# Calculation of The Sensitivity and transitivity of $f(x,y)=\begin{pmatrix}1-a|y|+bx\\ x\end{pmatrix}$

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**المستخلص :**

في هذا البحث عملنا على دراسة السلوك الفوضوي للدالة $f(x,y)=\begin{pmatrix}1-a|y|+bx\\ x\end{pmatrix}$ وذلك من خلال دراسة خاصيتي الحساسية المعتمدة على الشروط الابتدائية والتعدي واستخدمنا لهذا الغرض برنامج (Matlab) ودرسنا معالم القيم حيث وجدنا إن التعدي للدالة متحقق عندما $1.2\le a\le 1.7$ و $-1.2\le b\le 0.2$ وعند هذه القيم تكون الدالة متعدية ولا تكون حساسة والقيم التي تكون فيها الدالة حساسة لا تكون متعدية.

**Abstract**

In this work ,We study the chaotic behavior for $f(x,y)=\begin{pmatrix}1-a|y|+bx\\ x\end{pmatrix}$ through employment sensitivity dependent on initial condition and transitive by using the software (Matlap) these are implement by varying the parameter of system. We found the parameters which make *f(x,y)* sensitive does not make him transitive and vise versa.

## 1-Introduction :

A dynamical system is chaotic on a given invariant set X for a flow φ when it satisfied certain properties. Thus to apply this concept we must first identify an invariant set of course X could be a very small set in the phase space. And then the assertion of chaos on X would not necessarily be of much practical important thus, a chaotic flow mixes things up and is hard to predict, although this definition of chaotic is reasonably useful, it is also important to note that the term "chaotic" in the literature is used with many definition some researches simply use the loose sense we first discussed and some require stronger condition than sensitive dependent, while it is clear from the quotes that Poincare and Lorenz had clear notions of sensitive dependent Li and Yorke first gave a mathematical definition of chaos in 1975 the definition that we use is due to AVS Lander and Yorke (1980). However all definition include element comparable to transitivity and sensitive dependent [Devaney 1986; Robinson 1999; Wiggins 2003]. While another definition include sensitive or positive Lypanov satisfies [Gulick, 1992]. The study of discrete map such as piece wise Linear map [Devaney, 1984; Lozi 1978; Cao and Liu 1998; Aharonov and Devaney and Elias 1997; Ashwin and Fu 2002] is an interesting contribution to the development of theory of dynamical system , with many possible application in science and engineering [scheizer and Hasler 1996; Abel and Bauer and Kerber and Schwarz 1997]. Discrete mathematical models arise directly from experiment or by the use of the Poincare for the study of continues models two of these models are the Henon [Henon 1976] and the Lozi [Lozi 1978] maps given by :

$$H_{a,b}(x,y) = \begin{pmatrix} 1 - ax + y \\ bx \end{pmatrix}$$

And

$$L_{a,b}(x,y) = \begin{pmatrix} 1 - a\ |x\ | + y \\ bx \end{pmatrix}$$

The $\boldsymbol{H_{a,b}}$ map gives a chaotic attractor called the Hennon attractor , which is obtain for **a=1.4** and **b=0.3** as shown in fig (a) . there are many poper that discuss the original Henon and Lozi map such as [maorotto, 1979l Gao and Liv 1998] , moreover, it is possible to change the form of the Henon mapping H to obtain other chaotic attractor.[lozi, 1978; Aziz alaovi and Carl Robert and Gelso Grebogi 2001, Zeraoulia Elhadj 2005].

Application of These maps include secure communication using the notions of chaos [Scheizer and Hasler, 1996; Abel and Bouer and Kerber and Schwarz 1997 ].

The Lozi map L is a 2-D non–invertible iterated map that given chaotic attractor called the Lozi attractor which is obtained for **a=1.4** and **b=0.3** as show in **fig (1b)** [Zeraoulia and Protte, 2007 ] introduce a new simple 2-D piecewise linear map given by :

$$f(x,y)=\begin{pmatrix}1-a|y|+bx\\ x\end{pmatrix}\ldots\ldots\ldots\ldots(3)$$

Where ***a*** and ***b*** are parameter and equation (3) is an interesting system has simple form, similar to the Lozi map [ Lozi, 1978].

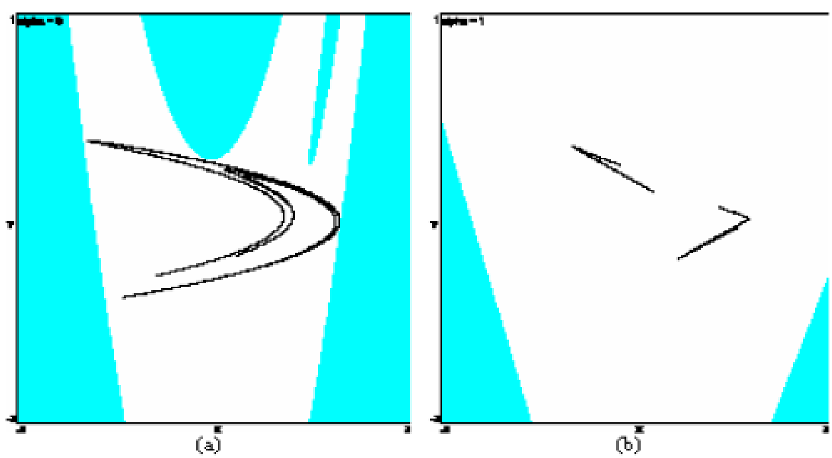

**Fig. (1): (a)** The original Hénon chaotic attractor obtained from the H mapping with its basin of attraction (white) for a = 1.4 and b = 0.3. (b) The original Lozi chaotic attractor obtained from the L mapping with its basin of attraction (white) for a = 1.4 and b = 0.3.

## 2-Elementary Fundamental definition :

In this section we refered to many definition which we needed in this work

### Definition . 2-1 [meiss , 2007]:

A set $A$ is invariant under a rule $f_t$ if $f_t(A)=A$ for all $t$ . that is for each $x\in A$, $f_t(X)\in A$ for any $t$.

### Definition. 2-2 [meiss, 2007]

A set $A$ is forward invariant if $f_t(A) \subset A$ for all $t > 0$

### Definition. 2-3 [meiss, 2007]

A point $X^*$ is an equilibrium of $\bar{X}=f(x)$, $x(0) = X_0$ if $f(X^*) = 0$

### Definition. 2-4 [meiss, 2007]

Let p be in the domain of f then p is fixed point of f if f(p)=p

### Def. 2-5 [Meiss, 2007]

An equilibrium is attractor if all the eigen value has negative real part.

### Def. 2-6 : [Meiss, 2007]

An equilibrium is saddle if it is hyperbolic but not attractor or a source.

### Def. 2-7 [Devaney,1984]

$f: J\rightarrow J$ has sensitive dependence on initial condition if there exist $\delta>0$ such that for any $x\in J$ and any neighborhood Nof x there exist $y \in N$ and $n\geq 0$, such that $|f^n(x) - f^n(y)| > \delta$

### Def. 2-8 [Devaney,1984 ]

$f: J\rightarrow J$ is said to be transitive if for every pair of open set U, VCJ there exist k>0 such that $f^k(u) \cap V \neq \varphi$.

### Def 2-9 [Devaney, 1984 ]

Let V be aset . $f: V\rightarrow V$ is said to be chaotic on V if

1- f has sensitive dependence on initial condition
2- f is transitive

3- periodic point are dense in V

**Theorem 2- 10[Gulick,1992]**

Suppose that J is a closed interval and f:J→J.Then f is transitive if and only if there is an x in J whose orbit is dense in J

## 3-Fixed Point of System [Zeraoulia and Prottle, 2007] :

Zeraoulia and Prottle are Studied the existence of fixed point and their determined stability type of the ***f*** mapping :

$$f(x,y)=\begin{pmatrix}1-a\,|\,y\,|+bx\\ x\end{pmatrix}$$ , where ***a, b*** real number

The left and right hand vector are equal :

if ***y=x*** and ***x = 1-a|y|+bx***

which implies that :

***x= 1-a|x|+bx*** ⇒ ***-1 = -x – a|x| +bx***

$$-1 = x(-1 + a + b) \Rightarrow x=\frac{-1}{+a+b-1},\; y=\frac{-1}{+a+b-1}$$

So it is easy to see that :

$$p1=(\frac{-1}{+a+b-1},\frac{-1}{+a+b-1})$$

And $$-1 = x(-1 - a + b) \Rightarrow x=\frac{-1}{-a+b-1},\; y=\frac{-1}{-a+b-1}$$

So $$p2=(\frac{-1}{-a+b-1},\frac{-1}{-a+b-1})$$

In The event $f(x,y)=\begin{pmatrix}\frac{1+a\,|\,y\,|+bx}{x}\end{pmatrix}$ has two fixed point P1, P2 They are given by :

$$p1=(\frac{-1}{+a+b-1},\frac{-1}{+a+b-1})$$ , which exist when b>-a+1

$$p2=(\frac{-1}{-a+b-1},\frac{-1}{-a+b-1}),$$

which exist when b< a+1

To determine the eigen value of $f(x,y)=\begin{pmatrix}1-a|y|+bx\\ x\end{pmatrix}$ they observe that at P1

det $\Delta$ *f(x,y)* = det$\begin{pmatrix}b & a\\ 1 & 0\end{pmatrix}$=-a

and the characteristic polynomial given as :

$$\left|\Delta f(x,y)-\lambda I\right|=\begin{vmatrix}b-\lambda & a\\ 1 & 0-\lambda\end{vmatrix}=\lambda^2-b\lambda-a=0$$

$$\lambda_{1,2}=\frac{+b\mp\sqrt{b^2+4a}}{2}$$

and the characteristic polynomial at P2

$$\left|\Delta f(x,y)-\lambda I\right|=\begin{vmatrix}b-\lambda & -a\\ 1 & 0-\lambda\end{vmatrix}=\lambda^2-b\lambda+a=0$$

$$\lambda_{1,2}=\frac{b\mp\sqrt{b^2-4a}}{2}$$

## Note that 3-1 :

The map f in (3) has the same fixed point as the Lozi map but different stability due to different in the Jacobean matrix [Zeraoulia and Sporotte, 2007].

## 4-Stability of $f(x,y)=\begin{pmatrix}1-a|y|+bx\\ x\end{pmatrix}$

At the fixed point P1 is :

(1) A repeller if a > 1. 0<b<a-1.

(2) A regular saddle if a>1, 0<b<a-1.

(3) A flip saddle if 0<a<1 , b>1-a.

While at the fixed point P2 :

(1) A regular attractor if 0<a<1 , $2\sqrt{a}$ <b<1+a

(2) A flip saddle if -1<a<0 , b < -1-a, b<a+1 or 0<a<1 , b<-1-a

(3) A flip attractor if $-1< a<0$ , $-1-a<b<1+a$ or $0<a<1$ , $-1-a<b<-2\sqrt{a}$

(4) A clock wise spiral attractor

if $0<a<1$ , $0<b<2\sqrt{a}$

(5) A counter – clock wise spiral attractor

if $0<a<1$ , $-2\sqrt{a}<b<0$

(6) A repeller if $a>1$ , $b < 1-a$

A schematic representation in fig. (2) where regions ( $bi_{0<=i<=7}$) have respectively the following boundaries :

$b_0=-a-1$, $b_1=a+1$, $b_2=1-a$, $b_3=a-1$, $b_4=-2\sqrt{-a}$ , $b_5=2\sqrt{-a}$ , $b_6=-2\sqrt{a}$ , $b_7=2\sqrt{a}$ .

Where the single number indicate the nature of single fixed point and two number indicate the nature of both fixed point [Zeraoulia and Sprottle , 2007].

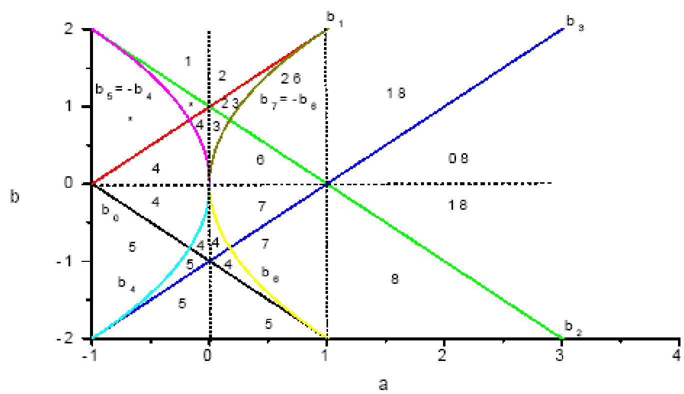


**Fig.(2)**: Stability of the fixed points of the map (3) in the ab-plane, where the numbers on the figure are as follow: 0: P1 repeller, 1: P1 regular saddle,2: P1 flip saddle, 3: P2 regular attractor, 4: P2 flip attractor, 5: P2 flip saddle, 6: P2 a clockwise spiral attractor, 7: P2 a counter-clockwise spiral attractor, 8: P2 repeller. * : There are no fixed points.

## 5- Calculation Chaotic of $f(x,y)=\begin{pmatrix}1-a|y|+bx\\ x\end{pmatrix}$:

In this section we study the chaotic of system (3) by calculate the sensitivity and `transitivity of them.

### 5-1 Calculate the Sensitivity of $f(x,y)=\begin{pmatrix}1-a|y|+bx\\ x\end{pmatrix}$:

We can calculate the sensitivity to initial condition of system (3) by varying. The control parameter (a, b) by using (Matlap ) to analysis of view for sensitivity dependent on initial condition.

Consider The system (3) we get sensitivity on initial condition (xi,yi) as follow :

**(1)** (0.05, 0) and (0.05, 0.05) with parameter **a=1.5 , b=1.2**

We get two orbit one of this orbil run away from the other see fig. (3)

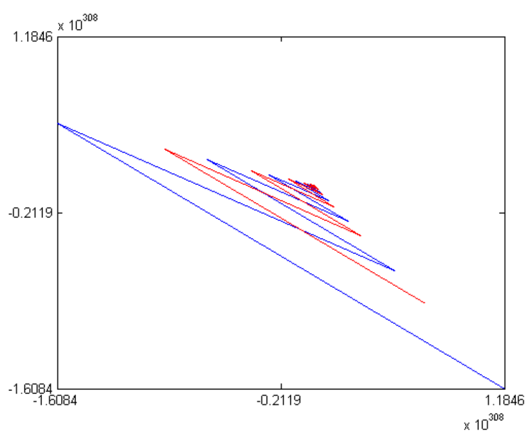


Fig. (3) :behavior of system f(x,y) when a=1.5, b=1.2 and initial point (0.05,0) and (0.05, 0.05) when one orbit is (red) and the other orbit is blue.

**(2)** (0.05, 0.05 ) and (0, 0.05) with parameter **a=1, b=0.2**

There is two orbil each of these orbil like ring and one of these orbit leave the other see fig. (4).

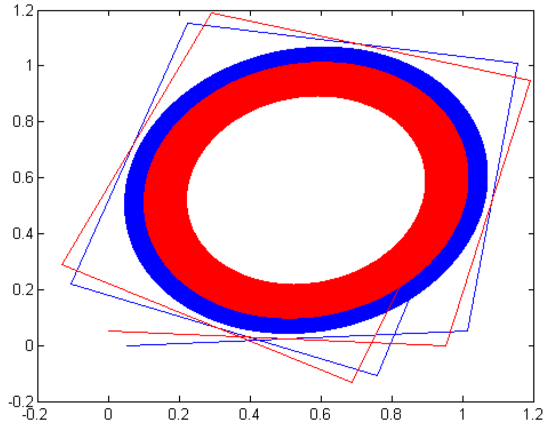


Fig.(4) : Phenomena sensitive dependent on initial point (0.05, 0.05)is (red)and (0, 0.05) is (blue) with parameter a=1, b=0.2.

**(3)** (1.01, 1) and (1.01, 1.05) with parameter **a=0.7 , b=0.2**

These two orbit like as spiral and each of these orbil run away from other.

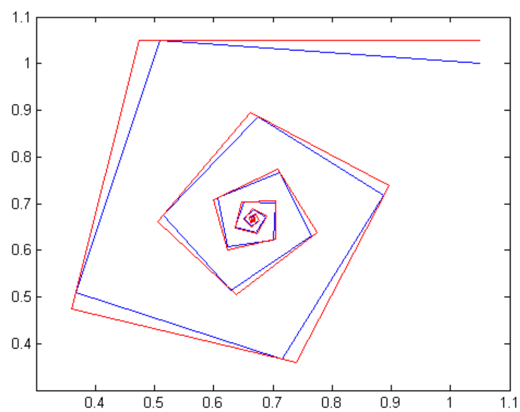


Fig.(5) : Phenomena sensitive dependent to initial point with a=0.7, b=0.2, initial point is (1.01, 1)orbit is (red)and (1.01, 1.05)orbit is (blue)

But when we change the value of parameter we did not get any sensitivity to initial condition since each of these orbil near from other see fig.(6) and fig.(7).

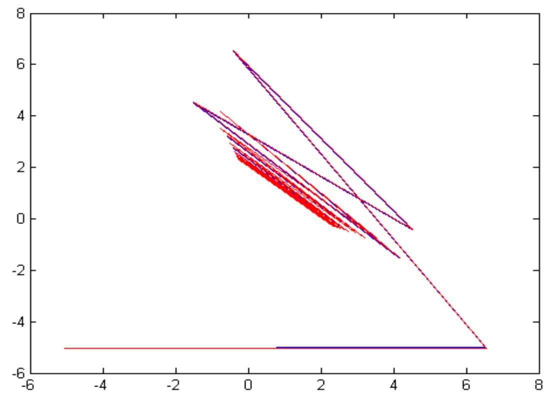


Fig.(6) : Phenomena not sensitive dependent to initial point with a=-0.5, b=-0.5,and initial point (-5, -5.05)orbit is (red) and (-5.05, -5.05)orbit is (blue)

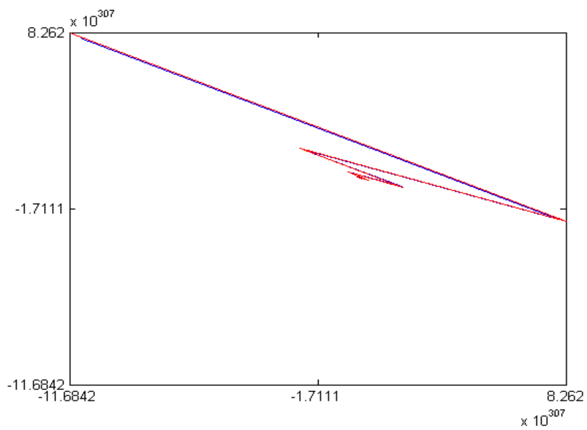


Fig. (7) :behavior of system f(x,y) when a=0.2, b=0.2 and initial point (2, 1.05)orbit is (red) and (2.05, 1.05)orbit is(blue)

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**5.2 Calculate the Transitivity of** $f(x,y)=\begin{pmatrix}1-a|y|+bx\\ x\end{pmatrix}$**:**

We can calculate the transitive of system (3) and examined by varying the control parameter **(a,b).** Matlap is used to analysis the transitivity.

When we study these Phenomena we have two case from system (3)

Case one .Transitive

Case two. There are local transitive we called them semi transitive

(1) a=1.3, b=0.2

There exists a point э the orbit of this point covered all the plan except small region . we say in this case semi transitive see fig(8).

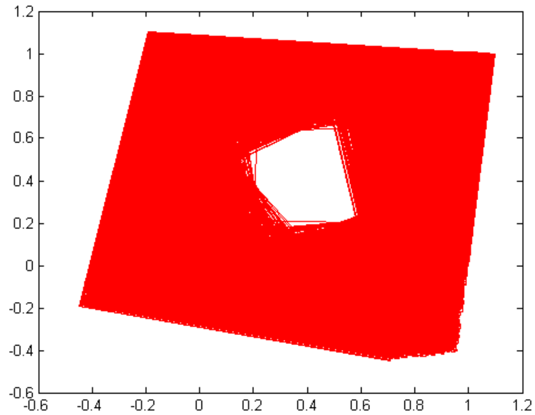



Fig. (8) :behavior of system f(x,y) is semi transitive with a=1.3, b=0.2 , the transitive region is (red) and not dense (white)

(2) a=0.4 , b=0.2

In this case we have transitivity since ∃x∈dom(f) э orb (f)=X see fig (9).

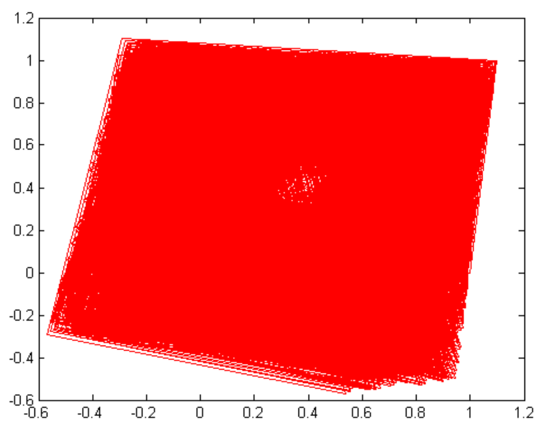



Fig. (9) :behavior of system f(x,y) is transitive with a=1.4, b=0.2 ,

(3) a=1.2 , b=1.2

Also, In this case we have semi transitivity see fig (10).

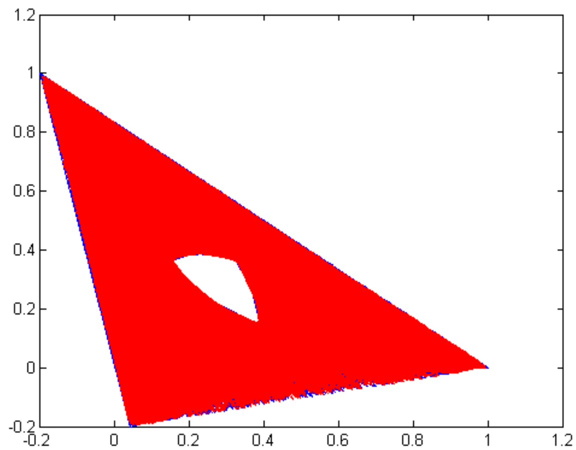



Fig.(10) : Phenomena of system semi transitive with a=1.2, b=1.2,

(4) a=1.2 , b=1.2

We have semi transitive since not all the region dense see fig(11).

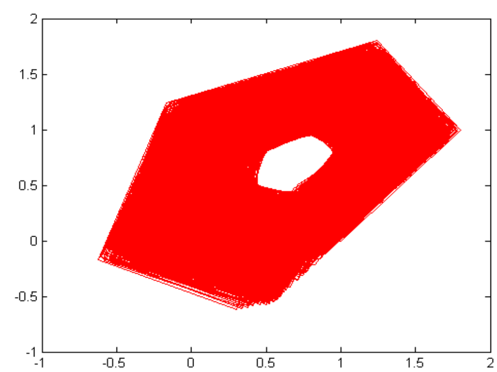



Fig.(11) : Phenomena of system semi transitive with a=1.2, b=-1.2,

(5) But when we change the value of parameter we did not have transitivity since there are not exist an orbit which orb(x)=X .

see(fig(12), fig(13),fig(14))

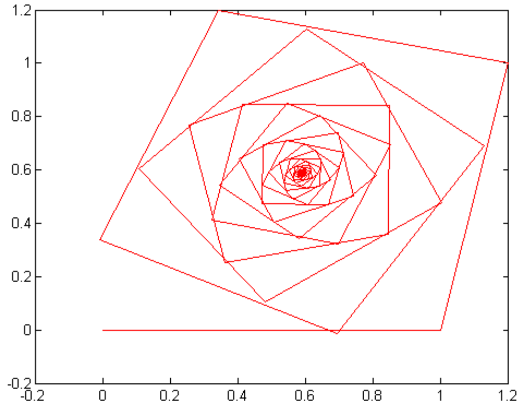



Fig. (12) :behavior of system f(x,y) with a=0.9, b=0.2 ,

Fig. (13) :behavior of system not transitive with a=1.5, b=-1.2 ,

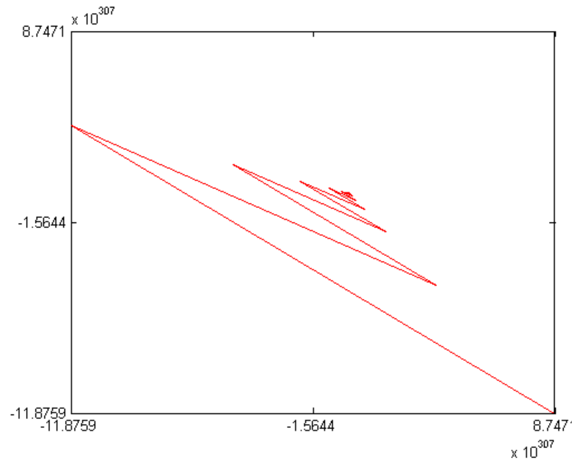
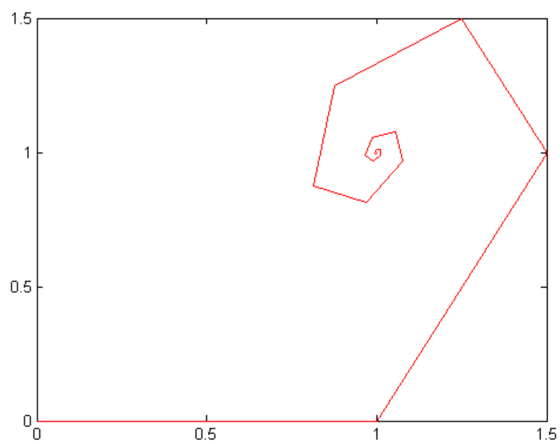



Fig. (14) :phenomena not transitive with a=0.5, b=0.5



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