Neural Network Trigonometric Approximation

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Abstract

In this paper, we define a weighted norm to construct a weighted $L_{p,\alpha}$-space for 2π-periodic functions. Then, we prove that any $f \in L_{p,\alpha}([-\pi,\pi]^m)$ is approximated by a feedforward neural network with sigmoidal hidden neuron in terms of $\omega_k(f,\delta)_{p,\alpha}$, That's what we called Neural Networks Trigonometric Approximation.

Keywords: Neural Network, Trigonometric Approximation, Modulus of Smoothness.

1. Introduction

In recent years, trigonometric polynomials have a main rule in approximation functions in L_p-space as well as other more general spaces. They are widely used to find neural networks as approximators for those functions, in for examples [1,10,11], for the huge importance for neural networks in different fields, and the essential need for approximated neural networks in different applications, such as [2,4,5].

Three-layer feedforward neural network is an important class of neural networks that can approximate the desired function well, see [3,8]. Most papers deal with the rate of approximation as a tool to understand the approximation capability, see [6,7,9].

In this work, we care to find a new space with a new norm to spot light on the relationship between the approximation error and the properties of the used neural network. For our space, a feedforward neural network (FNN) of three layer can be existed to approximate the present function well.

2. Preliminaries

First of all, we need to define a three layer FNNs with one hidden layer as follow [11]

$$"N(x)=\sum_{i=1}^{m} C_i \emptyset\left(\sum_{j=1}^{d} w_{ij}.x_j+\varphi_i\right), x\in R^d, d\geq 1" \qquad (1)$$

that generates one output of that *d* inputs, with thresholds $\varphi_i \in R$, connection weights $w_{ij} \in R^d$ and $\emptyset$ is the activation function. We use the sigmoid activation function that satisfies

$$\emptyset(t)=\begin{cases}1, & as\ t\to+\infty\\ 0, & as\ t\to-\infty\end{cases}$$

Given $m\in N$, $\boldsymbol{x}=(x_i)_{i=1}^m \in N_0^m$, where $|\boldsymbol{x}|=\sum_{i=1}^m |x_i|$, $\boldsymbol{rx}=\sum_{i=1}^m r_i x_i$. We define the weighted L_p-space $L_{p,\alpha}([-\pi,\pi]^m), p\geq 1, \alpha\geq\frac{1}{p}$ such that for any $f\in L_{p,\alpha}([-\pi,\pi]^m)$, we define the norm

$$\|f\|_{p,\alpha}=(2\pi)^{-m}\left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}|f(\boldsymbol{x})|^p|\boldsymbol{x}|^{-\alpha p}dx_1\dots dx_m\right)^{\frac{1}{p}},$$

We estimate the trigonometric approximation in terms of modulus of smoothness

$$\omega_k(f,\delta)_p=\sup_{\|\mathbf{t}\|\leq\delta}\left\|\Delta_{\mathbf{t}}^k(\mathbf{t})\right\|_p,$$

where,

$$\Delta_{\mathbf{t}}^k f(\mathbf{x})=\sum_{j=0}^k(-1)^{k-j}\binom{k}{j}f(\mathbf{x}+(\tfrac{\boldsymbol{k}}{\mathbf{2}}-\boldsymbol{j})\mathbf{t}),$$

Our main result is as follow:

Theorem 1. For $\in L_{p,\alpha}([-\pi,\pi]^m), p\geq 1, \alpha\geq\frac{1}{p}$, there exist FNN of the form (1) with $x\in[-\pi,\pi]^m, C_i\in R$, such that

$$\|N(x)-f(x)\|_{p,\alpha}\leq\omega_m(f,\delta)_{p,\alpha}$$

3. Auxiliary Lemma

In the lemma below, it is useful to construct a FNN that approximate trigonometric function od m-dimension. FNN (that has piecewise linear and activation function hidden layer units of sigmoid type) constructed by Suzuki [8] and then developed by Wang and Xu[10] can be more accurate and suitable for $L_{p,\alpha}([-\pi,\pi]^m)$ in the next lemma.

Lemma: For $\sigma \in N$, we can find networks of three layers $NS_\sigma(\boldsymbol{rx})$ and $NC_\sigma(\boldsymbol{rx})$ to approximate respectively $\sin(\boldsymbol{rx})$ and $cos(\boldsymbol{rx})$, each one has $q$hidden layer units , where $q = 4|\boldsymbol{r}|\sigma$ related to $NL_\sigma(\boldsymbol{rx})$ s.t.

$$NS_\sigma(\boldsymbol{rx}) = 2(-1)^{|r|} sin\frac{\pi}{4\sigma}\sum_{k=0}^{q-1}(-1)^{q-1-k}\binom{q-1}{k} cos\left(\boldsymbol{x} + \left(\frac{q-1}{2}\right)\boldsymbol{r}\right) NP_\sigma(\boldsymbol{rx})$$

and

$$NC_\sigma(\boldsymbol{rx}) = (-1)^{|r|} - 2(-1)^{|r|} sin\frac{\pi}{4\sigma}\sum_{k=0}^{q-1}(-1)^{q-1-k}\binom{q-1}{k} sin\left(\boldsymbol{x} + \left(\frac{q-1}{2}\right)\boldsymbol{r}\right) NP_\sigma(\boldsymbol{rx})$$

where

$$NL_\sigma(\boldsymbol{rx}) = \frac{1}{(1+e^{-rx})}$$

then

$$\|\sin(\boldsymbol{rx}) - NS_\sigma(\boldsymbol{rx})\|_{p,\alpha} = \|\cos(\boldsymbol{rx}) - NC_\sigma(\boldsymbol{rx})\|_{p,\alpha} \le c(p)\frac{1}{|\boldsymbol{r}|^\alpha}$$

Proof:

It is enough to prove the first estimate,

$$\|\sin(\boldsymbol{rx}) - NS_\sigma(\boldsymbol{rx})\|_{p,\alpha} = (2\pi)^{-m}\left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}|\sin(\boldsymbol{rx}) - NS_\sigma(\boldsymbol{rx})|^p|\boldsymbol{rx}|^{-\alpha p}dx_1 \dots dx_m\right)^{\frac{1}{p}}$$

$$\le (2\pi)^{-m}(2)^{-m}\left\{\int_0^{\pi}\cdots\int_0^{\pi}|\sin(\boldsymbol{rx})|^p|\boldsymbol{rx}|^{-\alpha p}dx_1 \dots dx_m\right\}^{\frac{1}{p}}$$

$$+2(-1)^{|\boldsymbol{r}|}(2\pi)^{-m}(2)^{-m}sin\frac{\pi}{4\sigma}\sum_{k=0}^{q-1}\left\{\int_0^{\pi}\cdots\int_0^{\pi}\left|(-1)^{q-1-k}\binom{q-1}{k}cos\left(\boldsymbol{x}+\left(\frac{q-1}{2}\right)\boldsymbol{r}\right)\right|^p\frac{|\boldsymbol{rx}|^{-\alpha p}}{(1+e^{-\boldsymbol{rx}})}dx_1\ldots dx_m\right\}^{\frac{1}{p}}$$

$$\leq 2^{p-1}\left\{(\pi)^{-m}\pi^m(|\boldsymbol{r}|\pi)^{-\alpha}+(\pi)^{-m}\sum_{k=0}^{q-1}\binom{q-1}{k}\pi^m(|\boldsymbol{r}|\pi)^{-\alpha}\right\}$$

$$\leq c(p)\frac{1}{|\boldsymbol{r}|^{\alpha}}$$

where

$$|\boldsymbol{rx}|=\left|\sum_{i=1}^{m}r_ix_i\right|\leq\sum_{i=1}^{m}|r_ix_i|\leq\sum_{i=1}^{m}|r_i|\pi=|\boldsymbol{r}|\pi\,.$$

since $\boldsymbol{x}\in[-\pi,\pi]^m$. ■

4. Proof of Theorem 1.

Let >0, "$t(x)=\sum_{|\boldsymbol{r}|<n}\big(a_r sin(rx)+b_r cos(rx)\big)$ ", then, by lemma 1, choose FNNs

$$N(x)=\sum_{|\boldsymbol{r}|<n}2(-1)^{|\boldsymbol{r}|}+\sum_{|\boldsymbol{r}|<n}(-1)^{|\boldsymbol{r}|}\,sin\frac{\pi}{4\sigma}\sum_{k=0}^{q-1}(-1)^{q-1-k}\binom{q-1}{k}\left\{cos\left(\boldsymbol{x}+\left(\frac{q-1}{2}\right)\boldsymbol{r}\right)-b_r sin\left(\boldsymbol{x}+\left(\frac{q-1}{2}\right)\boldsymbol{r}\right)\right\}NP_{\sigma}(\boldsymbol{rx}),$$

such that

$$\|N-t\|_{p,\alpha}\leq\varepsilon$$

Let $t_n^*(x)$ be the the nth best approximation using trigonometric transformation of N, s.t.

$$\|N-t_n^*(x)\|_{p,\alpha}\leq\varepsilon$$

Now, by triangle inequality

$$\|f-N\|_{p,\alpha} \leq \|N-t_n^*(x)\|_{p,\alpha} + \|f-t_n^*(x)\ \|_{p,\alpha} \leq \varepsilon + \omega_m(f,\delta)_{p,\alpha}$$

by taking $\omega_m(f,\delta)_{p,\alpha} \geq \varepsilon$, the proof is done.■

5. Conclusions and Future Work:

We work on estimating the rate of function approximation in weighted space with a three layer neural network using trigonometric approximation and module of smoothness. This paper shows the development of the upper bound estimating approximation. In the future, it is useful to think about estimating the lower bound approximation.

6. References:

[1] Cao F. L., Zhang Y. Q. and Zhang W.G., 2007.Neural Network with Single Hidden Layer and the Best Polynomial Approximation, Acta Mathematica Sinica, 2005, 50(2):385-392.

[2] Dineva, A, Várkonyi-Kóczy, A R, Tar J and Piur, V.,2015. Intelligent Neural Network Design for Nonlinear Control using Simultaneous Perturbation Stochastic Approximation (SPSA) Optimization.

[3] Hornik K. , Stinchombe M. and White H., 1989, Multilayer Feedforword Networks are Universal Approximator, 1989, 2:303-314.

[4] Ferrari S., Stengel R.S., 2005. Smooth Function Approximation Using Neural Networks, IEEE TRANSACTIONS ON NEURAL NETWORKS, VOL. 16, NO. 1, JANUARY 2005, PP 24-38.

[5] Jun Ye , 2014, Hybrid trigonometric compound function neural networks for tracking control of a nonholonomic mobile robot. October 2014, Volume 7, Issue 4, pp 235-244

[6] Lin, S Cao, F., 2015. Simultaneous approximation by spherical neural networks, Neurocomputing, Volume 175, Part A,348–354.

[7] Li, F and Xu, Z., 2007. The Essential Order of Simultaneous Approximation for Neural Networks, Applied Mathematics and Computations 194 , 120-127.

[8] Suzuki, S., 1998. Constructive Function Approximation by Three Layer Artificial Neural Networks, Neural Networks 11, 1049-1058.

[9] Wang J and Xu Z., 2010. New Study on Neural Networks: The Essential Order of Approximation, Neural Networks 23, 618_624.

[10] Wang J and Xu Z., 2011. Neural Networks and the Best Trigonometric Approximation,JSSC & Springer-Verlag Berlin Heidelberg, 24, 401-412.

[11] Wang, J., Chen, B. & Yang, C.,2012 Approximation of Algebraic and Trigonometric Polynomials by Feedforward Neural Networks. Neural Comput & Applic (2012) 21: 73. doi:10.1007/s00521-011-0617-3.