Fully Dual-Stable Modules

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Abstract.
An R-module M is fully stable if $\alpha(\mathrm{N}) \subset \mathrm{N}$ for each submodule N and R-homomorphism α of N into M. In this paper we study the dual concept of full stability. Duo property of modules being a necessary condition for both full stability and full dual stability, and quasi-projectivity is sufficient condition for duo to be fully dual stable modules. Several properties and characterizations of full dual stability are investigated.

Key words. dual-stable submodules, fully dual-stable modules, duo modules, quasi projective modules, Hopfian and Co-Hopfian modules .

1. Introduction

Throughout, all rings are associative with identity, and modules are left unitary (unless otherwise stated). Let M be an R-module. A submodule N of M is called stable if $\alpha(\mathrm{N}) \subset \mathrm{N}$ for each submodule N and R-homomorphism α of N into M. In case each submodule of M is stable, M is called fully stable, this is equivalent to saying that every cyclic submodule of M is stable. It is clear that full stability is closed under submodules but not, homomorphic images. Also it is clear that every fully stable R-module is duo, and quasi-injectivity makes the converse true [1].

This work (in section two) is studying the dual concept of fully stable modules . Let M be an R-module. A submodule N of M is called dual stable if $\mathrm{N} \subset \ker(\alpha)$ for each submodule N and R-homomorphism α of M into M/N. If each submodule of M is dual stable, then M is called fully dual stable. We show that full dual stability is closed under homomorphic images, but not, submodules. We show that, over a commutative ring, every cyclic module is fully dual stable, this gives that the two concepts, full stability and full dual stability are comparable.

It is known that duo modules are generalized Hopfian and weakly Co-Hopfian. We show that full dual stability is Hopfian.

In section three, we investigate dual stability in certain class of modules. Every direct summand of duo modules is dual stable. Also we see that dual stability satisfies a transitive relation, that is, if L is dual stable of N and N is dual stable of M where $L \subset \mathrm{N} \subset \mathrm{M}$, then L is dual stable of M, and the converse is true in case N is direct summand, also we consider d-stability of submodules in quasi-projective modules, we see that homomorphic image of quasi-projective modules by dual-stable submodules is quasi-projective. Furthers quasi-projectivity guarantees that, for modules to be fully dual-stable, it is enough each cyclic submodule is dual-stable. At the rest of the section, we construct a fully dual stable module which is not quasi-projective.

2. Fully dual stable modules

We start by introducing the dual concept of fully stable modules.

Definition 2.1. Let M be an R-module. A submodule N of M is called dual- stable (simply, d-stable) if $N \subset Ker(\alpha)$ for each R-homomorphism $\alpha : M \to M/N$. In case each submodule of M is d-stable, M is called fully dual stable (simply, fully d-stable) . A ring R is fully d-stable if it is a fully d-stable R-module.

Examples and Remarks 2.2.

(a) It is clear that every simple R-module is fully d-stable.

(b) Z is fully d-stable Z-module. For, let $f : Z \to Z/nZ$ be any Z-homomorphism and $f(n) = nf(1) = n\overline{x_0}$ for some x_0, $0 \le x_0 \le n-1$. Hence $f(n) = n\overline{x_0} = 0$, so $n \in \ker f$ implies that $nZ \subset \ker f$. However Z is not fully stable [1].

(c) Q is not fully d-stable Z-module since. Consider the Z- homomorphism , $\alpha : Q \to Q/Z$ defined by $\alpha(r) = \frac{r}{3} + Z$, $Z \not\subset \ker f$.

(d) It is well-known that the quotient of the Z-module $Z_{(p^\infty)}$ by every proper submodule is isomorphic to $Z_{(p^\infty)}$, thus if $\alpha : Z_{(p^\infty)} \to Z_{(p^\infty)}/Z_{(p^k)}$ is an isomorphism then $Z_{(p^k)} \not\subset \ker\alpha$., ($k \ge 1$).This shows that $Z_{(p^\infty)}$ is not fully d-stable, while it is a fully stable module[1].

(e) From (b) and (d), it follows that the concepts of full stability and full d-stability are completely independent, while Q is neither fully stable [1] nor fully d-stable and, from (a), simple module are both fully stable and fully d- stable..

(f) Recall that an R - module M is called duo if each submodule of M is fully invariant ,that is, $f(N) \subset N$ for each R-endomorphism f of M and each submodule N. For let N be a submodule of a fully d-stable R - module M, and f be an R-endomorphism of M, set $\alpha = \pi \circ f : M \to M/N$ where $\pi : M \to M/N$ is the natural epimorphism of M onto M/N. Then $N \subset \ker\alpha = f^{-1}(N)$. This shows that duo property is a necessary condition for full d-stability. Note that the converse of this statement may not be true generally, for example the Z-module $Z_{(p^\infty)}$ is duo, but not fully d-stable.

(g) It is clear that a submodule of a fully stable R - module is fully stable, unlike for fully d-stable (see example (2.5 b)). On the other hand, homomorphic image of fully stable R - module need not be fully stable [1]. However we shall see that full d-stability is closed under every homomorphic image, and hence every direct summand of fully d-stable R - module is fully d-stable.

Proof: Let N be a submodule of a fully d-stable R - module M and $\alpha : M/N \to (M/N)/(K/N)$ be an R-homomorphism ,where K is a submodule of M containing N. Put $\varphi = \beta \circ \alpha \circ \pi : M \to M/K$ where $\pi : M \to M/N$ is the natural epimorphism of M onto M/N and $\beta : (M/N)/(K/N) \to M/K$ is an isomorphism. Then $K \subset \ker\varphi = \pi^{-1}(\ker\alpha)$, that is $K/N \subset \ker\alpha$. ◊

(h) For any R - module M, if R has no zero divisors, then the torsion submodule $T(M)$ is d-stable. If $\alpha : M \to M/T(M)$, and $x \in T(M)$, then there exists $0 \ne r \in R$ such that $rx = 0$, hence $\alpha(rx) = 0$ then $r\alpha(x) = 0$. If $\alpha(x) = y + T(M)$, then $ry + T(M) = 0$, that is , $ry \in T(M)$, then there exists $0 \ne s \in R$ such that $s(ry) = 0$, hence $(sr)y = 0$ with $sr \ne 0$ (since R has no zero divisors) , that is, $y \in T(M)$ which means $\alpha(x) = 0$. ◊

In the next we consider a sufficient conditions for a duo R -module to be fully d-stable. Note that a sufficient condition for duo R -module to be fully stable is quasi-injectivity.[1]

Recall that an R -module M is quasi-projective if for each submodule N of M and each R -homomorphism $\alpha : M \to M/N$, there exists an R -endomorphism f of M such that $\alpha = \pi \circ f$ where $\pi : \mathrm{M} \to \mathrm{M}/\mathrm{N}$ is the natural epimorphism.[2]

Proposition 2.3. Every fully invariant submodule of a quasi-projective R -module is d-stable and hence every quasi-projective duo R -module is fully d-stable.

Proof: Let N be a fully invariant submodule of a quasi-projective R -module M , and $\alpha : M \to M/N$ be an R-homomorphism. Then there is an R -endomorphism f of M such that $\alpha = \pi \circ f$, but $f(\mathrm{N}) \subset \mathrm{N}$, so $\mathrm{N} \subset f^{-1}(\mathrm{N}) = \ker \alpha$. This shows that N is d-stable. ◊

We shall see later that fully d-stable modules may not be quasi-projective. Next, we give an example to show that full d-stability is not closed under submodule. First, we need the following corollary of the above proposition.

Corollary 2.4. Every commutative ring R is fully d-stable. ◊

This corollary and remark (2.2 g), imply that any cyclic module over a commutative ring is fully d-stable. The following example shows that corollary (2.4) is not true for non commutative ring and hence cyclic modules over a non commutative ring need not be fully d-stable.

Examples 2.5.

(a) Let R be a commutative ring , $S = \mathrm{M}_{2\times 2}(R)$, the ring of all 2×2 matrices over R and $\mathrm{I} = \left\{ \begin{bmatrix} a & 0 \\ b & o \end{bmatrix} \mid a,b \in R \right\}$,then I is a left ideal of S . Define $f : S \to S$, defined by $f(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then f is an S -endomorphism , but $f(\mathrm{I}) \not\subset \mathrm{I}$, so S is not duo and hence not fully d-stable, even though S is quasi-projective. ◊

(b) Let $\mathrm{M} = \mathrm{Z} \oplus \mathrm{Z}$, $R = \mathrm{Z} \oplus \mathrm{M}$, we define a multiplication on R by $(x,m)\cdot(y,n) = (xy, xn + ym)$, by corollary (2.4), it is fully d-stable $R-$module. $0 \oplus \mathrm{M}$ is a submodule of R which is not duo (so not fully d-stable) module. Note that If R is a proper subring of a ring S , then the left $R-$module S is not duo ([8], proposition 1.6) and hence not fully d-stable.

In the following, we discuss full d-stability of free modules. First of all the following corollary discuss full d-stability of free modules of rank one, and the proof follows from remark (2.2 f) and proposition (2.3).

Corollary 2.6. A ring R is fully d-stable if and only if R is duo. ◊

For free modules of rank more than one we have the following.

Proposition 2.7 . For any ring R , any free $R-$module with rank greater than one is not fully d-stable.

Proof: Let M be a free R-module and X be a basis for M with $|\mathrm{X}| \geq 2$. For any two distinct elements x_1, x_2 of X, we define an endomorphism f of M as follows:
$f(x) = x\, if\ x \in \mathrm{X}, and\ x \neq x_i, i = 1,2, f(x_1) = x_2, and\ f(x_2) = x_1$.
Now if N_i is a submodule of M generated by $\mathrm{X} - \{x_i\}$ $(i = 1,2)$ then $f(\mathrm{N}_1) = \mathrm{N}_2 \not\subset \mathrm{N}_1$. This shows that M is not duo and hence not fully d-stable. ◊

It is well-known that any finitely generated torsion free module over a P. I. D. is free, (see [6], IV 6.5), so by proposition (2.7) , corollary (2.4) and remark (2.2 g), we have the following.

Corollary 2.8. A finitely generated torsion free module M over a P.I.D. R, is fully d-stable if and only if it is isomorphic to R. ◊

We close this section by further properties of fully d-stable modules. First, recall a lemma and certain concepts that appear in [8].

Lemma 2.9[8]. Let R be a ring . Then an R-module M is duo if and only if for each $R-$ endomorphism f of M and each m in M, there exists r in R such that $f(m) = rm$.

Definitions 2.10 [3]. Let M be an R-module.

(a) M is called Hopfian (Co- Hopfian) if every surjective (injective) endomorphism of M is an isomorphism.

(b) M is called generalized Hopfian (weakly Co- Hopfian) if every surjective (injective) endomorphism of M has a small kernel (has an essential image).

It is known that duo modules are generalized Hopfian and weakly Co-Hopfian, but neither Hopfian nor Co-Hopfian, we shall use lemma (2.9) to give a characterization of fully d-stable modules . First we consider the following necessary condition.

Theorem 2.11. Let M be a fully d-stable R-module, N a submodule of M and $\alpha : \mathrm{M} \to \mathrm{M}/\mathrm{N}$ any R-homomorphism. Then for each $m \in \mathrm{M}$ there exists r in R such that $\alpha(m) = rm + \mathrm{N}$ (r depends on α and m).

Proof: Define $\lambda : \mathrm{M}/\mathrm{N} \to \mathrm{M}/\mathrm{N}$, by $\lambda(m + \mathrm{N}) = \alpha(m)$.Full d-stability of M implies that $\mathrm{N} \subset \ker \alpha$, so if $m \in \mathrm{N}$, then $\alpha(m) = \bar{0}$, this shows that λ is well-defined and it is an easy matter to check that λ is an R-homomorphism. By (1.2 f and g), M/N is fully d-stable and hence duo. Lemma (2.9) implies that for each $(m + \mathrm{N}) \in \mathrm{M}/\mathrm{N}$ there is r in R such that $\lambda(m + \mathrm{N}) = rm + \mathrm{N}$, and hence $\alpha(m) = rm + \mathrm{N}$. ◊

It is clear that if $\alpha : \mathrm{M} \to \mathrm{M}/\mathrm{N}$ has the property that for each $m \in \mathrm{M}$ there exists r in R such that $\alpha(m) = rm + \mathrm{N}$, then $\mathrm{N} \subset \ker \alpha$. So we have the following characterization of full d-stability .

Corollary 2.12. An R-module M is fully d-stable if and only if for each submodule N each R-homomorphism $\alpha : \mathrm{M} \to \mathrm{M}/\mathrm{N}$ has the property that for each $m \in \mathrm{M}$ there exists r in R such that $\alpha(m) = rm + \mathrm{N}$ (r depends on α and m). ◊

Now, we modify lemma (2.9) with extra conditions to characterize duo modules in such a way that the existence of the element r in R do for all elements of module.

Proposition 2.13. Let R be an integral domain, and M a torsion free R-module. Then M is duo if and only if for each R-endomorphism f of M, there exists r in R such that $f(m) = rm$ for all $m \in \mathrm{M}$.

Proof: It is enough to prove the necessity condition. Assume that M is duo and f is an R-endomorphism of M, then by the lemma(2.9) , for each element x in M there is r in R such that $f(x) = rx$. Now if x, y are two distinct elements of M and $f(x) = rx, f(y) = sy, r, s \in R$, then we have the following two cases.

Case(I). $Rx \cap Ry \neq 0$. Let $z(\neq 0) \in Rx \cap Ry$, assume $f(z) = tz, t \in R$, then $z = ux = vy$ for some $u, v \in R$. Hence : $tz = f(ux) = urx$ implies that $tux = urx$ and so $t = r$. Similarly $t = s$. Then $r = s$.

Case(II). $Rx \cap Ry = 0$, assume $f(x + y) = t(x + y)\, where\ t \in R$, but $f(x + y) = f(x) + f(y) = rx + sy$, hence $(t - r)x = (s - t)y$ implies $t - r = s - t$, then $r = s$. Therefore $r = s$ in any case, that is, $f(x) = rx$ for all x in M. ◊

Corollary 2.14. Let R be an integral domain and M a torsion-free R-module. Then M is duo if and only if $End(\mathrm{M}) \cong R$. (where $End(\mathrm{M})$ is the endomorphism ring of M) ◊

Corollary 2.15. Let R be an integral domain, M a fully d-stable R- module and N a submodule of M such that $\mathrm{M/N}$ is torsion-free. Then for each homomorphism $\alpha : \mathrm{M} \to \mathrm{M/N}$ there is r in R such that $\alpha(m) = rm + \mathrm{N}$ for all $m \in \mathrm{M}$.

Proof: Recall the proof of theorem (2.11), λ is an R- endomorphism of the torsion-free duo module $\mathrm{M/N}$, then by proposition (2.13) there exists r in R such that $\lambda(m + \mathrm{N}) = r(m + \mathrm{N})$ for all $m + \mathrm{N} \in \mathrm{M/N}$. Therefore $\alpha(m) = rm + \mathrm{N}$ for all $m \in \mathrm{M}$. ◊

Proposition 2.16. Every fully d-stable R-module is Hopfian (and hence generalized Hopfian) and weakly co-Hopfian .

Proof. Let M be a fully d-stable R-module and let f be an endomorphism of M and surjective. Then $\mathrm{M}/\ker f$ is isomorphic to M, let $\alpha : \mathrm{M} \to \mathrm{M}/\ker f$ be an isomorphism , so $\ker \alpha = 0$ and by full d-stability of M, $\ker f \subset \ker \alpha$ hence f is an isomorphism. M is weakly co-Hopfian since it is duo. ◊

Note that fully d-stable modules may not be co-Hopfian, since the Z-module Z is fully d-stable which is not co-Hopfian.

We shall use lemma (2.9) , to justify conditions that prevent a module to be fully d-stable, see the following.

Proposition 2.17. Let R be a commutative ring (with identity) which has at least one non-invertible and not zero divisor element r. If M is a torsion free divisible R-module, then M is not fully d-stable.

Proof. Since M is divisible, then for each $x \in \mathrm{M}$ there exists $y \in \mathrm{M}$ such that $x = ry$ (r is the element of R with above conditions) , hence $f : \mathrm{M} \to \mathrm{M}$ defined by $f(x) = y$ is a well defined endomorphism (note that M is torsion free) . Now let $0 \neq x_0$ be fixed in M , if M is duo then $f(x_0) = sx_0$ for some $s \in R$ but $x_0 = rf(x_0)$ by de which implies $x_0 = rsx_0$, definition of f hence $(1-rs)x_0 = 0$, then $1 = rs$ (M is torsion free), a contradiction with the hypothesis, r is not invertible. Therefore M is not duo, so not fully d-stable. ◊

Corollary 2.18. Let R be an integral domain, which is not a field , M is an injective , torsion free module over R , then M is not fully d-stable. ◊

3. dual-stability in quasi-projective modules

In this section we shall investigate d-stability in the class of quasi-projective modules. We mentioned before that every simple module is fully d-stable, for semisimple modules we have the following.

Proposition 3.1. Every direct summand of duo module is d-stable (and hence every semisimple duo module is fully d-stable).

Proof: Let N a direct summand of a duo R-module M and $\alpha : \mathrm{M} \to \mathrm{M}/\mathrm{N}$ an R-homomorphism. Then $\mathrm{M} = \mathrm{N} \oplus L$ for some submodule L of M .So the natural epimorphism $\pi : \mathrm{M} \to \mathrm{M}/\mathrm{N}$ splits . Let $\varphi : \mathrm{M}/\mathrm{N} \to \mathrm{M}$ be an R- homomorphism such that $\pi \circ \varphi = 1_{\mathrm{M}/\mathrm{N}}$, then $f = \varphi \circ \alpha$ is an R-endomorphism of M . Hence $f(\mathrm{N}) \subset \mathrm{N}$, and $\alpha(\mathrm{N}) = 0$. This shows that N is d-stable. ◊

Proposition 3.2. If M is a fully d-stable R-module and $\mathrm{M} = \mathrm{A} \oplus \mathrm{B} = \mathrm{A} \oplus C$, then $\mathrm{B} = C$.

Proof. Let $\alpha : \mathrm{M} \to \mathrm{M}/\mathrm{B}$ defined by $\alpha(a + c) = c + \mathrm{B}$ for each $a \in \mathrm{A}$ and $c \in C$. Then $\ker \alpha = \mathrm{A} \oplus (\mathrm{B} \cap C)$. it is clear that $\mathrm{B} \subset \ker \alpha$ only if $\mathrm{B} \subset C$. By a similar fashion $C \subset \mathrm{B}$ and hence $\mathrm{B} = C$. ◊

In the following example we show that this is not a general case . Let M be the non cyclic $\mathrm{Z}-$module of order 4, it has three cyclic submodules of order 2 and M is a direct sum of any two of them, then by proposition (3.3), M cannot be fully d-stable and note that M is semisimple.

In the following we give some properties of d-stable direct summand.

Proposition 3.3. Let N be a d-stable direct summand of an R-module M and L a submodule of N , then L is d-stable in N if and only if L is d-stable in M .

Proof: Assume that L is d-stable in N and $\alpha : \mathrm{M} \to \mathrm{M}/L$ an R-homomorphism. Define $\beta : \mathrm{M}/L \to \mathrm{M}/\mathrm{N}$ by $\beta(x + L) = x + \mathrm{N}$ for all x in M , hence $\mathrm{N} \subset \ker(\beta \circ \alpha) = \alpha^{-1}(\mathrm{N}/L)$ and then $\alpha(\mathrm{N}) \subset \mathrm{N}/L$. So if $\delta = \alpha|_{\mathrm{N}}$, then $\delta : \mathrm{N} \to \mathrm{N}/L$ and $L \subset \ker \delta$, but $\ker \delta = (\ker \alpha) \cap \mathrm{N}$ implies $L \subset \ker \alpha$. This shows that L is d-stable in M . Conversely, assume that L is d-stable in M and $\alpha : \mathrm{M} \to \mathrm{M}/L$ is an R-homomorphism, let β be the natural projection of M onto N , then $\alpha \circ \beta : \mathrm{M} \to \mathrm{M}/L$, hence $L \subset \ker(\alpha \circ \beta) = \ker \alpha$. Thus L is d-stable in N . ◊

Note that the direct summand property in proposition (3.3) used only in the sufficient condition. So the following corollary clarifies the transivity of d-stability .

Corollary 3.4. Let A, N and K be submodules of an R-module M with $\mathrm{A} \subset \mathrm{N} \subset \mathrm{K}$. If A is d-stable in N and N is d-stable in K, then A is d-stable in K. ◊

Now, we investigate the effect of quasi-projectivity on full d-stability and conversely. We mentioned in section one that, quasi-projectivity is a sufficient condition of duo modules to be fully d-stable, but it is not the case of necessary condition(see lemma (3.7) and example (3.8)).

First, note that a homomorphic image of a quasi-projective module may not be quasi-projective , for if M is an R-module which is not quasi-projective, it is well-known that M is an epimorphic image of a free (and hence quasi-projective) module. But, "If N is a fully invariant submodule of a quasi- projective module M, then M/N is likewise quasi-projective" [9]. This statement, and the fact that any submodule of a fully d-stable module is fully invariant remark (2.2 f) leads to the following result.

Corollaries 3.5. A homomorphic image of a fully d-stable quasi-projective module is likewise quasi-projective. ◊

It is known that an R-module is fully stable if and only if each cyclic submodule is stable. For fully d-stable modules, this is not the case, but in the following we show this is the case under quasi-projectivity.

Proposition 3.6. Let M be a quasi-projective R-module. Then M is fully d-stable if and only if every cyclic submodule of M is d-stable.

Proof: By proposition (2.3) it is enough to prove that M is duo. Assume that every cyclic submodule of M is d-stable, let f be an R-endomorphism of M , $x \in \mathrm{M}$, and π_x be the natural epimorphism of M onto M/Rx put $\alpha = \pi_x \circ f$, then by assumption $\alpha(x) = 0$, so there exists $r \in R$ such that $f(x) = rx$ Thus lemma (2.9) implies that M is duo . ◊

Now, it is natural to look for a fully d-stable R-module which is not quasi-projective. First consider the following lemma.

Lemma 3.7. If M is an R-module having exactly three nontrivial submodules, $\mathrm{N}_1, \mathrm{N}_2$ and $\mathrm{N}_1 \cap \mathrm{N}_2$, with M/N_1 *not isomorphic to* M/N_2 , then M is a fully d-stable module which is not quasi-projective .

Proof: To prove full d-stability of M, we have to check the d-stability of all its submodules. Note that M/N_1 is simple R-module, so any R-homomorphism $\alpha : \mathrm{M} \to \mathrm{M}/\mathrm{N}_1$ is either trivial or surjective, if $\alpha = 0$, then it is clear that $\mathrm{N}_1 \subset \ker\alpha$. If α is surjective, then there are four cases for $\ker\alpha$. Case(I): If $\ker\alpha = \mathrm{N}_2$, then $\mathrm{M}/\mathrm{N}_2 \cong \mathrm{M}/\mathrm{N}_1$ which contradicts the hypothesis. Case(II): If $\ker\alpha = \mathrm{N}_1 \cap \mathrm{N}_2$, then $\mathrm{M}/(\mathrm{N}_1 \cap \mathrm{N}_2) \cong \mathrm{M}/\mathrm{N}_1$ and hence $\mathrm{M}/(\mathrm{N}_1 \cap \mathrm{N}_2)$ is simple, which is impossible since $\mathrm{N}_1 \cap \mathrm{N}_2$ is not maximal in M . Case(III): If $\ker\alpha = 0$, then M is simple which is absurd. In case (IV) whence $\ker\alpha = \mathrm{N}_1$ and hence N_1 is d-stable submodule of M . In a similar fashion N_2 is d-stable in M . If $\alpha : \mathrm{M} \to \mathrm{M}/(\mathrm{N}_1 \cap \mathrm{N}_2)$ and $\ker\alpha = 0$, then $\mathrm{M} \cong \mathrm{M}/(\mathrm{N}_1 \cap \mathrm{N}_2)$, which impossible, in the other case $(\mathrm{N}_1 \cap \mathrm{N}_2) \subset \ker\alpha$ and this shows that $\mathrm{N}_1 \cap \mathrm{N}_2$ is d-stable in M . This completes that M is fully d-stable.

Now, to prove that M is not quasi-projective. Note that M is hopfian, since it is fully d-stable and proposition (2.16). Hence any endomorphism of M is either isomorphism or not surjective. Let

$\alpha : M \to M/(N_1 \cap N_2)$ be an R-homomorphism and f R- endomorphism of M such that $\pi \circ f = \alpha$ (where π is the natural epimorphism of M onto $M/(N_1 \cap N_2)$), if f is not surjective there are three cases (no loss of generality if we assume that $\alpha \neq 0$).

Case (I) $\operatorname{Im} f = N_1$, then $M/\ker f \cong N_1$ which is impossible (it is an easy matter to check of all possibilities of $\ker f$).

Case (II) if $\operatorname{Im} f = N_2$, by a similar argument of case(I) we have impossible case.

Case (III) if $\operatorname{Im} f = N_1 \cap N_2$, then $\pi \circ f = 0$, which is absurd.

Now, if f is an isomorphism, then α must be surjective, that is, if we choose α not surjective, then there is no f such that $\pi \circ f = \alpha$ and hence M is not quasi-projective . It is clear that $M = N_1 + N_2$, hence $M/N_1 \cong N_2/(N_1 \cap N_2)$, let $\varphi : M/N_1 \to N_2/(N_1 \cap N_2)$ be an isomorphism .We can consider φ as a nontrivial homomorphism from M/N_1 into $M/(N_1 \cap N_2)$, which is not surjective. If $\nu : M \to M/N_1$ is the natural epimorphism, then $0 \neq \alpha = \varphi \circ \nu : M \to M/(N_1 \cap N_2)$ and not surjective. This ends the proof. ◊

In [4], a module with conditions of the above lemma is asked, as an exercise to prove it is not quasi-projective, this exercise is duo to [7] , as a dual concept.

The natural question, now, is there a module satisfying the conditions of the above lemma. The answer is yes, by the following example.

Example 3.8. Referring to an example of Hallett [5], where R is an algebra over $Z/2Z$ having basis $\{e_1, e_2, e_3, n_1, n_2, n_3, n_4\}$ with the following multiplication table:

	e_1	e_2	e_3	n_1	n_2	n_3	n_4
e_1	e_1	0	0	n_1	n_2	0	0
e_2	0	e_2	0	0	0	0	0
e_3	0	0	e_3	0	0	n_3	n_4
n_1	0	n_1	0	0	0	0	0
n_2	0	0	n_2	0	0	0	0
n_3	n_3	0	0	0	0	0	0
n_4	0	n_4	0	0	0	0	0

Let $A = Re_1 + Re_2$, $B = Re_1 + Rn_4$, $C = Re_2 + Rn_3$, $D = B \cap C = Rn_3 + Rn_4$ $E = R(n_3 + n_4)$, $M = A/E$, $N_1 = B/E$, $N_2 = C/E$ and $N_1 \cap N_2 = D/E$, then M is an R-module with exactly three nontrivial submodules N_1, N_2 *and* $N_1 \cap N_2$ satisfying the conditions of lemma (3.7) .

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