T_i Spaces with Respect to Weak Forms of ω −Open Sets, for $i = 0,1,2$

By

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Abstract. *In this paper we introduce the associative separation axioms of the weak ω −open sets defined in [5], and then give some new theorems about them.*

Key words. *Weak separation axioms, weak ω* −open sets, *weak T_0spaces, weak T_1spaces, weak T_2spaces.*

1. Introduction and Preliminaries

Through out this paper , (X,T) stands for topological space. Let (X,T) be a topological space and A a subset of X. A point x in X is called ***condensation point*** of A if for each U in T with x in U, the set $U \cap A$ is un countable [3]. In 1982 the ω −closed set was first introduced by H. Z. Hdeib in [3], and he defined it as: A is $\boldsymbol{\omega}$ ***−closed*** if it contains all its condensation points and the $\boldsymbol{\omega}$ ***−open*** set is the complement of the ω −closed set. Equivalently. A sub set W of a space (X,T), is ω −open if and only if for each $x \in W$, there exists $U \in T$ such that $x \in U$and $U \backslash W$ is countable. The collection of all ω −open sets of (X,T)denoted T_ω form topology on X and it is finer than T. Several characterizations of ω −closed sets were provided in [1, 3, 4, 6].

In 2009 in [5] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced and investigated new notions called $\alpha - \omega$ −open, pre $-\omega$ −open, $b - \omega$ −open and $\beta - \omega$ −open sets which are weaker than ω −open set. Let us introduce these notions in the following definition:

Definition 1.1. [5] A subset A of a space X is called

1. $\boldsymbol{\alpha - \omega}$ ***−open*** if $A \subseteq int_\omega\left(cl\left(int_\omega(A)\right)\right)$.

2. ***pre* −*ω* −*open*** if $A \subseteq int_{\omega}\left(cl(A)\right)$.

3. ***b* −*ω* −*open*** if $A \subseteq int_{\omega}\left(cl(A)\right) \cup cl\left(int_{\omega}(A)\right)$.

4. ***β* − *ω* −*open*** if $A \subseteq cl\left(int_{\omega}\left(cl(A)\right)\right)$.

For a subset A of X, the ***ω* − *interior*** of the set A defined as the union of all $\omega -$ open sets contained in A, and denoted by $int_{\omega}(A)$. The closure of A will be denoted by $cl(A)$.

Remark 1.2. [5] Any ω −open (resp. $\alpha - \omega$ −open, $pre - \omega$ −open, $b - \omega$ −open and $\beta - \omega$ −open) sets need not be open (resp. α −open, pre −open, b −open and β −open) as can be seen in the following example:

In [5] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced relationships among the weak open sets above by the lemma below:

Lemma 1.3. [5] In any topological space:

1. Any open set is ω −open.

2. Any ω −open set is $\alpha - \omega$ −open.

3. Any $\alpha - \omega$ −open set is $pre - \omega$ −open.

4. Any $pre - \omega$ −open set is $b - \omega$ −open.

5. Any $b - \omega$ −open set is $\beta - \omega$ −open.

The converse is not true [5].

Remark 1.4. [5] The intersection of two $pre - \omega$ −open, (resp. $b - \omega$ −open and $\beta - \omega$ −open) sets need not be $pre - \omega$ −open, (resp. $b - \omega$ −open and $\beta - \omega$ −open) sets. As can be seen in the following example:

Example 1.5. [5] Let $X = \mathbb{R}$ with the usual topology T. Let $A = Q$ and $B = (R \backslash Q) \cup \{1\}$, then A and B are $pre - \omega$ −open, but $A \cap B = \{1\}$, is not $\beta - \omega$ −open since $cl(int_{\omega}(cl(\{1\}))) = cl(int_{\omega}(\{1\})) = cl(\{\emptyset\}) = \emptyset$.

Lemma 1.6. [5] The intersection of an $\alpha - \omega$ −open (resp. $pre - \omega$ −open, $b - \omega$ −open and $\beta - \omega$ −open) subset of any topological space and an open subset is $\alpha - \omega$ −open (resp. $pre - \omega$ −open, $b - \omega$ −open and $\beta - \omega$ −open) set.

Theorem 1.7. The union of an $\alpha-\omega-$closed (resp. $pre-\omega-$ closed, $b-\omega-$ closed and $\beta-\omega-$ closed) subset of any topological space and a closed subset is $\alpha-\omega-$ closed (resp. $pre-\omega-$ closed, $b-\omega-$ closed and $\beta-\omega-$ closed) set.

Proof:

Let A be an $\alpha-\omega-$closed subset of a topological space X and B is a closed subset of X. Then A^c is $\alpha-\omega-$open subset of X and B^c is an open subset of X. Then by Lemma 1.6 we have $A^c \cap B^c$ is an is $\alpha-\omega-$open subset of X and $(A^c \cap B^c)^c$ is an $\alpha-\omega-$closed subset of X Therefore $(A^c \cap B^c)^c = A \cup B$ is $\alpha-\omega-$closed subset of X ◉

Theorem 1.8. [5] If $\{A_\alpha : \alpha \in \Delta\}$ is a collection of $\alpha-\omega-$open (resp. $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) subsets of the topological space (X,T), then $\cup_{\alpha\in\Delta} A_\alpha$ is $\alpha-\omega-$open (resp. $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) set.

Theorem 1.9. If $\{A_\alpha : \alpha \in \Delta\}$ is a collection of $\alpha-\omega-$closed (resp. $pre-\omega-$closed, $b-\omega-$closed and $\beta-\omega-$closed) subsets of the topological space (X,T), then $\cap_{\alpha\in\Delta} A_\alpha$ is $\alpha-\omega-$closed (resp. $pre-\omega-$closed, $b-\omega-$closed and $\beta-\omega-$closed) set.

Proof:

Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of $\alpha-\omega-$closed subsets of X, then ${A_\alpha}^c$ (the complement set of A_α) is $\alpha-\omega-$open set for each $\alpha \in \Delta$. Then by Theorem 1.8 we have $\cup_{\alpha\in\Delta} {A_\alpha}^c$ is $\alpha-\omega-$open set . Therefore $(\cup_{\alpha\in\Delta} {A_\alpha}^c)^c = \cap_{\alpha\in\Delta} A_\alpha$, is $\alpha-\omega-$closed subsets of X. A similar proof for the other cases ◉

Definition 1.10. [5] A space (X,T) is called a ***door space*** if every subset of X is either open or closed.

Lemma 1.11. [5] If (X,T) is a door space, then every $pre-\omega-$open set is $\omega-$open.

Theorem 1.12. Let A be a $\beta-\omega-$open set in the topological space (X,T), then A is $b-\omega-$open , whenever X is door space.

Proof:

Let A be a $\beta-\omega-$open subset of X. If A is open then by Lemma 1.4 it is $b-\omega-$open. Then if A is closed we get $A \subseteq cl(int_{\omega}(A)) \subseteq int_{\omega}(cl(A)) \cup cl(int_{\omega}(A))$. Thus A is $b-\omega-$open set in X ◉

Definition 1.13. [5] A subset A of a space X is called

1. An $\boldsymbol{\omega-t-}$***set***, if $int(A) = int_{\omega}(cl(A))$.

2. An $\boldsymbol{\omega-B-}$***set*** if $A = U \cap V$, where U is an open set and V is an $\omega-t-$set.

3. An $\boldsymbol{\omega-t_{\alpha}-}$***set***, if $int(A) = int_{\omega}(cl(int_{\omega}(A)))$.

4. An $\boldsymbol{\omega-B_{\alpha}-}$***set*** if $A = U \cap V$, where U is an open set and V is an $\omega-t_{\alpha}-$set.

5. An $\boldsymbol{\omega-}$***set*** if $A = U \cap V$, where U is an open set and $int(V) = int_{\omega}(V)$.

Definition 1.14. Let (X,T) be topological space. It said to be satisfy

1. The $\boldsymbol{\omega-}$***condition*** if every $\omega-$open set is $\omega-$set.

2. The $\boldsymbol{\omega-B_{\alpha}-}$***condition*** if every $\alpha-\omega-$open set is $\omega-B_{\alpha}-$set.

3. The $\boldsymbol{\omega-B-}$***condition*** if every $pre-\omega-$open is $\omega-B-$set.

Now let us introduce the following lemma from [5].

Lemma 1.15. [5] For any subset A of a space X, We have

1. A is open if and only if A is $\omega-$open and $\omega-$set.

2. A is open If and only if A is $\alpha-\omega-$open and $\omega-B_{\alpha}-$set.

3. A is open if and only if A is $pre-\omega-$open and $\omega-B-$set.

Lemma 1.16. [5] Let (X,T) be a topological space, and let $A \subseteq X$. If A is $b-\omega-$open set such that $int_{\omega}(A) = \emptyset$, then A is $pre-\omega-$open.

Definition 1.17. Let X be a topological space. We say that a subset A of X is $\boldsymbol{\omega-}$***compact*** [2] (resp. $\boldsymbol{\alpha-\omega-}$***compact*** **,** $\boldsymbol{pre-\omega-}$***compact,*** $\boldsymbol{b-\omega-}$***compact*** and $\boldsymbol{\beta-\omega-}$***compact***) if for each cover of $\omega-$open (resp. $\alpha-\omega-$open , $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) sets from X contains a finite subcover for A.

Definition 1.18. A function $f:(X,\sigma) \rightarrow (Y,\tau)$ is called ***ω-continuous*** **(** resp**.** $\boldsymbol{\alpha-}$***ω-continuous*** **,** $\boldsymbol{pre-\omega-}$***continuous,*** $\boldsymbol{b-\omega-}$***continuous and*** $\boldsymbol{\beta-\omega-}$***continuous*** **)**, if for each $x \in X$, and each

ω −open (resp. $\alpha-\omega$ −open, $pre-\omega$ −open, $b-\omega$ −open and $\beta-\omega$ −open) set V containing $f(x)$, there exists an ω −open (resp. $\alpha-\omega$ −open, $pre-\omega$ −open, $b-\omega$ −open and $\beta-\omega$-open,) set U containing x, such that $f(U)\subset V$.

2. weak T_0 spaces

In this article, let us introduce the weak T_0 spaces with some relations, propositions and theorems.

Definition 2.1. Let X be a topological space. If for each $x\neq y\in X$, either there exists a set U, such that $x\in U$, $y\notin U$,or there exists a set U such that $x\notin U, y\in U$. Then X called

1. $\boldsymbol{\omega-T_0}$ space, whenever U is ω −open set in X.

2. $\boldsymbol{\alpha-\omega-T_0}$ space, whenever U is $\alpha-\omega$ −open set in X.

3. $\boldsymbol{pre-\omega-T_0}$ space, whenever U is pre $-\omega$ −open set in X.

4. $\boldsymbol{b-\omega-T_0}$ space, whenever U is $b-\omega$ −open set in X.

5. $\boldsymbol{\beta-\omega-T_0}$ space, whenever U is $\beta-\omega$ −open set in X.

Using Lemma 1.3 we can write the following proposition:

Proposition 2.2. Let (X,T) be a topological space.

1. If (X,T) is T_0, then it is $\omega-T_0$.

2. If (X,T) is $\omega-T_0$, then it is $\alpha-\omega-T_0$.

3. If (X,T) is $\alpha-\omega-T_0$, then it is $pre-\omega-T_0$

4. If (X,T) is $pre-\omega-T_0$, then it is $b-\omega-T_0$.

5. If (X,T) is $b-\omega-T_0$, then it is $\beta-\omega-T_0$.

Remark 2.3. The converse of the above theorem is not true as we see in the following example:

Example 2.4. Let $X=\{1,2,3\}$ with the topology $T=\{\emptyset,X,\{1\}\}$. It is clear that (X,T) is $\omega-T_0$ space but not T_0 space.

Theorem 2.5. Let (X,T) be a door space. Then we have:

1. Every $pre-\omega-T_0$ space is $\omega-T_0$.

2. Every $\beta-\omega-T_0$ space is $b-\omega-T_0$.

Proof:

Directly from Definition 2.1, Lemma 1.11 and Theorem 1.12 ◉

Theorem 2.6. Let (X, T), be a topological space.

1. If (X, T) is $\omega - T_0$ topological space satisfies the ω −condition, then it is T_0 topological space.

2. If (X, T) is $\alpha - \omega - T_0$ topological space satisfies the $\omega - B_\alpha$ −condition, then it is T_0 topological space.

3. If (X, T) is $pre - \omega - T_0$ topological space satisfies the $\omega - B$ −condition, then it is T_0 topological space.

Proof:

Directly from Definition 2.1, Definition 1.14 and Lemma 1.15 ◉

Proposition 2.7. If (X, T) is $b - \omega - T_0$ topological space with the property that any $b - \omega$ −open subset has empty ω −interior. Then it is $pre - \omega - T_0$.

Proof:

Directly from Definition 2.1 and Lemma 1.16 ◉

One can summarize the theorems above by Figure 1.

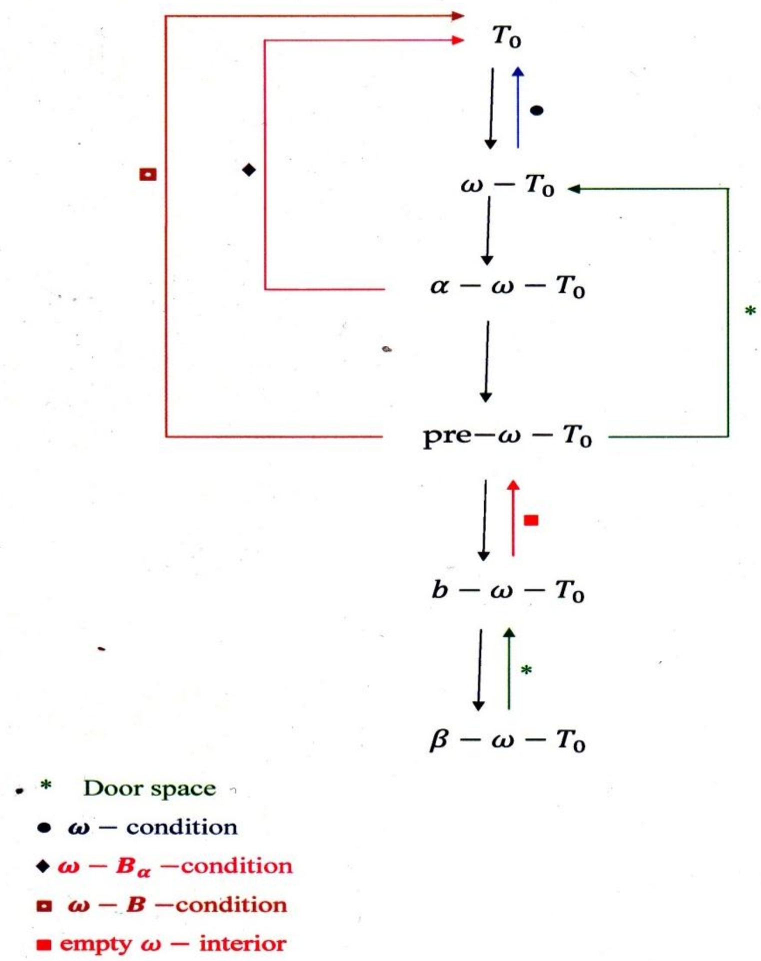


Figure 1

3. weak T_1 space

Weak types of $\omega - T_1$ spaces is the subject of this article. Also we introduce some related results.

Definition 3.1. Let X be a topological space. For each $x \neq y \in X$, there exists a set U, such that $x \in U, y \notin U$, and there exists a set V such that $y \in V, x \notin V$, then X is called

1. $\boldsymbol{\omega - T_1}$ ***space*** if U is open and V is ω −open sets in X.

2. $\boldsymbol{\alpha - \omega - T_1}$ ***space*** if U is open and V is $\alpha - \omega$ −open sets in X.

3. $\boldsymbol{\omega^* - T_1}$ ***space*** [3] if U and V are ω −open sets in X.

4. $\boldsymbol{\alpha - \omega^* - T_1}$ ***space*** if U is ω-open and V is $\alpha - \omega$ −open sets in X.

5. $\boldsymbol{\alpha - \omega^{**} - T_1}$ ***space*** if U and V are $\alpha - \omega$ −open sets in X.

6. $\boldsymbol{pre - \omega - T_1}$ ***space*** if U is open and V is $pre - \omega$ −open sets in X.

7. $\boldsymbol{pre - \omega^* - T_1}$ ***space*** if U is ω-open and V is $pre - \omega$ −open sets in X.

8. $\boldsymbol{\alpha - pre - \omega - T_1}$ ***space*** if U is $\alpha - \omega -$ open and V is $pre - \omega$ −open sets in X.

9. $\boldsymbol{pre - \omega^{**} - T_1}$ ***space*** if U and V are $pre - \omega$ −open sets in X.

10. $\boldsymbol{b - \omega - T_1}$ ***space*** if U is open and V is $b - \omega$ −open sets in X.

11. $\boldsymbol{b - \omega^* - T_1}$ ***space*** if U is ω −open and V is $b - \omega$ −open sets in X.

12. $\boldsymbol{\alpha - b - \omega - T_1}$ ***space*** if U is $\alpha - \omega$ −open and V is $b - \omega$ −open sets in X.

13.$\boldsymbol{pre - b - \omega - T_1}$ ***space*** if U is $pre - \omega$ −open and V is $b - \omega$ −open sets in X.

14. $\boldsymbol{b - \omega^{**} - T_1}$ ***space*** if U and V are $b - \omega$ −open sets in X.

15. $\boldsymbol{\beta - \omega - T_1}$ ***space*** if U is open and V is $\beta - \omega$ −open sets in X.

16. $\boldsymbol{\beta - \omega^* - T_1}$ ***space*** if U is ω −open and V is $\beta - \omega$ −open sets in X.

17. $\boldsymbol{\alpha - \beta - \omega - T_1}$ ***space*** if U is $\alpha - \omega$ −open and V is $\beta - \omega$ −open sets in X.

18.$\boldsymbol{pre - \beta - \omega - T_1}$ ***space*** if U is $pre - \omega$ −open and V is $\beta - \omega$ −open sets in X.

19. $\boldsymbol{\beta - \omega^{**} - T_1}$ ***space*** if U and V are $\beta - \omega$ −open sets in X.

20. $\boldsymbol{b - \beta - \omega - T_1}$ ***space*** if U is $b - \omega$ −open and V is $\beta - \omega$ −open sets in X

Theorem 3.2. Let X be a topological space,

1. X is $\omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is closed and $\{y\}$ is ω −closed set in X.

2. X is $\alpha - \omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is closed and $\{y\}$ is $\alpha - \omega$ −closed set in X.

3. X is $\omega^* - T_1$ space if and only if for each $x \in X$, $\{x\}$ is ω −closed set in X.

4. X is $\alpha - \omega^* - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is ω −closed and $\{y\}$ is $\alpha - \omega$ −closed set in X.

5. X is $\alpha - \omega^{**} - T_1$ space if and only if for each $x \in X$, $\{x\}$ is $\alpha - \omega$ −closed set in X.

6. X is $pre - \omega - T_1$space if and only if for each $x \neq y \in X$, $\{x\}$ is closed and $\{y\}$ is $pre -$ ω −closed set in X.

7. X is $pre - \omega^* - T_1$space if and only if for each $x \neq y \in X$, $\{x\}$ is ω −closed and $\{y\}$ is $pre -$ ω −closed set in X.

8. X is $\alpha - pre - \omega - T_1$space if and only if for each $x \neq y \in X$, $\{x\}$ is $\alpha - \omega$ −closed and $\{y\}$ is $pre -$ ω −closed set in X.

9. X is $pre - \omega^{**} - T_1$space if and only if for each $x \in X$, $\{x\}$ is $pre - \omega$ −closed set in X.

10. X is $b - \omega - T_1$space if and only if for each $x \neq y \in X$, $\{x\}$ closed and $\{y\}$ is $b -$ ω −closed set in X.

11. X is $b - \omega^* - T_1$space if and only if for each $x \neq y \in X$, $\{x\}$ is ω −closed and $\{y\}$ is $b-$ ω −closed set in X.

12. X is $\alpha - b - \omega - T_1$space if and only if for each $x \neq y \in X$, $\{x\}$ is $\alpha - \omega$ −closed and $\{y\}$ is $b-$ ω −closed set in X.

13. X is $pre - b - \omega - T_1$space if and only if for each $x \neq y \in X$, $\{x\}$ is pre −closed and $\{y\}$ is $b-$ ω −closed set in X.

14. X is $b - \omega^{**} - T_1$ space if and only if for each $x \in X$, $\{x\}$ is $b - \omega$ −closed set in X.

15. X is $\beta - \omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$is closed and $\{y\}$ is $\beta -$ ω −closed set in X.

16. X is $\beta - \omega^* - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is ω −closed and $\{y\}$ is $\beta -$ ω −closed set in X.

17. X is $\alpha - \beta - \omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is $\alpha - \omega$ −closed and $\{y\}$ is $\beta -$ ω −closed set in X.

18. X is $pre-\beta-\omega-T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is $pre-\omega-$closed and $\{y\}$ is $\beta-\omega-$closed set in X.

19. X is $\beta-\omega^{**}-T_1$ space if and only if for each $x \in X, \{x\}$ is $\beta-\omega-$closed set in X.

20. X is $b-\beta-\omega-T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is $b-\omega-$closed and $\{y\}$ is $\beta-\omega-$closed set in X.

Proof of (4):

Let $x \in X$. If $y \in X$, such that $y \neq x$, then there exist an $\omega-$open set U_y containing y but not x, and $\alpha-\omega-$open set U_x containing x but not y. Hence $y \in U_y \subset \{x\}^c$. Therefore $\{x\}^c = \cup_{y\in\{x\}^c} U_y$, which is $\omega-$open set and $\{x\}$ is $\omega-$closed set in X. Also $\{y\}$ is $\alpha-\omega-$closed set in X. In fact $x \in U_x \subset \{y\}^c$, which implies $\{y\}^c = \cup_{x\in\{y\}^c} U_x$. Then because U_x is $\alpha-\omega-$open set, for each $x \in \{y\}^c$, so $\{y\}^c$ is $\alpha-\omega-$open set, and $\{y\}$ is $\alpha-\omega-$closed set

Now for the converse, let $x \neq y \in X$, $U_x = X\backslash\{y\}$ is $\alpha-\omega-$open set containing x but not y, and $U_y = X\backslash\{x\}$ is $\omega-$open set, containing y but not x. Thus X is $\alpha-\omega^*-T_1$ space.

A similar proof for 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, and 20 ◉

Theorem 3.3. For any topological space.

1. Any T_1 is $\omega-T_1$ space.

2. Any $\omega-T_1$ is $\alpha-\omega-T_1$ space.

3. Any $\omega-T_1$ is ω^*-T_1 space.

4. Any ω^*-T_1 is $\alpha-\omega^*-T_1$ space.

5. Any $\alpha-\omega^*-T_1$ is $pre-\omega^*-T_1$ space.

6. Any $\alpha-\omega-T_1$ is $\alpha-\omega^*-T_1$ space.

7. Any $\alpha-\omega-T_1$ is $pre-\omega-T_1$ space.

8. Any $pre-\omega-T_1$ is $\beta-\omega-T_1$ space.

9. Any $\alpha-\omega^*-T_1$ is $\alpha-\omega^{**}-T_1$ space.

10. Any $\alpha-\omega^{**}-T_1$ is $\alpha-pre-\omega-T_1$ space.

11. Any $\alpha-pre-\omega-T_1$ is $pre-\omega^{**}-T_1$ space.

12. Any $pre-\omega^{**}-T_1$ is $pre-b-\omega-T_1$ space.

13. Any $pre-b-\omega-T_1$ is $b-\omega^{**}-T_1$ space.

14. Any $b-\omega^{**}-T_1$ is $b-\beta-\omega-T_1$ space.

15. Any $b-\beta-\omega-T_1$ is $\beta-\omega^{**}-T_1$ space.

16. Any $pre-\omega^*-T_1$ is $\alpha-pre-\omega-T_1$ space.

17. Any $pre-\omega^*-T_1$ is $b-\omega^*-T_1$ space.

18. Any $\alpha-pre-\omega-T_1$ is $\alpha-b-\omega-T_1$ space.

19. Any $b-\omega^*-T_1$ is $\alpha-b-\omega-T_1$ space.

20. Any $pre-\omega-T_1$ is $b-\omega-T_1$ space.

21. Any $\beta-\omega-T_1$ is $\beta-\omega^*-T_1$ space.

22. Any $\beta-\omega^*-T_1$ is $\alpha-\beta-\omega-T_1$ space.

23. Any $\alpha-\beta-\omega-T_1$ is $pre-\beta-\omega-T_1$ space.

24. Any $pre-\omega-T_1$ is $pre-\omega^*-T_1$ space.

25. Any $b-\omega-T_1$ is $\beta-\omega-T_1$ space.

26. Any $b-\omega^*-T_1$ is $\beta-\omega^*-T_1$ space.

27. Any $\alpha-b-\omega-T_1$ is $\alpha-\beta-\omega-T_1$ space.

28. Any $\alpha-b-\omega-T_1$ is $pre-b-\omega-T_1$ space.

29. Any $pre-b-\omega-T_1$ is $pre-\beta-\omega-T_1$ space.

30. Any $pre-\beta-\omega-T_1$ is $b-\beta-\omega-T_1$ space.

31. Any $b-\omega-T_1$ is $b-\omega^*-T_1$ space.

Proof:

Easy. By using Lemma 1.3 ◉

Remark 3.4. The converse of the theorem above is not satisfied in general. As we see in the following examples.

Example 3.5. Let $X=\{1,2,3\}$ with the topology $T=\{\emptyset,X,\{1\},\{3\},\{1,3\}\}$. (X,T) is $\omega-T_1$ space, but not T_1.

To have equivalence between the weak T_1s spaces, we shall introduce the following theorems:

Theorem 3.6. Let (X,T) be a door space. Then we have:

1. Every $pre-\omega-T_1$ space is $\omega-T_1$.

2. Every $pre-\omega^*-T_1$ space is ω^*-T_1.

3. Every $\alpha - pre - \omega - T_1$ space is $\alpha - \omega^* - T_1$,

4. Every $pre - b - \omega - T_1$ space is b $-\omega^* - T_1$.

5. Every $pre - \beta - \omega - T_1$ space is $\beta - \omega^* - T_1$.

6. Every $pre - \omega^{**} - T_1$ space is $\omega^* - T_1$.

7. Every $b - \beta - \omega - T_1$ space is $b - \omega^{**} - T_1$.

8. Every $\beta - \omega^{**} - T_1$ space is $b - \omega^{**} - T_1$.

9. Every $pre - \beta - \omega - T_1$ space is $pre - b - \omega - T_1$.

10. Every $\alpha - \beta - \omega - T_1$ space is $\alpha - b - \omega - T_1$.

11. Every $\beta - \omega^* - T_1$ space is $b - \omega^* - T_1$.

12. Every $\beta - \omega - T_1$ space is $b - \omega - T_1$.

Proof:

Directly from Lemma 1.11 and Theorem 1.12 ◉

Using Definition 1.14 and Lemma 1.15 we can prove the following important theorem

Theorem 3.7. For any topological space (X, T).

A. Let (X, T), satisfies the ω −condition.

1. If (X, T) is $\omega - T_1$, then it is T_1.

2. If (X, T) is $\omega^* - T_1$, then it is T_1.

3. If (X, T) is $\alpha - \omega^* - T_1$, then it is $\alpha - \omega - T_1$.

4. If (X, T) is $pre - \omega^* - T_1$, then it is $pre - \omega - T_1$.

5. If (X, T) is $b - \omega^* - T_1$, then it is $b - \omega - T_1$.

6. If (X, T) is $\beta - \omega^* - T_1$, then it is $\beta - \omega - T_1$.

B. Let (X, T), satisfies the $\omega - B_\alpha$ −condition.

1. If (X, T) is is $\alpha - \omega - T_1$, then it is T_1.

2. If (X, T) is $\alpha - \omega^* - T_1$, then it is $\omega - T_1$.

3. If (X, T) is $\alpha - \omega^{**} - T_1$, then it is T_1.

4. If (X, T) is $\alpha - pre - \omega - T_1$, then it is $pre - \omega - T_1$.

5. If (X, T) is $\alpha - b - \omega - T_1$, then it is $b - \omega - T_1$.

6. If (X, T) is $\alpha - \beta - \omega - T_1$, then it is $\beta - \omega - T_1$.

C. Let (X, T), satisfies the $\omega - B$ −condition.

1. If (X, T) is $pre - \omega - T_1$, then it is T_1.

2. If (X, T) is $pre - \omega^* - T_1$, then it is $\omega - T_1$.

3. If (X, T) is $pre - \omega^{**} - T_1$, then it is T_1.

4. If (X, T) is $pre - b - \omega - T_1$, then it is $b - \omega - T_1$.

5. If (X, T) is $pre - \beta - \omega - T_1$, then it is $\beta - \omega - T_1$.

6. If (X, T) is $\alpha - pre - \omega - T_1$, then it is $\alpha - \omega - T_1$.

Proposition 3.8. Let (X, T) be a topological space with the property that any $b - \omega -$open subset has empty ω −interior.

1. If (X, T) is $b - \omega - T_1$, then it is $pre - \omega - T_1$.

2. If (X, T) is $b - \omega^* - T_1$, then it is $pre - \omega^* - T_1$.

3. If (X, T) is $\alpha - b - \omega - T_1$, then it is $\alpha - pre - \omega - T_1$.

4. If (X, T) is $pre - b - \omega - T_1$, then it is $pre - \omega^{**} - T_1$.

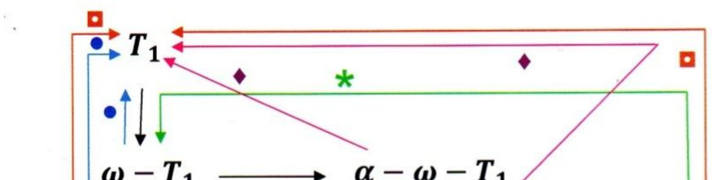
5. If (X, T) is $b - \omega^{**} - T_1$, then it is $pre - \omega^{**} - T_1$.

6. If (X, T) is $b - \beta - \omega - T_1$, then it is $pre - \beta - \omega - T_1$.

Proof:

Directly from Lemma 1.16 ◉

One can summarize the relationships among weak T_1s spaces by Figure 2.



Definition 3.9. A topological space (X,T) is $\boldsymbol{\omega}$ ***−symmetric*** if for x and y in the space X, $x \in cl_{\omega}(\{y\})$ implies $y \in cl_{\omega}(\{x\})$.

Proposition 3.10. Let X be a door, ω −symetric topological space . Then for each $x \in X$, the set $\{x\}$ is ω −closed.

Proof:

Let $x \neq y \in X$, since X is a door space so $\{y\}$ is open or closed set in X. When $\{y\}$ is open, so it is ω −open, let $V_y = \{y\}$. Whenever $\{y\}$ is ω −closed , $x \notin \{y\} = cl_\omega(\{y\})$. Since X is ω −symetric we get $y \notin cl_\omega(\{x\})$. Put $V_y = X \backslash cl_\omega(\{x\})$, then $x \notin V_y$ and $y \in V_y$, and V_y is ω −open set in X. Hence we get for each $y \in X\backslash\{x\}$ there is an ω −open set V_y such that $x \notin V_y$ and $y \in V_y$. Therefore $X\backslash\{x\} = \bigcup_{y \in X\backslash\{x\}} V_y$ is ω −open, and $\{x\}$ is ω −closed ◉

Proposition 3.11. Let (X, T) be an $\omega - T_1$ (resp. $\omega^* - T_1$, $\alpha - \omega - T_1$, $\alpha - \omega^* - T_1$, $b - \omega - T_1$, $b - \omega^* - T_1$, $pre - \omega - T_1$, $pre - \omega^* - T_1$, $\beta - \omega - T_1$, $\beta - \omega^* - T_1$) topological space, then it is ω −symetric space.

Proof:

Assume $y \notin cl_\omega(\{x\})$, so $x \neq y$, then since X is $\omega - T_1$ there is an open set U containing x but not y, so $x \notin cl_\omega(\{y\})$. This completes the proof ◉

Theorem 3.12. The topological door space is ω − symmetric if and only if it is $\omega^* - T_1$.

Proof:

Let (X, T) be a door ω − symmetric space. Then using Proposition 3.١٠ for each $x \in X$, $\{x\}$ is ω −closed set in X. Then by (3) of Theorem 3.2, we get that (X, T) is $\omega^* - T_1$. On the other hand, assume (X, T) is $\omega^* - T_1$, then directly by Proposition 3.11 (X, T) is ω − symmetric space ◉

4. Weak $\omega - T_2$ spaces

In this article we will define weak types of $\omega - T_2$ spaces and introduce some results about it.

Definition 4.1. Let X be a topological space. And for each $x \neq y \in X$, there exist two disjoint sets U and V with $x \in U$ and $y \in V$, then X is called:

1. ***$\omega - T_2$ space*** if U is open and V is ω −open sets in X.
2. ***$\alpha - \omega - T_2$ space*** if U is open and V is $\alpha - \omega$ −open sets in X.
3. ***$\omega^* - T_2$ space*** if U and V are ω −open sets in X.

4. $\boldsymbol{\alpha-\omega^{*}-T_2}$ ***space*** if U is ω −open and V is $\alpha-\omega$ −open sets in X.

5. $\boldsymbol{\alpha-\omega^{**}-T_2}$ ***space*** if U and V are $\alpha-\omega$ −open sets in X.

6. $\boldsymbol{pre-\omega-T_2}$ ***space*** if U is open and V is $pre-\omega$ −open sets in X.

7. $\boldsymbol{pre-\omega^{*}-T_2}$ ***space*** if U is ω −open and V is $pre-\omega$ −open sets in X.

8. $\boldsymbol{\alpha-pre-\omega-T_2}$ ***space*** if U is α −open and V is $pre-\omega$ −open sets in X.

9. $\boldsymbol{pre-\omega^{**}-T_2}$ ***space*** if U and V are $pre-\omega$ −open sets in X.

10. $\boldsymbol{b-\omega-T_2}$ ***space*** if U is open and V is $b-\omega$ −open sets in X.

11. $\boldsymbol{b-\omega^{*}-T_2}$ ***space*** if U is ω −open and V is $b-\omega$ −open sets in X.

12. $\boldsymbol{\alpha-b-\omega-T_2}$ ***space*** if U is $\alpha-\omega$ −open and V is $b-\omega$ −open sets in X.

13. $\boldsymbol{pre-b-\omega-T_2}$ ***space*** if U is $pre-\omega$ −open and V is $b-\omega$ −open sets in X.

14. $\boldsymbol{b-\omega^{**}-T_2}$ ***space*** if U and V are $b-\omega$ −open sets in X.

15. $\boldsymbol{\beta-\omega-T_2}$ ***space*** if U is open and V is $\beta-\omega$ −open sets in X.

16. $\boldsymbol{\beta-\omega^{*}-T_2}$ ***space*** if U is ω −open and V is $\beta-\omega$ −open sets in X.

17. $\boldsymbol{\alpha-\beta-\omega-T_2}$ ***space*** if U is $\alpha-\omega$ −open and V is $\beta-\omega$ −open sets in X.

18. $\boldsymbol{pre-\beta-\omega-T_2}$ ***space*** if U is $pre-\omega$ −open and V is $\beta-\omega$ −open sets in X.

19. $\boldsymbol{\beta-\omega^{**}-T_2}$ ***space*** if U and V are $\beta-\omega$ −open sets in X.

20. $\boldsymbol{b-\beta-\omega-T_2}$ ***space*** if U is $b-\omega$ −open and V is $\beta-\omega$ −open sets in X.

Remark 4.2. We can restate Theorem 3.3 for the weak T_2s spaces.

Theorem 4.3. For any door topological space we have:

1. Every $pre-\omega-T_2$ space is $\omega-T_2$.

2. Every $pre-\omega^{*}-T_2$ space is $\omega^{*}-T_2$.

3. Every $\alpha-pre-\omega-T_2$ space is $\alpha-\omega^{*}-T_2$.

4. Every $pre-b-\omega-T_2$ space is b $-\omega^{*}-T_2$.

5. Every $pre-\beta-\omega-T_2$ space is $\beta-\omega^{*}-T_2$.

6. Every $pre-\omega^{**}-T_2$ space is $\omega^{*}-T_2$.

7. Every $b-\beta-\omega-T_2$ space is $b-\omega^{**}-T_2$.

8. Every $\beta-\omega^{**}-T_2$ space is $b-\omega^{**}-T_2$.

9. Every $pre-\beta-\omega-T_2$ space is $pre-b-\omega-T_2$.

10. Every α- $\beta - \omega - T_2$ space is $\alpha - b - \omega - T_2$.

11. Every $\beta - \omega^* - T_2$ space is $b - \omega^* - T_2$.

12. Every $\beta - \omega - T_2$ space is $b - \omega - T_2$.

Proof:

Directly from Lemma 1.11 and Theorem 1.12 ◉

Theorem 4.4. For any topological space (X, T).

A. Let (X, T), satisfies the ω −condition.

1. If (X, T) is $\omega - T_2$, then it is T_2.

2. If (X, T) is $\omega^* - T_2$, then it is T_2.

3. If (X, T) is $\alpha - \omega^* - T_2$, then it is $\alpha - \omega - T_2$.

4. If (X, T) is $pre - \omega^* - T_2$, then it is $pre - \omega - T_2$.

5. If (X, T) is $b - \omega^* - T_2$, then it is $b - \omega - T_2$.

6. If (X, T) is $\beta - \omega^* - T_2$, then it is $\beta - \omega - T_2$.

B. Let (X, T), satisfies the $\omega - B_\alpha$ −condition.

1. If (X, T) is $\alpha - \omega - T_2$, then it is T_2.

2. If (X, T) is $\alpha - \omega^* - T_2$, then it is $\omega - T_2$.

3. If (X, T) is $\alpha - \omega^{**} - T_2$, then it is T_2.

4. If (X, T) is $\alpha - pre - \omega - T_2$, then it is $pre - \omega - T_2$.

5. If (X, T) is $\alpha - b - \omega - T_2$, then it is $b - \omega - T_2$.

6. If (X, T) is $\alpha - \beta - \omega - T_2$, then it is $\beta - \omega - T_2$.

C. Let (X, T), satisfies the $\omega - B$ −condition.

1. If (X, T) is $pre - \omega - T_2$, then it is T_2.

2. If (X, T) is $pre - \omega^* - T_2$, then it is $\omega - T_2$.

3. If (X, T) is $pre - \omega^{**} - T_2$, then it is T_2.

4. If (X, T) is $pre - b - \omega - T_2$, then it is $b - \omega - T_2$.

5. If (X, T) is $pre - \beta - \omega - T_2$, then it is $\beta - \omega - T_2$.

6. If (X, T) is $\alpha - pre - \omega - T_2$, then it is $\alpha - \omega - T_2$.

Proof:

Using Definition 4.1 Definition 1.14 and Lemma 1.15 ◉

Proposition 4.5. Let (X, T) be a topological space with the property that any $b - \omega -$open subset has empty $\omega -$interior.

1. If (X, T) is $b - \omega - T_2$, , then it is $pre - \omega - T_2$.

2. If (X, T) is $b - \omega^* - T_2$, , then it is $pre - \omega^* - T_2$.

3. If (X, T) is $\alpha - b - \omega - T_2$, , then it is $\alpha - pre - \omega - T_2$.

4. If (X, T) is $pre - b - \omega - T_2$, , then it is $pre - \omega^{**} - T_2$.

5. If (X, T) is $b - \omega^{**} - T_2$, , then it is $pre - \omega^{**} - T_2$.

6. If (X, T) is $b - \beta - \omega - T_2$, , then it is $pre - \beta - \omega - T_2$.

Proof:

Directly from Lemma 1.16 ◉

One can summarize the relationships among weak T_2s spaces by a figure coincide with Figure 2.

Theorem 4.6. Let (X, τ) and (Y, σ) be two topological spaces, and $f: (X, \tau) \longrightarrow (Y, \sigma)$ be injective map.

1. If f is $\omega -$continuous, and Y is $\omega^* - T_2$, then X is also $\omega^* - T_2$.

2. If f is $\alpha - \omega -$continuous, and Y is $\alpha - \omega^{**} - T_2$, then X is also $\alpha - \omega^{**} - T_2$.

3. If f is $pre - \omega -$continuous, and Y is $pre - \omega^{**} - T_2$, then X is also $pre - \omega^{**} - T_2$.

4. If f is $b - \omega -$continuous, and Y $b - \omega^{**} - T_2$, then X is also $b - \omega^{**} - T_2$.

5. If f is $\beta - \omega -$continuous, and Y is $\beta - \omega^{**} - T_2$, then X is also $\beta - \omega^{**} - T_2$.

Proof of (2):

Let us prove one case and the others are similar. Let Y be $\alpha - \omega^{**} - T_2$ space, to prove X is $\alpha - \omega^{**} - T_2$, let $x, y \in X$ with $x \neq y$, Since f is injective, so $f(x) \neq f(y)$. And since Y is $\alpha - \omega^{**} - T_2$, there exist $\alpha - \omega -$open sets V and U such that $f(x) \in U$ and $f(y) \in V$ with $U \cap V = \emptyset$. Let $G = f^{-1}(U)$ and $H = f^{-1}(V)$. Since f is $\alpha - \omega -$continuous, so G and H are $\alpha - \omega -$open sets in X, with $x \in G$, and $y \in H$. Also $G \cap H = f^{-1}(U) \cap f^{-1}(V)$ $= f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$. Hence X is $\alpha - \omega - T_2$ space ◉

Theorem 4.7. Let (X, τ) and (Y, σ) be two topological spaces, and $f: (X, \tau) \longrightarrow (Y, \sigma)$ be injective map.

1. If X satisfies $\omega -$condition, f is $\omega -$continuous, and Y is $\omega - T_2$, then X is also $\omega - T_2$.

2. If X satisfies $\omega - B_\alpha$ −condition, f is $\alpha - \omega$ −continuous, and Y is $\alpha - \omega - T_2$, then X is also $\alpha - \omega - T_2$.

3. If X satisfies $\omega - B_\alpha$ −condition, f is $\alpha - \omega$ −continuous, and Y is $\alpha - \omega^* - T_2$, then X is also $\alpha - \omega^* - T_2$.

4. If X satisfies , $\omega - B$ −condition, f is $pre - \omega$ −continuous, and Y is $pre - \omega - T_2$, then X is also $pre - \omega - T_2$.

5. If X is a door space or satisfies $\omega - B$ −condition, f is $pre - \omega$ −continuous, and Y is $pre - \omega^* - T_2$, then X is also $pre - \omega^* - T_2$.

6. If X is a door space or satisfies $\omega - B$ −condition, f is $pre - \omega$ −continuous, and Y is $\alpha - pre - \omega - T_2$, then X is also $\alpha - pre - \omega - T_2$.

7. If X is a door space, f is $\beta - \omega$ −continuous, and Y is $b - \beta - \omega - T_2$, then X is also $b - \beta - \omega - T_2$.

Proof of (7):

Let Y be $b - \beta - \omega - T_2$, and let $x, y \in X$ with $x \neq y$, Since f is injective, so $f(x) \neq f(y)$. And since Y is $b - \beta - \omega - T_2$there exist $b - \omega$ −open set U and $\beta - \omega$ −open set V such that $f(x) \in U$ and $f(y) \in V$ with $U \cap V = \emptyset$. Let $G = f^{-1}(U)$ and $H = f^{-1}(V)$. $\beta - \omega$ −continuity implies G and H are $\beta - \omega$ −open sets in X. When X is a door space we can consider one of the two $\beta - \omega$ −open sets as a $b - \omega$ −open with $x \in G$, and $y \in H$. Also $G \cap H = f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$. Hence X is $b - \beta - \omega - T_2$ space ◉

Proposition 4.8. Let (X, T) be a topological space.

1. If X is an $\omega - T_2$ space, $x \notin Y$ and Y is ω −compact subset of X. Then there exist disjoint sets U is open and V is ω −open in X such that U containing x and V containing Y.

2. If X is an $\omega^* - T_2$ space, $x \notin Y$and Y is ω −compact subset of X. Then there exist disjoint sets U and V are ω −open in X such that U containing x and V containing Y.

3. If X is an $\alpha - \omega - T_2$ space, $x \notin Y$and Y is $\alpha - \omega$ −compact subset of X. Then there exist disjoint sets U is open and V is $\alpha - \omega$ −open in X such that U containing x and V containing Y.

4. If X is an $\alpha - \omega^* - T_2$, space, $x \notin Y$and Y is $\alpha - \omega$ −compact subset of X. Then there exist disjoint sets U is ω −open and V is $\alpha - \omega$ −open in X such that U containing x and V containing Y.

5. If X is a $pre-\omega-T_2$space, $x\notin Y$ and Y is $pre-\omega-$compact subset of X. Then there exist disjoint sets U is open and V is $pre-\omega-$open in X such that U containing x and V containing Y.

6. If X is a $pre-\omega^*-T_2$space, $x\notin Y$and Y is $pre-\omega-$compact subset of X. Then there exist disjoint sets U is $\omega-$open and V is $pre-\omega-$open in X such that U containing x and V containing Y.

7. If X is a $b-\omega-T_2$space, $x\notin Y$and Y is $b-\omega-$compact subset of X. Then there exist disjoint sets U is open and V is $b-\omega-$open in X such that U containing x and V containing Y.

8. If X is a $b-\omega^*-T_2$space, $x\notin Y$and Y is $b-\omega-$compact subset of X. Then there exist disjoint sets U is $\omega-$open and V is $b-\omega-$open in X such that U containing x and V containing Y.

9. If X is a $\beta-\omega-T_2$space, $x\notin Y$and Y is $\beta-\omega-$compact subset of X. Then there exist disjoint sets U is open and V is $\beta-\omega-$open in X such that U containing x and V containing Y.

10. If X is a $\beta-\omega^*-T_2$space, $x\notin Y$and Y is $\beta-\omega-$compact subset of X. Then there exist disjoint sets U is $\omega-$open and V is $\beta-\omega-$open in X such that U containing x and V containing Y.

Proof of (3):

Let $x\notin Y$. Assume $y\in Y$, since X is an $\alpha-\omega-T_2$ space , so there exist two disjoint sets U_y open and V_y $\alpha-\omega-$open in X with $x\in U_y$, and $y\in V_y$, so $Y\subset\cup_{y\in Y}V_y$. Since Y is an $\alpha-\omega-$compact so there exist $y_1,y_2,\dots,y_n$, such that $Y\subset\cup_{i=1}^{n}V_{y_i}$. Let $V=\cup_{i=1}^{n}V_{y_i}$, V is $\alpha-\omega-$open set containing Y, and $U=\cap_{i=1}^{n}U_{y_i}$ is open set containing x. U and V are disjoint because if there is $z\in U\cap V$, then $z\in V_{y_i}$ for some i and $z\in U_{y_i}$ for each i. This contradicts U_{y_i} and V_{y_i} are disjoint. Similarly we can prove the other cases ◉

As a consequence of the proof of the theorem above one can get the following corollary.

Corollary 4.9. Let (X,T) be a topological space. If X is an $\omega-T_2$ space, $x\in X$ and Y is compact set not containing x. Then there exist disjoint sets U open containing Y and V $\omega-$open containing x.

Theorem 4.10. For any topological space.

1. Every $\omega-$compact subset of $\omega-T_2$ space is closed.

2. Every $\alpha-\omega-$compact subset of $\alpha-\omega-T_2$ space is closed.

3. Every $\omega-$compact subset of ω^*-T_2 space is $\omega-$closed.

4. Every $\alpha-\omega$ −compact subset of $\alpha-\omega^{*}-T_2$ space is ω − closed.

5. Every $pre-\omega$ −compact subset of $pre-\omega-T_2$ space is closed.

6. Every *pre*−ω-compact subset of $pre-\omega^{*}-T_2$ space is ω − closed.

7. Every $b-\omega$ −compact subset of $b-\omega-T_2$ space is closed.

8. Every $b-\omega$ −compact subset of $b-\omega^{*}-T_2$ space is ω − closed.

9. Every $\beta-\omega$ −compact subset of $\beta-\omega-T_2$ space is closed.

10. Every $\beta-\omega$ −compact subset of $\beta-\omega^{*}-T_2$ space is ω −closed.

Proof of (2):

Let Y be an $\alpha-\omega$ −compact subset of the $\alpha-\omega-T_2$ space X. To prove Y is closed, we shall prove $X\backslash Y$ is open. Let $x_0 \in X\backslash Y$, but X is $\alpha-\omega-T_2$, so for each $y \in Y$ there are disjoint sets U_y and V_y such that U_y is open set containing x_0 and V_y is $\alpha-\omega$ −open set containing y. The collection $\{V_y, y \in Y\}$ is a cover for Y consists of $\alpha-\omega$ −open sets in X. Since Y is $\alpha-\omega$ −compact so we can find a finite subcover V for Y, $V=\cup_{i=1}^{n} V_{y_i}$. Let $U=\cap_{i=1}^{n} U_{y_i}$. Note that U is open set and V is $\alpha-\omega$ −open set in X, also they are disjoint. If $z \in V$ then there is i, such that $z \in V_{y_i}$ and $z \notin U$, therefore U is an open set containing x_0 disjoint from Y. Hence $X\backslash Y$ is open and Y is closed . Similarly we can prove the other statements ◉

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