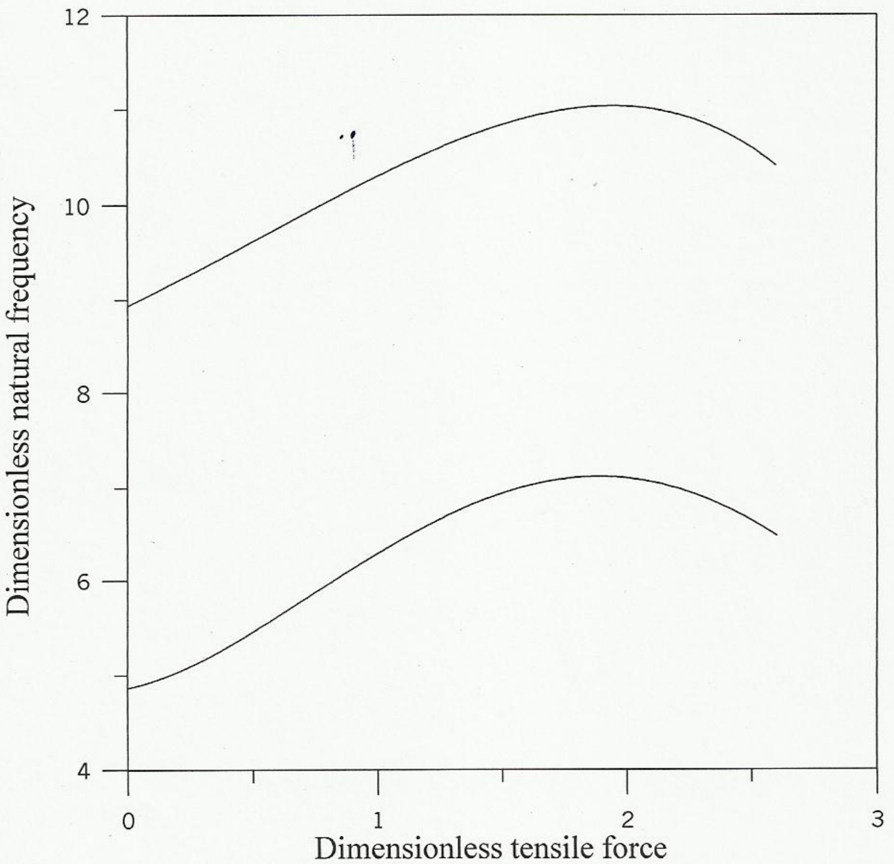


Figure (5): Effect of tensile dimensionless axial force on natural frequency, $\Delta = 0$, end stiffness =1.

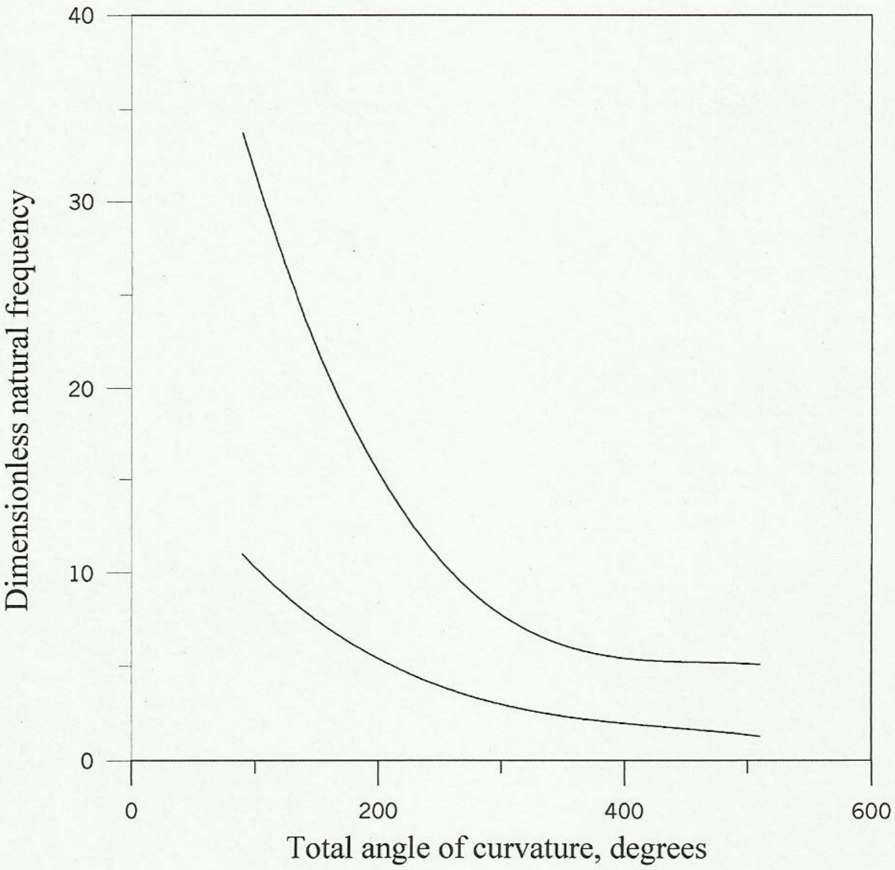


Figure (6): Effect of total angle of curvature on natural frequency, $\Delta = 0$, end stiffness =1, exciting force =1.

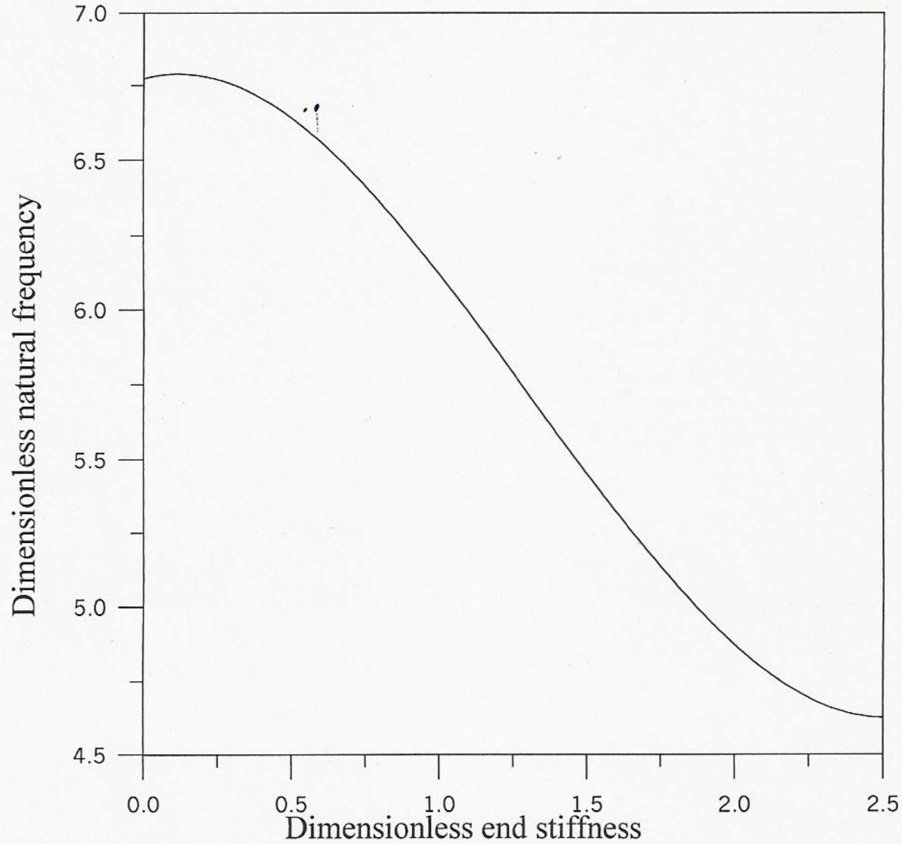


Figure (7): Effect of dimensionless end on natural frequency, $\Delta = 0$, α=180^0, exciting force =1.