New Characterization Of Kernel Set in Topological Spaces

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ABSTRACT:

In this paper we introduce a kernelled point, boundary kernelled point and derived kernelled point of a subset A of X, and using these notions to define kernel set of topological spaces. Also we introduce kr-topological space .The investigation enables us to present some new separation axioms between T_0 and T_1-spaces.

Key: kernelled point, boundary kernelled point, derived kernelled point, Kernel set, weak separation axioms, R_0-space.

1. INTRODUCTION AND PRELIMINARIES

In the recent papers kernel of a set A (ker(A))of a topological space defined as the intersection of all open superset of A. [2],[3].

In this paper we introduce that $x \in X$ is a kernelled point of a subset A of X (Briefly $x \in \ker(A)$) . Also we present the notions boundary kernelled point of A denoted it $x \in kr_{bd}(A)$, and x is derived kernelled point of A denoted it $kr_{dr}(A)$, we obtain that the kernel of a set A in topological space(X,T) is a union of A itself with the set of all boundary kernelled points (Briefly $kr_{bd}(A)$). Also it is a union of A itself with the set of all derived kernelled points (Briefly $kr_{dr}(A)$), and we gave some result of R_0-space[1], [2], [4]], by using these notions.

Also in this paper we introduce kr-topological space iff kernel of a set A of X is an open set. Via this kind of topological space we give a new characterization of separation axioms lying between T_0 and T_1-spaces.

***Definition* 1.1 [1, 2, 4]**

A topological space (X,T) is called an R_o-space if for each open set U and $x \in U$ then $cl\{x\} \subseteq U$.

***Lemma* 1.2 [2]**

Let (X,T) be topological space then $x \in cl\{y\}\ iff\ y \in \ker\{x\}$. for each $x \neq y \in X$

***Theorem* 1.3 [5]**

A topological space (X, T) is a T_1-space if and only if for each $x \in X$ then $ker\{x\} = \{x\}$.

2. Kernel set

***Definition* 2.1**

Let (X,T) be a topological space. A point x is said to be kernelled point of A(Briefly $x \in \ker(A)$) iff for each F closed set contains x, then $F \cap A \neq \emptyset$.

***Definition* 2.2**

Let (X,T) be a topological space. A point x is said to be boundary kernelled point of A(Briefly $x \in \mathrm{kr_{bd}}(A)$) iff for each F closed set contains x, then $F \cap A \neq \emptyset$ and $F \cap A^c \neq \emptyset$.

***Definition* 2.3**

Let (X,T) be a topological space. A point x is said to be derived kernelled point of A (Briefly $x \in \mathrm{kr_{dr}}(A)$) iff for each F closed set contains x, then $A \cap F / \{x\} \neq \emptyset$.

***Definition* 2.4**

We can define $\ker\{x\}$ as follows $\ker\{x\} = \{y : x \in F_y, F_y^c \in T\}$.

***Theorem* 2.5**

Let (X,T) be a topological space and $x \neq y \in X$. Then x is a kernelled point of $\{y\}$ iff y is an adherent point of {x}.

Proof

Let x be a kernelled point of $\{y\}$. Then for every closed set F such that $x \in F$implies $y \in F$,then $y \in \bigcap\{F : x \in F\}$, this means $y \in cl\{x\}$. Thus y is an adherent point of {x}.

Conversely

Let y be an adherent point of $\{x\}$. Then for every open set U such that $y \in U$implies $x \in U$,then $x \in \bigcap\{U : y \in U\}$, this means $y \in ker\{x\}$. Thus x is a kernelled point of {y}.

***Theorem* 2.6**

Let (X,T) be a topological space and $A \subseteq X$ and let $\mathrm{kr_{dr}}(A)$ be the set of all kernelled derived point of A, then $\ker(A) = A \cup \mathrm{kr_{dr}}(A)$.

Proof

Let $x \in A \cup \mathrm{kr_{dr}}(A)$ and if $x \in \mathrm{kr_{dr}}(A)$, then for every closed set F intersects A (in a point different from$x$). Therefore $x \in \ker\{x\}$. Hence $\mathrm{kr_{dr}}(A) \subseteq \ker(A)$, it follows that $A \cup \mathrm{kr_{dr}}(A) \subseteq \ker(A)$.

To demonstrate the reverse inclusion, we let x be a point of $\ker(A)$. If $x \in A$, then $x \in A \cup \mathrm{kr_{dr}}(A)$. Suppose that $x \notin A$. Since $x \in \ker(A)$, then for every closed set F containing x implies $F \cap A \neq \emptyset$, this means $A \cap F / \{x\} \neq \emptyset$. Then $x \in \mathrm{kr_{dr}}(A)$, so that $x \in A \cup \mathrm{kr_{dr}}(A)$. Hence $\ker(A) \subseteq A \cup \mathrm{kr_{dr}}(A)$. Thus $\ker(A) = A \cup \mathrm{kr_{dr}}(A)$.

***Theorem* 2.7**

Let (X,T) be a topological space and $A \subseteq X$ and let $\mathrm{kr_{bd}}(A)$ be the set of all kernelled boundary point of A, then $\ker(A) = A \cup \mathrm{kr_{bd}}(A)$.

Proof

Let $x \in A \cup \mathrm{kr_{bd}}(A)$ and if $x \in \mathrm{kr_{bd}}(A)$, then for every closed set F intersects A. Therefore $x \in \ker\{x\}$. Hence $\mathrm{kr_{bd}}(A) \subseteq \ker(A)$, it follows that $A \cup \mathrm{kr_{bd}}(A) \subseteq \ker(A)$.

To demonstrate the reverse inclusion, we let x be a point of $\ker(A)$. If $x \in A$, then $x \in A \cup \mathrm{kr_{bd}}(A)$. Suppose that $x \notin A$.implies $x \in A^c$. Since $x \in \ker(A)$, then for every closed set F containing x implies $F \cap A \neq \emptyset$ and $F \cap A^c \neq \emptyset$. Then $x \in \mathrm{kr_{bd}}(A)$, so that $x \in A \cup \mathrm{kr_{bd}}(A)$. Hence $\ker(A) \subseteq A \cup \mathrm{kr_{bd}}(A)$. Thus $\ker(A) = A \cup \mathrm{kr_{dr}}(A)$.

***Theorem* 2.8**

Let (X, T) be a topological space and A is a subset of X. then A is an open set iff every x kernelled point of A is an interior point of A.

Proof

Let A be an open set, then $\ker(A) = A = int(A)$, implies every kernelled point is an interior point.

Conversely

Let every x kernelled point of A is an interior point of A. Then $\ker(A) \subseteq int(A)$. Hence $int(A) \subseteq A \subseteq \ker(A)$, implies $int(A) = A = \ker(A)$. Thus A is an open set

***Corollary* 2.9**

A subset A of X is an open set iff for each x kernelled point then $x \notin cl(A^c)$.

Proof

By theorem 2.8.

***Theorem* 2.10**

A subset A of X is a closed set iff for each $\ker(A^c) \cap cl(A) = \emptyset$.

Proof

Let A is a closed set. Then A^c is an open set, implies $A^c = \ker(A^c)$ [By theorem 2.8]. Hence $A = cl(A)$. Thus$\ker(A^c) \cap cl(A) = \emptyset$.

Conversely

Let $\ker(A^c) \cap cl(A) = \emptyset$, then for each $x \in \ker(A^c)$, implies $x \notin cl(A)$, implies $x \in ext(A)$. Therefore $x \in int(A^c)$. Hence by theorem 2.8, A^c is an open set. Thus A is a closed set.

***Corollary* 2.11**

Every interior point is a kernelled point.

Proof

Clearly.

***Theorem* 2.12**

A topological space(X, T) is an R_0-space iff every adherent point of $\{x\}$ is a kernelled point of $\{x\}$.

Proof

Let (X, T) be an R_0-space. Then for each $x \in X$, $\ker\{x\} = cl\{x\}$[By theorem 1.2]. Thus every adherent point of $\{x\}$ is a kernelled point of $\{x\}$

Conversely

Let every adherent point of $\{x\}$ is a kernelled point of $\{x\}$ and let $U \subseteq T$, $x \in U$. Then $cl\{x\} \subseteq \ker\{x\}$ for each $x \in X$. Since $ker\{x\} = \cap\{U: U \in T, x \in U\}$, implies $cl\{x\} \subseteq U$ for each U open set contains x. Thus (X, T) is an R_0-space.

***Theorem* 2.13**

A topological space (X, T) is T_0-space iff for each $x \neq y \in X$, either x is not kernelled point of $\{y\}$ or y is not kernelled point of $\{x\}$.

Proof

Let a topological space (X, T) is T_0-space. Then for each $x \neq y \in X$ there exist an open set U such that $\in U$, $y \notin U$ (say), implies $y \in U^c$. Hence U^c is a closed, then y is not kernelled point of $\{x\}$. Thus either x is not kernelled point of $\{y\}$ or y is not kernelled point of $\{x\}$.

Conversely

Let for each $x \neq y \in X$, either x is not kernelled point of $\{y\}$ or y is not kernelled point of $\{x\}$. Then there exist a closed set F such that $x \in F$, $F \cap \{y\} = \emptyset$ or $y \in F$, $F \cap \{x\} = \emptyset$, implies $x \notin F^c$, $y \in F^c$ or $x \in F^c$, $y \notin F^c$. Hence F^c is an open set. Thus (X, T) is T_0-space.

***Theorem* 2.14**

A topological space(X, T) is an T_1-space iff $kr_{dr}\{x\} = \emptyset$, for each $x \in X$.

Proof

Let (X, T) be an T_1-space. Then for each $x \in X$, $\ker\{x\} = \{x\}$[By theorem 1.3]. since $kr_{dr}\{x\} = \ker\{x\} - \{x\}$. Thus $kr_{dr}\{x\} = \emptyset$

Conversely

Let $kr_{dr}\{x\} = \emptyset$. By theorem 2.5, $\ker\{x\} = \{x\} \cup kr_{dr}\{x\}$, implies $\ker\{x\} = \{x\}$. Hence by theorem 1.3, (X, T) is aT_1-space.

***Theorem* 2.15**

A topological space (X, T) is T_1-space iff for each $x \neq y \in X$, x is not kernelled point of $\{y\}$ and y is not kernelled point of $\{x\}$.

Proof

Let a topological space (X, T) is T_1-space. Then for each $x \neq y \in X$ there exist open sets U , V such that $\in U$, $y \notin U$ and $y \in V$, $x \notin V$, implies $x \in V^c$, $\{y\} \cap V^c = \emptyset$ and $y \in U^c$, $\{x\} \cap U^c = \emptyset$. Hence U^candV^c are a closed sets, then y is not kernelled point of $\{x\}$. Thus x is not kernelled point of $\{y\}$ and y is not kernelled point of $\{x\}$.

Conversely

Let for each $x \neq y \in X$, x is not kernelled point of $\{y\}$ and y is not kernelled point of $\{x\}$. Then there exist a closed sets F_1, F_2 such that $x \in F_1$, $F_1 \cap \{y\} = \emptyset$ and $y \in F_2, F_2 \cap \{x\} = \emptyset$, implies $x \in F_2^c$, $y \notin F_2^c$ and $y \in F_1^c$, $x \notin F_1^c$. Hence F_1^c and F_2^c are open sets. Thus (X, T) is T_1-space.

3. kr −spaces

***Definition* 3.1**

A topological space (X, T) is said to be kr-space iff for each subset A of X then $\ker(A)$ is an open set.

***Definition* 3.2**

A topological kr-space (X, T) is said to be T_k-space iff for each subset $x \in X$, then $kr_{dr}\{x\}$ is an open set.

***Theorem* 3.3**

In topological kr-space (X, T), every T_1-space is T_k-space.

Proof

Let (X, T) be a T_1-space. Then for each $x \in X$, $\ker\{x\} = \{x\}$[By theorem 2.10]. As $kr_{dr}\{x\} = \ker\{x\} - \{x\}$, implies $kr_{dr}\{x\} = \emptyset$. Thus (X, T) is a T_k-space.

***Theorem* 3.4**

In topological kr-space (X, T), everyT_k-space is a T_0-space.

Proof

Let (X, T) be a T_k-space and let $x \neq y \in X$. Then$kr_{dr}\{x\}$ is an open set. Therefore there exist two cases:

i) $y \in kr_{dr}\{x\}$ is an open set. Since $x \notin kr_{dr}\{x\}$. Thus (X, T) is a T_0-space

ii)$y \notin kr_{dr}\{x\}$, implies $y \notin \ker\{x\}$. But $\ker\{x\}$ is an open set. Thus(X, T) is a T_0-space.

***Definition* 3.5**

A topological kr-space (X,T) is said to be T_L-space iff for each $x \neq y \in X$, $\ker\{x\} \cap \ker\{y\}$ is degenerated (empty or singleton set).

***Theorem* 3.6**

In topological kr-space (X,T), every T_1-space is T_L-space.

Proof

Let (X,T) be a T_1-space. Then for each $x \neq y \in X$, $\ker\{x\} = \{x\}$ and $\ker\{y\} = \{y\}$ [By theorem 1.3], implies $\ker\{x\} \cap \ker\{y\} = \emptyset$. Thus (X,T) is a T_L-space..

***Theorem* 3.7**

In topological kr-space (X,T), every T_L-space. is a T_0-space.

Proof

Let (X,T) be a T_L-space. Then for each $x \neq y \in X$, $\ker\{x\} \cap \ker\{y\}$ is degenerated (empty or singleton set). Therefore there exist three cases:

i) $\ker\{x\} \cap \ker\{y\} = \emptyset, implies (X,T)$ is a T_0-space

ii)$\ker\{x\} \cap \ker\{y\} = \{x\}$ *or* $\{y\}$, implies $y \notin \ker\{x\}$ or $x \notin \ker\{y\}$, implies (X,T) is a T_0-space.

iii) $\ker\{x\} \cap \ker\{y\} = \{z\}, z \neq x \neq y, z \in X$, implies $y \notin \ker\{x\}$ and $x \notin \ker\{y\}$, implies (X,T) is a T_0-space.

***Definition* 3.8**

A topological kr-space (X,T) is said to be T_N-space iff for each $x \neq y \in X$, $\ker\{x\} \cap \ker\{y\}$ is empty or {x} or {y}.

***Theorem* 3.9**

In topological kr-space (X,T), every T_1-space is $T_N -$ space.

Proof

Let (X,T) be a T_N-space. Then for each $x \neq y \in X$, $\ker\{x\} = \{x\}$ and $\ker\{y\} = \{y\}$ [By theorem 1.3], implies $\ker\{x\} \cap \ker\{y\} = \emptyset$. Thus (X,T) is a T_N-space.

***Theorem* 3.10**

In topological kr-space (X,T), every T_N-space. is a T_0-space.

Proof

Let (X,T) be a T_N-space. Then for each $x \neq y \in X$, $\ker\{x\} \cap \ker\{y\}$ is degenerated (empty or singleton set). Therefore there exist two cases:

i) $\ker\{x\} \cap \ker\{y\} = \emptyset, implies (X,T)$ is a T_0-space

ii)$\ker\{x\} \cap \ker\{y\} = \{x\}$ *or* $\{y\}$, implies $y \notin \ker\{x\}$ or $x \notin \ker\{y\}$, implies (X,T) is a T_0-space.

***Theorem* 3.11**

A topological kr-space (X,T) is T_2-space iff for each $x \neq y \in X$, then $\ker\{x\} \cap \ker\{y\} = \emptyset$

Proof

Let a topological kr-space (X,T) is T_2-space. Then for each $x \neq y \in X$ there exist disjoint open sets U, V such that $x \in U$, and $y \in V$. Hence $\ker\{x\} \subseteq U$ and $\ker\{y\} \subseteq V$. Thus $\ker\{x\} \cap \ker\{y\} = \emptyset$

Conversely

Let for each $x \neq y \in X$, $\ker\{x\} \cap \ker\{y\} = \emptyset$. Since (X,T) be a topological kr-space, this means kernel is an open set. Thus (X,T) is T_2-space.

***Theorem* 3.12**

A topological kr-space (X,T) is a regular space iff for each F closed set and $x \notin F$, then $\ker(F) \cap \ker\{x\} = \emptyset$

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Proof

By the same way of proof of theorem 3.11

***Theorem* 3.13**

A topological kr-space (X,T) is a normal space iff for each disjoint closed sets G, H, then $\ker(G) \cap \ker(H) = \emptyset$

Proof

By the same way of proof of theorem 3.11

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