PAPER • OPEN ACCESS

Toeplitz Determinant whose Its Entries are the Coefficients for Class of Non-Bazilevi´c Functions

To cite this article: Saba N. Al-Khafaji *et al* 2020 *J. Phys.: Conf. Ser.* **1660** 012091

Toeplitz Determinant whose Its Entries are the Coefficients for Class of Non-Bazilevi´c Functions

[1]Saba N. Al-khafaji, [2]Ali Al-Fayadh, [3]Ahmed Hadi Hussain, [4]Sameer Annon Abbas

[1]Department of Mathematics, Faculty of Computer Science and Mathematics, University of Kufa.
[2]Department of Mathematics and Computer Applications, Al-Nahrain University-Iraq
[3]University of Babylon, College of Engineering Al-musayab, Department of Energy Engineering
[4]Iraqi Ministry of Education, The Directorate of Education of the Governorate Babylon

sabanf.mc11p@uokufa.edu.iq
aalfayadh@yahoo.com
met.ahmed.hadi@uobabylon.edu.iq
sameer.annon@bab.epedu.gov.iq

Abstract. The famous Toeplitz matrix is a matrix in which each descending diagonals form left to right is constant, this mean $T = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_{-1} & a_0 & \ldots & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ \ldots & \cdots & a_{-1} & a_0 \end{pmatrix}$.Mathematician, engineers, and physicists are interested into this matrix for their computational properties and appearances in various areas: C^*-dynamical systems [1], dynamical systems [6], operator algebra [2], Pseudospectrum and signal processing [10]. The object of this research is to define a new class Non-Bazilevi´c functions $\mathcal{N}_\delta$ in unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ related to exponential function. As well as, we obtained coefficient estimates and an upper bound for the second and third determinant of the Toeplitz matrix such that the entries these matrix are belong to this class.

Keywords. Toeplitz matrix, Non-Bazilevi´c function, Exponential Function.

1. Introduction

Let $f(z)$ is analytic function in unit disk $\mathbb{D} = \{\text{z} \in \mathbb{C}: |\text{z}| < 1\}$, with

$$f(\text{z}) = \sum_{n=0}^{\infty} a_n\, z^n\,, \qquad (1)$$

it seems that the Toeplitz determinants are much less studied and the investigation for many classes of functions still not known.

The Toeplitz determinant which is defined as following:

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & a_{n+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & a_{n+1} \\ a_{n+q-1} & \cdots & a_{n+1} & a_n \end{vmatrix}, \qquad (q \geq 1; k \geq 1). \quad (2)$$

Recently, M. Jahangiri in [8] and Thomas and Halim in [11] gave a sharp estimations for $T_2(2), T_2(3)$ and $T_3(2)$ for close-to-convex and starlike classes of functions.

In the present work, we investigate to the sharp estimates for the Toeplitz determinant in the cases of $q = 3, n = 1$ and $q = 2, n = 2$, when f is Non-Bazilevic function i.e.

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \qquad T_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_2 \\ a_3 & a_2 & a_1 \end{vmatrix}.$$

Assume $a_1 = 1$, we have

$$T_3(1) = a_3(a_2^2 - a_3) - a_2(a_2 - a_2 a_3) + (1 - a_2^2).$$

Now, by use the triangle inequality for both side, we get

$$T_3(1) \leq |a_3||a_2^2 - a_3| + |a_2||a_2 - a_2 a_3| + |1 - a_2^2|. \quad (3)$$

Let $\mathcal{A}$ refer to the class of functions $f(z)$ which are analytic and have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{D}) \quad (4)$$

in $\mathbb{D}$ and satisfies the two normalization conditions as follows: $f(0) = 0$ and $f'(0) - 1 = 0$.

Let S denote to the subclass of $\mathcal{A}$ whose elements are regular functions.

Also, let P is the class of Carthéodory functions which is analytic belongs in $\mathbb{D}$ have the form

$$p(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, \quad (5)$$

which satisfies the two properties $p(0) = 1$ and $Re\{p(z)\} > 0$.

If the functions $f(z)$ and $h(z)$ are analytic in $\mathbb{D}$, it call $f(z)$ is a subordinate to $h(z)$ if there exist a Schwarz function $w(z)$ analytic in $\mathbb{D}$ also $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = h(w(z))$ (see [4]).

Furthermore, if $h(z)$ be regular, then $f(z) \prec h(z) \Leftrightarrow f(0) = h(0)$ and $f(\mathbb{D}) \subset h(\mathbb{D})$.

Definition 1.1 Let $f(z) \in \mathcal{A}$ which is given by (1). Then $f(z)$ is called star like function if and only is satisfies the condition

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

Minda and Ma in [7] presented a unique format of different subclasses of S by using subordination which are described by the quantity $\frac{zf'(z)}{f(z)}$ lying in the right half-plane of the domain. The authors considered the class of functions ϕ contained in P, mapping from $\mathbb{D}$ onto domains starlike related with $\phi(0) = 1$ and $\phi'(0) > 0$. For $\Gamma \in \phi$ and symmetric related with the real-axis and

$$S^*(\Gamma) = \left\{f \in \mathcal{A}: \frac{zf'(z)}{f(z)} \prec \Gamma(z)\right\}.$$

Let $\psi(z) = e^z$ contained in P and the function $\psi \in \phi$ such that $\psi_0(\mathbb{D}) = \{w \in \mathbb{C}: |log\, w| < 1\}$ is a starlike with respect to one, symmetric with respect to the real-axis and $\psi_0'(0) > 0$. Then $\psi_0 \in \phi$.

In this work, we applying the subordinate to define a class of Non- Bazilević functions by dependence the exponential function e^z as follows:

Definition 1.2 A function $f(z) \in \mathcal{A}$ contains in the class $\mathcal{N}_\delta$ if it satisfying the condition:

$$f'(z)\left(\frac{z}{f(z)}\right)^{\delta} \prec e^z, \quad (z \in \mathbb{D}; 0 < \delta < 1). \quad (6)$$

It is easy to check the equivalences:

$f \in \mathcal{N}_\delta$ if and only if $\left| log f'(z) \left(\frac{z}{f(z)}\right)^\delta \right| < 1.$

2. Main Results

Lemma 2.1 [3] If $p(z) \in P$, then $|d_n| \leq 2, n = 1,2, \dots$.

Lemma 2.2 [5] Let $p(z) \in P$. Then

$2d_2 = d_1^2 + (4 - d_1^2)\xi$

and

$$4d_3 = d_1^3 + 2d_1(4 - d_1^2)\xi - d_1(4 - d_1^2)\xi^2 + 2(4 - d_1^2)(1 - |\xi|^2)\eta,$$

for some ξ and η satifiying $|\xi| \leq 1$, $|\eta| \leq 1$ and $d_1 \in [0, 2]$.

Theorem 2.3 If the function $f(z) \in \mathcal{N}_\delta$**.** Then

$$|a_2| \leq \frac{1}{k_1},$$

$$|a_3| \leq \frac{1}{k_2} + \frac{q_1}{8k_1^2 k_2},$$

where $k_{i-1} = (i - \delta)$, for $i = 2,3;$ $q_1 = -4 + 7\delta - 2\delta_1^2$ and $\delta \in (0,1)$.

Proof: For $f(z)$ belongs to the class $\mathcal{N}_\delta$, then there exists $w(z)$ called a Schwarz function with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f'(z)\left(\frac{z}{f(z)}\right)^\delta = e^{w(z)}. \qquad (7)$$

Now, replacing the values of $f(z)$ and $f'(z)$ in (7) to get

$$\left(1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right)\left(\frac{z}{z + \sum_{n=2}^{\infty} a_n z^n}\right)^\delta = e^{w(z)}, \qquad (8)$$

In [9] defined the relationship between Schwarz and Carthéodory functions as

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots,$$

we remark that $p \in P$ and

$$w(z) = \frac{h(z) - 1}{h(z) + 1} = \frac{d_1 z + d_2 z^2 + d_3 z^3 + \cdots}{2 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots}.$$

On the other side, we calculate the function $e^{w(z)}$ in the right part of (8)

$$e^{w(z)} = 1 + w(z) + \frac{\left(w(z)\right)^2}{2!} + \frac{\left(w(z)\right)^3}{3!} + \cdots. \qquad (9)$$

Some simplifying on (9), we obtain

$$e^{w(z)} = 1 + \frac{d_1}{2}z + \left(\frac{d_2}{2} - \frac{d_1^2}{8}\right)z^2 + \left(\frac{d_1^3}{48} - \frac{d_1 d_2}{4} + \frac{d_3}{2}\right)z^3 + \ \dots\ . \qquad (10)$$

Now,

$$(1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots) = \left(\frac{z + a_2 z^2 + a_3 z^3 + \cdots}{z}\right)^\delta e^{w(z)}. \qquad (11)$$

By applying the binomial expansion on the right side from (11) under to the next condition: $|\sum_{n=2}^{\infty} n a_n z^{n-1}| < \delta$.

Simplify to get

$$1+2a_2z+3a_3z^2+4a_4z^3+\cdots=\left(\frac{1+a_2z+a_3z^2+a_4z^3+\cdots}{1}\right)^{\delta}e^{w(z)}$$
$$=\Bigg(1+\delta a_2\mathrm{z}+(\delta a_3+\frac{\delta(\delta-1)!}{2!}a_2^2)\mathrm{z}^2$$
$$+(\delta a_4+\delta(\delta-1)!\,a_2a_3+\frac{\delta(\delta-1)(\delta-2)!}{3!}a_2^3)\mathrm{z}^3+\cdots\Bigg)\left[1+\frac{d_1}{2}z+\left(\frac{d_2}{2}-\frac{d_1^2}{8}\right)z^2\right.$$
$$\left.+\left(\frac{d_1^3}{48}-\frac{d_1d_2}{4}+\frac{d_3}{2}\right)z^3+\cdots\right]$$
$$1+2a_2z+3a_3z^2+4a_4z^3+\cdots=$$
$$=1+(\delta a_2+\frac{d_2}{2})\mathrm{z}+(\delta a_3+\frac{\delta(\delta-1)}{2!}a_2^2+\frac{d_2}{2}a_2+\frac{d_2}{2}-\frac{d_1^2}{8})\mathrm{z}^2$$
$$+\cdots \tag{12}$$

We comparing between the coefficients of the variables z, z^2 and z^3 in the equation (12) we obtain

$$a_2=\frac{d_1}{2k_1},$$
$$a_3=\frac{d_2}{2k_2}+\frac{q_1d_1^2}{8k_1^2k_2},$$

Applying lemma 2.1, we completes the proof.

Theorem 2.4 If the function $f(z)\in\mathcal{N}_\delta$. Then

$$|T_2(2)|\le|a_3^2-a_2^2|\le\left|\frac{q_1^2+k_1^4+4q_1\,k_1^2k_2}{4k_1^4k_2^2}-\frac{1}{k_1^2}\right|,$$

where $k_{i-1}=(i-\delta)$ for $i=2,3;\ q_1=-4+7\delta-2\delta^2$, and $\delta\in(0,1)$.

Proof: Using the coefficients in the Theorem 2.3, we have

$$|a_3^2-a_2^2|=\left|\frac{d_2^2}{4k_2^2}+\frac{q_1d_2d_1^2}{8k_1^2k_2^2}+\frac{q_1^2d_1^4}{64k_1^4k_2^2}-\frac{d_1^2}{4k_1^2}\right|.$$

Now, we use lemma 2.2 to get

$$|a_3^2-a_2^2|=\left|\frac{(q_1^2+k_1^4+4q_1\,k_1^2k_2\,)d_1^4}{64k_1^4k_2^2}-\frac{d_1^2}{4k_1^2}+\frac{(q_1+2)d_1^2\xi X}{16k_1^2k_2^2}+\frac{\xi^2X^2}{16k_2^2}\right|,\ldots \tag{13}$$

for simplicity we assume that $X=(4-d_1^2)$.

Applying the triangle inequality for the equation (13) and suppose that $d_1=d$ where $0\le d\le 2$, we get

$$|a_3^2-a_2^2|\le\left|\frac{(q_1^2+k_1^4+4q_1\,k_1^2k_2\,)d^4}{64k_1^4k_2^2}-\frac{d^2}{4k_1^2}\right|+\frac{(q_1+2)d^2\,|\xi|X}{16k_1^2k_2^2}+\frac{|\xi|^2X^2}{16k_2^2}:=\Phi(d,|\xi|).$$

Differentiating $\Phi(d,|\xi|)$ partially with respect to ξ we have

$$\frac{\partial\Phi(d,|\xi|)}{\partial\xi}=\frac{(q_1+2)d^2\,X}{16k_1^2k_2^2}+\frac{2|\xi|X^2}{16k_2^2}$$

Next,

$$|a_3^2-a_2^2|=\left|\frac{(q_1^2+k_1^4+4q_1\,k_1^2k_2\,)d^4}{64k_1^4k_2^2}-\frac{d^2}{4k_1^2}\right|+\frac{(q_1+2)d^2\,X}{16k_1^2k_2^2}+\frac{X^2}{16k_2^2}:=\Phi(d,1).$$

It's easy to see the following:

$\Phi'(d,|\xi|)>0$ for any $\xi\in[0,1]$;

$\Phi(d,|\xi|)\le\Phi(d,1)$.

Then

$$|a_3^2-a_2^2|\le\left|\frac{(q_1^2+k_1^4+4q_1\,k_1^2k_2\,)d^4}{64k_1^4k_2^2}-\frac{d^2}{4k_1^2}\right|+\frac{(q_1+2)(4-d_1^2)d^2}{16k_1^2k_2^2}+\frac{16-8d^2\,+d^4}{16k_2^2}.$$

Finally, it is easy to check that this inequality has a maximum value when $d = 2$ is

$$\left|a_3^2 - a_2^2\right| \leq \left|\frac{q_1^2 + k_1^4 + 4q_1\, k_1^2 k_2}{4k_1^4 k_2^2} - \frac{1}{k_1^2}\right|.$$

Remark 1: If $\delta = \frac{1}{2}$ in Theorem 2.4, we have

$$\left|a_3^2 - a_2^2\right| \leq 0.53612.$$

Remark 2: If the limit of δ equal to the zero in Theorem 2.4, we have

$$\left|a_3^2 - a_2^2\right| \leq 0.1388888.$$

Theorem 2.5 If the function $f(z) \in \mathcal{N}_\delta$. Then

$$|T_3(1)| \leq \left|1 + \frac{8k_1^2 k_2 - 4q_1 k_1^2 + 4k_2 q_1 - q_1^2 - 4k_1^4}{4k_1^4 k_2^2} - \frac{2}{k_1^2}\right|,$$

where $k_{i-1} = (i - \delta)$ for $i = 2,3$; $q_1 = -4 + 7\delta - 2\delta^2$, and $\delta \in (0,1)$.

Proof: We note that the inequality (3) be equivalent to

$|T_3(1)| = \left|1 + 2a_2^2(a_3 - 1) - a_3^2\right|$.

As well as, since $f(z)$ belongs to the class $\mathcal{N}_\delta$, we will use the coefficients a_2 and a_3 obtained in Theorem 2.3 as follows

$$|T_3(1)| = \left|1 + \frac{d_1}{2k_1}\left(\frac{d_2}{2k_2} + \frac{q_1 d_1^2}{8k_1^2 k_2} - 1\right) - \left(\frac{d_2^2}{4k_2^2} + \frac{q_1 d_1^2 d_2}{8k_1^2 k_2^2} + \frac{q_1^2 d_1^4}{64 k_1^4 k_2^2}\right)\right|$$

$$= \left|1 + \frac{(2k_2 - q_1)d_1^2 d_2}{8k_1^2 k_2^2} + \frac{(4k_2 - q_1)q_1 d_1^4}{64k_1^4 k_2^2} - \frac{d_1^2}{2k_1^2} - \frac{d_2^2}{4k_2^2}\right|.$$

Now, by applying lemma 2.2 with $d_1 = d$ for $d \in [0,2]$, we obtain

$$|T_3(1)| = \left|1 + \frac{(8k_1^2 k_2 - 4q_1 k_1^2 + 4k_2 q_1 - q_1^2 - 4k_1^4)d^4}{64k_1^4 k_2^2} - \frac{d^2}{2k_1^2} + \frac{\xi^2(4-d^2)^2}{16k_1^2} + \frac{(2k_2 - q_1 - 2k_1^2)\xi d^2(4-d^2)}{16k_1^2 k_2^2}\right|.$$

We applying the triangle inequality, we obtain

$$|T_3(1)| \leq \left|1 + \frac{(8k_1^2 k_2 - 4q_1 k_1^2 + 4k_2 q_1 - q_1^2 - 4k_1^4)d^4}{64k_1^4 k_2^2} - \frac{d^2}{2k_1^2}\right| + \frac{|\xi|^2(4-d^2)^2}{16k_1^2} + \frac{(2k_2 - q_1 - 2k_1^2)|\xi| d^2(4-d^2)}{16k_1^2 k_2^2}.$$

Now, applying the fact $|\xi| \leq 1$, we get

$$|T_3(1)| \leq \left|1 + \frac{(8k_1^2 k_2 - 4q_1 k_1^2 + 4k_2 q_1 - q_1^2 - 4k_1^4)d^4}{64k_1^4 k_2^2} - \frac{d^2}{2k_1^2}\right| + \frac{(4-d^2)^2}{16k_1^2} + \frac{(2k_2 - q_1 - 2k_1^2)d^2(4-d^2)}{16k_1^2 k_2^2} := \Phi(d,1).$$

Finally, by using Lemma 2.1 we show that the $\Phi(d,1)$ has maximum value when $d = 2$ is

$$|T_3(1)| \leq \left|1 + \frac{8k_1^2 k_2 - 4q_1 k_1^2 + 4k_2 q_1 - q_1^2 - 4k_1^4}{4k_1^4 k_2^2} - \frac{2}{k_1^2}\right|.$$

Remark 3: If the function $(z) \in \mathcal{N}_{\frac{1}{2}}$. Then

$|T_3(1)| \leq 0.153580$.

Remark 4: If the function $(z) \in \mathcal{N}_0$. Then

$|T_3(1)| \leq 0.722222$.

References

[1] A. Antonevich, M. Belousov and A. Lebedev, Functional differential equations:II.C*-applications, Longman Scientific and Technical, Harlow, 1998.

[2] A. B¨ottcher and B. Silbermann, Invertibility and asymptotics of Toeplitz matrices, 1983, Akademie-Verlag.Berlin.
[3] P. L. Duren, Univalent Functions, Springer, New York, 1983.
[4] P. Duren, Subordination in Complex Analysis, Lecture Notes in Mathematics, Springer, Berlin, Germany, 599 (1977), 22-29.
[5] R. J. Libera and E. J. Zlotkiewicz,Coefficient bounds for the invers of function with derivative in P, Proceedings of the American Mathematical Society 87. 2(1983), 251-257.
[6] [6] G. L. David, Introduction to Dynamic Systems: Theory, Models, and Applications, 1979, John Wiley & Sons
[7] W. C. Ma, D. Minda, Aunified treatment of some special classes of univalent functions. In: Proceedings of the Conference on Complex Analysis (Tianjin, 1992). Lecture Notes in Analysis, pp. 157-169.
[8] V. Radhika, S. Sivasubramanian, G. Murugo. and J.M. Jahangiri, Toeplitz determinants whose elements are the coefficints functions with bounded boundary rotation, J. Copmlex Analysis,(2016), pp.496-704.
[9] H.M. Srivastava, S. Owa,(Eds.), Current Topics in Analytic Function Theory; World Scientific Publishing Company: London, UK, 1992.
[10] U. Steimel, Fast computation of Toeplitz forms under narrowband conditions with applications to statistical signal processing, Signal Process., 1 (1979), no. 2, 141-158.
[11] D. K. Thomas and S. Abdul Halim, Toeplitz Matrices Whose Elements are the Coefficients of Starlike and Close-to-Convex Functions, Bull. Malays. Math. Sci. Soc.,2016.