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A Move Recent Review of The Integral Equations and Their Applications

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A Move Recent Review of The Integral Equations and Their Applications

Sahar Muhsen Jaabar[1], Ahmed Hadi Hussain[2*]

[1]University of Babylon /College of Education For Pure Sciences / Department of Mathematics, Babil, Iraq
[2]University of Babylon /College of Engineering Al-Musayab / Department of Energy Engineering Babil, Iraq

Emails: pure.saher.mohsen@uobabylon.edu.iq
met.ahmed.hadi@uobabylon.edu.iq

Abstract. In many branches of pure analysis, Integral Equations are one of the most useful techniques, such as functional analysis theories and stochastic processes. It is one of the most significant branches of mathematical analysis, in many fields of mechanics and mathematical physics,. In this research, we will address the integral equations in many physical issues and their applications. They are also associated with mechanical vibration problems, analytic function theory, orthogonal systems, quadratic form theory of infinitely many variables.

Key words: Integral equation, Volterra integral eq., Fredholm integral eq.

1. Introduction

In physics and other applied fields, various physical problems result in initial value problems or boundary value problems.[1],[2] and [3] Although it is equivalent to framing the problems in the form of differential equations (ordinary and partial) or in the form of integral equations, for two main reasons, it is preferred to choose the integral form. First the solution to the integral equation is much simpler than the problems with the original boundary value or the initial value. The second reason is that integral equations are better suited than differential equations for approximate methods.[4] Moreover, Integral equations are developed for the solution of differential equations as a representation formula. With the help of initial and boundary conditions, differential equations can be replaced by an integral equation. As a result, the boundary conditions themselves are met by each solution of the integral equation.Historically, Fourier (1768-1830) is the initiator of the theory of integral equations. Du Bois-Reymond first suggested the term integral equation in 1888. Du Bois-Reymond describes an integral equation and understands an equation in which, under one or more signs of definite integration, the unidentified function takes place. In the late eighteenth and early ninetieth centuries, Laplace, Fourier, Poission, Liouville and qualified studies of some basic form of integral equation. Pioneering systematic research goes back to the work of Volterra, Fredholm and Hilbert in the late 19th and early 20th centuries.In a paper published in 1903 in the Acta Mathematica, The basics of the Fredholm integral equation theory were presented by Fredholm's. Nearly overnight, this paper became popular, and soon took its rightful place among the gems of modern mathematics. Hilbert followed the renowned paper by Fredholm with a series of papers in the Nachrichten of the Gottingen Academy..[5],[6] and [7]

1.1 ABEL'S Problem

[4], In 1826, Abel obtained an integral equation (see References.[8],[9] and [10]) by considering the motion of a material point $p(x,y)$ under the action of force of gravity moving in vertical plane (ξ,η) along some smooth curve. It is required to establish the curve such that the material point P, starting from rest at $p(x,y)$ reaches the point $Q(\xi,\eta)$ at any instant. Let $\mathfrak{T}$ be the time taken by the particle from P to the lowest point O, the origin of coordinates and axes. Let $\widehat{OQ}=S$, then the velocity of particle at Q is

$$\frac{d\mathfrak{s}}{d\mathrm{t}}=-\sqrt{2\mathscr{g}(x-\xi)}$$

$$[\mathrm{t}]_0^{\mathfrak{T}}=-\int_P^Q\frac{d\mathfrak{s}}{\sqrt{2\mathscr{g}(x-\xi}}\Longrightarrow\mathfrak{T}=\int_0^P\frac{d\mathfrak{s}}{\sqrt{2\mathscr{g}(x-\xi)}}$$

Now, we take $d\mathfrak{s}=\mathfrak{U}(\xi)d\xi$, and then $\mathfrak{T}=\int_0^x\frac{\mathfrak{U}(\xi)d\xi}{\sqrt{2\mathscr{g}(x-\xi)}}$ then the above relation shapes as: $f(x)=\int_0^x\frac{1}{\sqrt{2\mathscr{g}(x-\xi)}}\mathfrak{U}(\xi)d\xi$

This leads to find the unknown function $\mathfrak{U}(\xi)$ and we get Able's integral equation. The actual development of integral equation theory began only at the end of the 19th century due to the obvious works of the Italian mathematician V. Volterra (1896), and mainly until 1900, when the Swedish mathematician . Fredholm published his work on the Dirichlet problem solution method. [6],[11] and [4] It is worth being familiar with the initial value problems and the boundary value actual problems before we take the classification of integral equations.

1.2 Initial Value Problem and Boundary value Problem

Definition(1.2.1) Initial Value Problem [4]**:** If under conditions involving a dependent variable and its derivative, an ODE is resolved at the same value of its independent variable, the problem under consideration is an initial value problem. For instance

$z'''+\mathscr{w}z''+2(\mathscr{w}^2-\mathscr{w})z=e^{\mathscr{w}}-\mathscr{w}$ With the circumstances $z(0)=1, z'(0)=-1, z''(0)=\mathcal{S}$

Definition(1.2.2) Boundary Value Actual Problem [4]**:** Under conditions involving a dependent variable and its derivative at two separate independent variable values, when an ODE is solved, then the problem under consideration is a boundary value problem.

Ex.: $y''+xy'+2y=e^{-x}$ with the condition $y(0)=1, y'(1)=-1$

1.3 Integral Equation [4],[12]

An integral equation consists of one or more integral signs, an unknown function appears (which is to be determined). If there are also derivatives of this unknown function in the equation, an integro-differential equation is called.

Ex: If $\mathscr{p}\le\mathrm{g}\le\mathscr{q}, \mathscr{p}\le\kappa\le\mathscr{q}$, then

$$\mathfrak{W}(\mathrm{g})=f(\mathrm{g})+\varsigma\int_{\mathscr{p}}^{\mathscr{q}}\mathfrak{H}(\mathrm{g},\kappa).\mathfrak{W}(\kappa)d\kappa \qquad \ldots[1.1(a)]$$

$$\mathfrak{W}(\mathrm{g})=\int_{\mathscr{p}}^{\mathscr{q}}\mathfrak{H}(\mathrm{g},\kappa).\mathfrak{M}(\kappa)d\kappa \qquad \ldots[1.1(b)]$$

$$\mathfrak{W}(\mathrm{g}) = \int_a^b \mathfrak{H}(\mathrm{g},\kappa).[\mathfrak{M}(\kappa)]^3 d\kappa \qquad ...[1.1(c)]$$

are integral equations. $\mathfrak{U}(x)$ is an Unidentified function, $f(\mathrm{g}), \mathfrak{H}(\mathrm{g},\kappa)$ are renowned features and $\varsigma, \wp, q$ are constant. The Functions involved may be complex valued functions of real variables g and

$\kappa.\frac{d\mathfrak{U}}{dx} + \varsigma\int_{\wp}^{q} \mathfrak{H}(\mathrm{g},\kappa).\mathfrak{W}(\kappa)d\kappa = f(\mathrm{g}), \qquad \wp \le x \le q \qquad ...[1.1(d)]$

is integro-differential equation of unknown function $\mathfrak{U}(\mathrm{g})$. The known bivariate function $\mathfrak{H}(\mathrm{g},\kappa)$ which is integrable in the domain $\wp \le \mathrm{g} \le q$, $\wp \le \kappa \le q$ is called The Equation's kernel.

Definition(1.3.1)[4]: If only linear activities on unknown functions are perfect in it an integral equation is called linear, which means it is the equation in which no non-linear functions of the unknown function are involved. Equations $[1.1(a)]$ and $[1.1(b)]$ are linear equations. An equation which is not linear is known as non-linear integral equation, as shown in equation $[1.1(c)]$. The most general linear integral equation may be shaped as:

$\mathfrak{V}(\mathfrak{m}).\mathfrak{U}(\mathfrak{m}) = \mathfrak{F}(\mathfrak{m}) + \lambda\int_a \mathfrak{K}(\mathfrak{m},\eta).\mathfrak{U}(\eta)d\eta \qquad ...(1.2)$

The functions $\mathfrak{V}, \mathfrak{F}, \mathfrak{K}$ are known functions and $\mathfrak{U}$ is to be determined, λ is a non-zero parameter, which may be real or complex. The importance of keeping λ separate from $\mathfrak{K}$ lies in the fact that it plays an essential role in the theoretical arguments for the problem under context.[13]

(1.3.2) Sorts of Integral Linear Equations [13]

1. First Kind, if $\mathfrak{V} = 0$, Eq. (1.2) reduces to $\mathfrak{F}(\mathfrak{m}) + \lambda\int_a \mathfrak{K}(\mathfrak{m},\eta).\mathfrak{U}(\eta)d\eta = 0$.
2. . First Kind, if $\mathfrak{V} = 1$, equation (1.2) provides $\mathfrak{U}(\mathfrak{m}) = \mathfrak{F}(\mathfrak{m}) + \lambda\int_a \mathfrak{K}(\mathfrak{m},\eta).\mathfrak{U}(\eta)d\eta$.
3. Third kind, if $\mathfrak{V} \neq 0$, Eq. (1.2) itself works.

1.4 Special Kinds of Kernels [4]:

The role of the known bivariate function $\mathfrak{K}(\mathfrak{m},\eta)$ is quite significant both from the problem and its solution point of view. Mainly, we shall come across with the following forms:

1- **Symmetric kernel**: The stone $\mathfrak{K}(\mathfrak{m},\eta)$ is commensurate (synthesis commensurate) is also called Hermitian) if $\mathfrak{K}(\mathfrak{m},\eta) = \overline{\mathfrak{K}}(\eta,\mathfrak{m})$ when the stripe represents the synthesis combine. For instance $e^{\eta\mathfrak{m}}$ is symmetric, while $tan^{-1}\left(\frac{\eta}{\mathfrak{m}}\right)$ is not a symmetric kernel.

2- **Separable or Degenerate Kernel**: If $\mathfrak{K}(\mathfrak{m},\eta) = \sum_{i=1}^{n}\mathfrak{G}_i(\mathfrak{m}).\mathfrak{H}_i(\eta)$ implies that $\mathfrak{K}$ has been represented as the sum of a limited number of terms, each of which is a functional product of $\mathfrak{m}$ only and η only, then such a kernel is called separable or degenerate kernel. Obviously, $\mathfrak{G}_i(\mathfrak{m})$ and $\mathfrak{H}_i(\eta)$ are linearly independent. A degenerate stone has a limited number of characteristic values.

3. **Difference Core**: A core of $\mathfrak{K}(\mathfrak{m} - \eta)$ is called difference kernel.

1.5 Characterization of Integral Equation [14]

In the following four classes , Integral equations are categorized into :
1- Fredholm integral equation.
2- Volterra integral equation.
3- Singular integral equation.
4- Convolution integral equation.

Fredholm Integral Equation: A linear integral equation of the form:

$$\mathfrak{V}(\mathfrak{m}).\mathfrak{U}(\mathfrak{m}) = \mathfrak{F}(\mathfrak{m}) + \mu \int_a^b \mathfrak{K}(\mathfrak{m},\eta).\mathfrak{U}(\eta)d\eta \qquad \text{... (1.3)}$$

Where a and b both are constants. Now, if in Eq. (1.3) we set $\mathfrak{V}(\mathfrak{m}) = 0$, then we get :

$$\mathfrak{F}(\mathfrak{m}) + \mu \int_a^b \mathfrak{K}(\mathfrak{m},\eta).\mathfrak{U}(\eta)d\eta = 0 \qquad \text{... (1.4)}$$

Next , if in Eq. (1.3) , $\mathfrak{V}(\mathfrak{m}) = 1$, we get

$$\mathfrak{U}(\mathfrak{m}) = \mathfrak{F}(\mathfrak{m}) + \mu \int_a^b \mathfrak{K}(\mathfrak{m},\eta).\mathfrak{U}(\eta)d\eta \qquad \text{... (1.5)}$$

. If in Eq. (1.5) , $\mathfrak{F}(\mathfrak{m}) = 0$, i.e.

$$\mathfrak{U}(\mathfrak{m}) + \mu \int_a^b \mathfrak{K}(\mathfrak{m},\eta).\mathfrak{U}(\eta)d\eta \qquad \text{... (1.6)}$$

it is known as homogeneous Fredholm integral Eq. of the second type.

Volterra Integral Equation: A linear Eq. of the shape:

$$\mathfrak{V}(\mathfrak{m}).\mathfrak{U}(\mathfrak{m}) = \mathfrak{F}(\mathfrak{m}) + \mu \int_a^{\mathfrak{m}} \mathfrak{K}(\mathfrak{m},\eta).\mathfrak{U}(\eta)d\eta \qquad \text{... (1.7)}$$

The Volterra integral equation is said to be the three-type when the integral variable upper limit, $\mathfrak{V}(\mathfrak{m}), \mathfrak{F}(\mathfrak{m})$ and $\mathfrak{K}(\mathfrak{m},\eta)$ are a known features and $\mathfrak{U}(\mathfrak{m})$ is Unidentified function . As obvious μ it is a parameter, and feature $\mathfrak{K}(\mathfrak{m},\eta)$ is the stone integral equation.
If we set $\mathfrak{V}(\mathfrak{m}) = 0$, i.e. Eq.(1.7) takes the form:

$$\mathfrak{F}(\mathfrak{m}) + \lambda \int_a^{\mathfrak{m}} \mathfrak{K}(\mathfrak{m},\eta).\mathfrak{U}(\eta)d\eta = 0 \qquad \text{... (1.8)}$$

If $\mathfrak{U}(\mathfrak{m}) = 1$, Eq.(1.7) takes the shape

$$\mathfrak{U}(\mathfrak{m}) = \mathfrak{F}(\mathfrak{m}) + \mu \int_a^{\mathfrak{m}} \mathfrak{K}(\mathfrak{m},\eta).\mathfrak{U}(\eta)d\eta \qquad \text{... (1.9)}$$

Further, if in Eq. (1.9), $\mathfrak{F}(\mathfrak{m}) = 0$, then it becomes

$$\mathfrak{U}(\mathfrak{m}) = \lambda \int_a^{\mathfrak{m}} \mathfrak{K}(\mathfrak{m},\eta).\mathfrak{U}(\eta)d\eta \qquad \text{... (1.10)}$$

and is called identical Volterra integral Eq. of the 2^{nd} kind.

Singular Integral Equation: An integral Eq. is a singular integral Eq., if either :

1- One or both of the integration boundaries are endless.

2- 2- Within the range of integration at one or more points, the Kernel becomes endless. . For instance,

$$\mathfrak{U}(r) = \mathfrak{F}(r) + \lambda \int_{\infty}^{-\infty} e^{-|x-\zeta|}.\mathfrak{U}(\zeta)d\zeta$$

and $\mathfrak{U}(r) = \lambda \int_a^x \frac{1}{(r-\zeta)^\alpha}.\mathfrak{U}(\zeta)d\zeta$, $\qquad 0 < \alpha < 1$

are both singular integral equations.

Detour integral equations: If the kernel $\mathfrak{K}(\mathfrak{m},\eta)$ is a function of one variable and is of the type $\mathfrak{K}(\mathfrak{m},\eta) = \mathfrak{K}(\mathfrak{m}-\eta)$, then the integral equations, i.e., Eq. (1.5) and (1.9) take the shape

$$\mathfrak{U}(\mathfrak{m}) = \mathfrak{F}(\mathfrak{m}) + \lambda \int_a^b \mathfrak{K}(\mathfrak{m}-\eta).\mathfrak{U}(\eta)d\eta \qquad \text{... (1.11)}$$

and $\mathfrak{U}(\mathfrak{m}) = \mathfrak{F}(\mathfrak{m}) + \lambda \int_a^{\mathfrak{m}} \mathfrak{K}(\mathfrak{m} - \eta).\mathfrak{U}(\eta)d\eta$ ... (1.12)
respectively , and are called integral equations of convolution type, then the convolution or faulting of $\mathfrak{U}_1$ and $\mathfrak{U}_2$ is expressed or defined by:
$\mathfrak{U}_1 * \mathfrak{U}_2 = \int_0^{\mathfrak{m}} \mathfrak{U}_1(\mathfrak{m} - \eta).\mathfrak{U}_2(\eta)d\eta = \int_0^{\mathfrak{m}} \mathfrak{U}_1(\eta).\mathfrak{U}_2(\mathfrak{m} - \eta)d\eta$ ... (1.13)

1.6 Solution of Integral Equations [4], [15]: A solution of an integral Eq. [such as equations (1.3) and (1.7) is a function $\mathfrak{U}(x)$, which when substituted in the Eq. reduces it to an identity. For example, for the integral Eq.

$$\mathfrak{U}(\mathfrak{m}) = 1 + \int_0^{\mathfrak{m}} \mathfrak{U}(\eta)d\eta$$

When we take the equations for different kernels, we need two specific formula of integration, which are as follow :

1-Formula for converting a multiple integral of order n into a single ordinary integral of order one:[4]
It is given below :

$$\int_0^{\mathfrak{m}} \mathfrak{U}(\eta)(d\eta)^n = \int_a^{\mathfrak{m}} \frac{(\mathfrak{m} - \eta)^{n-1}}{(n-1)!}\mathfrak{U}(\eta)d\eta \qquad \ldots (1.14)$$

[$(d\eta)^n$ is also written as $d\eta^n$]

2. Leibnittz's Rule of Differentiation dawn the Sign of Integration:[4]
If $\mathfrak{F}(\mathfrak{m}, \eta)$ and $\frac{\partial \mathfrak{F}}{\partial \mathfrak{m}}$ are continuous features of both $\mathfrak{m}$, then

$$\frac{d}{d\mathfrak{m}} \int_{\mathfrak{G}(\mathfrak{m})}^{\mathfrak{H}(\mathfrak{m})} \mathfrak{F}(\mathfrak{m}, \eta)d\eta = \int_{\mathfrak{G}(\mathfrak{m})}^{\mathfrak{H}(\mathfrak{m})} \frac{\partial \mathfrak{F}}{\partial \mathfrak{m}} d\eta + \mathfrak{F}[\mathfrak{m}, \mathfrak{H}(\mathfrak{m})]\frac{d\mathfrak{H}}{d\mathfrak{m}} - \mathfrak{F}[\mathfrak{m}, \mathfrak{G}(\mathfrak{m})]\frac{d\mathfrak{G}}{d\mathfrak{m}} \qquad \ldots (1.15)$$

If $\mathfrak{G}(\mathfrak{m})$ and $\mathfrak{H}(\mathfrak{m})$ are constant functions, then Eq. (1.15) reduce to

$$\frac{d}{d\mathfrak{m}} \int_{\mathfrak{G}}^{\mathfrak{H}} \mathfrak{F}(\mathfrak{m}, \eta)d\eta = \int_{\mathfrak{G}}^{\mathfrak{H}} \frac{\partial \mathfrak{F}}{\partial \mathfrak{m}} d\eta \qquad \ldots (1.16)$$

Example(1.6.1)[16] : Prove that $\mathfrak{U}(x) = e^x\left(2x - \frac{2}{3}\right)$ is answer of Fredholm integral Eq. $\mathfrak{U}(x) + 2\int_0^1 e^{x-\xi}.\mathfrak{U}(\xi)d\xi = 2xe^x$
Sol:We have $\mathfrak{U}(x) = e^x\left(2x - \frac{2}{3}\right)$

$$\mathfrak{U}(\xi) = e^{\xi}\left(2\xi - \frac{2}{3}\right)$$

Then, the L.H.S. of the given integral Eq.

$$= e^x\left(2x - \frac{2}{3}\right) + 2\int_0^1 e^{x-\xi}.e^{\xi}\left(2\xi - \frac{2}{3}\right)d\xi = e^x\left(2x - \frac{2}{3}\right) + 2\,e^x\left[\xi^2 - \frac{2\xi}{3}\right]_0^1$$

$$= e^x\left(2x - \frac{2}{3}\right) + 2e^x\left[1 - \frac{2}{3}\right] = 2xe^x$$

L.H.S.=R.H.S. $\Longrightarrow$ Hence proved

Example(1.6.2)[4]: Display that the features $\Re(\mathfrak{s}) = \mathfrak{s}e^{\mathfrak{s}}$ is a sol. Of the Volterra integral Eq.

$$\Re(\mathfrak{s}) = \sin(\mathfrak{s}) + 2\int_0^x \cos(\mathfrak{s}-\omega).\mathfrak{U}(\omega)d\omega$$

Sol: Since $\Re(\mathfrak{s}) = \mathfrak{s}e^{\mathfrak{s}}$
$\mathfrak{U}(\omega) = \omega e^{\omega}$
Then , the R.H.S. of given integral Eq. $= \sin(\mathfrak{s}) + 2\int_0^x \cos(\mathfrak{s}-\omega).\omega.e^{\omega}d\omega = \sin(\mathfrak{s}) + 2\int_0^x \omega.e^{\omega}[cos(\mathfrak{s}-\omega)]\,d\omega$ (Now integrating by parts)

$$= \sin(\mathfrak{s}) + 2\left[\omega\frac{e^{\omega}}{2}\big(cos(\omega-\mathfrak{s}) + sin(\omega-\mathfrak{s})\big)\right]_0^x - 2\int_0^x \frac{e^{\omega}}{2}\big(cos(\omega-\mathfrak{s}) + sin(\omega-\mathfrak{s})\big)\,d\omega$$

$$= \sin(\mathfrak{s}) + \mathfrak{s}e^{\mathfrak{s}} - \int_0^x e^{\omega}\,cos(\omega-\mathfrak{s})d\omega - \int_0^x e^{\omega}\,sin(\omega-\mathfrak{s})d\omega$$

$$= \sin(\mathfrak{s}) + \mathfrak{s}e^{\mathfrak{s}} - \left[\frac{e^{\omega}}{2}\big(cos(\omega-\mathfrak{s}) + sin(\omega-\mathfrak{s})\big)\right]_0^x - \left[\frac{e^{\omega}}{2}\big(sin(\omega-\mathfrak{s}) + cos(\omega-\mathfrak{s})\big)\right]_0^x$$

$$= \sin(\mathfrak{s}) + \mathfrak{s}e^{\mathfrak{s}} - \left[\frac{e^{\mathfrak{s}}}{2} - \frac{1}{2}\big(cos(\mathfrak{s}) - sin(\mathfrak{s})\big)\right] - \left[\frac{e^{\mathfrak{s}}}{2}(-1) - \frac{1}{2}\big(-sin(\mathfrak{s}) - cos(\mathfrak{s})\big)\right] = \mathfrak{s}e^{\mathfrak{s}} = \mathfrak{U}(\mathfrak{s})$$

L.H.S.=R.H.S.$\Longrightarrow$ Hence proved

2. Application to Ordinary Differential Equations [4],[14],[17]

2.1 Introduction

The review for a representation formula to substitute an ODE (with a problem of initial value or boundary value) always leads to an integral equation. More specifically, an initial value actual problem is found to be converted to a Volterra integral equation. Finally, it makes it easy for us to find the solution to the integral equation thus obtained.

2.2 Method of Conversion of An Initial Value actual Problem to A Volterra integral Eq. [4]

Let an ODE.

$$\frac{d^n\mathscr{y}}{dx^n} + \alpha_1(x)\frac{d^{n-1}\mathscr{y}}{dx^{n-1}} + \alpha_2(x)\frac{d^{n-2}\mathscr{y}}{dx^{n-2}} + \cdots + \alpha_n(x)\mathfrak{Y} = \varphi(x) \quad ...(2.1)$$

With the initial conditions

$\mathscr{y}(a) = q_0$, $\mathscr{y}'(a) = q_1$, …, $\mathscr{y}^{(n-1)}(a) = q_{n-1}$ ... (2.2)

Where $\alpha_1(x), \alpha_2(x), \dots, \alpha_1(x)$, $\varphi(x)$ are characterized and are significant in $a \le \mathcal{X} \le b$.

Let $\mathfrak{U}(x)$ be an unknown function such that

$$\frac{d^n\mathscr{y}}{dx^n} = \mathfrak{U}(x) \quad ...(2.3)$$

Integrating Eq. (2.3) from a to x and after many steps, we get

$$\mathfrak{U}(x) = \mathfrak{F}(x) + \int_a^x \mathfrak{K}(x,t).\mathfrak{U}(t)dt \quad ...(2.4)$$

We have found a relation between a linear differential Eq. (eq.2.1) and a Volterra integral equation which establishes that an initial value actual problem is modify to a Volterra integral equation.

Example(2.2.1)[4]: Transform the initial value problem

$$\frac{d^2z}{d\vartheta^2} + \vartheta z = 1 \, , z(0) = 0 \ , z'(0) = 0$$

to a Volterra integral Eq. of the 2[nd] type.

Sol: The given differential Eq. is $\frac{d^2y}{dx^2} + xy = 1$ ... (1)

Subject to initial conditions : $z(0) = 0$ ... (2)

$z'(0) = 0$ ... (3)

Let $\frac{d^2z}{d\vartheta^2} = \mathfrak{U}(\vartheta)$ ... (4)

Now , integrating Eq. (4), we get

$\left[\frac{dz}{d\vartheta}\right]_0^x = \int_0^x \mathfrak{U}(\vartheta) d\vartheta$ or $\frac{dz}{d\vartheta} - z'(0) = \int_0^x \mathfrak{U}(\vartheta) d\vartheta$

which on using Eq. (3), becomes :

$$\frac{dz}{d\vartheta} = \int_0^x \mathfrak{U}(\vartheta) d\vartheta \quad \text{... (5)}$$

or $\frac{dz}{d\vartheta} = \int_0^x \mathfrak{U}(\mathfrak{n}) d\mathfrak{n}$ ... (6)

Now, integrating Eq. (5) and using Eq. (2), we obtain

$$z(\vartheta) - z(0) = \int_0^x \mathfrak{U}(\vartheta)\, (d\vartheta)^2$$

$$\Rightarrow z(\vartheta) = \int_0^x \mathfrak{U}(\mathfrak{n})\, (d\mathfrak{n})^2$$

which on applying Eq. (1.14) gives

$$z(\vartheta) = \int_0^x (\vartheta - \mathfrak{n}) \mathfrak{U}(\mathfrak{n})\, d\mathfrak{n} \quad \text{... (7)}$$

Finally, putting $\frac{d^2z}{d\vartheta^2}$ from Eq. (4) and y from Eq. (7) into Eq. (1) , we get

$\mathfrak{U}(\vartheta) + \vartheta \left[\int_0^x (\vartheta - \mathfrak{n}) \mathfrak{U}(\mathfrak{n})\, d\mathfrak{n}\right] = 1$

or $\mathfrak{U}(\vartheta) = 1 - \int_0^x (\vartheta - \mathfrak{n}) \mathfrak{U}(\mathfrak{n})\, d\mathfrak{n}$

which is the required Eq.

3. Conclusion

This study was an integral equation review. In pure and applied mathematics, it is one of the most paramount and informative mathematical methods. In many physical problems, it has enormous applications.[18],[19] The progress of science has led to the creation of many physical regulations , which often appear as differences when reaffirmed in mathematics. With equations. Differential equations can mathematically describe engineering problems. When solving practical problems, differential equations play a very important role.

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