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Bivariate Generalized Double Weighted Exponential Distribution

In this article we suggest a bivariate generalized double weighted exponential distribution with discussion some of its properties , such as joint probability density function and its marginal , joint reliability function , the mathematical expectation , the marginal moment generating function and, we use the maximum likelihood method to estimate its parameters.

Keywords

Generalized double weighted exponential distribution , Conditional probability density function , Joint reliability function.

1. Introduction

Abed Al-Kadim and Hantoosh [1] introduced the double weighted distribution and double weighted exponential (DWE) distribution.

So that our object of this article is to display a bivariate generalized double weighted exponential (BGDWE) distribution , which is a special case of the multivariate distributions . Its marginals are generalized double weighted exponential (GDWE) distribution by using the method similar to those used by Marshall and Olkin [2], Sarhan and Balakrishnan [3] defined a new bivariate distribution using generalized distribution and exponential distribution and derived some properties of this new distribution , Al-Khedhairi and El-Gohary [4] presented a class of bivariate Gompertz distributions , Kundu and Gupta [5] proposed the bivariate generalized exponential distribution , El-Sherpieny et al. [6] presented a new bivariate distribution with generalized gompertz marginals and Davarzani et al. [7] studied the bivariate lifetime geometric distribution in presence of cure fractions.

Plan of the Article:

In this article, we define the BVGDWE distribution and discuss its different properties in Section 2. Section 3 present the reliability analysis. In Section 4 we introduce the mathematical expectation. In Section 5 we derive the marginal moment generating function. Section 6 obtains the parameter estimation using MLE. Finally, a conclusion for the results is given in Section 7.

2. Bivariate Generalized Double Weighted Exponential Distribution

Suppose Y is a non-negative random variable with probability density function (PDF) ,then the double weighted exponential distribution by using probability density function is:

$$f_{DWE}(y) = \frac{[w(y)\, f(y)]\, f(y)}{\mu_w} = \frac{w(y)\,[f(y)]^2}{\mu_w},\ y > 0\ \ and\ \ \mu_w = E[w(y)\, f(y)] < \infty$$

The first weight is $w(y) = y$ and the second is $f(y)$, where $f(y)$ is probability density function of exponential distribution.

Then

$$f_{DWE}(y;\lambda) = 4\lambda^2 y e^{-2\lambda y}\ \ ,\ y > 0\ ,\lambda > 0 \tag{1}$$

also the cumulative distribution function is:

$$F_{DWE}(y;\lambda) = 1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y} \tag{2}$$

The univariate GDWE distribution has the following PDF and CDF respectively for $y > 0$;

$$f_{GDWE}(y;\alpha,\lambda) = 4\alpha\lambda^2\ y e^{-2\lambda y}\left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha-1} \tag{3}$$

$$F_{GDWE}(y;\alpha,\lambda) = \left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha} \tag{4}$$

where $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters respectively.Suppose that $D_1 \sim$ GDWE(α_1,λ) , $D_2 \sim$ GDWE(α_2,λ) and $D_3 \sim$ GDWE(α_3,λ) and they are mutually independent . Here $'\sim'$ means is distributed GDWE . Define $Y_1 = \max(D_1,D_3)$ and $Y_2 = \max(D_2,D_3)$. Then we say that the bivariate vector (Y_1,Y_2) has a bivariate generalized double weighted exponential distribution with the shape parameters α_1,α_2 and α_3 and the scale parameter λ . We will denote it by BGDWE$(\alpha_1,\alpha_2,\alpha_3,\lambda)$.

2.1. The Joint Cumulative Distribution Function

We now introduce the joint distribution of random variables Y_1 and Y_2 considered the following theorem of the joint CDF of the BGDWE$(\alpha_1,\alpha_2,\alpha_3,\lambda)$.

Theorem 2.1. [8] If $(Y_1,Y_2) \sim$ BGDWE$(\alpha_1,\alpha_2,\alpha_3,\lambda)$, then the joint CDF of (Y_1,Y_2) for $y_1 > 0$, $y_2 > 0$, is:

$$F_{BGDWE}(y_1,y_2) = \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1}\left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2} \times \left(1 - 2\lambda t e^{-2\lambda t} - e^{-2\lambda t}\right)^{\alpha_3} \tag{5}$$

where $t = \min(y_1,y_2)$

Proof.

Since $\quad F(y_1,y_2) = P(Y_1 \le y_1, Y_2 \le y_2)$

we get $\quad F(y_1,y_2) = P(\max(D_1,D_3) \le y_1, \max(D_2,D_3) \le y_2)$

$$= P(D_1 \le y_1, D_2 \le y_2, D_3 \le \min(y_1,y_2))$$

where D_j $(j = 1,2,3)$ are mutually independent , we readily obtain

$$F_{BGDWE}(y_1,y_2) = P(D_1 \le y_1)P(D_2 \le y_2)P(D_3 \le \min(y_1,y_2))$$
$$= F_{GDWE}(y_1;\alpha_1,\lambda)F_{GDWE}(y_2;\alpha_2,\lambda)F_{GDWE}(t;\alpha_3,\lambda) \tag{6}$$

Substituting (4) into (6) we obtain (5) which completes the proof of the theorem 2.1.

2.2. The Joint Probability Density Function

Lemma 2.2. If $(Y_1, Y_2) \sim BGDWE(\alpha_1, \alpha_2, \alpha_3, \lambda)$, then the joint PDF of (Y_1, Y_2) for $y_1 > 0, y_2 > 0$, is:

$$f_{BGDWE}(y_1, y_2) = \begin{cases} f_1(y_1, y_2) & if\ 0 < y_1 < y_2 < \infty \\ f_2(y_1, y_2) & if\ 0 < y_2 < y_1 < \infty \\ f_3(y, y) & if\ 0 < y_1 = y_2 = y < \infty \end{cases} \tag{7}$$

where

$$\begin{aligned} f_1(y_1, y_2) &= f_{GDWE}(y_1; \alpha_1 + \alpha_3, \lambda)\, f_{GDWE}(y_2; \alpha_2, \lambda) \\ &= (\alpha_1 + \alpha_3) 16\lambda^4 y_1 e^{-2\lambda y_1} \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1 + \alpha_3 - 1} \\ &\quad \times \alpha_2 y_2 e^{-2\lambda y_2} \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2 - 1} \end{aligned} \tag{8}$$

$$\begin{aligned} f_2(y_1, y_2) &= f_{GDWE}(y_1; \alpha_1, \lambda)\, f_{GDWE}(y_2; \alpha_2 + \alpha_3, \lambda) \\ &= \alpha_1 (\alpha_2 + \alpha_3) 16\lambda^4 y_1 e^{-2\lambda y_1} \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1 - 1} \\ &\quad \times y_2 e^{-2\lambda y_2} \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2 + \alpha_3 - 1} \end{aligned} \tag{9}$$

$$\begin{aligned} f_3(y, y) &= \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} f_{GDWE}(y; \alpha_1 + \alpha_2 + \alpha_3, \lambda) \\ &= (\alpha_3) 4\lambda^2 y e^{-2\lambda y} \left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha_1 \alpha_2 + \alpha_3 - 1} \end{aligned} \tag{10}$$

Proof.

Let us first suppose that $y_1 < y_2$. Then , $F_{BGDWE}(y_1, y_2)$ in (5) will be denoted by $F_1(y_1, y_2)$ and becomes

$$F_1(y_1, y_2) = \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1 + \alpha_3} \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2}$$

By taking $\frac{\partial^2 F_1(y_1, y_2)}{\partial y_1 \partial y_2} = f_1(y_1, y_2)$,we get equation (8). By the same way we find $f_2(y_1, y_2)$ when $y_2 < y_1$. But $f_3(y, y)$ cannot be derived in a similar way. Using the facts that:

$$\int_0^\infty \int_0^{y_2} f_1(y_1, y_2)\, dy_1 dy_2 + \int_0^\infty \int_0^{y_1} f_2(y_1, y_2)\, dy_2 dy_1 + \int_0^\infty f_3(y, y) dy = 1 \tag{11}$$

Let

$$T_1 = \int_o^\infty \int_0^{y_2} f_1(y_1, y_2)\, dy_1 dy_2 \quad and \quad T_2 = \int_0^\infty \int_0^{y_1} f_2(y_1, y_2)\, dy_2 dy_1$$

Then

$$\begin{aligned} T_1 &= \int_o^\infty \int_0^{y_2} (\alpha_1 + \alpha_3) 16\lambda^4 y_1 e^{-2\lambda y_1} \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1 + \alpha_3 - 1} \\ &\quad \times \alpha_2 y_2 e^{-2\lambda y_2} \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2 - 1} dy_1\, dy_2 \\ &= \int_o^\infty \alpha_2 4\lambda^2 y_2 e^{-2\lambda y_2} \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} dy_2 \end{aligned} \tag{12}$$

Similarly

$$T_2 = \int_o^\infty \alpha_1 4\lambda^2 y_1 e^{-2\lambda y_1} \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} dy_1 \tag{13}$$

By substituting (12) and (13) in equation (11) , we get

$$\begin{aligned} \int_0^\infty f_3(y, y) dy &= \int_o^\infty (\alpha_1 + \alpha_2 + \alpha_3) 4\lambda^2 y e^{-2\lambda y} \\ &\quad \times \left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} dy \end{aligned}$$

$$-\int_o^\infty \alpha_2 4\lambda^2 y e^{-2\lambda y}\left(1-2\lambda y e^{-2\lambda y}-e^{-2\lambda y}\right)^{\alpha_1+\alpha_2+\alpha_3-1}\ dy$$
$$-\int_o^\infty \alpha_1 4\lambda^2 y e^{-2\lambda y}\left(1-2\lambda y e^{-2\lambda y}-e^{-2\lambda y}\right)^{\alpha_1+\alpha_2+\alpha_3-1}\ dy$$

This is

$$f_3(y,y)=\alpha_3 4\lambda^2 y e^{-2\lambda y}\left(1-2\lambda y e^{-2\lambda y}-e^{-2\lambda y}\right)^{\alpha_1+\alpha_2+\alpha_3-1} \quad \blacksquare \qquad (١٤)$$

2.3. Marginal Probability Density Function

The following theorem gives the marginal density function of Y_1 and Y_2 .

Theorem 2. 3. The marginal probability density functions of Y_i $(i=1,2)$ is given by

$$f_{Y_i}(y_i)=(\alpha_i+\alpha_3)4\lambda^2 y_i e^{-2\lambda y_i}\left(1-2\lambda y_i e^{-2\lambda y_i}-e^{-2\lambda y_i}\right)^{\alpha_i+\alpha_3-1} \qquad (15)$$
$$=f_{GDWE}(y_i;\alpha_i+\alpha_3,\lambda)\ ,\ y_i>0\ ,(i=1,2)$$

Proof.

The marginal cumulative distribution function of Y_i , say $F_{Y_i}(y_i)$, written as:

$$F_{Y_i}(y_i)=P(\,Y_i\le y_i)$$
$$=P(\max(D_i\,,D_3)\le y_i)$$
$$=P(D_i\le y_i\,,D_3\le y_i)$$

and since D_i is independent of D_3 , we simply have

$$F_{Y_i}(y_i)=\left(1-2\lambda y_i e^{-2\lambda y_i}-e^{-2\lambda y_i}\right)^{\alpha_i}\ \left(1-2\lambda y_i e^{-2\lambda y_i}-e^{-2\lambda y_i}\right)^{\alpha_3}$$
$$=\left(1-2\lambda y_i e^{-2\lambda y_i}-e^{-2\lambda y_i}\right)^{\alpha_i+\alpha_3} \qquad (16)$$
$$=F_{GDWE}(y_i;\alpha_i+\alpha_3,\lambda)\quad y_i>0\ ,i=1,2$$

By differentiating w.r.t. y_i , we get (15) .

2.4. Conditional Probability Density Functions

We present the conditional probability density functions of Y_1 and Y_2 by using the marginal probability density functions in the following theorem.

Theorem 2.4. The conditional probability density functions of Y_i , given $Y_j=y_j$ denoted by $f_{Y_i/Y_j}(y_i/y_j)$, $i,j=1,2\ ;i\neq j$, is:

$$f_{Y_i/Y_j}(y_i/y_j)=\begin{cases} f^{(1)}_{Y_i/Y_j}(y_i/y_j) & if\ y_i<y_j \\ f^{(2)}_{Y_i/Y_j}(y_i/y_j) & if\ y_j<y_i \\ f^{(3)}_{Y_i/Y_j}(y_i/y_j) & if\ y_i=y_j=y \end{cases} \qquad (17)$$

where

$$f^{(1)}_{Y_i/Y_j}(y_i/y_j)=\frac{(\alpha_i+\alpha_3)\alpha_j\ 4\lambda^2 y_i\ e^{-2\lambda y_i}\ \left(1-2\lambda y_i e^{-2\lambda y_i}-e^{-2\lambda y_i}\right)^{\alpha_i+\alpha_3-1}}{(\alpha_j+\alpha_3)\left(1-2\lambda y_j\ e^{-2\lambda y_j}-e^{-2\lambda y_j}\right)^{\alpha_3}} \qquad (18)$$

$$f^{(2)}_{Y_i/Y_j}(y_i/y_j)=\alpha_i 4\lambda^2 y_i e^{-2\lambda y_i}\left(1-2\lambda y_i e^{-2\lambda y_i}-e^{-2\lambda y_i}\right)^{\alpha_i-1} \qquad (19)$$

and

$$f^{(3)}_{Y_i/Y_j}(y_i/y_j)=\frac{\alpha_3\left(1-2\lambda\, y e^{-2\lambda y}-e^{-2\lambda y}\right)^{\alpha_1}}{(\alpha_2+\alpha_3)} \qquad (20)$$

Proof.
We get (18),(19) and (20), using the joint PDF of (Y_1, Y_2) given in (7) and $f_{Y_i}(y_i)$ in (15) in the following formula :

$$f_{Y_i/Y_j}(y_i/y_j) = \frac{f_{Y_i/Y_j}(y_i/y_j)}{f_{Y_j}(y_j)} \quad , i \neq j = 1,2 \tag{21}$$

3. Reliability Analysis [9]

We discuss some reliability measures , the joint reliability function , joint hazard function and joint reversed hazard function .

3.1. The Joint Reliability Function

In the following Proposition , we find the joint reliability function of Y_1 and Y_2 .

Proposition 3.1. The joint reliability function of Y_1 and Y_2 is given by:

$$R_{BGDWE}(y_1, y_2) = \begin{cases} R_1(y_1, y_2) & if\ y_1 < y_2 \\ R_2(y_1, y_2) & if\ y_2 < y_1 \\ R_3(y, y) & if\ y_1 = y_2 = y \end{cases} \tag{22}$$

then

$$\begin{aligned} R_1(y_1, y_2) = 1 &- \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1+\alpha_3} \\ &- \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2+\alpha_3} \\ &+ \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1+\alpha_3} \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2} \end{aligned} \tag{23}$$

$$\begin{aligned} R_2(y_1, y_2) = 1 &- \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1+\alpha_3} \\ &- \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2+\alpha_3} \\ &+ \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1} \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2+\alpha_3} \end{aligned} \tag{24}$$

and

$$\begin{aligned} R_3(y, y) = 1 &- \left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha_1+\alpha_3} \\ &- \left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha_2+\alpha_3} + \left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{\alpha_1+\alpha_2+\alpha_3} \end{aligned} \tag{25}$$

Proof.
The joint reliability function of Y_1 and Y_2 is:

$$R_{BGDWE}(y_1, y_2) = 1 - \left[F_{Y_1}(y_1) + F_{Y_2}(y_2) - F_{BGDWE}(y_1, y_2)\right] \tag{26}$$

substituting from equation (16) and (5) in equation (26) , we get

$$\begin{aligned} R_{BGDWE}(y_1, y_2) = 1 &- \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1+\alpha_3} \\ &- \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2+\alpha_3} \\ &+ \left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)^{\alpha_1} \left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)^{\alpha_2} \\ &\times \left(1 - 2\lambda t e^{-2\lambda t} - e^{-2\lambda t}\right)^{\alpha_3} \end{aligned}$$

where $t = min(y_1, y_2)$,
if $y_1 < y_2$, we have obtain the expression of given in (23),
if $y_2 < y_1$, we have obtain the expression of given in (24) and
if $y_1 = y_2 = y$ we have obtain the expression of given in (25).

3.2. Joint Hazard Function

Let (Y_1, Y_2) be two random variables with probability density function $f_{BGDWE}(y_1, y_2)$. defined joint hazard function as:

$$h_{BGDWE}(y_1, y_2) = \frac{f_{BGDWE}(y_1, y_2)}{R_{BGDWE}(y_1, y_2)} \tag{27}$$

Then , the joint hazard function is:

$$h_{BGDWE}(y_1, y_2) = \begin{cases} h_1(y_1, y_2) & if\ y_1 < y_2 \\ h_2(y_1, y_2) & if\ y_2 < y_1 \\ h_3(y, y) & if\ y_1 = y_2 = y \end{cases} \tag{28}$$

$if\ y_1 < y_2$, then

$$h_1(y_1, y_2) = \frac{f_1(y_1, y_2)}{R_1(y_1, y_2)} \tag{29}$$

where $f_1(y_1, y_2)$ from equation (8) and $R_1(y_1, y_2)$ from equation (23),

$if\ y_2 < y_1$, then

$$h_2(y_1, y_2) = \frac{f_2(y_1, y_2)}{R_2(y_1, y_2)} \tag{30}$$

where $f_2(y_1, y_2)$from equation (9) and $R_2(y_1, y_2)$from equation (24),

$if\ y_1 = y_2 = y$, then

$$h_3(y, y) = \frac{f_3(y, y)}{R_3(y, y)} \tag{31}$$

where $f_3(y, y)$ from equation (10) and $R_3(y, y)$ from equation (25).

3.3. Joint Reversed Hazard Function

The joint reversed hazard function is defined as the ratio of the PDF and the corresponding CDF .

1. The joint reversed hazard function of (Y_1, Y_2) is defined as:

$$r_{BGDWE}(y_1, y_2) = \frac{f_{BGDWE}(y_1, y_2)}{F_{BGDWE}(y_1, y_2)} \tag{32}$$

so that

$$r_{BGDWE}(y_1, y_2) = \begin{cases} r_1(y_1, y_2) & if\ y_1 < y_2 \\ r_2(y_1, y_2) & if\ y_2 < y_1 \\ r_3(y, y) & if\ y_1 = y_2 = y \end{cases} \tag{33}$$

then

$$r_1(y_1, y_2) = (\alpha_1 + \alpha_3)\, \alpha_2 16\lambda^4 y_1\, y_2\, e^{-2\lambda(y_1+y_2)} \times \left[\left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)\left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)\right]^{-1} \tag{34}$$

$$r_2(y_1, y_2) = \alpha_1(\alpha_2 + \alpha_3) 16\lambda^4 y_1\, y_2\, e^{-2\lambda(y_1+y_2)} \times \left[\left(1 - 2\lambda y_1 e^{-2\lambda y_1} - e^{-2\lambda y_1}\right)\left(1 - 2\lambda y_2 e^{-2\lambda y_2} - e^{-2\lambda y_2}\right)\right]^{-1} \tag{35}$$

and

$$r_3(y, y) = \alpha_3 4\lambda^2 y\, e^{-2\lambda y} \left(1 - 2\lambda y e^{-2\lambda y} - e^{-2\lambda y}\right)^{-1} \tag{36}$$

2.The gradient vector of the joint reversed hazard function is given by:

$r(y_1, y_2) = \left(r_{Y_1}(y_1), r_{Y_2}(y_2)\right)$, where

$$r_{Y_i}(y_i) = \frac{f_{Y_i}(y_i)}{F_{Y_i}(y_i)} = \frac{\partial}{\partial y_i} \ln F_{Y_i}(y_i) \quad , i = 1,2 \text{ , then} \tag{37}$$

$$r_{Y_i}(y_i) = (\alpha_i + \alpha_3) 4\lambda^2 y_i \; e^{-2\lambda y_i} \left(1 - 2\lambda y_i e^{-2\lambda y_i} - e^{-2\lambda y_i}\right)^{-1}, i = 1,2 \tag{38}$$

4. The Mathematical Expectation

In the following Proposition , we can derive the mathematical expectation of Y_i , $(i = 1,2)$.

Proposition 4.1. If $Y_i \sim$ GDWE$(\alpha_i + \alpha_3, \lambda)$, then the r^{th} moment of Y_i as following:

$$E(Y_i^r) = (\alpha_i + \alpha_3) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{\alpha_i+\alpha_3-1-j+k} \binom{\alpha_i+\alpha_3-1}{j} \binom{j}{k} (2\lambda)^{2+k}$$
$$\times \frac{\Gamma(r+k+2)}{(2\lambda\alpha_i + 2\lambda\alpha_3 - 2\lambda j + 2\lambda k)^{(r+k+2)}} \quad , i = 1,2 \tag{39}$$

Proof.

$$E(Y_i^r) = \int_0^\infty y_i^r \; f_{Y_i}(y_i) \; dy_i$$
$$= \int_0^\infty (\alpha_i + \alpha_3) \, 4\lambda^2 y_i^{r+1} \, e^{-2\lambda y_i} \left(\left(1 - 2\lambda y_i e^{-2\lambda y_i}\right) - e^{-2\lambda y_i}\right)^{\alpha_i+\alpha_3-1} dy_i$$

Since $0 < \left(\left(1 - 2\lambda y_i e^{-2\lambda y_i}\right) - e^{-2\lambda y_i}\right) < 1$ for $y_i > 0$, then by using the binomial series expansion we have

$$\left(\left(1 - 2\lambda y_i e^{-2\lambda y_i}\right) - e^{-2\lambda y_i}\right)^{\alpha_i+\alpha_3-1} = \sum_{j=0}^{\infty} \left(1 - 2\lambda y_i e^{-2\lambda y_i}\right)^j$$
$$\times (-1)^{\alpha_i+\alpha_3-1-j} \binom{\alpha_i+\alpha_3-1}{j} \left(e^{-2\lambda y_i}\right)^{\alpha_i+\alpha_3-1-j} \tag{40}$$

also
$$\left(1 - 2\lambda y_i e^{-2\lambda y_i}\right)^j = \sum_{k=0}^{\infty} (-1)^k \binom{j}{k} \left(2\lambda y_i e^{-2\lambda y_i}\right)^k \tag{41}$$

then

$$E\left(Y_i^r\right) = (\alpha_i + \alpha_3) \, 4\lambda^2 \int_0^\infty y_i^{r+1} \, e^{-2\lambda y_i} \sum_{j=0}^{\infty} (-1)^{\alpha_i+\alpha_3-1-j} \binom{\alpha_i+\alpha_3-1}{j}$$
$$\times \left(e^{-2\lambda y}\right)^{\alpha_i+\alpha_3-1-j} \sum_{k=0}^{\infty} (-1)^k \binom{j}{k} \left(2\lambda y_i e^{-2\lambda y_i}\right)^k dy_i$$
$$= (\alpha_i + \alpha_3) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{\alpha_i+\alpha_3-1-j+k} \binom{\alpha_i+\alpha_3-1}{j} \binom{j}{k} (2\lambda)^{2+k}$$
$$\times \int_0^\infty y_i^{r+1+k} \, e^{-(2\lambda\alpha_i + 2\lambda\alpha_3 - 2\lambda j + 2\lambda k) y_i} \; dy_i$$

$$E\left(Y_i^r\right) = (\alpha_i + \alpha_3) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{\alpha_i+\alpha_3-1-j+k} \binom{\alpha_i+\alpha_3-1}{j} \binom{j}{k} (2\lambda)^{2+k}$$

$$\times \frac{\Gamma(r+k+2)}{(2\lambda\alpha_i+2\lambda\alpha_3-2\lambda j+2\lambda k)^{(r+k+2)}}$$

Then the r^{th} moment of Y_i is:

$$E(Y_i^r) = (\alpha_i + \alpha_3)\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{\alpha_i+\alpha_3-1-j+k} \binom{\alpha_i+\alpha_3-1}{j} \binom{j}{k} (2\lambda)^{2+k}$$

$$\times \frac{\Gamma(r+k+2)}{(2\lambda\alpha_i+2\lambda\alpha_3-2\lambda j+2\lambda k)^{(r+k+2)}} \quad , i = 1,2 \qquad \blacksquare$$

5. The Marginal Moment Generating Function

We find the marginal moment generating function of Y_i , $(i = 1,2)$ in the following lemma

Lemma 5.1. If $Y_i \sim$ GDWE$(\alpha_i + \alpha_3, \lambda)$, then the marginal moment generating function of Y_i , $(i = 1,2)$ as following:

$$M_{Y_i}(t_i) = (\alpha_i + \alpha_3)\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} (-1)^{\alpha_i+\alpha_3-1-j+k} \binom{\alpha_i+\alpha_3-1}{j} \binom{j}{k} (2\lambda)^{2+k}$$

$$\times \frac{t_i^r \Gamma(r+k+2)}{r!\,(2\lambda\alpha_i+2\lambda\alpha_3-2\lambda j+2\lambda k)^{(r+k+2)}} \quad , i = 1,2 \qquad (42)$$

Proof.

$$M_{Y_i}(t_i) = E(e^{t_i y_i})$$

$$= \int_0^{\infty} e^{t_i y_i} \, f_{Y_i}(y_i) \, dy_i$$

$$= \sum_{r=0}^{\infty} \frac{t_i^r}{r!} \int_0^{\infty} y_i^r \, f_{Y_i}(y_i) \, dy_i$$

$$= \sum_{r=0}^{\infty} \frac{t_i^r}{r!} E(Y_i^r)$$

$$= (\alpha_i + \alpha_3)\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} (-1)^{\alpha_i+\alpha_3-1-j+k} \binom{\alpha_i+\alpha_3-1}{j} \binom{j}{k} (2\lambda)^{2+k}$$

$$\times \frac{t_i^r \Gamma(r+k+2)}{r!\,(2\lambda\alpha_i+2\lambda\alpha_3-2\lambda j+2\lambda k)^{(r+k+2)}} \quad , i = 1,2 \qquad \blacksquare$$

6. Maximum Likelihood Estimation

To estimate the unknown parameters of the BGDWE distribution , we use the method of maximum likelihood estimators (MLEs) .

Let $\left((Y_{11}, Y_{21}), (Y_{12}, Y_{22}), \dots, (Y_{1n}, Y_{2n})\right)$ is a random sample from BGDWE$(\alpha_1, \alpha_2, \alpha_3, \lambda)$, where

$$n_1 = (i; Y_{1i} < Y_{2i}), n_2 = (i; Y_{1i} > Y_{2i}), n_3 = (i; Y_{1i} = Y_{2i} = Y_i), n = \sum_{k=1}^{3} \mathrm{n}_k \qquad (43)$$

By using the equations (8) , (9) ,(10) and (43), we find that the likelihood of the sample as following:

$l(\alpha_1, \alpha_2, \alpha_3, \lambda) = \prod_{i=1}^{n_1} f_1\,(y_{1i}, y_{2i}) \prod_{i=1}^{n_2} f_2\,(y_{1i}, y_{2i}) \prod_{i=1}^{n_3} f_3\,(y_i\,, y_i)$

The log-likelihood function becomes:

$$
\begin{aligned}
L(\alpha_1, \alpha_2, \alpha_3, \lambda) = {} & n_1 ln(\alpha_1 + \alpha_3) + n_1 \ln(4) + 2n_1 ln(\lambda) + \sum_{i=1}^{n_1} ln(y_{1i}) \\
& -2\lambda \sum_{i=1}^{n_1} (y_{1i}) + (\alpha_1 + \alpha_3 - 1) \\
& \times \sum_{i=1}^{n_1} ln\left(1 - 2\lambda y_{1i}\, e^{-2\lambda y_{1i}} - e^{-2\lambda y_{1i}}\right) + n_1 \ln(\alpha_2) + n_1 \ln(4) \\
& +2n_1 ln(\lambda) + \sum_{i=1}^{n_1} ln(y_{2i}) - 2\lambda \sum_{i=1}^{n_1} (y_{2i}) + (\alpha_2 - 1) \\
& \times \sum_{i=1}^{n_1} ln\left(1 - 2\lambda y_{2i}\, e^{-2\lambda y_{2i}} - e^{-2\lambda y_{2i}}\right) + n_2 \ln(\alpha_1) + n_2 \ln(4) \\
& +2n_2 ln(\lambda) + \sum_{i=1}^{n_2} ln(y_{1i}) - 2\lambda \sum_{i=1}^{n_2} (y_{1i}) + (\alpha_1 - 1) \\
& \times \sum_{i=1}^{n_2} ln\left(1 - 2\lambda y_{1i}\, e^{-2\lambda y_{1i}} - e^{-2\lambda y_{1i}}\right) + n_2 \ln(\alpha_2 + \alpha_3) \\
& +n_2 \ln(4) + 2n_2 ln(\lambda) + \sum_{i=1}^{n_2} ln(y_{2i}) - 2\lambda \sum_{i=1}^{n_2} (y_{2i}) \\
& +(\alpha_2 + \alpha_3 - 1) \sum_{i=1}^{n_2} ln\left(1 - 2\lambda y_{2i}\, e^{-2\lambda y_{2i}} - e^{-2\lambda y_{2i}}\right) + n_3 ln(\alpha_3) \\
& +n_3 ln(4) + 2n_3 ln(\lambda) + \sum_{i=1}^{n_3} ln(y_i) - 2\lambda \sum_{i=1}^{n_3} (y_i) \\
& +(\alpha_1 + \alpha_2 + \alpha_3 - 1) \sum_{i=1}^{n_3} ln\left(1 - 2\lambda y_i\, e^{-2\lambda y_i} - e^{-2\lambda y_i}\right) \qquad (44)
\end{aligned}
$$

Taking the first partial derivatives of (44) with respect to $\alpha_1, \alpha_2, \alpha_3$ and λ , and setting the results equal zero:

$$\frac{\partial L}{\partial \alpha_1}=\frac{n_1}{(\alpha_1+\alpha_3)}+\sum_{i=1}^{n_1} ln\left(1-2\lambda y_{1i}\, e^{-2\lambda y_{1i}}-e^{-2\lambda y_{1i}}\right)+\frac{n_2}{\alpha_1}$$

$$+\sum_{i=1}^{n_2} ln\left(1-2\lambda y_{1i}\, e^{-2\lambda y_{1i}}-e^{-2\lambda y_{1i}}\right)+\sum_{i=1}^{n_3} ln\left(1-2\lambda y_i\, e^{-2\lambda y_i}-e^{-2\lambda y_i}\right) \quad (45)$$

$$\frac{\partial L}{\partial \alpha_2}=\frac{n_1}{\alpha_2}+\sum_{i=1}^{n_1} ln\left(1-2\lambda y_{2i}\, e^{-2\lambda y_{2i}}-e^{-2\lambda y_{2i}}\right)+\frac{n_2}{(\alpha_2+\alpha_3)}$$

$$+\sum_{i=1}^{n_2} ln\left(1-2\lambda y_{2i}\, e^{-2\lambda y_{2i}}-e^{-2\lambda y_{2i}}\right)+\sum_{i=1}^{n_3} ln\left(1-2\lambda y_i e^{-2\lambda y_i}-e^{-2\lambda y_i}\right) \quad (46)$$

$$\frac{\partial L}{\partial \alpha_3}=\frac{n_1}{(\alpha_1+\alpha_3)}+\sum_{i=1}^{n_1} ln\left(1-2\lambda y_{1i}\, e^{-2\lambda y_{1i}}-e^{-2\lambda y_{1i}}\right)+\frac{n_2}{(\alpha_2+\alpha_3)}$$

$$+\sum_{i=1}^{n_2} ln\left(1-2\lambda y_{2i}\, e^{-2\lambda y_{2i}}-e^{-2\lambda y_{2i}}\right)+\frac{n_3}{\alpha_3}$$

$$+\sum_{i=1}^{n_3} ln\left(1-2\lambda y_i\, e^{-2\lambda y_i}-e^{-2\lambda y_i}\right) \quad (47)$$

$$\frac{\partial L}{\partial \lambda}=\frac{4n_1}{\lambda}-2\sum_{i=1}^{n_1}(y_{1i})+(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}\frac{4\lambda(y_{1i})^2 e^{-2\lambda y_{1i}}}{\left(1-2\lambda y_{1i}\, e^{-2\lambda y_{1i}}-e^{-2\lambda y_{1i}}\right)}$$

$$-2\sum_{i=1}^{n_1}(y_{2i})+(\alpha_2-1)\sum_{i=1}^{n_1}\frac{4\lambda(y_{2i})^2\, e^{-2\lambda y_{2i}}}{\left(1-2\lambda y_{2i}\, e^{-2\lambda y_{2i}}-e^{-2\lambda y_{2i}}\right)}+\frac{4n_2}{\lambda}-2\sum_{i=1}^{n_2}(y_{1i})$$

$$+(\alpha_1-1)\sum_{i=1}^{n_2}\frac{4\lambda(y_{1i})^2\, e^{-2\lambda y_{1i}}}{\left(1-2\lambda y_{1i}\, e^{-2\lambda y_{1i}}-e^{-2\lambda y_{1i}}\right)}-2\sum_{i=1}^{n_2}(y_{2i})+(\alpha_2+\alpha_3 1)$$

$$\times\sum_{i=1}^{n_2}\frac{4\lambda(y_{2i})^2\, e^{-2\lambda y_{2i}}}{\left(1-2\lambda y_{2i}\, e^{-2\lambda y_{2i}}-e^{-2\lambda y_{2i}}\right)}+\frac{2n_3}{\lambda}-2\sum_{i=1}^{n_3}(y_i)$$

$$+(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}\frac{4\lambda(y_i)^2\, e^{-2\lambda y_i}}{\left(1-2\lambda y_i\, e^{-2\lambda y_i}-e^{-2\lambda y_i}\right)} \quad (48)$$

These equations cannot easy to solve , but numerically by using the statistical software , to get the MLEs of the unknown parameters.

7. Conclusion

This article introduced the bivariate generalized double weighted exponential distribution whose marginals are generalized double weighted exponential distribution . Some statistical properties of this distribution . It is observed that the MLEs of the unknown parameters can be obtained by solving four non-linear equations using numerical technique.

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الخلاصة

في هذا البحث نقترح التوزيع الاسي الموزون المضاعف المعمم للمتغيرين مع مناقشة بعض خواصة ، مثل دالة الكثافة الاحتمالية المشتركة والهامشية ، دالة الموثوقية المشتركة ، التوقع الرياضي ، الدالة المولدة للعزوم الهامشية وفي النهاية ، نستخدم طريقة الامكان الاعظم لتقدير معلماته.

الكلمات المفتاحية

التوزيع الاسي الموزون المضاعف المعمم ، دالة الكثافة الاحتمالية الشرطية ، دالة الموثوقية المشتركة ، مقدرات الامكان الاعظم .