Characterizations of Continuity and Compactness with Respect to Weak Forms of ω-Open Sets

By

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Abstract. *We use the weak ω −open sets defined by T. Noiri, A. Al-Omari, M. S. M. Noorani in [5], to define new weak types of continuity and compactness and prove some theorems about them.*

Key words. *Weak open set, weak continuity, weak compactness, ω −open set.*

1. Introduction and Preliminaries

Through out this paper , (X,T) stands for topological space. Let (X,T) be a topological space and A a subset of X. A point x in X is called ***condensation point*** of A if for each U in T with x in U, the set $U \cap A$ is un countable [3]. In 1982 the ω −closed set was first introduced by H. Z. Hdeib in [3], and he defined it as: A is ***ω −closed*** if it contains all its condensation points and the ***ω −open*** set is the complement of the ω −closed set. Equivalently. A subset W of a space (X,T), is ω −open if and only if for each $x \in W$, there exists $U \in T$ such that $x \in U$and $U \setminus W$ is countable. The collection of all ω −open sets of (X,T)denoted T_ω form topology on X and it is finer than T. Several characterizations of ω −closed sets were provided in [1, 3, 4, 6]. For a subset A of X, the closure of A and the $\omega-$ interior of A will be denoted by $cl(A)$ and $int_\omega(A)$ respectively. The $\boldsymbol{\omega}-$ ***interior*** of the set A defined as the union of all $\omega-$ open sets contained in A.

In 2009 in [5] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced and investigated new notions called $\alpha-\omega$ −open, pre $-\omega$ −open, $b-\omega$ −open and $\beta-\omega$ −open sets which are weaker than ω −open set. Let us introduce these notions in the following definition:

Definition 1.1. [5] A subset A of a space X is called

1. $\boldsymbol{\alpha}-\boldsymbol{\omega}$ ***−open*** if $A\subseteq int_\omega\left(cl(int_\omega(A))\right)$.

2. ***pre*** $-\boldsymbol{\omega}$ ***−open*** if $A\subseteq int_\omega(cl(A))$.

3. ***b*** $-\boldsymbol{\omega}$ ***−open*** if $A\subseteq int_\omega(cl(A))\cup cl(int_\omega(A))$.

4. $\boldsymbol{\beta}-\boldsymbol{\omega}$ ***−open*** if $A\subseteq cl\left(int_\omega(cl(A))\right)$.

Remark 1.2. [5] Any ω −open (resp. $\alpha-\omega$ −open, $pre-\omega$ −open, $b-\omega$ −open and $\beta-\omega$ −open) sets need not be open (resp. α −open, pre −open, b −open and β −open) as can be seen in the following example:

Example 1.3. Let $X=\{a,b\}$ with the topology $T=\{X,\emptyset,\{a\}\}$. Then $\{b\}$ satisfying the above remark.

In [5] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced relationships among the weak open sets above by the lemma below:

Lemma 1.4. [5] In any topological space:

1. Any open set is ω −open.

2. Any ω −open set is $\alpha-\omega$ −open.

3. Any $\alpha-\omega$ −open set is $pre-\omega$ −open.

4. Any $pre-\omega$ −open set is $b-\omega$ −open.

5. Any $b-\omega$ −open set is $\beta-\omega$ −open.

The converse is not true [5].

Remark 1.5. [5] The intersection of two $pre-\omega$ −open, (resp. $b-\omega$ −open and $\beta-\omega$ −open) sets need not be $pre-\omega$ −open, (resp. $b-\omega$ −open and $\beta-\omega$ −open) sets. As can be seen in the following example:

Example 1.6. [5] Let $X=\mathbb{R}$ with the usual topology T. Let $A=Q$ and $B=(R\backslash Q)\cup\{1\}$, then A and B are $pre-\omega$ −open, but $A\cap B=\{1\}$, is not $\beta-\omega$ −open since $cl(int_{\omega}(cl(\{1\})))=cl(int_{\omega}(\{1\}))$ $=cl(\{\emptyset\})=\emptyset$.

Lemma 1.7. [5] The intersection of an $\alpha-\omega$ −open (resp. $pre-\omega$ −open, $b-\omega$ −open and $\beta-\omega$ −open) subset of any topological space and an open subset is $\alpha-\omega$ −open (resp. $pre-\omega$ −open, $b-\omega$ −open and $\beta-\omega$ −open) set.

Theorem 1.8. The union of an $\alpha-\omega$ −closed (resp. $pre-\omega-$ closed, $b-\omega-$ closed and $\beta-\omega-$ closed) subset of any topological space and a closed subset is $\alpha-\omega-$ closed (resp. $pre-\omega-$ closed, $b-\omega-$ closed and $\beta-\omega-$ closed) set.

Proof:

Let A be an $\alpha-\omega-$closed subset of a topological space X and B is a closed subset of X. Then A^c is $\alpha-\omega-$open subset of X and B^c is an open subset of X. Then by Lemma 1.7 we have $A^c \cap B^c$ is an is $\alpha-\omega-$open subset of X and $(A^c \cap B^c)^c$ is an $\alpha-\omega-$closed subset of X Therefore $(A^c \cap B^c)^c = A \cup B$ is $\alpha-\omega-$closed subset of X

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Theorem 1.9. [5] If $\{A_\alpha : \alpha \in \Delta\}$ is a collection of $\alpha-\omega-$open (resp. $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) subsets of the topological space (X,T), then $\cup_{\alpha\in\Delta} A_\alpha$ is $\alpha-\omega-$open (resp. $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) set.

Theorem 1.10. If $\{A_\alpha : \alpha \in \Delta\}$ is a collection of $\alpha-\omega-$closed (resp. $pre-\omega-$closed, $b-\omega-$closed and $\beta-\omega-$closed) subsets of the topological space (X,T), then $\cap_{\alpha\in\Delta} A_\alpha$ is $\alpha-\omega-$closed (resp. $pre-\omega-$closed, $b-\omega-$closed and $\beta-\omega-$closed) set.

Proof:

Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of $\alpha-\omega-$closed subsets of X, then ${A_\alpha}^c$ (the complement set of A_α) is $\alpha-\omega-$open set for each $\alpha \in \Delta$. Then by Theorem 1.9 we have $\cup_{\alpha\in\Delta} {A_\alpha}^c$ is $\alpha-\omega-$open set . Therefore $(\cup_{\alpha\in\Delta} {A_\alpha}^c)^c = \cap_{\alpha\in\Delta} A_\alpha$, is $\alpha-\omega-$closed subsets of X. Similarly for the other cases ◉

Definition 1.11. [5] A space (X,T) is called a ***door space*** if every subset of X is either open or closed.

Example 1.12. The space (X,T) for $X=\{a,b\}$, and $T=\{X,\emptyset,\{a\}\}$, is a door space.

Lemma 1.13. [5] If (X,T) is a door space, then every $pre-\omega$ −open set is ω −open.

Theorem 1.14. Let A be a $\beta-\omega$ −open set in the topological space (X,T), then A is $b-\omega$ −open , whenever X is door space.

Proof:

Let A be a β −ω −open subset of X. If A is open then by Lemma 1. 4 it is $b-\omega$ −open. Then if A is closed we get $A\subseteq cl(int_{\omega}(A)) \subseteq int_{\omega}(cl(A))\cup cl(int_{\omega}(A))$. Thus A is b −ω −open set in X ◉

Definition 1.15. [5] A subset A of a space X is called

1. An $\boldsymbol{\omega-t}$ −***set***, if $int(A)=int_{\omega}(cl(A))$.

2. An $\boldsymbol{\omega-B}$ −***set*** if $A=U\cap V$, where U is an open set and V is an $\omega-t$ −set.

3. An $\boldsymbol{\omega-t_{\alpha}}$ −***set***, if $int(A)=int_{\omega}(cl(int_{\omega}(A)))$.

4. An $\boldsymbol{\omega-B_{\alpha}}$ −***set*** if $A=U\cap V$, where U is an open set and V is an $\omega-t_{\alpha}$ −set.

5. An $\boldsymbol{\omega}$ −***set*** if $A=U\cap V$, where U is an open set and $int(V)=int_{\omega}(V)$.

Definition 1.16. Let (X,T) be topological space. It said to be satisfy

1. The $\boldsymbol{\omega}$ −***condition*** if every ω −open set is ω −set.

2. The $\boldsymbol{\omega-B_{\alpha}}$ −***condition*** if every $\alpha-\omega$ −open set is $\omega-B_{\alpha}$ −set.

3. The $\boldsymbol{\omega-B}$ −***condition*** if every $pre-\omega$ −open is $\omega-B$ −set.

Now let us introduce the following lemma from [5].

Lemma 1.17. [5] For any subset A of a space X, We have

1. A is open if and only if A is ω −open and ω −set.

2. A is open If and only if A is $\alpha-\omega$ −open and $\omega - B_{\alpha}$ −set.

3. A is open if and only if A is $pre-\omega$ −open and $\omega - B$ −set.

2. Decomposition of Continuity

Let us now use the weak ω −open sets to define a decomposition of continuity. Also we introduce some theorems about this notion.

Definition 2.1. A function $f:(X,\sigma) \to (Y,\tau)$ is called ***ω-continuous* (** resp**. *α − ω-continuous* , *pre −ω −continuous*, *b −ω −continuous and β − ω −continuous*)**, if for each $x \in X$, and each ω −open (resp. $\alpha-\omega$ −open, pre $-\omega$ −open, b $-\omega$ −open and $\beta-\omega$ −open) set V containing $f(x)$, there exists an ω −open (resp. $\alpha-\omega$ −open, pre $-\omega$ −open, b $-\omega$ −open and $\beta-\omega$-open,) set U containing x, such that $f(U) \subset V$.

Using the definition above we can get the following theorem:

Proposition 2.2. A function $f:(X,\sigma) \to (Y,\tau)$ is ω −continuous (resp. $\alpha-\omega$ −continuous , pre $-\omega$ −continuous, b $-\omega$ −continuous and $\beta-\omega$ −continuous) if and only if for each ω −open (resp. $\alpha-\omega$ −open, pre $-\omega$ −open, b $-\omega$ −open and $\beta-\omega$ −open) set V in Y, $f^{-1}(V)$ is ω −open (resp. $\alpha-\omega$ −open, pre $-\omega$ −open, b $-\omega$ −open and $\beta-\omega$ −open) set in X.

Proof:

Let f be an ω −continuous map from X to Y, and let $x \in X$,and V be an ω −open subset of Y containing $f(x)$. We must show that $f^{-1}(V)$ is ω −open subset of X containing x, so we shall prove $int_{\omega}(f^{-1}(V)) = f^{-1}(V)$, let $x \in f^{-1}(V)$, then by the ω −continuity of f we can find an ω −open set U in X and containing x, such that $f(U) \subset V$, then $U \subset f^{-1}(V)$, which is true for any $x \in f^{-1}(V)$. This implies $f^{-1}(V)$ is ω −open subset of X. For the opposite side, let us assume that the inverse image of any ω −open set is also an ω −open to prove f is ω −continuous map. Let $x \in X$ and let V be an ω −open subset of Y containing $f(x)$, by the hypothesis $f^{-1}(V)$ is ω −open subset of X, so for any $x \in f^{-1}(V)$, $f(f^{-1}(V)) \subset V$, and f is ω −continuous. By the same way we can prove the other cases ◉

Theorem 2.3. Let (X, σ) and (Y, τ) be two topological spaces such that X satisfies the $\omega - B_{\alpha}$ −condition, and $f: (X, \sigma) \longrightarrow (Y, \tau)$ be a map. If f is $\alpha - \omega$ −continuous then it is ω −continuous.

Proof:

Let $f: (X, \sigma) \longrightarrow (X, \tau)$ be an $\alpha - \omega$ −continuous, to prove it is ω −continuous, let $x \in X$ and V be an ω −open (so it is $\alpha - \omega$ −open) set containing $f(x)$. Since f is $\alpha - \omega$ −continuousso there exists an $\alpha - \omega$ −open subset U of X containing x such that $f(U) \subset V$. Then since X satisfies the $\omega - B_{\alpha}$ −condition we have U is an ω −open of X containing x such that $f(U) \subset V$. This implies f is ω −continuous

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Theorem 2.4. Let (X,σ) and (Y,τ) be two topological spaces such that X is door space, and $f:(X,\sigma)\longrightarrow(Y,\tau)$ be a map.

1. If f is $pre-\omega$ $-$continuous then it is ω $-$continuous.

2. If f is $\beta-\omega$ $-$continuous then it is $b-\omega$ $-$continuous.

Proof:

By the same way as the proof of Theorem 2.3, using Lemma 1.4, Lemma 1.13 and Theorem 1.14, we can prove this theorem ◉

Theorem 2.5. Let : (X,σ) and (Y,τ) be two topological spaces that satisfy the ω $-$condition then the map $f:(X,\sigma)\longrightarrow(Y,\tau)$ is continuous if and only if it is ω $-$continuous.

Proof:

Let $f:(X,\sigma)\longrightarrow(X,\tau)$ be a continuous map at , $x\in X$ and V be an ω $-$open set in Y and containing $f(x)$. Since X satisfy ω $-$condition, so V is also open in Y. And by the continuity of f there is an open set U (also it is ω $-$open) with $f(U)\subset V$. For the converse let f be an ω $-$continuous map and V be an open set in Y and containing $f(x)$, so it is also ω $-$open and by the ω $-$continuity of f , there is an ω $-$open set U in X containing $x$with $f(U)\subset V$, and since X satisfies the ω $-$condition U is an open set therefore f is continuous ◉

Remark 2.6. Theorem 2.5. is not true in general. It mean if $f:(X,\tau)\longrightarrow(Y,\sigma)$ is ω $-$continuous, then it is not necessarily continuous. As we see in the following example.

Example 2.7. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$, $Y = \{d, e, f\}$, $\sigma = \{\emptyset, Y, \{d\}\}$, and let $f: (X, \tau) \longrightarrow (Y, \sigma)$be a map defined by $f(a) = f(b) = d,\ f(c) = e$. f is ω −continuous but not continuous.

Note that since X and Y are countable, so any subset of them is ω −open. If $x = a$, we have $f(x) = d$. $V_1 = \{d\}$, $V_2 = \{d, e\}$, $V_3 = \{d, f\}$, and $V_4 = Y$ are ω −open sets in Y containing $f(x)$, so there exist $U_1 = \{a, b\}$, $U_2 = \{a, c\}$, $U_3 = \{a\}$, and $U_4 = X$ such that $f(U_1) \subset V_1$, $f(U_2) \subset V_2$, $f(U_3) \subset V_3$ and $f(U_4) \subset V_4$. Similarly for $x = b$, and $x = c$, Therefore f is ω −continuous map.

Next f is not continuous. Let $x = b,\ f(x) = d,$ if $V = \{d\}$, then when $U = X$, we have $f(U) = \{d, e\} \not\subset \{d\} = V$. Hence f is not continuous map.

Remark 2.8. As the proof of Theorem 2.5 we can prove the following Theorem 2.9 and Theorem 2.10.

Theorem 2.9. Let (X, σ) and (Y, τ) be two topological spaces that satisfy the $\omega - B_\alpha$ −condition then the map $f: (X, \sigma) \longrightarrow (Y, \tau)$ is continuous if and only if it is $\alpha - \omega$ −continuous.

Theorem 2.10. Let (X, σ) and (Y, τ) be two topological spaces that satisfy the $\omega - B$ −condition then the map $f: (X, \sigma) \longrightarrow (Y, \tau)$ is continuous if and only if it is $pre - \omega$ −continuous.

Theorem 2.11. Let (X, σ) and (Y, τ) be two door topological spaces and $f: (X, \sigma) \longrightarrow (Y, \tau)$ be a map. Then

1. f is $pre - \omega$ −continuous if and only if it is ω −continuous.

2. f is $\beta - \omega$ −continuous if and only if it is $b - \omega$ −continuous.

Proof of (1):

Let f be a $pre-\omega$ −continuous, and let V be an ω −open set in Y and containing $f(x)$, therefore it is $pre-\omega$ −open and since f is $pre-\omega$ −continuous, there is a $pre-\omega$ −open set U in X containing x and $f(U)\subset V$. Since X is a door space U is also an ω −open set. For the converse let f be an ω −continuous map and V be a $pre-\omega$ −open set in Y. Then since Y is door space we get V is ω −open , and by the ω −continuity of f there exists an ω −open set U in X containing x (also $pre-\omega$ −open) with $f(U)\subset V$. And so f is a ω −continuous.

Similarly we can prove (2) ◉

3. Weak ω −Compactness

In this article we shall introduce weak ω −compactness. It is defined that every cover by such weak open sets contains a finite subcover. So let us state new definitions for the weak new types of ω −compact sets, and prove several rather simple theorems about it.

Definition 3.1. Let X be a topological space. We say that a subset A of X is ***ω −compact*** [2] **(** resp. ***α − ω −compact*** , ***pre −ω −compact***, ***b − ω −compact*** and ***β − ω −compact*** **)** if for each cover of ω −open (resp. $\alpha-\omega$ −open , $pre-\omega$ −open, $b-\omega$ −open and $\beta-\omega$ −open) sets from X contains a finite subcover for A.

Theorem 3. 2. In any topological space

1. Any $\beta-\omega$ −compact set is $b-\omega$ −compact.

2. Any $b-\omega$ −compact set is $pre-\omega$ −compact.

3. Any $b-\omega$ −compact set is $\alpha-\omega$ −compact.

4. Any $b-\omega$ −compact set is ω −compact.

5. Any $b-\omega$ −compact set is compact.

6. Any $\beta-\omega$ −compact set is $pre-\omega$ −compact.

7. Any $pre-\omega$ −compact set is $\alpha-\omega$ −compact.

8. Any $pre-\omega$ −compact set is ω −compact.

9. Any $pre-\omega$ −compact set is compact.

10. Any $\beta-\omega$ −compact set is $\alpha-\omega$ −compact.

11. Any $\alpha-\omega$ −compact set is ω −compact.

12. Any $\alpha-\omega$ −compact set is compact.

13. Any $\beta-\omega$ −compact set is ω −compact.

14. Any ω −compact set is compact.

15. Any $\beta-\omega$ −compact set is compact.

Proof of (15):

Let X be a topological space , and let A be a $\beta-\omega$ −compact sub set of X , to prove A is compact, let C be an open cover for A. Since we can consider C as a cover of $\beta-\omega$ −open sets, in addition to that it is an open cover and A is $\beta-\omega$ −compact subset of X. Then C contains a finite open subcover , Thus X is compact set.

Similarly we can prove the other cases using Lemma 1.4 ◉

Remark 3.3. The converse of the above theorem is not true in general, as we see in the following example.

Example 3.4. Consider the usual topology T for $\mathbb{R}$. The subset $A=\mathbb{Q}\cap[-\sqrt{2},\sqrt{2}]$ is ω −compact but not $b-\omega$ −compact. Since

$\left\{\left(-\sqrt{2}-\frac{1}{n},2\right]\right\}_{n\in\mathbb{N}}$ is $b-\omega-$open cover for A, but it not have a finite sub cover for A.

Theorem 3.5. Let (X,T) be a topological space

1.If (X,T) is door space, then any $\omega-$compact set is $pre-\omega-$compact.

2. If (X,T) is door space, then any $b-\omega-$compact set is $\beta-\omega-$compact.

3. If (X,T) satisfies the $\omega-$condition, then any compact set is $\omega-$compact.

4. If (X,T) satisfies the $\omega-B_{\alpha}-$condition, then any compact set is $\alpha-\omega-$compact.

5. If (X,T) satisfies the $\omega-B-$condition, then any compact set is $pre-\omega-$compact.

Proof:

1. Let X be a topological door space, and let A be an $\omega-$compact sub set of X , and C be a cover of $pre-\omega-$open subsets of X . Since X is a door space so we can consider C as a cover of $\omega-$open sets. And by the $\omega-$compactness of X, C contains a finite sub cover of $pre-\omega-$open sets. Hence A is $pre-\omega-$compact.

Similarly we can prove (2).

3. Let X be a topological space satisfies the $\omega-$condition, and A be a compact subset of X, to prove A is $\omega-$compact, let C be a cover of $\omega-$open sets for A. Since X satisfies the $\omega-$condition, we can consider C as a cover of open sets and by the compactness of A , C contains a finite subcover of open(also $\omega-$open) sets for A. This implies X is $\omega-$compact.

Similarly we can prove (4) and (5) ◉

Theorem 3.6. An ω −closed (resp. $\alpha-\omega$ −closed , pre −ω −closed, b −ω −closed and $\beta-\omega$ −closed) sub set of ω −compact (resp. $\alpha-\omega$ −compact , pre −ω −compact, b −ω −compact and $\beta-\omega$ −compact) subspace is ω −compact (resp. $\alpha-\omega$ −compact , pre −ω −compact, $b-\omega$ −compact and β −ω −compact).

Proof:

Let Y be an ω-compact subspace of the topological space X, and let F be an $\omega-$ closed subset of Y. Let $C=\{G_\lambda,\lambda\in\Lambda\}$ be a cover of ω −open sets for F. Then $C\cup(Y\backslash F)=D$ is a cover of ω −open sets for Y. Since Y is ω −compact there is a finite sub cover $\acute{D}$ of D for Y, and hence without $Y\backslash F$, a cover for F (because F and $Y\backslash F$ are disjoint). So we have shown that a finite sub collection of C cover F. Thus F is $\omega-$ compact. Similarly we can prove the other cases ◉

Theorem 3.7. Let $f:X\longrightarrow Y$ be an ω-continuous (resp. $\alpha-\omega-$ continuous, pre $-\omega-$ continuous, $b-\omega-$ continuous, and β $-\omega-$ continuous) map from the ω −compact (resp. $\alpha-\omega$ −compact , pre −ω −compact, $b-\omega$ −compact, and β −ω −compact) space X onto a topological space Y. Then Y is ω −compact (resp. $\alpha-\omega$ −compact , pre −ω −compact , $b-\omega$ −compact and β −ω −compact) space..

Proof:

Let $f:X\longrightarrow Y$ be an ω −continuous map from the ω −compact space X on to Y. Let $\{Y_\lambda,\lambda\in\Lambda\}$ be a cover of ω −open sets for Y, then since f is ω −continuous map so $\{f^{-1}(Y_\lambda), \lambda \in \Lambda\}$ is a cover of ω −open sets for X. Since X is ω −compact so it has a finite subcover $\{f^{-1}(Y_{\lambda_i}), i = 1,2,\ldots,n\}$. Then by the surjection of f we get $\{Y_{\lambda_i}, i = 1,2,\ldots,n\}$ is an ω −open cover for Y. Hence Y is ω −compact. With a simple modification to that prove one can prove the other cases

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