DOUBLE WEIGHTED LOMAX DISTRIBUTION

Kareema Abed Al-Kadim [*] , Mohannad Mohammad Fadhil [**]

Mathematics Department , College of Education of pure Sciences ,

University of Babylon , Iraq

[*]E-mail : kareema.kadim@yahoo.com , [**]E-mail : mohannadmm12@gmail.com

Summary

This paper deals with a new distribution called double weighted Lomax distribution and the statistical properties of this model, such as , mode , moments are studied. As well as studying the maximum likelihood estimators of this distribution.

الخلاصة

هذا البحث يتناول دراسة توزيع جديد يسمى توزيع لوماكس الموزون المضاعف ويتم دراسة الخواص الاحصائية لهذا النموذج مثل المنوال ، العزوم . فضلا عن دراسة مقدر الامكان الاعظم لهذا التوزيع.

Keywords

Weighted distribution, Double weighted Lomax distribution, Moment generating function, Maximum likelihood estimation.

الكلمات المفتاحية

التوزيع الموزون، توزيع لوماكس الموزون المضاعف، الدالة المولدة للعزوم، تقدير الامكان الاعظم.

1. Introduction

The theory of weighted distributions provides a unifying approach for the problems of model specification and data interpretation . Also the weighted distribution , is used in many fields of real life such as medicine ,ecology , reliability , and so on.

The concept of weighted distribution can be traced to the work of Fisher (1934) in connection with his studies on how method of ascertainment can influence the form of distribution of recorded observations . Later it was introduced and formulated in general terms by Rao (1965) in connection to modeling statistical data. The usefulness and application of the weighted distribution to biased samples in various area includes medicine , ecology and branching process in Patil and Rao (1978), Gupta and Keating (1985) , Gupta and Kirmani (1990).

There are many researches for weighted distribution Kersey (2010) presented the weighted inverse Weibull distribution and Beta-inverse Weibull distribution, Das and Roy (2011) introduced the length biased weighted generalized Rayleigh distribution , also they presented the length-biased from of the weighted Weibull distribution see Das and Roy (2011) , Ye (2012) introduced the properties of weighted generalized Beta distribution of the second kind , Al-khadim and Hantoosh (2013) introduced the double weighted distribution and discussed the statistical properties of double weighted exponential distribution , Rashwan (2013) presented the double weighted Rayleigh distribution properties and estimation .

Suppose X be a non-negative random variable (rv) with probability density function (pdf) f(x) . Let the weight function be w(x) which is a non-negative function. Then the weighted density function $f_w(x)$ is obtained as:

$$f_w(x) = \frac{w(x)\, f(x)}{\mu_w} \quad , \quad x > 0 \text{ and } \mu_w = E\big(w(x)\big) < \infty \tag{1}$$

when we use weighted distributions as a tool in the selection of suitable models for observed data it is the choice of the weight function that fits the data. Since $w(x) = x^c$, $c > 0$, then the resulting distribution is called size-biased distribution and if $c = 1$ and 2 ,we get the length-biased and area-biased distributions respectively.

This paper is to introduce the double weighted Lomax (DWL) distribution , and derive the statistical properties of this distribution.

2. Double Weighted Distribution

Definition 2.1. Suppose X is a non-negative random variable with probability density function , then the double weighted (DW) distribution by using probability density function is defined as:

$$f_w(x) = \frac{[w(x)\, f(x)]\, f(x)}{\mu_w} = \frac{[w(x)]\, f^2(x)}{\mu_w} \quad , \qquad x > 0 \tag{2}$$

where $\mu_w = \mathrm{E}[w(x)\, f(x)]$

$$= \int_0^\infty [w(x)\, f(x)]\ \ f(x)\ dx$$

Assuming that $\mu_w = \mathrm{E}[w(x)\, f(x)] < \infty$ i.e the first moment of $[w(x)\, f(x)]$ exists. and the first weight is $w(x)$ and the second is $f(x)$, where $f(x)$ is probability density function.

3. Double Weighted Lomax Distribution

Consider the weight function $w(x) = x^n$, $n > 0$, and the pdf of Lomax distribution (Rajab et al.2013) is given by:

$$f(x; \alpha, \lambda, \mu) = \frac{\alpha}{\lambda}\left(1 + \frac{x-\mu}{\lambda}\right)^{-(\alpha+1)} \quad , \quad x \geq \mu\,, \mu \geq 0, \alpha > 0, \lambda > 0 \tag{3}$$

so

$$\begin{aligned}
\mu_w &= \mathrm{E}[\, w(x)\, f(x)] \\
&= \int_\mu^\infty [w(x)\, f(x)]\ f(x)\, dx \\
&= \int_\mu^\infty x^n\ \frac{\alpha}{\lambda}(1 + \frac{x-\mu}{\lambda})^{-(\alpha+1)}\ \ \frac{\alpha}{\lambda}(1 + \frac{x-\mu}{\lambda})^{-(\alpha+1)}\ \ dx \\
&= \int_\mu^\infty x^n\ \frac{\alpha^2}{\lambda^2}(1 + \frac{x-\mu}{\lambda})^{-2(\alpha+1)}\ \ dx
\end{aligned}$$

$$\mu_w = \int_0^\infty (\mu + \lambda z)^n\ \frac{\alpha^2}{\lambda}(1 + z)^{-2(\alpha+1)}\ \ dz$$

Now by considering $(\mu + \lambda z)^n = \mu^n\left(1 + \frac{\lambda}{\mu} z\right)^n$ and comparing it with the binomial theorem $(1 + a)^n = \sum_{i=0}^{n} \binom{n}{i}\ a^i$, then

$$\mu_w = \int_0^\infty \mu^n\ \sum_{i=0}^{n} \binom{n}{i}\ \frac{\lambda^i}{\mu^i}\ z^i\ \frac{\alpha^2}{\lambda}(1 + z)^{-2(\alpha+1)}\ dz$$

$$\mu_w = \alpha^2 \sum_{i=0}^{n} \binom{n}{i} \mu^{n-i} \lambda^{i-1} \beta(i+1, 2\alpha - i + 1) \quad (4)$$

Then the probability density function of the DWL distribution with $w(x) = x^n$ is given by:

$$f_w(x;\alpha,\lambda,\mu,n) = \frac{x^n \frac{\alpha^2}{\lambda^2}\left(1+\frac{x-\mu}{\lambda}\right)^{-2(\alpha+1)}}{\alpha^2 \sum_{i=0}^{n} \binom{n}{i} \mu^{n-i} \lambda^{i-1} \beta(i+1, 2\alpha-i+1)} \quad , x \geq \mu, \mu \geq 0, \alpha, \lambda, \mathrm{n} > 0 \quad (5)$$

Suppose $n = 1$, then $w_1(x) = x$,then the probability density function of DWL distribution with $w_1(x) = x$ becomes:

$$f_{w_1}(x;\alpha,\lambda,\mu) = \frac{x \frac{\alpha^2}{\lambda^2}\left(1+\frac{x-\mu}{\lambda}\right)^{-2(\alpha+1)}}{\alpha^2 \sum_{i=0}^{1} \binom{1}{i} \mu^{1-i} \lambda^{i-1} \beta(i+1, 2\alpha-i+1)}$$

since $\alpha^2 \sum_{i=0}^{1} \binom{1}{i} \mu^{1-i} \lambda^{i-1} \beta(i+1, 2\alpha - i + 1) = \alpha^2[\frac{1}{\lambda} \mu \beta(1, 2\alpha + 1) + \beta(2, 2\alpha)]$

$$= \frac{2\alpha^2\mu + \alpha\lambda}{4\alpha\lambda + 2\lambda}$$

we have

$$f_{w_1}(x;\alpha,\lambda,\mu) = \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2} \; x \left(1 + \frac{x-\mu}{\lambda}\right)^{-2(\alpha+1)} \quad , \quad x \geq \mu, \mu \geq 0, \alpha >, \lambda > 0 \quad (6)$$

where α is a shape parameter , λ is a scale parameter and μ is a location parameter. Also the cumulative distribution function(cdf) of DWL distribution with $w_1(x) = x$ is given by:

$$F_{w_1}(x;\alpha,\lambda,\mu) = \int_\mu^x f_{w_1}(x;\alpha,\lambda,\mu) \; dx$$

$$= \int_\mu^x \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2} \; x \left(1 + \frac{x-\mu}{\lambda}\right)^{-2(\alpha+1)} dx$$

$$= \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\left(\frac{-\lambda}{(2\alpha+1)}\left[x\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha+1)} - \mu\right] - \frac{\lambda^2}{(2\alpha+1)2\alpha}\left[\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha)} - 1\right]\right)$$

then $F_{w_1}(x;\alpha,\lambda,\mu) = \dfrac{\lambda+2\alpha\mu-2\alpha x\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha+1)} - \lambda\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha)}}{2\alpha\mu+\lambda}$ (7)

and the limits of the probability density function are given in(6) as follows:

1. $\lim_{x \to \mu} f_{w_1}(x;\alpha,\lambda,\mu) = \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2} \lim_{x \to \mu}(x) \lim_{x \to \mu}\left(1+\frac{x-\mu}{\lambda}\right)^{-2(\alpha+1)}$

$$= \frac{4\alpha^2\mu+2\alpha\mu}{2\alpha\lambda\mu+\lambda^2} \quad (8)$$

2. $\lim_{x \to \infty} f_{w_1}(x;\alpha,\lambda,\mu) = \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2} \frac{\lim_{x\to\infty} x}{\lim_{x\to\infty}\left(1+\frac{x-\mu}{\lambda}\right)^{2(\alpha+1)}}$

$$= \frac{1}{\infty} \Rightarrow 0 \quad (9)$$

The plot of the pdf of DWL distribution is:

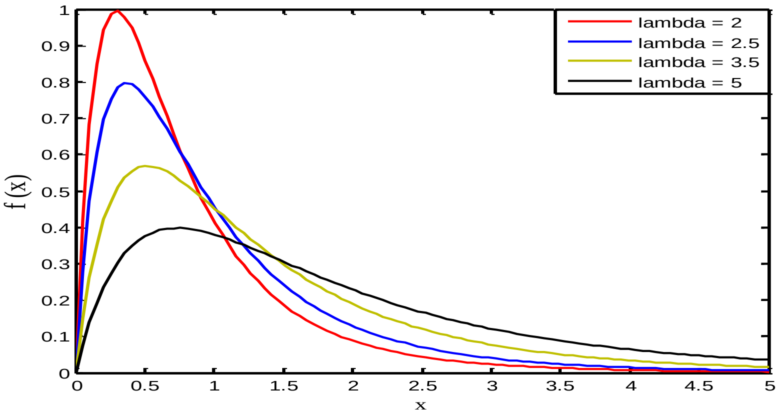

Figure 1. The pdf of DWL distribution using $w_1(x) = x$ with fixed $\alpha = 2.9$, $\mu = 0.001$ and λ takes the values $(2, 2.5, 3.5, 5)$.
we note that the pdf of DWL distribution decreases when the value of λ increases.

The plot of the cdf of DWL distribution is shown below:

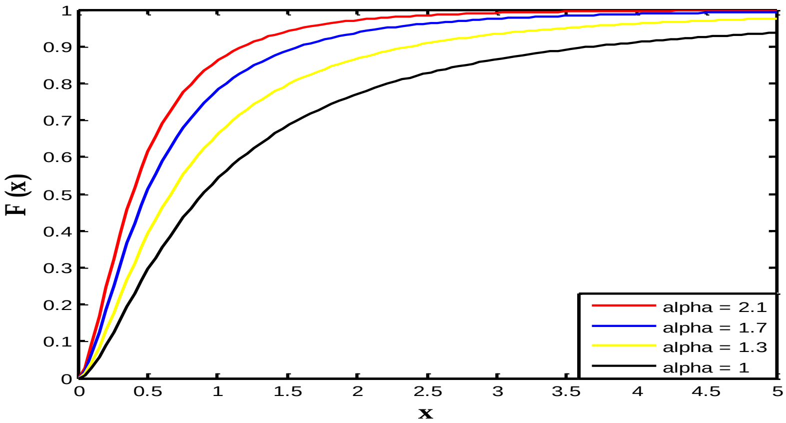

Figure 2. The cdf of DWL distribution using $w_1(x) = x$ with fixed $\lambda = 0.9, \mu = 0.001$ and α takes the values $(1, 1.3, 1.7, 2.1)$.
The figure shows that the cdf is non-creasing with increasing x.

4. Statistical Properties

Statistical properties was studied of the DWL distribution with $w_1(x) = x$ in this section.

4.1. Reliability Function

The reliability function or survival function of DWL distribution with $w_1(x) = x$ is given by:

$$R_{w_1}(x; \alpha, \lambda, \mu) = 1 - F_{w_1}(x; \alpha, \lambda, \mu)$$

$$= \frac{2\,\alpha\,x\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha+1)} + \lambda\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha)}}{2\alpha\mu+\lambda} \tag{10}$$

The plot of the reliability function of DWL distribution is:

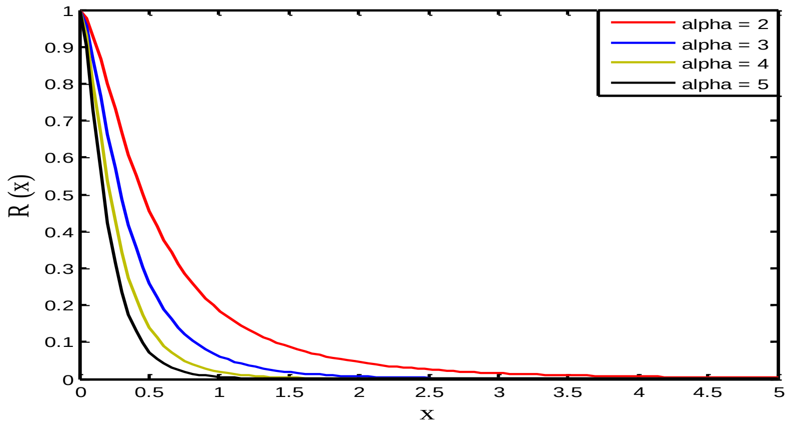

Figure 3. The reliability function of DWL distribution using $w_1(x) = x$ with fixed λ =1, $\mu = 0.001$ and α takes the values $(2, 3, 4, 5)$. Therefore, there is an inverse relationship between the shape parameter α and the reliability function.

4.2. Hazard Function

The hazard function of DWL distribution with $w_1(x) = x$ is given by:

$$h_{w_1}(x; \alpha, \lambda, \mu) = \frac{f_{w_1}(x;\alpha,\lambda,\mu)}{R_{w_1}(x;\alpha,\lambda,\mu)}$$

$$= \frac{4\alpha^2+2\alpha}{2\alpha\lambda\left(1+\frac{x-\mu}{\lambda}\right)+x^{-1}\,\lambda^2\left(1+\frac{x-\mu}{\lambda}\right)^2} \tag{11}$$

The plot of the hazard function of DWL distribution:

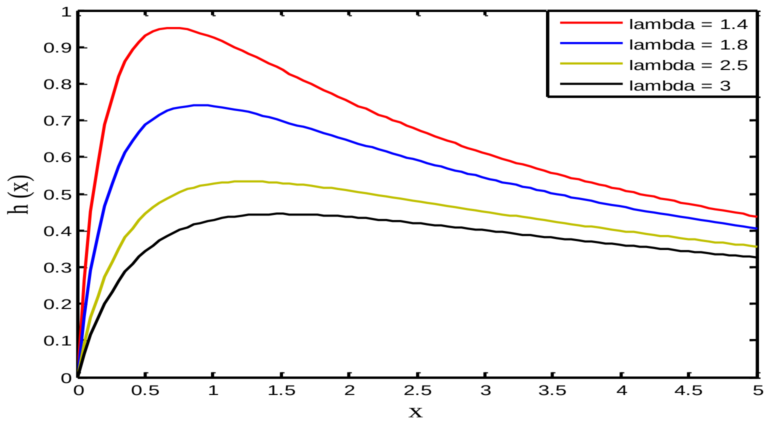

Figure 4. The hazard function of DWL distribution using $w_1(x) = x$ with fixed $\alpha = 1.5$, $\mu = 0.001$ and λ takes the values $(1.4, 1.8, 2.5, 3)$.

We note that the hazard function of DWL distribution decreases when the value of λ increases.

4.3. Reverse Hazard Function

The reverse hazard function of DWL distribution with $w_1(x) = x$ is given by:

$$\psi_{w_1}(x; \alpha, \lambda, \mu) = \frac{f_{w_1}(x;\alpha,\lambda,\mu)}{F_{w_1}(x;\alpha,\lambda,\mu)}$$

$$= \frac{(4\alpha^2+2\alpha)\; x\; \left(1+\frac{x-\mu}{\lambda}\right)^{-2(\alpha+1)}}{\lambda^2+2\alpha\lambda\mu-2\alpha\lambda x\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha+1)} - \lambda^2\left(1+\frac{x-\mu}{\lambda}\right)^{-2\alpha}} \tag{12}$$

The plot of the reverse hazard function of DWL distribution is:

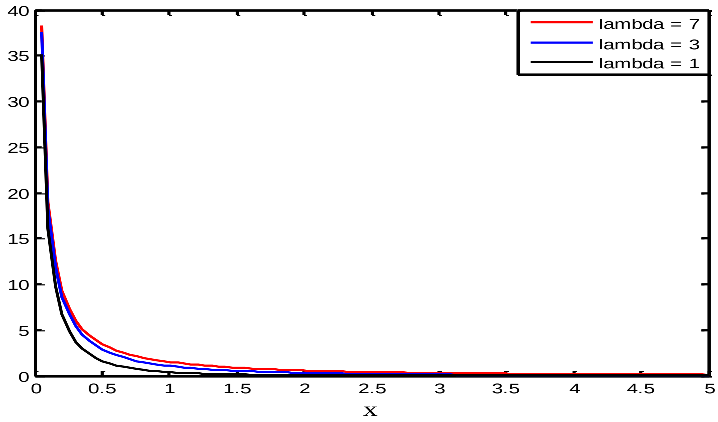

Figure 5. The reverse hazard function of DWL distribution using $w_1(x) = x$ with fixed $\alpha = 2$, $\mu = 0.001$ and λ takes the values $(1, 3, 7)$.

4.4. Cumulative Hazard Function

The cumulative hazard function of DWL distribution with $w_1(x) = x$ is given by:

$$H_{w_1}(x;\alpha,\lambda,\mu) = -\ln\left(R_{w_1}(x;\alpha,\lambda,\mu)\right) = -\ln\left(\frac{2\alpha x\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha+1)}+\lambda\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha)}}{2\alpha\mu+\lambda}\right) \quad (13)$$

The plot of the cumulative hazard function of DWL distribution is shown below:

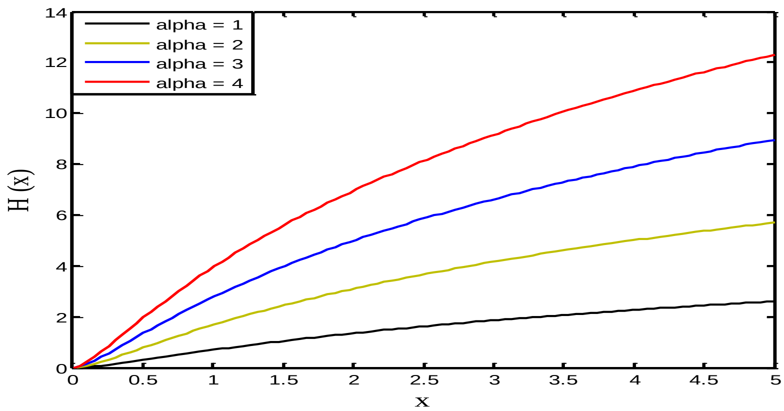

Figure 6. The cumulative hazard function of DWL distribution using $w_1(x) = x$ with fixed $\lambda = 1, \mu = 0.001$ and α takes the values $(1, 2, 3, 4)$.

4.5. Mode of DWL Distribution

Theorem 4.5.1. The mode of DWL distribution with $w_1(x) = x$ is as follows:

$$x_0 = \frac{\lambda-\mu}{1+2\alpha}\ , \quad \text{Where } 1 + 2\alpha \neq 0$$

Proof. Taking logarithm on $f_{w_1}(x;\alpha,\lambda,\mu)$ in (6) we get

$$\ln[f_{w_1}(x;\alpha,\lambda,\mu)] = \ln(4\alpha^2 + 2\alpha) - \ln(2\alpha\lambda\mu + \lambda^2) + \ln(x)$$

$$-2(\alpha+1)\ln\left(1+\frac{x-\mu}{\lambda}\right) \quad (14)$$

Differentiating equation (14) with respect to x , we obtain

$\frac{\partial}{\partial x}\ln[f_{w_1}(x;\alpha,\lambda,\mu)] = \frac{\lambda-x-m-2\alpha x}{x(\lambda+x-\mu)}$, Now $\frac{\partial}{\partial x}\ln[f_{w_1}(x;\alpha,\lambda,\mu)] = 0$

Implies $\frac{\lambda-x_0-m-2\alpha x_0}{x_0(\lambda+x-\mu)} = 0$ (15)

so $x_0 = \frac{\lambda-\mu}{1+2\alpha}$, where $1+2\alpha \neq 0$

x_0 represents the mode of DWL distribution where $w_1(x) = x$ and the derivative in equation (15) is equal to zero at x_0. ■

4.6. Moments of DWL Distribution

In this subsection we discuss the r^{th} moment of DWL distribution with $w_1(x) = x$.

Theorem 4.6.1. The r^{th} moment about the mean of DWL distribution where $w_1(x) = x$ is given by:

$$E(X-\mu)^r = \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda}\sum_{j=0}^{r}\sum_{i=0}^{r+1-j}(-1)^j\binom{r}{j}\binom{r+1-j}{i}\mu^{r+1-i}\lambda^i\beta(i+1,2\alpha-i+1) \quad (16)$$

$r = 1,2,\ldots$

and the r^{th} moment about the origin of DWL distribution where $w_1(x) = x$ is given by:

$$E(X^r) = \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda}\sum_{i=0}^{r+1}\binom{r+1}{i}\mu^{r+1-i}\lambda^i\beta(i+1,2\alpha-i+1)\ , r = 1,2,\ldots \quad (17)$$

Proof.

Using equation (6), the r^{th} moment about the mean is given by:

$$E(X-\mu)^r = \int_\mu^\infty (x-\mu)^r f_{w_1}(x;\alpha,\lambda,\mu)\, dx$$

$$= \int_\mu^\infty (x-\mu)^r \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\, x\left(1+\frac{x-\mu}{\lambda}\right)^{-2(\alpha+1)} dx$$

$$= \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\int_\mu^\infty \sum_{j=0}^{r}(-1)^j\binom{r}{j}x^{r-j}\mu^j\, x\left(1+\frac{x-\mu}{\lambda}\right)^{-2(\alpha+1)} dx$$

$$= \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\int_0^\infty \sum_{j=0}^{r}(-1)^j\binom{r}{j}\mu^j(\mu+\lambda z)^{r+1-j}(1+z)^{-2(\alpha+1)}\lambda dz$$

$$= \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda}\int_0^\infty \sum_{j=0}^{r}(-1)^j\binom{r}{j}\mu^j\mu^{r+1-j}\sum_{i=0}^{r+1-j}\binom{r+1-j}{i}\frac{\lambda^i}{\mu^i}z^i(1+z)^{-2(\alpha+1)}dz$$

$$= \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda}\sum_{j=0}^{r}\sum_{i=0}^{r+1-j}(-1)^j\binom{r}{j}\binom{r+1-j}{i}\mu^{r+1-i}\lambda^i\beta(i+1,2\alpha-i+1)$$

Thus the r^{th} moment about the mean is:

$$E(X-\mu)^r = \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda}\sum_{j=0}^{r}\sum_{i=0}^{r+1-j}(-1)^j\binom{r}{j}\binom{r+1-j}{i}\mu^{r+1-i}\lambda^i$$

$$\times\ \beta(i+1,2\alpha-i+1)\ ,\quad r = 1,2,\ldots$$

where $E(X) = \mu$, then if $\mu = 0$. Hence the r^{th} moment about the origin is:

$$E(X^r) = \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda} \sum_{i=0}^{r+1} \binom{r+1}{i} \mu^{r+1-i} \lambda^i \ \beta(i+1, 2\alpha - i + 1) \ , \ r = 1,2,\ldots \qquad \blacksquare$$

Now by using (17) the expressions for the first four raw moments were obtained by putting $r = 1, 2, 3, 4$ respectively.

$$E(X^1) = \frac{2\alpha\mu^2+2\lambda\mu}{2\alpha\mu+\lambda} + \frac{2\lambda^2}{(2\alpha\mu+\lambda)(2\alpha-1)} = \boldsymbol{Mean} \qquad (18)$$

$$E(X^2) = \frac{2\alpha\mu^3+3\lambda\mu^2}{2\alpha\mu+\lambda} + \frac{6\lambda^2\mu}{(2\alpha\mu+\lambda)(2\alpha+1)} + \frac{6\lambda^3}{(2\alpha\mu+\lambda)(2\alpha+1)(2\alpha-2)} \qquad (19)$$

$$E(X^3) = \frac{2\alpha\mu^4+4\lambda\mu^3}{2\alpha\mu+\lambda} + \frac{12\lambda^2\mu^2}{(2\alpha\mu+\lambda)(2\alpha-1)} + \frac{24\lambda^3\mu}{(2\alpha\mu+\lambda)(2\alpha-1)(2\alpha-2)} + \frac{24\lambda^4}{(2\alpha\mu+\lambda)(2\alpha-1)(2\alpha-2)(2\alpha-3)} \qquad (20)$$

$$E(X^4) = \frac{2\alpha\mu^5+5\lambda\mu^4}{2\alpha\mu+\lambda} + \frac{20\lambda^2\mu^3}{(2\alpha\mu+\lambda)(2\alpha-1)} + \frac{60\lambda^3\mu^2}{(2\alpha\mu+\lambda)(2\alpha-1)(2\alpha-2)} + \frac{120\lambda^4\mu}{(2\alpha\mu+\lambda)(2\alpha-1)(2\alpha-2)(2\alpha-3)} + \frac{100\lambda^3}{(2\alpha\mu+\lambda)(2\alpha-1)(2\alpha-2)(2\alpha-3)(2\alpha-4)} \qquad (21)$$

Also from equations (16) and (17) we can find variance , coefficient of variation, skewness and kurtosis is as follows:

The $\boldsymbol{Variance}$ is given by

$$var(X) = E(X-\mu)^2 = \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda} \sum_{j=0}^{2} \sum_{i=0}^{3-j} (-1)^j \binom{2}{j} \binom{3-j}{i} \mu^{3-i} \lambda^i \times \beta(i+1, 2\alpha - i + 1) \qquad (22)$$

The $\boldsymbol{Coefficient\ of\ Variation}$ is given by

$$CV = \frac{\sigma}{\mu} = \frac{\sqrt{var(X)}}{\mu = E(X)} = \frac{\sqrt{\frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda} \sum_{j=0}^{2} \sum_{i=0}^{3-j} (-1)^j \binom{2}{j}\binom{3-j}{i} \mu^{3-i} \lambda^i \ \beta(i+1,2\alpha-i+1)}}{\frac{2\alpha\mu^2+2\lambda\mu}{2\alpha\mu+\lambda} + \frac{2\lambda^2}{(2\alpha\mu+\lambda)(2\alpha-1)}} \qquad (23)$$

The $\boldsymbol{Coefficient\ of\ Skewness}$ is given by

$$CS = \frac{E(X-\mu)^3}{[var(X)]^{\frac{3}{2}}} = \frac{\frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda} \sum_{j=0}^{3} \sum_{i=0}^{4-j} (-1)^j \binom{3}{j}\binom{4-j}{i} \mu^{4-i} \lambda^i \ \beta(i+1,2\alpha-i+1)}{\left[\frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda} \sum_{j=0}^{2} \sum_{i=0}^{3-j} (-1)^j \binom{2}{j}\binom{3-j}{i} \mu^{3-i} \lambda^i \ \beta(i+1,2\alpha-i+1)\right]^{\frac{3}{2}}} \qquad (24)$$

The $\boldsymbol{Coefficient\ of\ Kurtosis}$ is given by

$$CK = \frac{E(X-\mu)^4}{[var(X)]^2} = \frac{\frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda} \sum_{j=0}^{4} \sum_{i=0}^{5-j} (-1)^j \binom{4}{j}\binom{5-j}{i} \mu^{5-i} \lambda^i \ \beta(i+1,2\alpha-i+1)}{\left[\frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda} \sum_{j=0}^{2} \sum_{i=0}^{3-j} (-1)^j \binom{2}{j}\binom{3-j}{i} \mu^{3-i} \lambda^i \ \beta(i+1,2\alpha-i+1)\right]^2} \qquad (25)$$

4.7. Moment Generating Function of DWL Distribution

Theorem 4.7.1. The moment generating function of DWL distribution where $w_1(x) = x$ is given by:

$$M_X(t) = \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda} \sum_{j=0}^{\infty} e^{t\mu}\, t^j \left[\frac{\mu\,\lambda^j\ \Gamma(2\alpha-j)+\lambda^{j+1}\,(j+1)\,\Gamma(2\alpha-j)}{\Gamma(2\alpha+2)}\right] \tag{26}$$

Proof.

$$M_X(t) = E[e^{tx}] = \int_\mu^\infty e^{tx}\ f_{w_1}(x;\alpha,\lambda,\mu)\ dx$$

$$= \int_\mu^\infty e^{tx}\ \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\ x\left(1+\frac{x-\mu}{\lambda}\right)^{-2(\alpha+1)} dx$$

$$= \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\int_0^\infty e^{t(\mu+\lambda z)}\,(\mu+\lambda z)\ (1+z)^{-2(\alpha+1)}\ \lambda dz$$

$$= \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda}\int_0^\infty e^{t\mu}\sum_{j=0}^{\infty}\frac{(t\lambda z)^j}{j!}\ \mu\sum_{k=0}^{1}\binom{1}{k}\frac{\lambda^k}{\mu^k}\, z^k\,(1+z)^{-2(\alpha+1)}\ dz$$

$$= \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda}\sum_{j=0}^{\infty}\sum_{k=0}^{1}\binom{1}{k} e^{t\mu}\frac{t^j}{j!}\,\mu^{1-k}\,\lambda^{j+k}\ \beta(j+k+1, 2\alpha-j-k+1)$$

$$= \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda}\sum_{j=0}^{\infty} e^{t\mu}\frac{t^j}{j!}\left[\mu\,\lambda^j\ \beta(j+1, 2\alpha-j+1)+\lambda^{j+1}\,\beta(j+2, 2\alpha-j)\right]$$

$$= \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda}\sum_{j=0}^{\infty} e^{t\mu}\, t^j\left[\frac{\mu\,\lambda^j\ \Gamma(2\alpha-j)+\lambda^{j+1}\,(j+1)\,\Gamma(2\alpha-j)}{\Gamma(2\alpha+2)}\right]$$

then $M_X(t) = \frac{4\alpha^2+2\alpha}{2\alpha\mu+\lambda}\sum_{j=0}^{\infty} e^{t\mu}\, t^j\left[\frac{\mu\,\lambda^j\ \Gamma(2\alpha-j)+\lambda^{j+1}\,(j+1)\,\Gamma(2\alpha-j)}{\Gamma(2\alpha+2)}\right]$ ■

5. Order Statistics of DWL Distribution

Order statistics make their appearance in many statistical theory and practice .The order statistics are random variables that satisfy $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$.

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from the DWL distribution with cdf $F_{w_1}(x,\boldsymbol{\vartheta})$ and pdf $f_{w_1}(x,\boldsymbol{\vartheta})$, where $\boldsymbol{\vartheta} = (\alpha,\lambda,\mu)$ then the probability density function of r^{th} order statistics $X_{(r)}$ is given by:

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!\,(n-r)!} f_{w_1}(x,\boldsymbol{\vartheta})\ \left[F_{w_1}(x,\boldsymbol{\vartheta})\right]^{r-1}\ \left[1-F_{w_1}(x,\boldsymbol{\vartheta})\right]^{n-r},\ x>0 \tag{27}$$

For $r = 1,2,\ldots,n$

by substituting (6) and (7) in above equation , then the probability density function of r^{th} order statistics for DWL distribution is given by:

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!\,(n-r)!}\ \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\ x\left(1+\frac{x-\mu}{\lambda}\right)^{-2(\alpha+1)}$$

$$\times\left[\frac{\lambda+2\alpha\mu-2\alpha x\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha+1)}-\lambda\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha)}}{2\alpha\mu+\lambda}\right]^{r-1}\left[\frac{2\alpha x\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha+1)}+\lambda\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha)}}{2\alpha\mu+\lambda}\right]^{n-r}$$

Therefore , the pdf of the largest order statistic $X_{(n)}$ is given by:

$$f_{X_{(n)}}(x) = n\ \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\ x\left(1+\frac{x-\mu}{\lambda}\right)^{-2(\alpha+1)}\left[\frac{\lambda+2\alpha\mu-2\alpha x\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha+1)}-\lambda\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha)}}{2\alpha\mu+\lambda}\right]^{n-1} \tag{28}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by:

$$f_{X_{(1)}}(x) = n\ \frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\ x\ \left(1+\frac{x-\mu}{\lambda}\right)^{-2(\alpha+1)}\left[\frac{2\alpha x\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha+1)}+\lambda\left(1+\frac{x-\mu}{\lambda}\right)^{-(2\alpha)}}{2\alpha\mu+\lambda}\right]^{n-1} \tag{29}$$

6. Entropy of DWL Distribution

The entropy of a random variable X with probability density $f(x)$ is a measure of variation of the uncertainty . A large value of entropy indicates the greater uncertainty in the data. It is an important concept in many fields of science, especially theory of communication, physics and probability. The Rényi entropy is defined as:

$$I_R(\rho) = \frac{1}{1-\rho} log\left[\int_0^\infty (f(x))^\rho\ dx\right] \tag{30}$$

where $\rho > 0$ and $\rho \neq 1$

Theorem 6.1. The Rényi entropy of DWL distribution where $w_1(x) = x$ is given by:

$$I_R(\rho) = \frac{1}{1-\rho}\ log\left[\left(\frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\right)^\rho \sum_{i=0}^{\rho}\binom{\rho}{i}\mu^{\rho-i}\ \lambda^{i+1}\beta(i+1, 2\alpha\rho-i+2\rho-1)\right] \tag{31}$$

Proof. Since

$$\begin{aligned}
I_R(\rho) &= \frac{1}{1-\rho} log\left[\int_\mu^\infty \left(f_{w_1}(x;\alpha,\lambda,\mu)\right)^\rho\ dx\right] \\
&= \frac{1}{1-\rho} log\left[\int_\mu^\infty \left(\frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\right)^\rho\ x^\rho\ \left(1+\frac{x-\mu}{\lambda}\right)^{-2\rho(\alpha+1)}\ dx\right] \\
&= \frac{1}{1-\rho} log\left[\left(\frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\right)^\rho \int_0^\infty (\mu+\lambda z)^\rho\ (1+z)^{-2\rho(\alpha+1)}\ \lambda\, dz\right] \\
&= \frac{1}{1-\rho} log\left[\left(\frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\right)^\rho \int_0^\infty \mu^\rho \sum_{i=0}^{\rho}\binom{\rho}{i}\frac{\lambda^i}{\mu^i} z^i\ (1+z)^{-2\rho(\alpha+1)}\ \lambda dz\right] \\
&= \frac{1}{1-\rho} log\left[\left(\frac{4\alpha^2+2\alpha}{2\alpha\lambda\mu+\lambda^2}\right)^\rho \sum_{i=0}^{\rho}\binom{\rho}{i}\mu^{\rho-i}\ \lambda^{i+1}\ \beta(i+1, 2\alpha\rho-i+2\rho-1)\right] \quad \blacksquare
\end{aligned}$$

7. Maximum Likelihood Estimation

The maximum likelihood estimates (MLEs) of the parameters was determine, in this section. Let $x_1, x_2, \dots, x_n$ be a random sample of size n from the DWL distribution and $\boldsymbol{\Theta} = (\alpha, \lambda, \mu)$. The likelihood function of (6) is given by:

$$l(\boldsymbol{\Theta}) = f(x_1, x_2, \dots, x_n; \boldsymbol{\Theta}) = \prod_{i=1}^{n} f(x_i; \boldsymbol{\Theta}) \tag{32}$$

By taking the logarithm of (32), we have the log- likelihood function

$$\mathrm{L} = \log l(\boldsymbol{\Theta}) = n \log\left(4\alpha^2 + 2\alpha\right) - n \log(2\alpha\lambda\mu + \lambda^2) + \sum_{i=1}^{n} \log(x_i)$$

$$-2(\alpha + 1) \sum_{i=1}^{n} \log\left(1 + \frac{x_i - \mu}{\lambda}\right) \quad (33)$$

Differentiating (33) with respect to α , λ and μ, then equating it to zero, we obtain the estimating equations are

$$\frac{\partial \mathrm{L}}{\partial \alpha} = \frac{n(4\alpha+1)}{(2\alpha^2+\alpha)} - \frac{2\mu n}{(2\alpha\mu+\lambda)} - 2\sum_{i=1}^{n} \log\left(1 + \frac{x_i - \mu}{\lambda}\right) \quad (34)$$

$$\frac{\partial \mathrm{L}}{\partial \lambda} = -\frac{n(2\alpha\mu+2\lambda)}{(2\alpha\lambda\mu+\lambda^2)} + \sum_{i=1}^{n} \frac{(2\alpha x_i - 2\alpha\mu - 2x_i - 2\mu)}{\left(\lambda^2 + \lambda(x_i - \mu)\right)} \quad (35)$$

$$\frac{\partial L}{\partial \mu} = -\frac{2\alpha n}{(2\alpha\mu+\lambda)} + \sum_{i=1}^{n} \frac{(2\alpha+2)}{(\lambda + x_i - \mu)} \quad (36)$$

It is more convenient to use quasi Newton algorithm to numerically maximize the log-likelihood function given in equation (33) to yield the maximum likelihood (ML) estimators α , λ and μ respectively.

8. Application

In this section , we provide application with real data to illustrate the importance of the DWL distribution . We consider $\mu = 0$ for the DWL distribution . We have considered a dataset corresponding to remission times (in months) of a random sample of 128 bladder cancer patients given in Lee and Wang (2003).

The data are given as follows : 0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73,2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Fitted the DWL distribution to the dataset using MLE , and compared the proposed DWL distribution with exponentiated transmuted generalized Raleigh (ETGR), Lindley (Li) and transmuted complementary Weibull geometric (TCWG) models.

Table1 shows the MLEs the model parameters . The model selection is carried out using the AIC (Akaike information criterion), the BIC (Bayesian information criterion), the CAIC (consistent Akaike information criteria) and the HQIC (Hannan-Quinn information criterion):

$$\text{AIC} = -2L(\widehat{\boldsymbol{\Theta}}) + 2q \quad , \quad \text{BIC} = -2L(\widehat{\boldsymbol{\Theta}}) + q\log(n)$$

$$\text{CAIC} = -2L(\widehat{\boldsymbol{\Theta}}) + \frac{2qn}{n-q-1} , \quad \text{HQIC} = -2L(\widehat{\boldsymbol{\Theta}}) + 2q\log\left(\log(n)\right)$$

where $L(\widehat{\boldsymbol{\Theta}})$ denotes the log-likelihood function evaluated at the maximum likelihood estimates , q is the number of parameters, and n is the sample size.
Here we let $\boldsymbol{\Theta}$ denotes the parameters, i.e. $\boldsymbol{\Theta} = (\alpha, \lambda)$ by puttig $\mu = 0$.

Table 1. MLE for the cancer data.

Models	**Estimates**					
	$\hat{\boldsymbol{\alpha}}$	$\hat{\boldsymbol{\lambda}}$	$\hat{\boldsymbol{\beta}}$	$\hat{\boldsymbol{\delta}}$	$\hat{\boldsymbol{\theta}}$	$\hat{\boldsymbol{\gamma}}$
DWL	1.6859	11.2012	---	---	---	---
ETGR	7.3762	0.118	0.0473	0.0494	---	---
Li	---	---	---	---	0.196	
TCWG	106.0694	0.2168	1.7115	---	---	0.0095

Table 2. The statistics AIC, BIC, CAIC and HQIC for the cancer data

Models	**Statistic**				
	L	**AIC**	**BIC**	**CAIC**	**HQIC**
DWL	**- ٤١٠.٨٦٨١**	**827.7362**	**836.2922**	**827.9297**	**831.2125**
ETGR	- 429.175	866.35	877.758	866.675	870.985
Li	- 419.529	841.06	843.8920	841.091	842.1987
TCWG	- 410.9975	829.995	841.403	830.32	834.63

In Table 2 , we compare the fits of the DWL model with the ETGR , Li and TCWG models. We note that the DWL model has the lowest values for the L , AIC, BIC, CAIC and HQIC statistics (for the cancer data) among the fitted models. So, the DWL model could be chosen as the best model. Therefore, These new distribution can be better modelled than other competitive lifetime models.

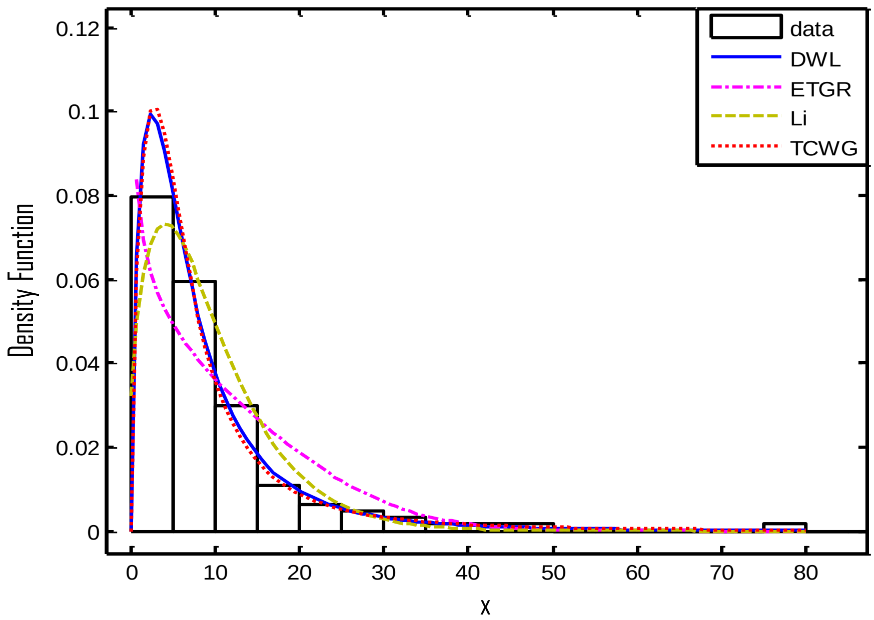

Figure 7. Estimated densities for bladder cancer data

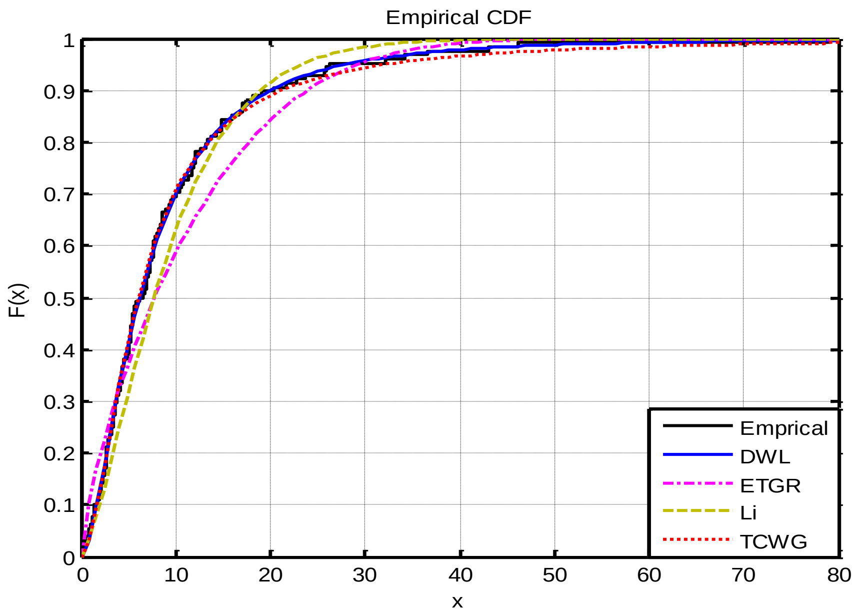

Figure 8. Estimated cdf for bladder cancer data

9. Conclusions

A double weighted Lomax distribution was presented in this paper and discussed some statistical properties of the new model . It can be observed that the MLE of the unknown parameters can be obtained method numerically, and we proved can that these new distribution can be better modelled than other competitive lifetime models.

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