The Cubic Transmuted Weibull Distribution

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Abstract

The lifetime distribution plays an important role in many real life fields such as biostatistics, reliability and survival analysis, so we try to contribute in finding a new lifetime distribution. We construct lifetime distribution, the cubic transmuted Weibull distribution, and discuss some of its statistical properties.

Keywords: Weibull distribution, cubic transmuted distribution, Reliability analysis, order statistic.

الخلاصة

يلعب توزيع الحياة دورا مهما في مجالات الحياة المختلفه كألأحصاءات الحياتيه, تحليل المعوليه والبقاء, لذا حاولنا المساهمه في ايجاد توزيع حياة جديد. فقمنا ببناء توزيع ويبل المحول التكعيبي مع مناقشة بعض خواصه الأحصائيه.

الكلمات المفتاحية: توزيع ويبل , توزيع ويبل المحول التكعيبي ، تحليل الموثوقية، رتبة الاحصاء.

1. Introduction

We analyze statistically the data of life phenomena under study after modulating these data. Sometimes the models available do not fit the data of many important and practical problems. That is, a non-parametric model may be recommended.

(Shaw and Buckley, **2009)** used the rank transmutation map RTM , a tool for the construction of new families of non-Gaussian distributions, to modulate a given base distribution for modifying the moments, like the skew and kurtosis. They introduced the quadratic rank transmutation map (QRTM) that was used by

1) (Merovci, 2013) in presentation the generalized Rayleigh distribution, and generalized the Lindley by (Merovci ,2013),

2) (Merovci ; *et al.*, ,2014) in derivation a generalization of the inverse Weibull distribution and some of its properties,

3) (Mahmoud and Mandouh , 2013) in finding the Transmuted Fréchet Distribution, some of their properties, and estimation the parameter using maximum likelihood and Bayesian methods,

4) (Ashour and Eltehiwy, 2013) in introducing the generalization of the exponentiated Lomax distribution and derivation some of its properties.

5) (Khan and King , 2013) in derivation the three parameter transmuted modified Weibull distribution , their moments maximum likelihood estimatin methods, and the order statistics of this distribution,

6) (Elbatal *et al.*, ,2013) in presentation the Transmuted Generalized Linear Exponential Distribution of four-parameter generalized version exponential distribution and some properties, and the maximum likelihood estimation.

7) (Elbatal and Aryal ,2013) in studying the transmuted additive Weibull distribution and some other distributions their order statistics and maximum likelihood estimation of its parameters,
8) (Aryal ,2013)in introducing the transmuted log-logistic distribution, and the moments, quartiles, mean deviations of the transmuted log-logistic distribution, and maximum likelihood estimators of its parameters.
9) (Elbatal and Elgarhy, 2013) in derivation the transmuted quasi Lindley distribution, and their moment and moment generating function, and weighted least squares and the maximum likelihood estimation of its parameters.
10) (Pal and Tiensuwan , 2014) in presentation the beta transmuted Weibull distribution, and some of its properties,
11) (Afify; *et al.*, 2014) in construction a new generalization of the complementary Weibull geometric distribution introduced by (Tojeiro et al., 2014) and estimation the model parameters using the maximum likelihood method,
12) (Khan *et al.*, 2014) in introducing the characteristic transmuted inverse Weibull distribution to compare it with many other generalizations of the two-parameter of this distribution , and discussion some of its properties and order statistics.
13) (Ebraheim, 2014) in studying a new generalization of the two parameter Weibull distribution, and its properties and the maximum likelihood estimation of its parameters,
14) (Hussian, 2014) in presentation a new generalized version Transmuted exponentiated gamma distribution and some of its properties. He derived a new four-parameter generalized version of the transmuted generalized linear exponential distribution,
14) (Merovcia and Pukab, 2014) in studying the transmuted Pareto distribution using the quadratic rank transmutation map studied by (Shaw,W.T. and Buckley,I.R. ,2009).

In this paper, we study the a new proposed distributions, cubic transmuted Weibull distribution CTWD, and discuss the order statistic some of its statistical properties.

2. Main Results

2.1 Definition: A random variable X is said to have a cubic transmuted distribution if its cumulative distribution function (cdf) is given by

$$F(x) = (1 + \lambda)\, G(x) - 2\lambda G^2(x) + \lambda G^3(x), |\lambda| \leq 1. \quad (1)$$

And the density function (pdf) is given by

$$f(x) = (1 + \lambda)\, g(x) - 4\lambda G(x)\, g(x) + 3\lambda G^2(x)\, g(x) \quad (2)$$

Where $G(x)$ is the cdf of the base distribution, according to the general formula of the transmuted distribution, (K. Abed AL-Kadeem ,Unpublished research) which is given as

$$F(x) = \begin{cases} (1 + \lambda)\, G(x) - 2\lambda G^2(x) + \cdots + \lambda G^n(x), & n \text{ is odd} \\ (1 + \lambda)\, G(x) - \lambda G^2(x) + \cdots + \lambda G^n(x), & n \text{ is even} \end{cases} \quad (3)$$

Observe that at $\lambda = 0$, $F(x) = G(x)$

Remark 2.2

Now to prove that $f(x)$ in (2) is pdf as follows:

1) $f(x) = (1 + \lambda)\, g(x) - 4\lambda G(x)\, g(x) + 3\lambda G^2(x)\, g(x)$

Let $u = G(x), du = G(x)\, g(x)dx$, then

$$f(x) = \int_0^1 [(1 + \lambda) - 4\lambda\, G(x) + 3\lambda G^2(x)]\, g(x)dx$$
$$= 1 + \lambda - 2\lambda + \lambda$$
$$= 1$$

Since that the $G(x) = p(X \le x)$ is probability distribution.

1) Now we want to prove that $f(x) > 0$

If $\lambda < 0$, and $G(x) = 1$ then $f(x) > 0$ when

$$\lambda G^3(x) - 2\lambda G^2(x) > -(1 + \lambda)\, G(x)$$

And then, $f(x)$ is also appositive function at $\lambda \le 1$,when

$$(1 + \lambda)\, G(x) + \lambda G^3(x) > 2\lambda G^2(x)$$

2.3 Cubic Transmuted Weibull Distribution

A random variable X is said to be a cubic transmuted Weibull distribution CTWD if its cumulative distribution function (cdf) is

$$F_{CTWD}(x;\alpha,\beta,\lambda) = (1 + \lambda)\left(1 - e^{-\alpha x^\beta}\right) - 2\lambda(1 - e^{-\alpha x^\beta})^2 + \lambda(1 - e^{-\alpha x^\beta})^3$$
$$= (1 - e^{-\alpha x^\beta})\left[1 + \lambda e^{-2\alpha x^\beta}\right] \quad (4)$$

for $x > 0$ with the scale and shape parameters $\alpha, \beta > 0$, respectively, and λ is the transmuted parameter.

And the pdf is

$$f_{CTWD}(x;\alpha,\beta,\lambda) = \beta\alpha x^{\beta-1} e^{-\alpha x^\beta}\left[1 - 2\lambda\, e^{-\alpha x^\beta} + 3\lambda e^{-2\alpha x^\beta}\right] \quad (5)$$

The shape of cdf and pdf of CTWD with selected parameter values are shown in figure 1 and figure 2 respectively.

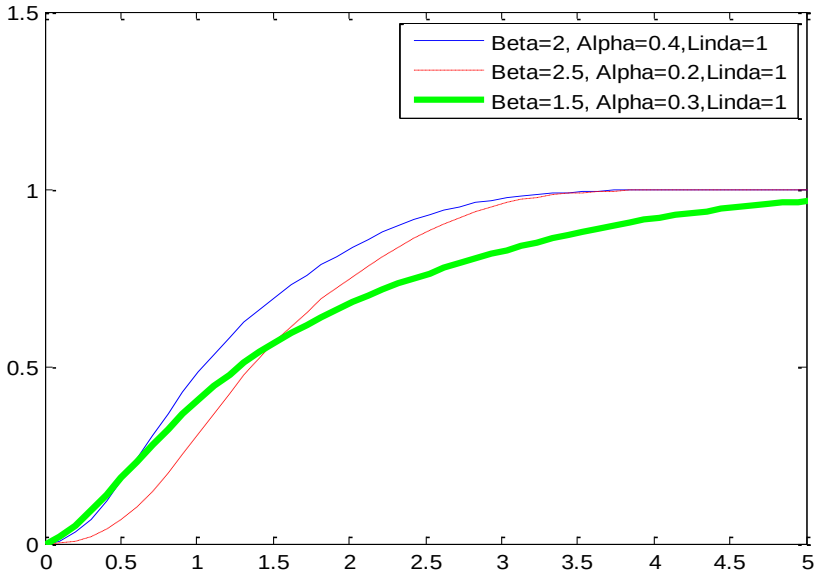

Figure 1 The cdf of the $CTWD$ at $\alpha = 0.4, 0.2, 0.3, \beta = 2, 2.5, 1.5, \lambda = 1$

figure 1, shows that the function $F_{CTW}(.)$is increasing with x increasing ,at different parameters values selected of the $\alpha, \beta,$ and fixed value $\lambda = 1$, to be fixed when $F_{CTW}(x) = 1$ at$x \geq 3$,

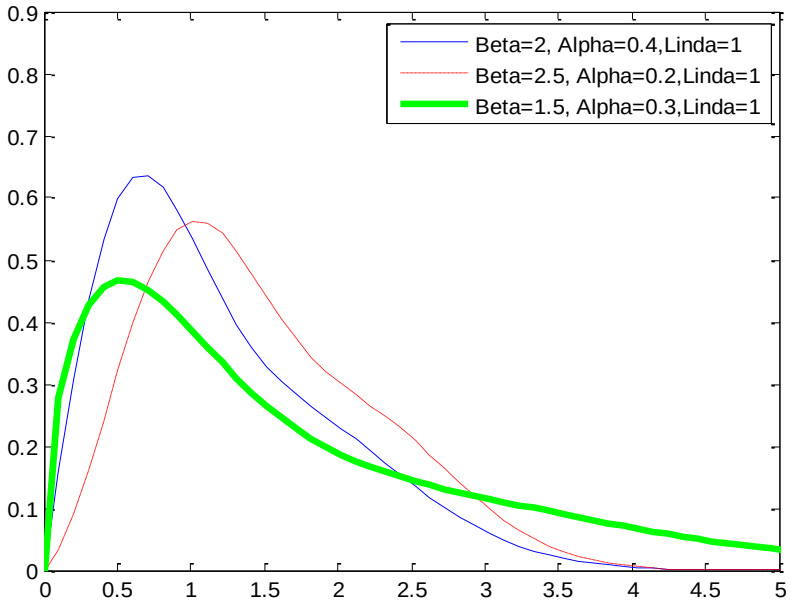

Figure 2 The pdf of the $CTWD$ at $\alpha = 0.4, 0.2, 0.3, \beta = 2, 2.5, 1.5, \lambda = 1$

And from figure 2, it is clear that the function $f_{CTW}(.)$ has one or two peaks at increasing x and at different values of the parameters. It has heavy tail skewed at the right. It has the shape of Weibull distribution at $\alpha = 0.3, \beta = 1.5, \lambda = 1$.

2.4 Special Cases

In the following table there are some cases (sub-models) from $CTWD$

Table 1 The Cubic Transmuted Weibull distribution and some of its sub-models

model	α	β	λ	Cumulative distribution function
WD	_	_	0	$\left(1 - e^{-\alpha x^{\beta}}\right)$
RD	_	2	0	$\left(1 - e^{-\alpha x^{2}}\right)$
ED	_	1	0	$(1 - e^{-\alpha x})$
CTWD	_	_	_	$(1 - e^{-\alpha x^{\beta}})\left[1 + \lambda e^{-2\alpha x^{\beta}}\right]$
CTRD	_	2	_	$(1 - e^{-\alpha x^{2}})\left[1 + \lambda e^{-2\alpha x^{2}}\right]$
CTED	_	1	_	$(1 - e^{-\alpha x})[1 + \lambda e^{-2\alpha x}]$

Where C= Cubic, T= Transmuted, W= Weibull, E = Exponential,R = Rayleigh.

2.5 Reliability Analysis

i. The Reliability Function (RF)

The reliability function (or survival function) of $CTWD$ is defined as

$$R_{CTWD}(x;\alpha,\beta\,\lambda) = e^{-\alpha x^{\beta}} - \lambda e^{-2\alpha x^{\beta}} + \lambda e^{-3\alpha x^{\beta}} \quad (6)$$

ii. The Hazard Rate Function (HF)

The hazard rate function of $CTWD$ is

$$h_{CTWD}(x;\alpha,\beta,\lambda) = \frac{\alpha\beta x^{\beta-1}\left[1 - 2\lambda\, e^{-\alpha x^{\beta}} + 3\lambda e^{-2\alpha x^{\beta}}\right]}{1 - \lambda e^{-\alpha x^{\beta}} + \lambda e^{-2\alpha x^{\beta}}} \quad (7)$$

Where $h_{CTWD}(x)$ is defined as the probability of failure per unit of time, distance or cycles. Some possible shape of $R_{CTWD}(x;\alpha,\beta\,\lambda)$ and $h_{CTWD}(x;\alpha,\beta,\lambda)$ with parameter values selected are shown in figure 3 and figure 4 respectively

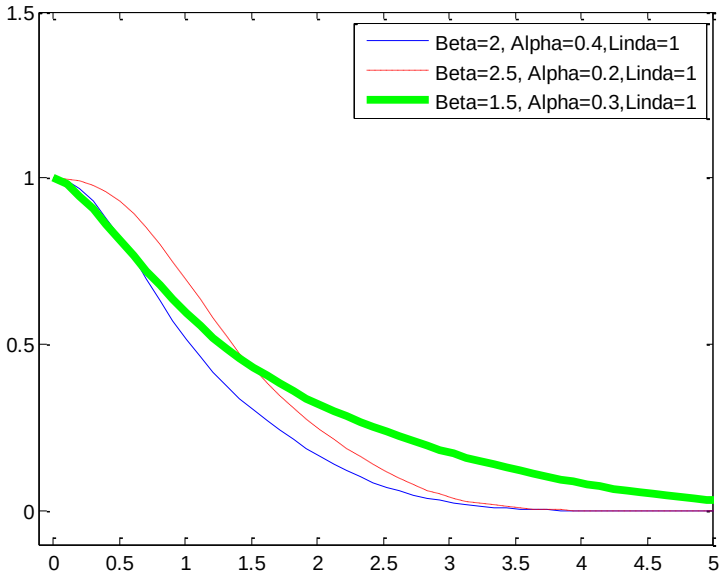

Figure3: The reliability function of the $CTWD$ at $\alpha = 0.4, 0.2, 0.3, \beta = 2, 2.5, 1.5, \lambda = 1$.

From figure 2, it is shown that the reliability function is decreasing with the increasing of x and different values of α, β and fixed value of λ.

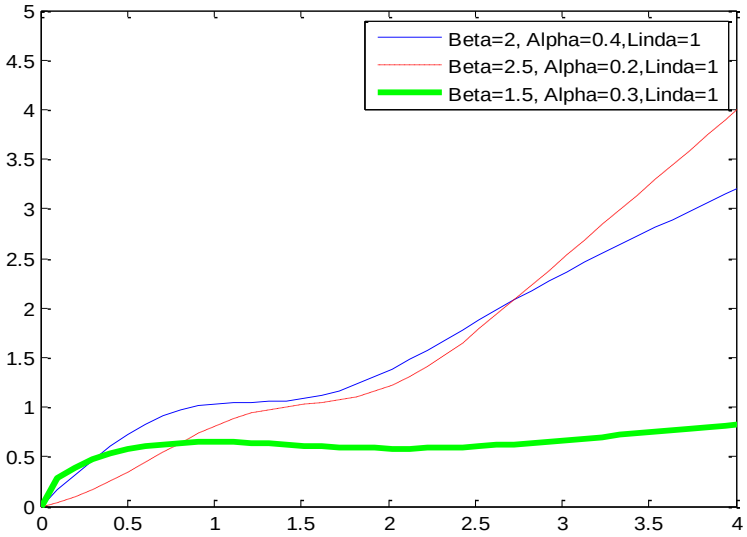

Figure4 The hazard rate function of the $CTWD$ at $\alpha = 0.4, 0.2, 0.3, \beta = 2, 2.5, 1.5, \lambda = 1$.

figure 4, it is shown that the hazard rate function is slowly increasing at increasing of x and different values of α, β and fixed value of λ.

Theorem 2.6 The hazard rate function of the cubic transmuted Weibull distribution has the following special cases:

1. If $\beta = 1$, then the failure rate is same as the hazard rate function of $CTED(x, \alpha, \lambda)$
2. If $\beta = 2$, then the failure rate is same as the hazard rate function of $CTRD(x; \alpha, \lambda)$
3. If $\lambda = 0$, then the failure rate is same as the hazard rate function of $WD(x; \alpha, \beta)$
4. If $\lambda = 0, \beta = 1$, then the failure rate is same as the hazard rate function of $ED(x; \alpha)$
5. If $\lambda = 0, \beta = 2$, then the failure rate is same as the hazard rate function of $RD(x; \alpha)$

Proof

If we substitute the parameters chosen in hazard function (HF) of the CTWD , equation (7), we get the following table:

Table 2 Hazard rate function for each of the parameters chosen

Parameters Chosen	Hazard Rate Function
β = 1	$h_{CTED}(x;\alpha,\lambda)=\frac{\alpha\left[1-2\lambda e^{-\alpha x}+3\lambda e^{-2\alpha x}\right]}{1-\lambda e^{-\alpha x}+\lambda e^{-2\alpha x}}$
β = 2	$h_{CTRD}(x;\alpha,\lambda)=\frac{2\alpha x\left[1-2\lambda e^{-\alpha x^2}+3\lambda e^{-2\alpha x^2}\right]}{1-\lambda e^{-\alpha x^2}+\lambda e^{-2\alpha x^2}}$
λ = 0	$h_{WD}(x;\alpha,\beta)=\alpha\beta x^{\beta-1}$
λ = 0,β = 1	$h_{ED}(x;\alpha)=\alpha$
λ = 0,β = 2	$h_{RD}(x;\alpha)=2\alpha$

iii. The Cumulative Hazard Rate Function (CHF)

The cumulative hazard function of $CTWD(x;\alpha,\beta,\lambda)$, is defined as

$$= -ln\,[1-F_{CTWD}(x;\alpha,\beta,\lambda)]$$

$$= -ln\left[e^{-\alpha x^{\beta}}-\lambda e^{-2\alpha x^{\beta}}+\lambda e^{-3\alpha x^{\beta}}\right] \quad (8)$$

Theorem 2.7 The cumulative hazard rate function of the cubic transmuted Weibull distribution $CTWD(x,\alpha,\beta,\lambda)$ has the following special cases:

1. When $\beta=1$, the cumulative failure rate is same as the $CTED(x,\alpha,\beta,\lambda)$
2. When $\beta=2$, the cumulative failure rate is same as the $CTRD(x,\alpha,\beta,\lambda)$
3. When λ = 0 the cumulative failure rate is same as the $WD(x;\alpha,\beta)$
4. When $\lambda=0,\beta=1$ the cumulative failure rate is same as the $ED(x;\alpha,\beta)$
5. When $\lambda=0,\beta=2$ the cumulative failure rate is same as the $RD(x;\alpha,\beta)$

Proof : Using the cumulative hazard function (CHF) of the CTWD , equation (8),then we get the special cases as in the following table

Table 3 Cumulative Hazard Function for each of the parameters chosen

Parameters Chosen	Cumulative Hazard Function
$\beta=1$	$H_{CTED}(x,\alpha,\beta,\lambda)=-ln\,[e^{-\alpha x}-\lambda e^{-2\alpha x}+\lambda e^{-3\alpha x}]$
$\beta=2$	$H_{CTRD}(x,\alpha,\beta,\lambda)=-ln\left[e^{-\alpha x^2}-\lambda e^{-2\alpha x^2}+e^{-3\alpha x^2}\right]$
λ = 0	$H_{WD}(x;\alpha,\beta)=-\ln\left[e^{-\alpha x^{\beta}}\right]=\alpha x^{\beta}$
$\lambda=0,\beta=1$	$H_{ED}(x;\alpha,\beta)=-\ln[e^{-\alpha x}]=\alpha x$
$\lambda=0,\beta=2$	$H_{RD}(x;\alpha,\beta)=-\ln\left[e^{-\alpha x^2}\right]=\alpha x^2$

2.8 Mode and maiden:

i. The mode

The mode of a probability distribution is given as:

$$\text{Mode} = \frac{\partial f_{TW}(x;\alpha,\beta,\lambda)}{\partial x} = 0$$

$$ln\, f_{TW}(x;\alpha,\beta,\lambda) = ln\left(\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\left(1 - 2\lambda\, e^{-\alpha x^{\beta}} + 3\lambda e^{-2\alpha x^{\beta}}\right)\right)$$

$$= \ln\alpha + ln\,\beta + (\beta-1)\,ln\,x - \alpha x^{\beta} + \ln\left(1 - 2\lambda\, e^{-\alpha x^{\beta}} + 3\lambda e^{-2\alpha x^{\beta}}\right)$$

The mode is

$$\frac{\partial f_{TW}(x;\alpha,\beta,\lambda)}{\partial x} = \frac{(\beta-1)}{x} - \alpha\beta x^{\beta-1} - \frac{2\lambda\alpha\beta x^{\beta-1}\, e^{-\alpha x^{\beta}}\left(1+3e^{-\alpha x^{\beta}}\right)}{\left(1-2\lambda\, e^{-\alpha x^{\beta}}+3\lambda e^{-2\alpha x^{\beta}}\right)}$$

By using Newton Raphson Method we get $x_{mode} = 0.530861$

ii. The Median

The median of a $CTWD$ with shape parameter β and scale parameter λ is

$$\frac{1}{2} = \int_0^m f_{tw}(x)dx = \left[(1 - e^{-\alpha x^{\beta}})\left(1 + \lambda e^{-2\alpha x^{\beta}}\right)\right]_0^m$$

$$\frac{1}{2} = (1 - e^{-\alpha m^{\beta}})\left(1 + \lambda e^{-2\alpha m^{\beta}}\right)$$

$$ln\,\frac{1}{2} = ln\left(1 - e^{-\alpha m^{\beta}}\right) + ln\left(1 + \lambda e^{-2\alpha m^{\beta}}\right)$$

$$-ln2 = ln\left(1 - e^{-\alpha m^{\beta}}\right) + ln\left(1 + \lambda e^{-2\alpha m^{\beta}}\right)$$

By using Newton Raphson Method we get $x_{medain} = 0.973485$.

2.9 Moment and Moment Generated Function:

i.Moment

Proposition 1

If X has $C\,TWD$ with $|\lambda| \leq 1$, then the rth moment of X about the origin is

$$\mathrm{E}(X^r) = \frac{1}{\alpha^{\frac{r}{\beta}}}\Gamma(1+\frac{r}{\beta})\left[1 - \frac{\lambda}{2^{\frac{r}{\beta}}} + \frac{\lambda}{3^{\frac{r}{\beta}}}\right] \quad (9)$$

And then mean and the variance of $CTWD$ are defined as

$$E(X) = \frac{1}{\alpha^{\frac{1}{\beta}}}\Gamma(1+\frac{1}{\beta})\left(1 - \frac{\lambda}{2^{\frac{1}{\beta}}} + \frac{\lambda}{3^{\frac{1}{\beta}}}\right) \quad (10)$$

$$V(X) = \frac{1}{\alpha^{\frac{2}{\beta}}}\left[\Gamma(1+\frac{2}{\beta})\left(1 - \frac{\lambda}{2^{\frac{2}{\beta}}} + \frac{\lambda}{3^{\frac{2}{\beta}}}\right) - \Gamma^2(1+\frac{1}{\beta})^2\left(1 - \frac{\lambda}{2^{\frac{1}{\beta}}} + \frac{\lambda}{3^{\frac{1}{\beta}}}\right)^2\right] \quad (11)$$

Proof

$$\mathrm{E}(X^r) = \int_0^{\infty} x^r\, \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\left(1 - 2\lambda\, e^{-\alpha x^{\beta}} + 3\lambda e^{-2\alpha x^{\beta}}\right)dx$$

$$Let\, u = \alpha x^{\beta} \xrightarrow{yields} x^{\beta} = \frac{u}{\alpha} \rightarrow x = \left(\frac{u}{\alpha}\right)^{\frac{1}{\beta}} \quad , du = \alpha\beta x^{\beta-1}\,dx$$

$$\mathrm{I} = \frac{1}{\alpha^{\frac{r}{\beta}}}\int_0^{\infty} u^{\frac{r}{\beta}}e^{-u}du - 2\lambda\frac{1}{\alpha^{\frac{r}{\beta}}}\int_0^{\infty} u^{\frac{r}{\beta}}e^{-2u}\,du + 3\lambda\frac{1}{\alpha^{\frac{r}{\beta}}}\int_0^{\infty} u^{\frac{r}{\beta}}e^{-3u}\,du$$

Let $y = 2u \rightarrow u = \frac{y}{2}$, $du = \frac{dy}{2}$ and $y_1 = 3u \rightarrow u = \frac{y_1}{3}$, $du = \frac{dy_1}{3}$

$$\mathrm{E}(X^r) = \frac{1}{\alpha^{\frac{r}{\beta}}}\Gamma(1+\frac{r}{\beta})\left(1-\frac{\lambda}{2^{\frac{r}{\beta}}}+\frac{\lambda}{3^{\frac{r}{\beta}}}\right)$$

Using (9), then when r=1, we get the mean

$$E(X) = \frac{1}{\alpha^{\frac{1}{\beta}}}\Gamma(1+\frac{1}{\beta})\left(1-\frac{\lambda}{2^{\frac{1}{\beta}}}+\frac{\lambda}{3^{\frac{1}{\beta}}}\right)$$

And we need $E(X^2)$ to find the variance

$$E(X^2) = \frac{1}{\alpha^{\frac{2}{\beta}}}\Gamma(1+\frac{2}{\beta})\left(1-\frac{\lambda}{2^{\frac{2}{\beta}}}+\frac{\lambda}{3^{\frac{2}{\beta}}}\right)$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{1}{\alpha^{\frac{2}{\beta}}}\left[\Gamma(1+\frac{2}{\beta})\left(1-\frac{\lambda}{2^{\frac{2}{\beta}}}+\frac{\lambda}{3^{\frac{2}{\beta}}}\right)-\Gamma^2(1+\frac{1}{\beta})^2\left(1-\frac{\lambda}{2^{\frac{1}{\beta}}}+\frac{\lambda}{3^{\frac{1}{\beta}}}\right)^2\right]$$

■

Proposition 2

If X has $CTWD(\alpha,\beta,\boldsymbol{\lambda})$ with$|\lambda| \leq 1$, then the rth central moment of X about the mean is

$$\mathrm{E}(X-\mu)^r = \sum_{j=0}^{r} \mathrm{C}_j^r \left(\frac{1}{\beta\sqrt{\alpha}}\right)^j (-\mu)^{r-j}\Gamma\left(1+\frac{r}{\beta}\right)\left[1-\frac{\lambda}{2^{\frac{r}{\beta}}}+\frac{\lambda}{3^{\frac{r}{\beta}}}\right]$$

(13)

And then the coefficients of variation CV, skewness CS, kurtosis CK , and of kurtosis of the $CTWD(\alpha,\beta,\lambda)$ are respectively as

$$CV. = \frac{\sqrt{\Gamma(1+\frac{2}{\beta})\left(1-\frac{\lambda}{2^{\frac{2}{\beta}}}+\frac{\lambda}{3^{\frac{2}{\beta}}}\right)-\Gamma^2(1+\frac{1}{\beta})^2\left(1-\frac{\lambda}{2^{\frac{1}{\beta}}}+\frac{\lambda}{3^{\frac{1}{\beta}}}\right)^2}}{\Gamma(1+\frac{1}{\beta})\left(1-\frac{\lambda}{2^{\frac{1}{\beta}}}+\frac{\lambda}{3^{\frac{1}{\beta}}}\right)}$$

(14)

$$CS = \frac{\left[\Gamma\left(1+\frac{3}{\beta}\right)\left(1-\frac{\lambda}{2^{\frac{3}{\beta}}}+\frac{\lambda}{3^{\frac{3}{\beta}}}\right)-3\Gamma\left(1+\frac{2}{\beta}\right)\left(1-\frac{\lambda}{2^{\frac{2}{\beta}}}+\frac{\lambda}{3^{\frac{2}{\beta}}}\right)\Gamma(1+\frac{1}{\beta})\left(1-\frac{\lambda}{2^{\frac{1}{\beta}}}+\frac{\lambda}{3^{\frac{1}{\beta}}}\right)+4\Gamma^3\left(1+\frac{1}{\beta}\right)^3\left(1-\frac{\lambda}{2^{\frac{1}{\beta}}}+\frac{\lambda}{3^{\frac{1}{\beta}}}\right)^3\right]}{\frac{1}{\alpha^{\frac{2}{\beta}}}\left[\Gamma(1+\frac{2}{\beta})\left(1-\frac{\lambda}{2^{\frac{2}{\beta}}}+\frac{\lambda}{3^{\frac{2}{\beta}}}\right)-\Gamma^2(1+\frac{1}{\beta})^2\left(1-\frac{\lambda}{2^{\frac{1}{\beta}}}+\frac{\lambda}{3^{\frac{1}{\beta}}}\right)^2\right]^3}$$

(15)

$$CK = \frac{\left[\Gamma(1+\frac{4}{\beta})\,\omega_4 - 4\,\Gamma(1+\frac{1}{\beta})\omega_1\Gamma(1+\frac{3}{\beta})\,\omega_2 + 6\Gamma^2\left(1+\frac{1}{\beta}\right)^2\,\omega_1{}^2\Gamma(1+\frac{2}{\beta})\,\omega_2 - 3\,\Gamma^4(1+\frac{1}{\beta})^4\,\omega_1{}^4\right]}{\left[\Gamma(1+\frac{2}{\beta})\,\omega_2 - \Gamma^2(1+\frac{1}{\beta})^2\,\omega_1{}^2\right]^2} - 3$$

(16)

$$\omega_1 = \left(1-\frac{\lambda}{2^{\frac{1}{\beta}}}+\frac{\lambda}{3^{\frac{1}{\beta}}}\right),\ \omega_2 = \left(1-\frac{\lambda}{2^{\frac{2}{\beta}}}+\frac{\lambda}{3^{\frac{2}{\beta}}}\right), \omega_3 = \left(1-\frac{\lambda}{2^{\frac{3}{\beta}}}+\frac{\lambda}{3^{\frac{3}{\beta}}}1\right),$$

$$\omega_4 = \left(1-\frac{\lambda}{2^{\frac{4}{\beta}}}+\frac{\lambda}{3^{\frac{4}{\beta}}}\right)$$

Proof

$$\mathrm{E}(X-\mu)^r = \int_0^\infty (x-\mu)^r \, \mathrm{f}(x;\alpha,\beta,\lambda)dx$$

$$= \int_0^\infty (x-\mu)^r \, \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} dx - 2\lambda \int_0^\infty (x-\mu)^r \, \alpha\beta x^{\beta-1} e^{-2\alpha x^\beta} dx + 3\lambda \int_0^\infty (x-\mu)^r \, \alpha\beta x^{\beta-1} e^{-3\alpha x^\beta} dx$$

$$Let\ u = \alpha x^\beta \xrightarrow{yields} x^\beta = \frac{u}{\alpha} \rightarrow x = \left(\frac{u}{\alpha}\right)^{\frac{1}{\beta}}, \quad du = \alpha\beta x^{\beta-1} dx$$

$$= \int_0^\infty ((\frac{u}{\alpha})^{\frac{1}{\beta}} - \mu)^r \, e^{-u} du - 2\lambda \int_0^\infty ((\frac{u}{\alpha})^{\frac{1}{\beta}} - \mu)^r \, e^{-2u} du + 3\lambda \int_0^\infty ((\frac{u}{\alpha})^{\frac{1}{\beta}} - \mu)^r \, e^{-3u} du$$

By using the "Binomial Theorem" we get

$$= \sum_{j=0}^r \mathrm{C}_j^r \, (\frac{1}{\beta\sqrt{\alpha}})^{r-j} (-\mu)^r \left[\int_0^\infty u^{\frac{r}{\beta}} e^{-u} du - 2\lambda \int_0^\infty u^{\frac{r}{\beta}} e^{-u} du + 3 \int_0^\infty u^{\frac{r}{\beta}} e^{-2u} du \right]^{r-j}$$

$$let\ y = 2u \rightarrow u = \frac{y}{2}, du = \frac{dy}{2}\ and\ y_1 = 3u \rightarrow u = \frac{y_1}{3}, du = \frac{dy_1}{3}$$

$$= \sum_{j=0}^r \mathrm{C}_j^r \, (\frac{1}{\beta\sqrt{\alpha}})^{r-j} (-\mu)^r \Gamma\left(1+\frac{r}{\beta}\right) \left[1 - \frac{\lambda}{2^{\frac{r}{\beta}}} + \frac{\lambda}{3^{\frac{r}{\beta}}} \right]$$

Now when $r = 2$, we get

$$E(X-\mu)^2 =$$

$$= \frac{1}{\alpha^{\frac{2}{\beta}}} \left[\Gamma(1+\frac{2}{\beta}) \left(1 - \frac{\lambda}{2^{\frac{2}{\beta}}} + \frac{\lambda}{3^{\frac{2}{\beta}}} \right) - \Gamma^2(1+\frac{1}{\beta})^2 \left(1 - \frac{\lambda}{2^{\frac{1}{\beta}}} + \frac{\lambda}{3^{\frac{1}{\beta}}} \right)^2 \right]$$

When $r = 3$

$$E(X-\mu)^3 = E(x^3 - 3x^2\mu + 3x\mu^2 + \mu^3)$$

$$= \frac{1}{\alpha^{\frac{3}{\beta}}} \left[\Gamma\left(1+\frac{3}{\beta}\right) \left(1 - \frac{\lambda}{2^{\frac{3}{\beta}}} + \frac{\lambda}{3^{\frac{3}{\beta}}} \right) - 3\Gamma\left(1+\frac{2}{\beta}\right) \left(1 - \frac{\lambda}{2^{\frac{2}{\beta}}} + \frac{\lambda}{3^{\frac{2}{\beta}}} \right) \Gamma(1+\frac{1}{\beta}) \left(1 - \frac{\lambda}{2^{\frac{1}{\beta}}} + \frac{\lambda}{3^{\frac{1}{\beta}}} \right) + 4\Gamma^3 \left(1+\frac{1}{\beta}\right)^3 \left(1 - \frac{\lambda}{2^{\frac{1}{\beta}}} + \frac{\lambda}{3^{\frac{1}{\beta}}} \right)^3 \right] \quad (17)$$

When $r = 4$

$$E(X-\mu)^4 = E(x^4 - 4\mu x^3 + 6\mu^2 x^2 - 4\mu^3 x + \mu^4)$$

$$= \frac{1}{\alpha^{\frac{4}{\beta}}} \left[\Gamma\left(1+\frac{4}{\beta}\right) \left(1 - \frac{\lambda}{2^{\frac{4}{\beta}}} + \frac{\lambda}{3^{\frac{4}{\beta}}} \right) - 4\,\Gamma\left(1+\frac{1}{\beta}\right) \left(1 - \frac{\lambda}{2^{\frac{1}{\beta}}} + \frac{\lambda}{3^{\frac{1}{\beta}}} \right) \Gamma\left(1+\frac{3}{\beta}\right) \left(1 - \frac{\lambda}{2^{\frac{3}{\beta}}} + \frac{\lambda}{3^{\frac{3}{\beta}}} \right) + 6\Gamma^2 \left(1+\frac{1}{\beta}\right)^2 \left(1 - \frac{\lambda}{2^{\frac{1}{\beta}}} + \frac{\lambda}{3^{\frac{1}{\beta}}} \right)^2 \Gamma\left(1+\frac{2}{\beta}\right) \left(1 - \frac{\lambda}{2^{\frac{2}{\beta}}} + \frac{\lambda}{3^{\frac{2}{\beta}}} \right) - 3\ \Gamma^4(1+\frac{1}{\beta})^4 \left(1 - \frac{\lambda}{2^{\frac{1}{\beta}}} + \frac{\lambda}{3^{\frac{1}{\beta}}} \right)^4 \right] \quad (18)$$

Then CV= $\frac{\sigma}{\mu}$

$$= \frac{\sqrt{\Gamma(1+\frac{2}{\beta}) \left(1 - \frac{\lambda}{2^{\frac{2}{\beta}}} + \frac{\lambda}{3^{\frac{2}{\beta}}} \right) - \Gamma^2(1+\frac{1}{\beta})^2 \left(1 - \frac{\lambda}{2^{\frac{1}{\beta}}} + \frac{\lambda}{3^{\frac{1}{\beta}}} \right)^2}}{\Gamma(1+\frac{1}{\beta}) \left(1 - \frac{\lambda}{2^{\frac{1}{\beta}}} + \frac{\lambda}{3^{\frac{1}{\beta}}} \right)} \quad CS = \frac{E(x-\mu)^3}{\sigma^3}$$

$$=\frac{\left[\Gamma\left(1+\frac{3}{\beta}\right)\left(1-\frac{\lambda}{2^{\frac{3}{\beta}}}+\frac{\lambda}{3^{\frac{3}{\beta}}}\right)-3\Gamma\left(1+\frac{2}{\beta}\right)\left(1-\frac{\lambda}{2^{\frac{2}{\beta}}}+\frac{\lambda}{3^{\frac{2}{\beta}}}\right)\Gamma(1+\frac{1}{\beta})\left(1-\frac{\lambda}{2^{\frac{1}{\beta}}}+\frac{\lambda}{3^{\frac{1}{\beta}}}\right)+4\Gamma^3\left(1+\frac{1}{\beta}\right)^3\left(1-\frac{\lambda}{2^{\frac{1}{\beta}}}+\frac{\lambda}{3^{\frac{1}{\beta}}}\right)^3\right]}{\alpha^{\frac{1}{\frac{2}{\beta}}}\left[\Gamma(1+\frac{2}{\beta})\left(1-\frac{\lambda}{2^{\frac{2}{\beta}}}+\frac{\lambda}{3^{\frac{2}{\beta}}}\right)-\Gamma^2(1+\frac{1}{\beta})^2\left(1-\frac{\lambda}{2^{\frac{1}{\beta}}}+\frac{\lambda}{3^{\frac{1}{\beta}}}\right)^2\right]^3}$$

$$CK=\frac{E(x-\mu)^4}{\sigma^4}-3$$

$$=\frac{\left[\Gamma(1+\frac{4}{\beta})\,\omega_4-4\,\Gamma(1+\frac{1}{\beta})\omega_1\Gamma(1+\frac{3}{\beta})\,\omega_2+6\Gamma^2\left(1+\frac{1}{\beta}\right)^2\,{\omega_1}^2\Gamma(1+\frac{2}{\beta})\,\omega_2-3\;\Gamma^4(1+\frac{1}{\beta})^4\,{\omega_1}^4\right]}{\left[\Gamma(1+\frac{2}{\beta})\,\omega_2-\Gamma^2(1+\frac{1}{\beta})^2\,{\omega_1}^2\right]^2}$$

■

ii. Moment Generating Function

Proposition 3

If X is a random variable that has the $CTWD\ (x;\alpha,\beta,\lambda)$ with $|\lambda| \le 1$, then the moment generating function of X is

$$M_x(t)= \sum_{r=0}^{k}\frac{\left(\frac{t}{\alpha^{\frac{1}{\beta}}}\right)^r}{r!}\,\Gamma\left(1+\frac{r}{\beta}\right)\left[1-\frac{\lambda}{2^{\frac{r}{\beta}}}+\frac{\lambda}{2^{\frac{3}{\beta}}}\right] \quad (20)$$

Proof

$$M_x(t)=\int_0^\infty e^{tx}f_{TW(x;\alpha,\beta,\lambda)}\,dx$$

$$=\int_0^\infty \sum_{r=0}^{k}\frac{(tx)^r}{r!}f_{TW(x;\alpha,\beta,\lambda)}\,dx$$

where $e^{tx}=\sum_{r=0}^{k}\frac{(tx)^r}{r!}$

$$=\alpha\beta\int_0^\infty\sum_{r=0}^{k}\frac{t^r(x)^{r+\beta-1}}{r!}e^{-\alpha x^\beta}dx-2\lambda\,\alpha\beta\int_0^\infty\sum_{r=0}^{k}\frac{t^r(x)^{r+\beta-1}}{r!}e^{-2\alpha x^\beta}\,dx+$$

$$3\lambda\alpha\beta\int_0^\infty\sum_{r=0}^{k}\frac{t^r(x)^{r+\beta-1}}{r!}e^{-3\alpha x^\beta}\,dx$$

Let $u=\alpha x^\beta \xrightarrow{yields} x^\beta=\frac{u}{\alpha} \rightarrow x=\left(\frac{u}{\alpha}\right)^{\frac{1}{\beta}} \rightarrow dx=\frac{1}{\alpha\beta}\left(\frac{u}{\alpha}\right)^{\frac{1}{\beta}-1}du$

Now compensation values above we get

$$M_x(t)=\alpha^{-\frac{r}{\beta}}\int_0^\infty\sum_{r=0}^{k}\frac{t^r u^{\frac{r}{\beta}}}{r!}e^{-u}\,du-2\lambda\alpha^{-\frac{r}{\beta}}\int_0^\infty\sum_{r=0}^{k}\frac{t^r u^{\frac{r}{\beta}}}{r!}e^{-2u}\,du+$$

$$3\lambda\alpha^{-\frac{r}{\beta}}\int_0^\infty\sum_{r=0}^{k}\frac{t^r u^{\frac{r}{\beta}}}{r!}e^{-3u}\,du$$

$$=\sum_{r=0}^{k}\frac{\left(\frac{t}{\alpha^{\frac{1}{\beta}}}\right)^r}{r!}\left[\int_0^\infty u^{\frac{r}{\beta}}e^{-u}\,du-2\lambda\int_0^\infty u^{\frac{r}{\beta}}e^{-2u}\,du+3\lambda\int_0^\infty u^{\frac{r}{\beta}}e^{-3u}\,du\right]$$

$$=\sum_{r=0}^{k}\frac{\left(\frac{t}{\alpha^{\frac{1}{\beta}}}\right)^r}{r!}\,\Gamma\left(1+\frac{r}{\beta}\right)\left[1-\frac{\lambda}{2^{\frac{r}{\beta}}}+\frac{\lambda}{2^{\frac{3}{\beta}}}\right]$$

■

2.10 Order Statistic:

Let $X_1, \dots, X_n$ be denote random samples from a $CTWD$ distribution, then the pdf of the rth order statistic is given by

$$f_{r,n}(x) = \frac{n!}{(i-1)!(n-i)!}\ \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\left(1\ -2\lambda\ e^{-\alpha x^{\beta}}+3\lambda e^{-2\alpha x^{\beta}}\right)\left[(1-\right.$$
$$\left.e^{-\alpha x^{\beta}}\right)\left(1+\ \lambda e^{-2\alpha x^{\beta}}\right)\Big]^{i-1}\left[1-\left[(1-\ e^{-\alpha x^{\beta}})\left(1+\ \lambda e^{-2\alpha x^{\beta}}\right)\right]\right]^{n-i}$$
$$=\frac{n!}{(i-1)!(n-i)!}\ \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\left[1\ -2\lambda\ e^{-\alpha x^{\beta}}+3\lambda e^{-2\alpha x^{\beta}}\right]\left[1-\ e^{-\alpha x^{\beta}}+\right.$$
$$\left.\lambda e^{-2\alpha x^{\beta}}-\ \lambda e^{-3\alpha x^{\beta}}\right]^{i-1}\left[\ e^{-\alpha x^{\beta}}-\ \lambda e^{-2\alpha x^{\beta}}+\ \lambda e^{-3\alpha x^{\beta}}\right]^{n-i}$$

(21)

Then the pdfs of the minimum, the maximum and the median are as

1) When i=1we have the pdf of the minimum

$$f_{1,n}(x) = n\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\left[1\ -2\lambda\ e^{-\alpha x^{\beta}}+3\lambda e^{-2\alpha x^{\beta}}\right]\left[\ e^{-\alpha x^{\beta}}-\ \lambda e^{-2\alpha x^{\beta}}+\right.$$
$$\left.\lambda e^{-3\alpha x^{\beta}}\right]^{n-1}$$

(22)

2) When i=n we have the pdf of the maximum

$$f_{n,n}(x) =\ n\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\left(1\ -2\lambda\ e^{-\alpha x^{\beta}}+3\lambda e^{-2\alpha x^{\beta}}\right)\left(1-\ e^{-\alpha x^{\beta}}+\right.$$
$$\left.\lambda e^{-2\alpha x^{\beta}}-\ \lambda e^{-3\alpha x^{\beta}}\right)^{n-1}$$

(23)

3) When $i = m + 1$we have the pdf of the median

$$f_{m+1,n}(x) = \frac{n!}{m!(n-m-1)!}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\left(1\ -2\lambda\ e^{-\alpha x^{\beta}}+3\lambda e^{-2\alpha x^{\beta}}\right)\left(1-\right.$$
$$\left.e^{-\alpha x^{\beta}}+\ \lambda e^{-2\alpha x^{\beta}}-\ \lambda e^{-3\alpha x^{\beta}}\right)^{m}\left(\ e^{-\alpha x^{\beta}}-\ \lambda e^{-2\alpha x^{\beta}}+\ \lambda e^{-3\alpha x^{\beta}}\right)^{n-m-1}$$

(24)

Theorem 2.11 Let $X_1, X_2, \dots, X_n$, are independently identically distributed ordered random variables from the cubic transmuted Weibull distribution having the minimum (first) order , the maximum (nth) order and the median $(m + 1)$, the probability density function have the same as of some distribution as follows:

1. $\beta = 1$, the order is same as of $CTED(x, \alpha, \lambda)$
2. $\beta = 2$, the order is same as of $CTRD(x, \alpha, \beta, \lambda)$
3. $\lambda = 0$ the order is same as of $WD(x; \alpha, \beta)$
4. $\lambda = 0, \beta = 1$ the order is same as of $ED(x; \alpha, \beta)$
5. $\lambda = 0, \beta = 2$ the order is same as of $RD(x; \alpha, \beta)$

Proof:

a) The pdf of the minimum when

1. $\beta = 1$ is

$$f_{1,n}(x) = n\alpha e^{-\alpha x}[1\ -2\lambda\ e^{-\alpha x}+3\lambda e^{-2\alpha x}][\ e^{-\alpha x}-\ \lambda e^{-2\alpha x}+\ \lambda e^{-3\alpha x}]^{n-1}$$

(25)

2. $\beta = 2$ is

$$f_{1,n}(x) = 2n\alpha x^2 e^{-\alpha x^2}\left[1 - 2\lambda\, e^{-\alpha x^2} + 3\lambda e^{-2\alpha x^2}\right]\left[e^{-\alpha x^2} - \lambda e^{-2\alpha x^2} + \lambda e^{-3\alpha x^2}\right]^{n-1} \quad (26)$$

3. $\lambda = 0$

$$f_{1,n}(x) = n\alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} \quad (27)$$

4. $\lambda = 0, \beta = 1$ is

$$f_{1,n}(x) = n\alpha e^{-\alpha x} \quad (28)$$

5. $\lambda = 0, \beta = 2$ is

$$f_{1,n}(x) = 2n\alpha x e^{-\alpha x^2} \quad (29)$$

b) The pdf of the maximum when

1. $\beta = 1$,is

$$f_{n,n}(x) = n\alpha e^{-\alpha x}(1 - 2\lambda\, e^{-\alpha x} + 3\lambda e^{-2\alpha x})(1 - e^{-\alpha x} + \lambda e^{-2\alpha x} - \lambda e^{-3\alpha x})^{n-1} \quad (30)$$

2. $\beta = 2$, is

$$f_{n,n}(x) = 2n\alpha x e^{-\alpha x^2}\left(1 - 2\lambda\, e^{-\alpha x^2} + 3\lambda e^{-2\alpha x^2}\right)\left(1 - e^{-\alpha x^2} + \lambda e^{-2\alpha x^2} - \lambda e^{-3\alpha x^2}\right)^{n-1} \quad (31)$$

3. $\lambda = 0$,is

$$f_{n,n}(x) = n\alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} \quad (32)$$

$\lambda = 0, \beta = 1$,is

$$f_{n,n}(x) = n\alpha e^{-\alpha x} \quad (33)$$

4. $\lambda = 0, \beta = 2$, is

$$f_{n,n}(x) = 2n\alpha x e^{-\alpha x^2} \quad (34)$$

c) The pdf of the median when

1. $\beta = 1$, is

$$f_{m+1,n}(x) = \frac{n!}{m!(n-m-1)!}\alpha e^{-\alpha x}(1 - 2\lambda\, e^{-\alpha x} + 3\lambda e^{-2\alpha x})(1 - e^{-\alpha x} + \lambda e^{-2\alpha x} - \lambda e^{-3\alpha x})^{m}(e^{-\alpha x} - \lambda e^{-2\alpha x} + \lambda e^{-3\alpha x})^{n-m-1} \quad (35)$$

2. $\beta = 2$, is

$$f_{m+1,n}(x) = \frac{n!}{m!(n-m-1)!} 2\alpha x e^{-\alpha x^2}\left(1 - 2\lambda\, e^{-\alpha x^2} + 3\lambda e^{-2\alpha x^2}\right)\left(1 - e^{-\alpha x^2} + \lambda e^{-2\alpha x^2} - \lambda e^{-3\alpha x^2}\right)^{m}\left(e^{-\alpha x^2} - \lambda e^{-2\alpha x^2} + \lambda e^{-3\alpha x^2}\right)^{n-m-1} \quad (36)$$

3. $\lambda = 0$, is

$$f_{m+1,n}(x) = \frac{n!}{m!(n-m-1)!}\alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} \quad (37)$$

4. $\lambda = 0, \beta = 1$, is

$$f_{m+1,n}(x) = \frac{n!}{m!(n-m-1)!}\alpha e^{-\alpha x} \quad (38)$$

5. $\lambda = 0, \beta = 2$,is

$$f_{m+1,n}(x) = \frac{n!}{m!(n-m-1)!} 2\alpha x e^{-\alpha x^2} \quad (39)$$

Conclusions

We conclude that

1) The pdf of the $CTWD$ has heavy tail skewed at the right. It has the shape of Weibull distribution at $\alpha = 0.3, \beta = 1.5, \lambda = 1$.
2) We can obtain W, E, R, CTW, CTE, CTR distributions from cubic transmuted Weibull distribution at certain assumptions of the parameters α, β, λ, as in Table 1,for example $\lambda = 0$ to get W distribution.
3) If we substitute the parameters chosen in hazard function (HF) of the CTWD , equation (7), we get the Table 2 that consists of the HF of $CTED, CTRD, ED, WD, RD$ respectively, for example $\boldsymbol{h}_{CTED}$
4) The cumulative hazard rate function of the cubic transmuted Weibull distribution $TCWD(x, \alpha, \beta, \lambda)$ has the following special cases as in **Table3**, like $\boldsymbol{H}_{CEWD}(x, \alpha, \beta, \lambda)$ at $\beta = 1$
5) The mode, median of $CTWD$ are $x_{mode} = 0.530861$, $x_{medain} = 0.973485$, respectively using Newton Raphson Method.
6) The pdf of the order statistics, minimum, maximum, median, of $CTWD$ is the same of the $CTED(x, \alpha, \lambda)$ when $\beta = 1$

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