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A Comparison between Some Bayesian Estimation Methods for Parameter Exponential Distribution by Using Data Type II Censoring

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Abstract. Due to the importance of exponential distribution, especially in the modeling and analysis of various data in several fields in mathematics and statistics, including the applied field (experiments) and the pure field (theories). Some tests do not provide sufficient information on the trial data, so they are called "censored data". The aim of this paper is to compare some Bayesian estimation methods for the exponential distribution of parameters using data type II censoring. We provided an estimate of the scale parameter (ESP) for the exponential distribution under data the control of type II by using the proposed Bayesian method and maximum probability. We also compared these methods using Mean square error (MSE). This study was conducted using simulation with different parameter values (θ) and different sample sizes (n = 10, 20, 50, 100). The calculation results showed that the best method of estimation is the proposed Bayesian method (BAY2), which uses the distribution of Chi-Square (n) in the previous information.

Keywords: Bayes Estimation, exponential distribution, Jeffery prior information, Maximum likelihood estimates.

1. Introduction

The exponential distribution is the most important mathematical models, which has very many applications in several life's areas in general and in mathematics in particular for both pure and applied where used in the modelling and analysis of data, in previous years, many researchers have reached to several results using this distribution see [1],[2] and [3].Usually, when the researcher performs some tests, the researcher may lack the ability to monitor and identify all the elements that have been selected on the basis of which the success of the experiment depends on its failure, the most important reasons: Some of them are related to the temporal factor of the experiment and some of them are related to constraints (such as financial cost and others) which negatively effect on the results. Therefore, when these effects are directly low with the number of failure observed states, the controlled experience of failure is better, more effective, and achieves less time and effort than the time-controlled experiment. After that, the researchers identified the experiment, which was controlled by failure, as surveillance type II. In this type of experiment, the test is terminated as soon as the number failures that pre-determined (r) is the number of units (n) that is tested. The researchers [4], [5], [6], [7],[8] and [9]presented the definition of predictive distribution in the life's areas and reached several results in this area. The methods of estimating Bayesian are studied in the classical exponential distribution (see [11]) and reached to the best methods of estimating through this distribution. In 2007, the researcher [12] also studied the Bayes estimation of the exponential distribution under double control of type II.Then the researchers used a set of previous distributions to estimate the parameter by exponential distribution. See [10]. The aim of this research is to compare some of the theoretical estimation methods for the exponential distribution of the parameters by using data type control II. We provided an estimate of the measurement parameter (ESP) of the exponential distribution under the data, and control of the second type that is using the proposed Bayes method and maximum probability. We also compared these methods using MSE (Mean Square Error). This study was conducted using simulation with different parameter values (θ) and different sample sizes (n= 10, 20, 50, 100).The calculated results showed that the best method of estimation is the proposed Bayesian method (BAY2), which uses the chi-Squared distribution (n) in the previous information.

2. Theoretical side

2.1. Exponential Distribution (E.D)

This distribution is commonly used to model waiting times between occurrences of rare events, lifetimes of electrical or mechanical devices. A continuous random variable X is said to have an Exponential distribution (E.D) with parameter (θ) if it has probability density function (p.d.f) and the cumulative distribution function (c.d.f) of (E.D) are given as follows respectively

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \;\; x > 0\,, \;\; \theta > 0. \tag{1}$$

$$F(x;\theta) = 1 - e^{-\frac{x}{\theta}}. \tag{2}$$

2.2. Reliability function

Let X be a continuous random variable of distribution has parameter set $\theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ and let $f(x;\theta)$ be its probability density function, where the survival time has parameter θ. The cdf of X is then

$$F(x;\theta) = p(X \le x) = \int_{-\infty}^{x} f(t;\theta)\, dt.$$

The survivor function or reliability function is defined as

$$R(x;\theta) = p(X > x) = 1 - F(x;\theta). \tag{3}$$

In other words, the survivor function is the probability of survival beyond time.

2.3. Type II Censoring Data

Using this type of data mainly for clinical situations and the idea here is to select (m) the units so that m < n, n represents the size of the sample being studied. And the possible function of this data category is defined by the following formula in ascending order [3]:

$$L = (\theta|t_1,t_2,\dots,t_m,m) = \frac{n!}{(n-m)!}\prod_{i=1}^{m} f(t_i)\,[1-F(t_0)]^{n-m} \quad . \tag{4}$$

Such that

$1-F(t_0) = R(t_0)$ represents the reliability function at time t_0 $f(t_i)$ represents density function of failure $(n-m)$ the number of units is a failed after time t_0 for exponential distribution and by using (1) and (2) then equation (4) becomes as follows

$$L = (\theta|t_1,t_2,\dots,t_m,m) = \frac{n!}{(n-m)!}\prod_{i=1}^{m}\left(\frac{1}{\theta}\right) e^{-e^{-\frac{t}{\theta}}}\left[1-(1-e^{-\frac{t_0}{\theta}})\right]^{n-m}$$

$$= \frac{n!}{(n-m)!}\,(\theta)^{-m}\, e^{-\frac{\sum_{i=1}^{m} t_i}{\theta}}\left[e^{-\frac{t_0}{\theta}}\right]^{n-m}. \tag{5}$$

When we take Log the two parties and from derivative second we get maximum likelihood estimator

$$\therefore\ \hat{\theta}_{MLE} = \frac{\sum_{i=1}^{m} t_i + (n-m)\,t_0}{m}. \tag{6}$$

3. Bayesian Estimation Methods

3.1. Standard Bayes Method (BAY1)

In this method the Bayes standard estimator is obtained in the case of the data under the control of type II. Then joint distribution function (pdf) as follows.

$$f(t_1,t_2,\dots,t_m,\theta) = g(t_1,t_2,\dots,t_m|\theta)\,g(\theta). \tag{7}$$

By using Jeffery Prior information then density function of the posterior distribution as follows.

$$g(\theta) \propto k\sqrt{I(\theta)}\ , \tag{8}$$

I (θ) represent Fisher information, k constant

$$g(\theta) = \frac{k\sqrt{n}}{\theta} \ , \ \theta > 0. \tag{9}$$

Then joint density function of T and θ described in equation (7) will be as follows

$$f(t_1, t_2, .., t_m, \theta) = \frac{n!k\sqrt{n}}{(n-m)!\,\theta^{m+1}} \ e^{-\frac{\left(\sum_{i=1}^{m} t_i + t_0(n-m)\right)}{\theta}} . \tag{10}$$

Hence from (10) we find marginal density function of ***T*** is given by

$$f(t_1, t_2, \ldots . t_m) = \int_0^{\infty} f(\underline{t}, \theta) \ d\theta$$

$$= \frac{n!k\sqrt{n}}{(n-m)!} \int_0^{\infty} \theta^{-m-1} \ e^{-\frac{\left(\sum_{i=1}^{m} t_i + t_0(n-m)\right)}{\theta}} \ d\theta, \tag{11}$$

By using transformation

$$u = \frac{\sum_{i=1}^{m} t_i + t_0(n-m)}{\theta} \implies \theta = \frac{\sum_{i=1}^{m} t_i + t_0(n-m)}{u} \implies d\theta = \frac{\sum_{i=1}^{m} t_i + t_0(n-m) \ du}{u^2}.$$

Hence $f(t_1, t_2, \ldots . t_m) = \frac{n!k\sqrt{n}}{(n-m)!\left(\sum_{i=1}^{m} t_i + t_0(n-m)\right)^m} \int_0^{\infty} u^{m-1} \ e^{-u} \ du$

$$\therefore f(t_1, t_2, \ldots . t_m) = \frac{k\sqrt{n} \ n! \ \Gamma(m)}{(n-m)! \left(\sum_{i=1}^{m} t_i + t_0(n-m)\right)^m} . \tag{12}$$

Hence, density function of the posterior distribution of θ is given by

$$H^*(\theta \,|t_1, t_2, \ldots, t_m) = \frac{f(t_1, t_2, \ldots t_m, \theta_1)}{f(t_1, t_2, \ldots . t_m)} = \frac{\left(\sum_{i=1}^{m} t_i + t_0(n-m)\right)^m}{\theta^{m+1} \ \Gamma(m)} \ e^{-\frac{\left(\sum_{i=1}^{m} t_i + t_0(n-m)\right)}{\theta}} . \tag{13}$$

Let $\delta = \sum_{i=1}^{m} t_i + t_0(n-m)$, then

$$H^*(\theta \,|t_1, t_2, \ldots ., t_m) = \frac{\delta^m}{\theta^{m+1} \ \Gamma(m)} \ e^{-\frac{\delta}{\theta}} . \tag{14}$$

By using the quadratic loss function $c\left(\hat{\theta}-\theta\right)^2$, Then Bayes' estimator will be the estimator that minimizes the posterior risk given b $Risk\ (\theta)=E\left[c\left(\hat{\theta},\theta\right)\right]=\int_0^{\infty}\left(\hat{\theta}-\theta\right)^2\ H^*(\theta\,|t_i)\ d\theta_1.$

$$Let\ \frac{\partial\ Risk\ (\ \theta\ )}{\partial\theta}=0.$$

Then $\hat{\theta}_{BAY1}=\int_0^{\infty}\theta\ H^*(\theta\,|t_i)\ d\theta=\int_0^{\infty}\frac{\delta^m}{\theta^{\,m}\,\Gamma(m)}\ e^{-\frac{\delta}{\theta}}\,d\theta.$

By using transformation $u=\frac{\delta}{\theta}\ \Longrightarrow\ \theta=\frac{\delta}{u}\ \Longrightarrow\ d\theta=\frac{\delta\ du}{u^2}$

$$\therefore\ \hat{\theta}_{BAY1}=\frac{\sum_{i=1}^{m}t_i+(n-m)\,t_0}{m-1}. \tag{15}$$

3.2. Bayes method proposed (BAY2)

In this method, a Bayes estimator for the measurement parameter will be found using Natural Conjugate Prior and let this distribution is Gamma distribution(1 , μ) as Prior information where pdf of the gamma distribution as follows [6 , 14] .

$$f(\,\theta\,)=\frac{\mu}{\theta^2}\ e^{-\frac{\mu}{\theta}}\ ,\theta>0. \tag{16}$$

Then joint density function of T and θ and by use equation (4) and (7) will be as follows

$$f(t_1,t_2,..,t_m,\theta)=\frac{n!}{(n-m)!\,\theta^m}\ e^{-\frac{\left(\sum_{i=1}^{m}t_i+\,t_0(n-m)\right)}{\theta}}\ \frac{\mu}{\theta^2}\ e^{-\frac{\mu}{\theta}}. \tag{17}$$

Let $\delta=\sum_{i=1}^{m}t_i+\ t_0(n-m)$, then

$$f(t_1,t_2,..,t_m,\theta)=\frac{n!\ \mu}{(n-m)!\ \theta^{m+1}}\ e^{-\frac{(\delta+\mu\ )}{\theta}}\ . \tag{18}$$

Hence from (18) we find marginal density function of $\boldsymbol{T}$ is given by

$$f(t_1\,,\dots t_m)=\int_0^{\infty}f\left(\underline{t}\,,\theta\right)\,d\theta=\frac{n!\ \mu}{(n-m)!}\int_0^{\infty}\frac{1}{\theta^{m+1}}\,e^{-\frac{(\delta+\mu\ )}{\theta}}d\theta\ .$$

By using transformation $u=\frac{\delta+\mu}{\theta}\Longrightarrow\theta=\frac{\delta+\mu}{u}\Longrightarrow d\theta=\frac{\delta+\mu}{u^2}\ du$

$$f(t_1, t_2, \ldots \ldots t_m) = \frac{n!\ \mu}{(n-m)!} \int_0^\infty \frac{u^{\,m+1}}{(\delta + \mu\ )^{m+1}} e^{-u}\ \frac{\delta + \mu}{u^2}\ du$$

$$\therefore f(t_1, t_2, \ldots \ldots t_m) = \frac{n!\ \mu\ \ \Gamma(m)}{(n-m)!\ (\delta + \mu\ )^m} \quad . \tag{19}$$

Hence, density function of the posterior distribution of $\boldsymbol{\theta}$ is given by

$$H^*(\theta\ |t_1, t_2, .., t_m) = \frac{f(t_1, t_2, \ldots t_m, \theta)}{f(t_1, t_2, \ldots t_m)} = \frac{(\delta + \mu\ )^m}{\theta^{\,m+1}\ \Gamma(m)}\ e^{-\frac{(\delta + \mu\ )}{\theta}} . \tag{20}$$

By using the quadratic loss function $c(\ \hat{\theta} - \theta\ )^2$, Then Bayes' estimator will be the estimator that minimizes the posterior risk given by

$\hat{\theta}_{BAY2} = \int_0^\infty \theta\ \ H^*(\theta\ |t_i)\ d\theta = \int_0^\infty \frac{(\delta + \mu\ )^m}{\theta^{\,m}\ \Gamma(m)}\ e^{-\frac{(\delta + \mu\ )}{\theta}}\ d\theta$ by using transformation $u = \frac{\delta + \mu}{\theta} \Longrightarrow$ $\theta = \frac{\delta + \mu}{u} \Longrightarrow d\theta = \frac{\delta + \mu}{u^2}\ du$

$$\therefore \hat{\theta}_{BAY2} = \frac{\delta + \mu}{m-1} = \frac{\sum_{i=1}^m t_i + t_0(n-m) + \mu}{m-1} \quad . \tag{21}$$

3.3. Bayes method proposed (BAY3)

Using Chi – square (n) distribution in Prior information at degree freedom (n) where the density function of Chi – square distribution as follows :

$$f(\ \theta\ ) = \frac{\mu^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\ \Gamma\left(\frac{n}{2}\right)}\ e^{-\frac{\mu}{\theta}}\ \ , I(0, \infty). \tag{22}$$

Thus the joint density function of T and θ will be as follows

$$f(t_1, .., t_m, \theta) = \frac{n!}{(n-m)!\ \theta^m}\ e^{-\frac{\left(\sum_{i=1}^m t_i + t_0(n-m)\right)}{\theta}}\ \frac{\mu^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\ \Gamma\left(\frac{n}{2}\right)}\ e^{-\frac{\mu}{\theta}}. \tag{23}$$

Let $\delta = \sum_{i=1}^m t_i + t_0(n-m)$, then

$$f(t_1, t_2, .., t_m, \theta) = \frac{n!\ \mu^{\frac{n}{2}-1}}{(n-m)!\ \theta^m\ 2^{\frac{n}{2}}\ \Gamma\left(\frac{n}{2}\right)}\ e^{-\frac{(\delta + \mu\ )}{\theta}} \quad . \tag{24}$$

Hence from (24) we find marginal density function of $\boldsymbol{T}$ is given by

$$f(t_1,\dots t_m)=\int_0^\infty f(\underline{t},\theta)\,d\theta=\frac{n!\;\mu^{\frac{n}{2}-1}}{(n-m)!\;2^{\frac{n}{2}}\;\Gamma\left(\frac{n}{2}\right)}\int_0^\infty\frac{1}{\theta^m}e^{-\frac{(\delta+\mu)}{\theta}}d\theta\,.$$

By using transformation $u=\frac{\delta+\mu}{\theta}\Longrightarrow\theta=\frac{\delta+\mu}{u}\Longrightarrow d\theta=\frac{\delta+\mu}{u^2}\,du$

$$f(t_1,t_2,\dots\dots t_m)=\frac{n!\;\mu^{\frac{n}{2}-1}}{(n-m)!\;2^{\frac{n}{2}}\;\Gamma\left(\frac{n}{2}\right)}\int_0^\infty\frac{u^m}{(\delta+\mu)^m}\,e^{-u}\,\frac{\delta+\mu}{u^2}\,du$$

$$\therefore f(t_1,t_2,\dots\dots t_m)=\frac{n!\;t^{\frac{n}{2}-1}\;\Gamma(m-1)}{(n-m)!\;2^{\frac{n}{2}}\;\Gamma\left(\frac{n}{2}\right)\;(\delta+t)^{m-1}}\,. \tag{25}$$

Hence, density function of the posterior distribution of $\boldsymbol{\theta}$ is given by

$$H^*(\theta\,|t_1,t_2,..,t_m)=\frac{f(t_1,t_2,\dots t_m,\theta)}{f(t_1,t_2,\dots t_m)}=\frac{(\delta+\mu)^{m-1}}{\theta^m\;\Gamma(m-1)}\;e^{-\frac{(\delta+\mu)}{\theta}}. \tag{26}$$

By using the quadratic loss function $c(\hat{\theta}-\theta)^2$. Then Bayes' estimator will be the estimator that minimizes the posterior risk given by

$$\hat{\theta}_{BAY3}=\int_0^\infty\theta\;H^*(\theta\,|t_i)\;d\theta=\int_0^\infty\frac{(\delta+\mu)^{m-1}}{\theta^{m-1}\;\Gamma(m-1)}\;e^{-\frac{(\delta+\mu)}{\theta}}\,d\theta.$$

And by using the same transformation the previous

$$\therefore\;\hat{\theta}_{BAY3}=\frac{(\delta+\mu)\,\Gamma(m-2)}{\Gamma(m-1)}=\frac{(\delta+\mu)}{m-2}\,. \tag{27}$$

4. Practical Aspect (Simulation):

Formulation of a model simulation includes the following essential and important steps for estimation of the scale parameter of exponential distribution that are respectively:

(P1). The initial values for the parameter θ. This step is important upon which later steps depend. Then we assume the initial values (θ =5.5, 6 , 6.5) for scale parameter θ of the of the exponential distribution.

(P2) Selected sample size (n).

We chose different sizes of the sample proportionally to the effect of sample size on the accuracy and efficiency of the results obtained from the estimation methods used, so we take the sizes (10, 30, 50, and 100).

(P3) The initial values for the time estimation of reliability function (t_0). We take three values of the time $t_0 = 1, 2, 3$.

(P4) Select values for the constants in the estimators. We take the value parameter μ = 1.

(P5) Step of Data Generation:

In this step, the generation of weighted exponential distribution data using the inverse method is as follows

$$F_w\,(t_i;\beta) = U_i = 1 - e^{-t_i(\theta-1)} \tag{28}$$

$$\text{And the } t_i = \frac{-\log\,(1-U_i)}{\theta}\ , \tag{29}$$

where $F_w\,(t_i;\theta)$ = The distribution function given in (2), U_i =Uniformly distributed random variable on (0,1).

(P6) Measure comparison :

We adopt the mean square error $MSE(\hat{\theta}) = \frac{\sum_{i=1}^{R}(\hat{\theta}_i-\theta)^2}{R}$. Where R=1000 is the number of replications .The tables below show the results of the estimation using the simulation. Program the simulation written by using (Matlab – 2011a).

6. Explanation Results (Conclusions)

The results of tables (1) to (4) show the following:

- Bayes method proposed estimation (BAY3) is the best, in all the size samples because it has the lowest (MSE).
- The second estimator (BAY2) is the best in the small size samples because it has the second lowest mean square error (MSE).
- We noted decreasing values of (MES) with increasing sample size of all cases , which corresponds with the statistical theory.
- (MLE) is in the last place by comparing with the other methods because it has achieved the highest level of (MSE) in all the sample sizes.

7. Recommendations

- Applying this study by using other convenient continuous distributions as (Gamma, Beta, Weibull,...).
- We recommend developing Bayes formulas that have been studied to other formats and under different loss functions.
- Testing the hypotheses theories for this study in the industrial fields and modeling machines failure times.

- **Table (1): Mean squared error for $\hat{\theta}$ where n=10**

θ	$t_\circ$	*MLE*	*BAY1*	*BAY2*	*BAY3*	*BEST*
5.5	1	0.02899825787	0.02852509742	0.02834999730	0.0281941948	BAY3
	2	0.02829045664	0.02826247087	0.02789703936	0.02778822244	BAY3
	3	0.02838586520	0.02817915176	0.02701127422	0.02661577431	BAY3
6	1	0.03427782795	0.03408518576	0.03279950530	0.03240564477	BAY3
	2	0.03366927449	0.03341126295	0.03213847877	0.03166696429	BAY3

	3	0.03425936917	0.03406473404	0.03277944305	0.03238321112	BAY3
6.5	1	0.03996978958	0.03971578213	0.03832698478	0.03784504312	BAY3
	2	0.04035306594	0.04014040555	0.03874413784	0.03831154716	BAY3
	3	0.04046677703	0.04040454652	0.03996714107	0.03988531573	BAY3

Table (2): Mean squared error for $\widehat{\theta}$ where n=30

θ	$t_\circ$	*MLE*	*BAY1*	*BAY2*	*BAY3*	*BEST*
5.5	1	0.02880808569	0.02875597439	0.02838914244	0.02832272952	BAY3
	2	0.02828501791	0.02802776103	0.02786235549	0.02747786154	BAY3
	3	0.02795479397	0.02787626175	0.02751314583	0.02741677139	BAY3
6	1	0.03312700381	0.03302887564	0.03263351754	0.03251511903	BAY3
	2	0.03425083404	0.03419008420	0.03378781551	0.03370886110	BAY3
	3	0.03411106222	0.03407232718	0.03383437383	0.03378924377	BAY3
6.5	1	0.04031846430	0.04025124255	0.03981466995	0.03972756650	BAY3
	2	0.04033862178	0.04027207781	0.03983539193	0.03974900517	BAY3
	3	0.04011716517	0.04007337271	0.03981527856	0.03976446891	BAY3

Table (3): Mean squared error for $\widehat{\theta}$ where n=50

θ	$t_\circ$	*MLE*	*BAY1*	*BAY2*	*BAY3*	*BEST*
5.5	1	0.02823476059	0.02819376006	0.02797734283	0.02793033339	BAY3
	2	0.02820932877	0.02837176728	0.02815466660	0.02811120546	BAY3
	3	0.02771114007	0.02765989433	0.02744553985	0.02738796392	BAY3
	1	0.03432245847	0.03428792211	0.03404921580	0.03400842276	BAY3

6	2	0.03405224148	0.03401234065	0.03377459722	0.03372826311	BAY3
	3	0.03405961403	0.03401985926	0.03378208951	0.03373590624	BAY3
6.5	1	0.04020936077	0.04018858938	0.04006055779	0.04003809676	BAY3
	2	0.04011970499	0.04007596295	0.03981786045	0.03976710290	BAY3
	3	0.03990381876	0.03985579722	0.03959840582	0.03954322578	BAY3

Table (4): Mean squared error for $\widehat{\theta}$ where n=100

θ	$t_\circ$	MLE	BAY1	BAY2	BAY3	BEST
5.5	1	0.02807538167	0.02805353512	0.02794658253	0.02792325156	BAY3
	2	0.02810994695	0.02808843608	0.02798141691	0.02795842682	BAY3
	3	0.02721432208	0.02719382618	0.02718660624	0.02715291967	BAY3
6	1	0.03425231166	0.03423452521	0.03411636561	0.03409704794	BAY3
	2	0.03371059863	0.03368750501	0.03357029405	0.03354558375	BAY3
	3	0.03375031406	0.03372760798	0.03361032722	0.03358601066	BAY3
6.5	1	0.04001046447	0.03996455540	0.03970681248	0.03965381541	BAY3
	2	0.03983280212	0.03980833463	0.03968091067	0.03965469280	BAY3
	3	0.03878913169	0.03840923894	0.03704368145	0.03641204655	BAY3

7. Appendix.

```
%%% Exponential Program%%%

n=30  ;
format long
theta=  ; %%% theta>0
R1=1000;
t0=  ;
M=1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for i=1:R1
u=rand(1,n);
tt=(-log(1-u))/(theta);
m1 = 0;
 for j=1:n
if tt<t0
 m1=m1+1;
else
end
 end
thetam1=(sum(tt)+(n-m1).*t0 )/ ( m1);
thetam11=(sum(tt)+(n-m1).*t0 )/ ( m1-1);
thetam111=(sum(tt)+(n-m1).*t0 + M)/ (m1-1);
thetam1111=(sum(tt)+(n-m1).*t0 + M)/ ( m1-2);
end
imse1=(sum(thetam1-theta).^2)/(R1-1) %%% MLE
imsebas2=(sum(thetam11-theta).^2)/(R1-1) %%% BAY1
imsebas3=(sum(thetam111-theta).^2)/(R1-1) %%% BAY2
imsebas4=(sum(thetam1111-theta).^2)/(R1-1) %%% BAY3
```

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