Some Generalizations of Fully Dual-Stable Modules

M.S.ABBAS
Department of Mathematics, College of Science, Mustansiriya University, Iraq
amsaj59@yahoo.com

A. M. A. Al-HOSAINY
Department of Mathematics, College of Science, Mustansiriya University, Iraq
zsene12@yahoo.com

Abstract
In a previous paper we introduce the concept of full d-stability, in this work several types of generalizations were introduced ; minimal (maximal) d-stable; fully pseudo d-stable and afd-stable module. A dual to the notion of terse module is, also, introduced namely d-terse and it is shown that it is coincide with fully pseudo d-stable.

Keywords: minimal d-stable, maximal d-stable, fully pseudo d-stable, d-terse and afd-stable modules.

1. Introduction

Throughout, all rings are associative having an identity and all modules are unital . R is a ring and M is a left R-module (simply we say module). In a previous paper[2] we introduced the concept of full d-stability as a dual to fully stability which was introduced by ABBAS [1]. Several types of generalizations of full d-stability are introduced and investigated in this work. In the second section an equivalent statement to the definition of fully d-stable module is proved, which will serve in the way of generalizations. Section 3 consists of minimal and maximal d-stability, their properties and related results. It is proved that in case of torsion free module over an integral domain maximal d-stability implies some generalization of quasi-projectivity. Fully pseudo d-stable module is introduced and investigated in section 4, the main result of this section (also in the case of torsion free module over integral domain) is " full d-stability implies some generalization of pseudo projectivity". Recall that" An R-module M is said to be *pseudo projective* if for any given R- module A and any two R-epimorphisms $f,g:\mathrm{M}\to\mathrm{A}$ there exists a homomorphism $h:\mathrm{M}\to\mathrm{M}$ such that $f=g\circ h$ ".[5]

In section 5 the notion of d-terse module (dual to terse module) is introduced, many properties of this new type of modules are proved, it is shown that d-terse module is exactly the fully pseudo d-stable module. In the last section other generalization of fully d-stable module namely almost fully d-stable (shortly afd-stable) module is defined and its properties are investigated and it is shown that for local modules full d-stability and afd-stability coincide .

2. Equivalent Condition and a lemma.

We start with an equivalent statement to the definition of a fully d-stable module. Recall that a module M is *fully d-stable* if for each submodule N and for each homomorphism $\alpha:M\to M/N$, $N\subset Ker(\alpha)$ [2] . A submodule N of a module M is *d-stable* if for each $\alpha:M\to M/N$, $N\subset Ker(\alpha)$ [2].

Theorem 2.1. Let M be an R-module. M is fully d-stable if and only if $\ker g\subset\ker f$ for each R- module A and any two R-homomorphisms $f,g:\mathrm{M}\to\mathrm{A}$ with g surjective .

Proof. <u>Necessity</u> , assume that M is fully d-stable and $f,g:\mathrm{M}\to\mathrm{A}$ with g surjective, where A is any R-module. Let $\mathrm{N}=\ker g$, then A is isomorphic to $\mathrm{M/N}$ (say $\varphi:\mathrm{A}\to\mathrm{M/N}$ is an isomorphism) , hence $\alpha=\varphi\circ f:\mathrm{M}\to\mathrm{M/N}$ and $\ker\alpha=\ker f$ (since φ is an isomorphism) . By hypothesis $\mathrm{N}\subset\ker\alpha=\ker f$. Therefore $\ker g\subset\ker f$.

Sufficiency If N is a submodule of M , and $\alpha : M \to M/N$ a homomorphism, then $\pi_N : M \to M/N$, the natural epimorphism, is surjective, hence by hypothesis $\ker \pi_N \subset \ker \alpha$, that is , $N \subset Ker(\alpha)$. ◊

A dualization of the above theorem gives an equivalent statement to the definition of fully stable modules. Recall that "a module M is *fully stable* if $f(N) \subset N$ for each submodule N and for each homomorphism $f : N \to M$ " [1], "a submodule N of a module M is *stable* if $f(N) \subset N$ for each homomorphism $f : N \to M$ " [1].

Theorem 2.2. Let M be a module, the following conditions are equivalent:

(i) Every submodule of M is stable.

(ii) For each module A and for any two homomorphisms $f, g : A \to M$, with g injective, $\operatorname{Im} f \subset \operatorname{Im} g$.

Proof: (i)$\Rightarrow$(ii). Let $f, g : A \to M$ with g injective, then if $N = \operatorname{Im} g$, there exists $h : N \to A$ such that $h \circ g = 1_A$, let $\varphi = f \circ h : N \to M$, by (i) N is stable, so we have $\varphi(N) \subset N$ which implies $f(h(N)) \subset N$,that is, $\operatorname{Im} f \subset \operatorname{Im} g$.
(ii)$\Rightarrow$(i) If N is a submodule of M , and $f : N \to M$ be a homomorphism , let $i : N \to M$ be the inclusion map (which is injective) then by (ii) $f(N) \subset i(N) = N$. ◊

We need the following lemma in later.

Lemma 2.3. If M is a fully d-stable R -module, and $\alpha : M \to M/N$ is an epimorphism, then

(i) $N = \ker \alpha$ and

(ii) $N \subset K$ implies $\alpha(K) = K/N$, where N and K are submodules of M .

Proof. (i) Let $\alpha : M \to M/N$ be an epimorphism, then there exists an isomorphism $\varphi : M/N \to M/\ker \alpha$, let $\beta : M \to M/N$, where $\beta = \varphi \circ \pi$, and π is the natural map of M onto M/N , then $\ker \beta = \pi^{-1}(\ker \varphi) = \pi^{-1}(\bar{0}) = N$, by fully d-stability , $\ker \alpha \subset \ker \beta = N$, but $N \subset \ker \alpha$ (also by fully d-stability). Therefore $N = \ker \alpha$.
(ii) $N \subset K$ implies $\theta : M/N \to M/K$ defined by $x + N \mapsto x + K$ is a well defined epimorphism , if $\alpha : M \to M/N$ is an epimorphism, then $\theta \circ \alpha : M \to M/K$ is an epimorphism too and by (i) $\ker(\theta \circ \alpha) = K$ which implies $\alpha^{-1}(\ker \theta) = K$, hence $\alpha^{-1}(K/N) = K$, so $\alpha(K) = K/N$ (since α is surjective). ◊

3. Minimal and Maximal d-Stable Modules

Now we will introduce some generalizations to the concepts of fully d-stable, duo and quasi-projective modules (other generalizations are coming in the following sections), and study properties and relationships.

Definitions 3.1. Let M be an R -module.

(i) M is said to be minimal(maximal) d-stable if each minimal(maximal) submodule of M is d-stable.

(ii) M is said to be minimal(maximal) quasi projective if for each minimal(maximal) submodule N of M and each $\alpha : \mathrm{M} \to \mathrm{M}/\mathrm{N}$, there exists an endomorphism h of M such that $\pi \circ h = \alpha$. where π is the natural map of M onto M/N.

(iii) M is said to be minimal(maximal) duo module if for each endomorphism f of M and each minimal(maximal) submodule N, $f(\mathrm{N}) \subset \mathrm{N}$ (or equivalently $\mathrm{N} \subset f^{-1}(\mathrm{N})$).

Similar to the general case, the relations between the concepts : duo, quasi-projective and fully d-stable, also hold in the minimal and maximal versions (see [2]), that is, we have: (a) Any minimal(maximal) d-stable module is minimal(maximal) duo; and (b) If a module is minimal(maximal) duo and minimal(maximal) quasi-projective, then it is minimal(maximal) d-stable.

Minimal and maximal d-stability hold for certain kind of modules. Recall that a module in which all its proper submodules are small is said to be *hollow* (dual to the *uniform* module in which all non zero submodules are large) See the following results.

Proposition 3.2. Let M be an R-module.

- **(a)** If M is uniform which is not isomorphic to a submodule of M/N for each minimal submodule N of M, then M is minimal d-stable.
- **(b)** If M is hollow, then it is maximal d-stable.
- **(c)** If M is local, then it is maximal d-stable.

Proof: (a) Assume that M is uniform and not minimal d-stable, let N be a minimal submodule of M which is not d-stable, then there exists a homomorphism $\varphi : \mathrm{M} \to \mathrm{M}/\mathrm{N}$ with $\mathrm{N} \not\subset \ker\varphi$, hence $\mathrm{N} \cap \ker\varphi$ is a proper submodule of N which implies $\ker\varphi = 0$, that is, φ is a monomorphism .

(b) Let N be a maximal submodule of M, and let $\varphi : \mathrm{M} \to \mathrm{M}/\mathrm{N}$, we may assume that $0 \neq \varphi$ and hence it is surjective, which implies $\mathrm{M}/(\ker\varphi) \cong \mathrm{M}/\mathrm{N}$, hence $\ker\varphi$ is maximal but $\ker\varphi \subset \mathrm{N} + \ker\varphi \neq \mathrm{M}$ implies $\mathrm{N} \subset \ker\varphi$.

(c) Let N be the unique maximal submodule of M, and let $\varphi : \mathrm{M} \to \mathrm{M}/\mathrm{N}$, we also assume that $0 \neq \varphi$ and then $\mathrm{M}/(\ker\varphi) \cong \mathrm{M}/\mathrm{N}$, hence $\ker\varphi$ is maximal and then $\ker\varphi = \mathrm{N}$. ◊

Note that a converse statement of part (a) of the above proposition generally holds (without the uniform and minimal conditions), that is: If N is a non trivial d-stable submodule of a module M, then M cannot be isomorphic to a submodule of M/N.

On the other hand, Q as a Z-module (which is uniform) is a minimal d-stable(trivially) but not fully d-stable module; $\mathrm{Z}/(pq\mathrm{Z})$ (p *and* q *distinct prime numbers*) is a fully d-stable Z-module, hence minimal d-stable which is not uniform . Q_p as a Z-module is a hollow and hence it is maximal d-stable module but not fully d-stable (see example 6.10).

If M is a simple R-module, then both $\mathrm{M} \oplus 0$ and $0 \oplus \mathrm{M}$ are maximal submodules in $\mathrm{M} \oplus \mathrm{M}$ which are not d-stable . On the other hand, if M (not necessarily local) has a unique maximal submodule, then it is maximal d-stable(see the proof of part (c)). The Z-module $\mathrm{Z}_{(p)}$ (the localization of Z at any prime number p) is such a module.

A more general result is in the following .

Lemma 3.3. If N and K are two submodules of a module M such that each of them is not contained in the other and such that $\mathrm{M}/\mathrm{N} \cong \mathrm{M}/\mathrm{K}$, then both of N and K are not d-stable.

Proof: Let $\varphi : \mathrm{M/N} \to \mathrm{M/K}$ be an isomorphism and $\pi : \mathrm{M} \to \mathrm{M/N}$ be the natural epimorphism, then $\alpha = \varphi \circ \pi : \mathrm{M} \to \mathrm{M/K}$ is a homomorphism and $\ker \alpha = \pi^{-1}(\ker \varphi) = \mathrm{N}$, hence by hypothesis $\mathrm{K} \not\subset \ker \alpha$ which implies K is not d-stable. Similarly N is not d-stable. ◊

Corollary 3.4. If N and K are two distinct maximal submodules of an R-module M such that $\mathrm{M/N} \cong \mathrm{M/K}$, then both of N and K are not d-stable. ◊

Example 3.5. Let $\mathrm{M} = (\mathrm{Z}/2\mathrm{Z}) \oplus (\mathrm{Z}/6\mathrm{Z})$. As a Z-module, M has three maximal submodules with six elements, which are not d-stable, and one d-stable maximal submodule having four elements. (easy check) ◊

The above proposition motivates introducing the following type of submodules and modules. A maximal submodule N of a module M will be called *u-maximal*, if $\mathrm{M/N}$ is not isomorphic to $\mathrm{M/K}$ for any other maximal submodule K. The module is said to be u-maximal if all its maximal submodules are u-maximal. In the light of these new notations and the previous results we can say in the help of corollary 3.4 that: *a module is maximal d-stable if and only if it is u-maximal.* Many examples of u-maximal modules exist; the Z-modules $\mathrm{Z},\ \mathrm{Z}/n\mathrm{Z},\ \mathrm{Z}_{(p)}$ *and* Q, also any local module. Any fully d-stable module is u-maximal. The module of example 3.5 is not u-maximal and if M is any simple module, then $\mathrm{M} \oplus \mathrm{M}$ is not u-maximal module. An easy check showing that a homomorphic image of u-maximal module is again u-maximal but submodule of u-maximal module need not be u-maximal.

An equivalent statement to the definition of minimal(maximal) d-stable module, is the following. The proof is similar to that of theorem 2.1.

Proposition 3.3. Let M be an R-module. M is minimal (maximal) d-stable if and only if for each R-module A and any two R-homomorphisms $f, g : \mathrm{M} \to \mathrm{A}$ with g surjective and $\ker g$ is minimal, $\ker g \subset \ker f$ (and $\ker g$ is maximal, $\ker g = \ker f$). ◊

Other results about the minimal and maximal d-stability are in the following.

Proposition 3.4. An R-module M is minimal quasi projective if and only if for each R-module A and any two R-homomorphisms $f, g : \mathrm{M} \to \mathrm{A}$ with g surjective and $\ker g$ is minimal, there exists an endomorphism h of M such that $g \circ h = f$.

Proof. <u>Necessity.</u> Assume that M is minimal quasi projective (in the sense of definition 3.1(ii)).
Let $f, g : \mathrm{M} \to \mathrm{A}$ be two R-homomorphisms with g surjective and $\ker g$ is minimal, let $\mathrm{N} = \ker g$, then A is isomorphic to $\mathrm{M/N}$.
If $\varphi : \mathrm{A} \to \mathrm{M/N}$ is the isomorphism such that $\varphi \circ g = \pi$, where π is the natural map of M onto $\mathrm{M/N}$, $\varphi \circ f : \mathrm{M} \to \mathrm{M/N}$ is a homomorphism, where N is a minimal submodule, then, by 3.1(ii), there exists an endomorphism h of M such that $\pi \circ h = \varphi \circ f \Rightarrow (\varphi \circ g) \circ h = \varphi \circ f \Rightarrow \varphi \circ (g \circ h) = \varphi \circ f \Rightarrow g \circ h = f$.
<u>Sufficiency.</u> Assume that the condition in the proposition holds , let N be a minimal submodule of M and $\alpha : \mathrm{M} \to \mathrm{M/N}$ be a homomorphism, set $\mathrm{A} = \mathrm{M/N}$, then by the hypothesis there exists an endomorphism h of M such that $\pi \circ h = \alpha$, that is , M is minimal quasi projective (3.1(ii)). ◊

Proposition 3.5. Let M be a torsion free module over an integral domain R. If M is maximal d-stable, then it is maximal quasi projective.

Proof: Let $\alpha:\mathrm{M}\to\mathrm{M/N}$, be a homomorphism where N is a maximal submodule of M, we may assume that $\alpha\neq 0$, for if $\alpha=0$, $f=0$ is an endomorphism with $\pi\circ f=\alpha$. Then $\mathrm{N}\subset\ker\alpha$ implies $\mathrm{N}=\ker\alpha$ (note that we assume). There exists $x_0\in\mathrm{M}-\mathrm{N}$, so $x_0\notin\ker\alpha$, hence $\alpha(x_0)=y+\mathrm{N}$ for some $y\notin\mathrm{N}$. N is maximal implies $y=r_0x_0+n$ for some $r_0\in R$ *and* $n\in\mathrm{N}$, hence $\alpha(x_0)=r_0x_0+n+\mathrm{N}=r_0x_0+\mathrm{N}$.

We claim that the desired endomorphism f is $f(x)=r_0x,\ x\in\mathrm{M}$.

Note that $\pi(f(x_0))=\pi(r_0x_0)=r_0x_0+\mathrm{N}=\alpha(x_0)$. If $x\in\mathrm{N}$, then $\alpha(x)=0$ and $\pi(f(x))=r_0x+N=0$, since $r_0x\in\mathrm{N}$. If $x\in\mathrm{M}-\mathrm{N}$, then $x=sx_0+m$, for some $s\in R, m\in\mathrm{N}$, (since N is maximal), hence

$$\alpha(x)=\alpha(sx_0+m)=s\alpha(x_0)=sr_0x_0+\mathrm{N}=r_0sx_0+\mathrm{N}=r_0sx_0+r_0m+\mathrm{N}$$

$=r_0(sx_0+m)+\mathrm{N}=rx+\mathrm{N}=\pi(f(x))$. Therefore $\pi\circ f=\alpha$. ◊

Corollary 3.8. Let M be a torsion free module over an integral domain R. If M is duo, then it is maximal d-stable if and only if it is maximal quasi projective. ◊

4. Fully Pseudo d-Stable module

In [1], ABBAS introduce the concept of fully pseudo-stable module as a generalization of full stability and investigated its properties and relations with the original concept. He defined : " A submodule N of an R-module M is called pseudo-stable if $f(\mathrm{N})\subset\mathrm{N}$ for each R-monomorphism $f:\mathrm{N}\to\mathrm{M}$. An R-module in which all submodules are pseudo-stable is called fully pseudo-stable ."[1]

In this section the concept of pseudo d-stability is introduced, investigating its characterizations and relationship with known concepts.

Definition 4.1. Let M be a module:

(i) M is said to be fully pseudo d-stable if for each R-module A and any two R- epimorphisms $f,g:\mathrm{M}\to\mathrm{A}$, $\ker f=\ker g$.

(ii) A submodule N of M is said to be pseudo d-stable if for each epimorphism $\alpha:\mathrm{M}\to\mathrm{M/N}$, $\ker\alpha=\mathrm{N}$.

(iii) M is said to be pseudo duo module if for each surjective endomorphism f of M and each submodule N, $f(\mathrm{N})\subset\mathrm{N}$.

(iv) M is said to be minimal pseudo d-stable if for each R-module A and any two R-epimorphisms $f,g:\mathrm{M}\to\mathrm{A}$,with $\ker g$ minimal, $\ker f=\ker g$.

(v) M is said to be minimal pseudo projective if each R-module A and any two R-epimorphisms $f,g:\mathrm{M}\to\mathrm{A}$,with $\ker g$ minimal, there exists a homomorphism $h:\mathrm{M}\to\mathrm{M}$ such that $f=g\circ h$.

It is clear from definition, that fully d-stable module is fully pseudo d-stable. Also the following results are either straight forward from definitions or have similar proofs to similar results in previous section and in [2], so we omit the proofs.

proposition 4.2. Let M be an R-module.

(i) If M is fully pseudo d-stable, then it is pseudo duo module.

(ii) If M is pseudo projective and duo module, then it is fully pseudo d-stable.

(iii) M is fully pseudo d-stable, if and only if each of its submodules is pseudo d -stable.

(iv) M is minimal pseudo projective if and only if for each R -module A and any two R -epimorphisms $f, g : \mathrm{M} \to \mathrm{A}$ with $\ker g$ is minimal, there exists an endomorphism h of M such that $g \circ h = f$. ◊

Proposition 4.3. A maximal submodule of a module is d-stable if and only if it is pseudo d-stable.

Proof: Necessity. Clear .
Sufficiency. Assume that M is a module, N a maximal submodule which is pseudo d-stable, and $\alpha : \mathrm{M} \to \mathrm{M}/\mathrm{N}$ is a homomorphism, since N is maximal, M/N is simple, hence α is either zero (in this case $\mathrm{N} \subset \mathrm{M} = \ker\alpha$) or is an epimorphism, in this case $\mathrm{N} \subset \ker\alpha$ since N is pseudo d-stable. Therefore N is d-stable. ◊

Corollary 4.4. Every fully pseudo d-stable module is maximal d-stable ◊

The converse of the above corollary is not true. Consider $Z_{(p)}$ (the localization of Z at a prime p) as a Z -module, it has a unique maximal submodule, namely $pZ_{(p)}$ and hence it is maximal d-stable (see the remark after proposition 3.2). On the other hand $Z_{(p)}\Big/(\frac{1}{q})Z \cong Z_{(p)}\Big/(\frac{1}{s})Z$ for any two primes q and s distinct from p , that is, $Z_{(p)}$ is not fully pseudo d-stable.

Corollary 4.5. Let M be a torsion free module over an integral domain R . If M is fully pseudo d-stable, then it is maximal quasi projective.

Proof: By corollary 3.8 and proposition 4.6. ◊

Theorem 4.6. Let M be a torsion free module over an integral domain R . If M is fully d-stable, then it is minimal pseudo projective.

Proof. Let $\alpha : \mathrm{M} \to \mathrm{M}/\mathrm{N}$ be an epimorphism , where N is a minimal submodule of M , we claim that M/N is torsion free R-module too, and hence (by corollary 2.12, [2]) , there exists r in R such that $\alpha(x) = rx + \mathrm{N}$ for all $x \in \mathrm{M}$. Therefore $f(x) = rx$ is an endomorphism of M satisfying $\pi \circ f = \alpha$, which implies that M is minimal pseudo projective.

Now we will prove our claim . Since M is fully d-stable and α is an epimorphism we have $\mathrm{N} = \ker\alpha$, that is, $\alpha(x) = \overline{0} \leftrightarrow x \in \mathrm{N}$. Let $x \in \mathrm{M} \setminus \mathrm{N}$, then $\alpha(x) \neq \overline{0}$, but $\alpha(x) = rx + \mathrm{N}$ for some r in R ([2], Theorem 2.11), which implies $\mathrm{N} \neq Rx$, and since N is minimal $Rx \cap \mathrm{N} = 0$ holds, hence $r \neq 0$ and $x + \mathrm{N} \neq \overline{0}$ implies $r(x + \mathrm{N}) \neq \overline{0}$ (note that M is torsion free by hypothesis). The claim is proved. ◊

5. d-terse module

A module M is said to be *terse* if distinct submodules of M are not isomorphic [6]. The relation between this concept and the concept of full stability was investigated in [1] and was proved that a module is terse if and only if it is fully pseudo stable. In this section we introduce the concept of d-terse module and investigate its properties and its relation with full d-stability, we will show that a module is d-terse if and only if it is fully pseudo d-stable .

First we give an equivalent statement to the definition of a terse module.

Proposition 5.1. A module M is terse if and only if for each two modules A, B and any two monomorphisms $f \in Hom(A, M)$, $g \in Hom(B, M)$, $A \cong B$ implies $\operatorname{Im} f = \operatorname{Im} g$.

Proof: ($\Rightarrow$) Assume that M is terse and A, B are two modules, $f \in Hom(A, M)$, $g \in Hom(B, M)$ are monomorphisms and $A \cong B$, then $\operatorname{Im} f$ and $\operatorname{Im} g$ are two isomorphic submodules of M, hence $\operatorname{Im} f = \operatorname{Im} g$ (by definition of terse module).
($\Leftarrow$) If M is not terse , then there exist two isomorphism distinct submodules, say A and B . Now if i *and* j are the inclusion maps of A and B into M (which are monomorphisms) with
$\operatorname{Im} i = A \neq B = \operatorname{Im} j$ but $A \cong B$. ◊

The above proposition motivates the following concept:

Definition 5.2. An R-module M is *d-terse*, if for each pair of epimorphisms $f : M \to A$ and $g : M \to B$, where A and B are any two isomorphic R-modules we must have $\ker f = \ker g$. A ring R is d-terse if it is d-terse R-module.

The first result of the definition is that we get a characterization for pseudo fully d-stable module, see the following.

Theorem 5.3. A module is d-terse if and only if it is fully pseudo d-stable.

Proof: Necessity. Let $f, g : M \to A$ be two epimorphisms, then $\ker f = \ker g$ (by definition 5.2). Hence M is pseudo fully d-stable.
Sufficiency. Assume that M is pseudo fully d-stable and $f : M \to A$, $g : M \to B$ are two epimorphisms with $\ker f \neq \ker g$ and $A \cong B$. Let $h : A \to B$ be an isomorphism, then $(h \circ f), g : M \to B$ are epimorphisms and $\ker(h \circ f) = \ker f \neq \ker g$, a contradiction (since M is pseudo fully d-stable). ◊

Proposition 5.4. A homomorphic image of a d-terse (fully pseudo d-stable) module is d-terse (fully pseudo d-stable) .

Proof: Let $h : M \to N$ be an epimorphisim, where M is a d-terse module, and let $f : N \to A, g : N \to B$ be two epimorphisms with $\ker f \neq \ker g$, then $\ker f \circ h \neq \ker g \circ h$ hence A and B are not isomorphic. Therefore N is d-terse. ◊

Corollary 5.5. If R is a d-terse ring then any cyclic R-module is d-terse. ◊

In the following we will prove a sufficient condition for full d-stability in certain type of modules.

Theorem 5.6. Every fully pseudo d-stable hollow module is fully d-stable.

Proof: Let M be a pseudo fully d-stable hollow module and let $f, g : M \to A$, with g epimorphism , hence A is a homomorphic image of M, so it is hollow too.
If f is an epimorphism we have nothing to prove, assume that $\operatorname{Im} f \neq A$, $\operatorname{Im} f$ is small in A, let $h = g - f$, then $h \in Hom(M, A)$ and $\operatorname{Im} h + \operatorname{Im} f = \operatorname{Im} g = A$, hence $\operatorname{Im} h = A$, that is, h is an epimorphism, so, $\ker g = \ker h$ (since M is a pseudo fully d-stable) . Now $x \in \ker g \Rightarrow x \in \ker h \Rightarrow g(x) - f(x) = 0 \Rightarrow f(x) = 0 \Rightarrow x \in \ker f$.

Hence $\ker g \subset \ker f$, therefore M is fully d-stable. ◊

Corollary 5.7. A hollow module is fully d-stable if and only if it is d-terse. ◊

A characterization of d-terse modules is in the following and its proof is immediate from the definition:

Proposition 5.8. A module is d-terse if and only if it has no distinct isomorphic factor modules. ◊

Recall that an R-module M is said to have the C_2 *condition, if any submodule of* M *which is isomorphic to a summand is, itself, a summand of* M (see [5]). It is clear that a terse module satisfies the C_2 condition As a dualization to this condition, there is the D_2 *conditions which stats: If* A *is a submodule of the* R*-module* M *such that* M/A *is isomorphic to a summand of* M*, then* A *is a summand of* M (see [5]). Now we prove this condition (D_2) for the d-terse modules

Proposition 5.9. A d-terse module satisfies D_2 .

Proof: Let N be a submodule of a d-terse module M, with M/N isomorphic to a direct summand of M ,say L. Let $\varphi:\mathrm{M}/\mathrm{N}\to L$ be an isomorphism and $\mathrm{M}=L\oplus K$, let $\pi:\mathrm{M}\to\mathrm{M}/\mathrm{N}$ be the natural epimorphism and $p:\mathrm{M}\to L$ be the natural projection.
Then $\varphi\circ\pi:\mathrm{M}\to L$ and $p:\mathrm{M}\to L$ are two epimorphosis, hence $\ker\varphi\circ\pi=\ker p$, since M is d-terse, but $\ker\varphi\circ\pi=\ker\pi=\mathrm{N}$ (since φ is an isomorphism) and $\ker p=K$, therefore $\mathrm{N}=K$ is a direct summand of M. ◊

As a consequence of the above proposition, every fully pseudo d-stable , and hence every fully d-stable module satisfies D_2 . The converse of the last statement is not true, for example $\mathrm{Z}\oplus\mathrm{Z}$, as a Z-module has D_2 ,since it is quasi-projective,(see [5]), but it is not fully d-stable.

Remark 5.10. It is known that every commutative ring is fully d-stable module over itself ([2], corollary 2.4), hence it is pseudo fully d-stable (by theorem 4.2 (i)), hence d-terse (by theorem 5.3), consequently every cyclic R-module is d-terse , if R is commutative, (by corollary 5.5).

In [2] we mentioned the concept of Hopfain, generalized Hopfain , … and proved that a fully d-stable module is Hopfain (hence generalized Hopfain), in the following we generalized these results for fully pseudo d-stable (d-terse) module, also, is Hopfain .

Proposition 5.11. A fully pseudo d-stable module **(**d-terse module) is Hopfain.

Proof. Let M be a d-terse module and f be a surjective endomorphism of M, then M is isomorphic to $\mathrm{M}/\ker f$, that is $\mathrm{M}/0$ is isomorphic to $\mathrm{M}/\ker f$, hence $\ker f=0$ (proposition 5.8), therefore f is an isomorphism. ◊

6. Almost fully d-stable module

In this section we introduce an other generalization of the concept of fully d-stable and investigate properties and relationships with previous concepts. Starting with definition.

Definition 6.1. A module with the property ,that any proper submodule is contained in a fully d-stable submodule will be called almost fully d-stable (briefly, afd-stable) module.

Remark 6.2. It is clear that any fully d-stable module is afd-stable, but not the converse . The module $Z_2 \oplus Z_2$ (as a Z-module) is afd-stable, since all its proper submodules are simple, but it is not fully d-stable. ◊

Remark 6.3. A homomorphic image of an afd-stable module is afd-stable, while a submodule of a afd-stable module need not be afd-stable.

Proof: The first part is clear since a homomorphic image of a fully d-stable module is fully d-stable[2]. For the second part see example (2.5, [2]) . ◊

Remark 6.4. Let M be a hollow afd-stable module. For any two proper submodules of M, there exists a fully d-stable submodule containing both. Consequently for any finite family of proper submodules of M, there exists a fully d-stable submodule containing all.

Proof: Assume that N *and* K are two proper submodules of M and A *and* B be fully d-stable submodules containing N *and* K resp. If $A + B = M$, then either $A = M$ or $B = M$,(since M is hollow) which implies M is the fully d-stable submodule containing N *and* K.

If $A + B \neq M$, then by the property of gfd-stable module, $A + B$ is contained in a fully d-stable submodule which contains both N *and* K too. ◊

Theorem 6.5. A non cyclic Noetherian hollow afd-stable module is fully d-stable.

Proof: If M is simple module , then it is fully d-stable. Let $0 \neq A$ be a proper submodule of M, let Γ be the family of all fully d-stable submodules containing A, then $\Gamma \neq \phi$, since M is gfd-stable, Γ has a maximal element say, N (since M is Noetherian).

If $N = M$, then M is fully d-stable. If $N \neq M$ and $x \in M \setminus N$ then Rx is contained in a fully d-stable submodule say, K.

If $N + K = M$, then $K = M$ (since M is hollow), hence M is fully d-stable.

If $N + K \neq M$, then $N + K$ is contained in a fully d-stable submodule , which contradicts the maximality of N. ◊

Remark 6.6. A module is finitely generated(cyclic) and hollow if and only if it is local . Recall that a module is local if it posses a largest submodule(see [8]) .

In the class of local modules the concepts of fully d-stable and gfd-stable coincide , to prove this we need first the following lemma.

Lemma 6.7. Let M be a local module, N its largest submodule and K any submodule of M different from N. If $\alpha : M \rightarrow M/K$ is a homomorphism, then $\alpha(N) \subset N/K$.

Proof: If K is any submodule of M, different from N, then by the above remark $K \subset N$ and N/K is a largest submodule of M/K. Let $\alpha : M \rightarrow M/K$ be a homomorphism.

Case1. α is not surjective, then $\alpha(N)$ is a proper subset of M/K, hence $\alpha(N) \subset N/K$

Case2. α is surjective , if $\alpha(N) = M/K$, then $\alpha^{-1}(\alpha(N)) = M$ which implies $N + \ker\alpha = M$, hence $\ker\alpha = M$ (since M is hollow and $N \neq M$) which means $\alpha = 0$ and $\alpha(N) \subset N/K$. If $\alpha(N) \neq M/K$, then by case1 $\alpha(N) \subset N/K$ too. ◊

Now we state and prove the above claim.

Theorem 6.8. A local module is fully d-stable if and only if it is afd-stable.

Proof: Necessity. Clear.

Sufficiency. Let N be the largest submodule of M, then either M is fully d-stable or N itself is fully d-stable. Let K be any submodule of M, different from N, and $\alpha : \mathrm{M} \to \mathrm{M}/\mathrm{K}$ be a homomorphism, then by the above lemma $\beta = \alpha|_{\mathrm{N}} : \mathrm{N} \to \mathrm{N}/\mathrm{K}$, hence $\mathrm{K} \subset \ker\beta = \mathrm{N} \cap \ker\alpha$, implies $\mathrm{K} \subset \ker\alpha$. By proposition (3.2 c) N is d-stable, too. Therefore M is fully d-stable. ◊

Corollary 6.9. A hollow Noetherian module is fully d-stable if and only if it is afd-stable.

Proof: For finitely generated case by (remark 6.6) For the infinitely generated case by theorem(6.5). ◊

Example 6.10. Let p be a prime number and let $Q_p := \left\{ \frac{a}{p^i} \middle| a \in \mathrm{Z} \wedge i \in \mathrm{N}- \right\}$ i.e. the set of all rational numbers whose denominator is a power of p (including $p^0 = 1$). Then Q_p is a subgroup of Q (as additive group) $\mathrm{Z} \subset Q_p$. Q_p/Z is Artinian but not Noetherian as a Z-module [4]. Note that Q_p/Z is a afd-stable module but not fully d-stable, since all its proper submodules are the chain $0 \subset|\frac{1}{p}+\mathrm{Z})\subset|\frac{1}{p^2}+\mathrm{Z})\subset|\frac{1}{p^3}+\mathrm{Z})\subset\ldots$ [4] , which are local and hence by theorem(6.8) are fully d-stable, that is the module Q_p/Z is a afd-stable module. On the other hand Q_p/Z is not fully d-stable since it is isomorphic to any of its factor modules. ◊

Proposition 6.11 Let M be a finitely generated u-maximal R-module. Then M is fully d-stable if and only if it is afd-stable.

Proof: The "if"" part is clear. Assume that M is finitely generated, u-maximal and afd-stable . If M is not fully d-stable then each of its maximal submodules is fully d-stable, let K be any submodule of M and let N be a maximal submodule containing K, then K is d-stable in N and N is d-stable in M. By the transitivity property of d-stability (see corollary 3.4 [2]), K is d-stable in M. Hence M is fully d-stable. ◊

In the end we summarize the relationships between the different concepts introduced in this paper by the following diagram:

duo
↑
afd-stable ← fully d-stable → minimal d-stable
↓
fully pseudo d-stable → maximal d-stable ↔ u-maximal
↕
d-terse

REFERENCES

1. M.S.ABBAS, (1990) On Fully Stable Modules, Ph.D. Thesis, University of Baghdad. Iraq.
2. M.S.ABBAS and ASAAD M. A. AL- HOSAINY,(2012) Fully Dual-Stable Modules, Archives Des Sciences Journal, Vol. 65, No. 12; Dec.
3. S. FEGELSTOCK and R. RADPAEL, (1986) Some Aspects of Relative Projectivity, Communication in Algebra, 14(7).
4. F. KASCH,(1982) MODULES and Rings, Academic Press.

5. SAAD H. MOHAMED and BRUNO J. MULLER, (1990) Continuous and Discrete Modules, London Mathematical Society Lecture Notes Series. 147.
6. A. K. TIWARY and B. M. PANDEYA, (1978) Pseudo Projective and pseudo Injective Modules, Indian J. of Pure Appl. Math., Vol.9, No.9.
7. WILLIAM D. WEAKLEY, (1987) Modules Whose Distinct Submodules are not Isomorphic, Communication in Algebra, 15(8).
8. ROBERT WISBAUER, (1991) Foundation of Module and Ring Theory, Gordon and Science Publishers, Reading.