Characterizations of Continuity and Compactness with Respect to Weak Forms of ω-Open Sets

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Abstract. In this paper *We use the weak ω −open sets defined by T. Noiri, A. Al-Omari, M. S. M. Noorani in [5], to define new weak types of continuity and compactness and prove some theorems about them.*

Key words. *Weak open set, weak continuity, weak compactness, ω −open set.*

1. Introduction and Preliminaries

Through out this paper , (X, T) stands for topological space. Let (X, T) be a topological space and A a subset of X. A point x in X is called ***condensation point*** of A if for each U in T with x in U, the set $U \cap A$ is un countable [3]. In 1982 the ω −closed set was first introduced by H. Z. Hdeib in [3], and he defined it as: A is ***ω −closed*** if it contains all its condensation points and the ***ω −open*** set is the complement of the ω −closed set. Equivalently. A subset W of a space (X, T), is ω −open if and only if for each $x \in W$, there exists $U \in T$ such that $x \in U$and $U \backslash W$ is countable. The collection of all ω −open sets of (X, T)denoted T_ω form topology on X and it is finer than T. Several characterizations of ω −closed sets were provided in [1, 3, 4, 6]. For a subset A of X, the closure of A and the ω − interior of A will be denoted by $cl(A)$ and $int_\omega(A)$ respectively. The ***ω − interior*** of the set A defined as the union of all ω − open sets contained in A.

In 2009 in [5] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced and investigated new notions called $\alpha-\omega-$open, $pre\ -\omega-$open, $b-\omega-$open and $\beta-\omega-$open sets which are weaker than $\omega-$open set. Let us introduce these notions in the following definition:

Definition 1.1. [5] A subset A of a space X is called

1. $\boldsymbol{\alpha-\omega-}$***open*** if $A \subseteq int_{\omega}\left(cl\left(int_{\omega}(A)\right)\right)$.

2. ***pre*** $\boldsymbol{-\omega-}$***open*** if $A \subseteq int_{\omega}\left(cl(A)\right)$.

3. $\boldsymbol{b-\omega-}$***open*** if $A \subseteq int_{\omega}\left(cl(A)\right) \cup cl\left(int_{\omega}(A)\right)$.

4. $\boldsymbol{\beta-\omega-}$***open*** if $A \subseteq cl\left(int_{\omega}\left(cl(A)\right)\right)$.

In [5] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced relationships among the weak open sets above by the lemma below:

Lemma 1.2. [5] In any topological space:

1. Any open set is $\omega-$open.

2. Any $\omega-$open set is $\alpha-\omega-$open.

3. Any $\alpha-\omega-$open set is $pre-\omega-$open.

4. Any $pre-\omega-$open set is $b-\omega-$open.

5. Any $b-\omega-$open set is $\beta-\omega-$open.

The converse is not true [5].

Remark 1.3. [5] The intersection of two $pre-\omega-$open, (resp. $b-\omega-$open and $\beta-\omega-$open) sets need not be $pre-\omega-$open, (resp. $b-\omega-$open and $\beta-\omega-$open) sets.

Lemma 1.4. [5] The intersection of an $\alpha-\omega$ −open (resp. $pre-\omega$ −open, $b-\omega$ −open and $\beta-\omega$ −open) subset of any topological space and an open subset is $\alpha-\omega$ −open (resp. $pre-\omega$ −open, $b-\omega$ −open and $\beta-\omega$ −open) set.

Remark 1.5. The union of an $\alpha-\omega$ −closed (resp. $pre-\omega-$ closed, $b-\omega-$ closed and $\beta-\omega-$ closed) subset of any topological space and a closed subset is $\alpha-\omega-$ closed (resp. $pre-\omega-$ closed, $b-\omega-$ closed and $\beta-\omega-$ closed) set.

Theorem 1.6. [5] If $\{A_\alpha : \alpha \in \Delta\}$ is a collection of $\alpha-\omega$ −open (resp. $pre-\omega$ −open, $b-\omega$ −open and $\beta-\omega$ −open) subsets of the topological space (X,T), then $\cup_{\alpha\in\Delta} A_\alpha$ is $\alpha-\omega$ −open (resp. $pre-\omega$ −open, $b-\omega$ −open and $\beta-\omega$ −open) set.

Corollary 1.7. If $\{A_\alpha : \alpha \in \Delta\}$ is a collection of $\alpha-\omega$ −closed (resp. $pre-\omega$ −closed, $b-\omega$ −closed and $\beta-\omega$ −closed) subsets of the topological space (X,T), then $\cap_{\alpha\in\Delta} A_\alpha$ is $\alpha-\omega$ −closed (resp. $pre-\omega$ −closed, $b-\omega$ −closed and $\beta-\omega$ −closed) set.

Definition 1.8. [5] A space (X,T) is called a ***door space*** if every subset of X is either open or closed.

Example 1.9. The space (X,T) for $X=\{a,b\}$, and $T=\{X,\emptyset,\{a\}\}$, is a door space.

Lemma 1.10. [5] If (X,T) is a door space, then every $pre-\omega$ −open set is ω −open.

Theorem 1.11. Let A be a $\beta-\omega$ −open set in the topological space (X,T), then A is $b-\omega$ −open , whenever X is door space.

Proof:

Let A be a $\beta-\omega-$open subset of X. If A is open then by Lemma 1. 4 it is $b-\omega-$open. Then if A is closed we get $A \subseteq cl(int_\omega(A)) \subseteq int_\omega(cl(A)) \cup cl(int_\omega(A))$. Thus A is $b-\omega-$open set in X ◉

Definition 1.12. [5] A subset A of a space X is called

1. An $\boldsymbol{\omega-t}$ **−*set***, if $int(A) = int_\omega(cl(A))$.

2. An $\boldsymbol{\omega-B}$ **−*set*** if $A = U \cap V$, where U is an open set and V is an $\omega-t-$set.

3. An $\boldsymbol{\omega-t_\alpha}$ **−*set***, if $int(A) = int_\omega(cl(int_\omega(A)))$.

4. An $\boldsymbol{\omega-B_\alpha}$ **−*set*** if $A = U \cap V$, where U is an open set and V is an $\omega-t_\alpha-$set.

5. An $\boldsymbol{\omega}$ **−*set*** if $A = U \cap V$, where U is an open set and $int(V) = int_\omega(V)$.

Definition 1.13. Let (X, T) be topological space. It said to be satisfy

1. The $\boldsymbol{\omega}$ **−*condition*** if every ω −open set is $\omega-$set.

2. The $\boldsymbol{\omega-B_\alpha}$ **−*condition*** if every $\alpha-\omega-$open set is $\omega-B_\alpha-$set.

3. The $\boldsymbol{\omega-B}$ **−*condition*** if every $pre-\omega-$open is $\omega-B-$set.

Now let us introduce the following lemma from [5].

Lemma 1.14. [5] For any subset A of a space X, We have

1. A is open if and only if A is $\omega-$open and $\omega-$set.

2. A is open If and only if A is $\alpha-\omega-$open and $\omega-B_\alpha-$set.

3. A is open if and only if A is $pre-\omega-$open and $\omega-B-$set.

2. Decomposition of Continuity

Let us now use the weak $\omega-$open sets to define a decomposition of continuity. Also we introduce some theorems about this notion.

Definition 2.1. A function $f:(X,\sigma)\to(Y,\tau)$ is called ***ω-continuous*** (resp. ***$\alpha-\omega$-continuous*** , ***pre $-\omega$ −continuous, b $-\omega$ −continuous and $\beta-\omega$ −continuous***), if for each $x\in X$, and each ω −open (resp. $\alpha-\omega$ −open, pre $-\omega$ −open, b $-\omega$ −open and $\beta-\omega$ −open) set V containing $f(x)$, there exists an ω −open (resp. $\alpha-\omega$ −open, pre $-\omega$ −open, b $-\omega$ −open and $\beta-\omega$-open,) set U containing x, such that $f(U)\subset V$.

Using the definition above we can get the following theorem:

Proposition 2.2. A function $f:(X,\sigma)\to(Y,\tau)$ is ω −continuous (resp. $\alpha-\omega$ −continuous , pre $-\omega$ −continuous, b $-\omega$ −continuous and $\beta-\omega$ −continuous) if and only if for each ω −open (resp. $\alpha-\omega$ −open, pre $-\omega$ −open, b $-\omega$ −open and $\beta-\omega$ −open) set V in Y , $f^{-1}(V)$ is ω −open (resp. $\alpha-\omega$ −open, pre $-\omega$ −open, b $-\omega$ −open and $\beta-\omega$ −open) set in X.

Proof:

Let f be an ω −continuous map from X to Y, and let $x\in X$,and V be an ω −open subset of Y containing $f(x)$. We must show that $f^{-1}(V)$ is ω −open subset of X containing x, so we shall prove $int_{\omega}(f^{-1}(V))=f^{-1}(V)$, let $x\in f^{-1}(V)$, then by the ω −continuity of f we can find an ω −open set U in X and containing x, such that $f(U)\subset V$, then $U\subset f^{-1}(V)$, which is true for any $x\in f^{-1}(V)$. This implies $f^{-1}(V)$ is ω −open subset of X. For the opposite side, let us assume that the inverse image of any ω −open set is also an ω −open to prove f is ω −continuous map. Let $x\in X$ and let V be an ω −open subset of Y containing $f(x)$, by the hypothesis $f^{-1}(V)$ is ω −open subset of X, so for any $x\in f^{-1}(V)$, $f(f^{-1}(V))\subset V$, and f is ω −continuous. By the same way we can prove the other cases ◉

Theorem 2.3. Let (X, σ) and (Y, τ) be two topological spaces such that X satisfies the $\omega - B_\alpha$ −condition, and $f: (X, \sigma) \longrightarrow (Y, \tau)$ be a map. If f is $\alpha - \omega$ −continuous then it is ω −continuous.

Proof:

Let $f: (X, \sigma) \longrightarrow (X, \tau)$ be an $\alpha - \omega$ −continuous, to prove it is ω −continuous, let $x \in X$ and V be an ω −open (so it is $\alpha - \omega$ −open) set containing $f(x)$. Since f is $\alpha - \omega$ −continuousso there exists an $\alpha - \omega$ −open subset U of X containing x such that $f(U) \subset V$. Then since X satisfies the $\omega - B_\alpha$ −condition we have U is an ω −open of X containing x such that $f(U) \subset V$. This implies f is ω −continuous

◉

Theorem 2.4. Let (X, σ) and (Y, τ) be two topological spaces such that X is door space, and $f: (X, \sigma) \longrightarrow (Y, \tau)$ be a map.

1. If f is $pre - \omega$ −continuous then it is ω −continuous.

2. If f is $\beta - \omega$ −continuous then it is $b - \omega$ −continuous.

Proof:

By the same way as the proof of Theorem 2.3, using Lemma 1.2, Lemma 1.10 and Theorem 1.11, we can prove this theorem ◉

Theorem 2.5. Let $:(X, \sigma)$ and (Y, τ) be two topological spaces that satisfy the ω −condition then the map $f: (X, \sigma) \longrightarrow (Y, \tau)$ is continuous if and only if it is ω −continuous.

Proof:

Let $f: (X, \sigma) \longrightarrow (X, \tau)$ be a continuous map at , $x \in X$ and V be an ω −open set in Y and containing $f(x)$. Since X satisfy ω −condition, so V is also open in Y. And by the continuity of f there is an open set U (also it is ω −open) with $f(U) \subset V$.

For the converse let f be an ω −continuous map and V be an open set in Y and containing $f(x)$, so it is also ω −open and by the ω −continuity of f, there is an ω −open set U in X containing $x$with $f(U) \subset V$, and since X satisfies the ω −condition U is an open set therefore f is continuous ◉

Remark 2.6. Theorem 2.5. is not true in general. It mean if $f:(X,\tau) \longrightarrow (Y,\sigma)$ is ω −continuous, then it is not necessarily continuous. As we see in the following example.

Example 2.7. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{c\}\}$, $Y = \{d,e,f\}$, $\sigma = \{\emptyset, Y, \{d\}\}$, and let $f:(X,\tau) \longrightarrow (Y,\sigma)$be a map defined by $f(a) = f(b) = d$, $f(c) = e$. f is ω −continuous but not continuous.

Note that since X and Y are countable, so any subset of them is ω −open. If $x = a$, we have $f(x) = d$. $V_1 = \{d\}$, $V_2 = \{d,e\}$, $V_3 = \{d,f\}$, and $V_4 = Y$ are ω −open sets in Y containing $f(x)$, so there exist $U_1 = \{a,b\}$, $U_2 = \{a,c\}$, $U_3 = \{a\}$, and $U_4 = X$ such that $f(U_1) \subset V_1$, $f(U_2) \subset V_2$, $f(U_3) \subset V_3$ and $f(U_4) \subset V_4$. Similarly for $x = b$, and $x = c$, Therefore f is ω −continuous map.

Next f is not continuous. Let $x = b$, $f(x) = d$, if $V = \{d\}$, then when $U = X$, we have $f(U) = \{d,e\} \not\subset \{d\} = V$. Hence f is not continuous map .

Theorem 2.8. Let (X,σ) and (Y,τ) be two topological spaces that satisfy the $\omega - B_\alpha$ −condition then the map $f:(X,\sigma) \longrightarrow (Y,\tau)$ is continuous if and only if it is $\alpha - \omega$ −continuous.

Theorem 2.9. Let (X,σ) and (Y,τ) be two topological spaces that satisfy the $\omega - B$ −condition then the map $f:(X,\sigma) \longrightarrow (Y,\tau)$ is continuous if and only if it is $pre - \omega$ −continuous.

Theorem 2.10. Let (X,σ) and (Y,τ) be two door topological spaces and $f:(X,\sigma)\longrightarrow(Y,\tau)$ be a map. Then

1. f is $pre-\omega-$continuous if and only if it is $\omega-$continuous.

2. f is $\beta-\omega-$continuous if and only if it is $b-\omega-$continuous.

Proof of (1):

Let f be a $pre-\omega-$continuous, and let V be an $\omega-$open set in Y and containing $f(x)$, therefore it is $pre-\omega-$open and since f is $pre-\omega-$continuous, there is a $pre-\omega-$open set U in X containing x and $f(U)\subset V$. Since X is a door space U is also an $\omega-$open set. For the converse let f be an $\omega-$continuous map and V be a $pre-\omega-$open set in Y. Then since Y is door space we get V is $\omega-$open, and by the $\omega-$continuity of f there exists an $\omega-$open set U in X containing x (also $pre-\omega-$open) with $f(U)\subset V$. And so f is a $\omega-$continuous.

Similarly we can prove (2) ⦿

3. Weak ω $-$Compactness

In this article we shall introduce weak $\omega-$compactness. It is defined that every cover by such weak open sets contains a finite subcover. So let us state new definitions for the weak new types of $\omega-$compact sets, and prove several rather simple theorems about it.

Definition 3.1. Let X be a topological space. We say that a subset A of X is ***$\omega-$compact*** [2] **(** resp. ***$\alpha-\omega-$compact***, ***$pre-\omega-$compact***, ***$b-\omega-$compact*** and ***$\beta-\omega-$compact*** **)** if for each cover of $\omega-$open (resp. $\alpha-\omega-$open , $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) sets from X contains a finite sub cover for A.

Theorem 3. 2. In any topological space , every $\beta-\omega-$compact set is compact

Proof :

Let X be a topological space , and let A be a $\beta-\omega-$compact sub set of X , to prove A is compact, let C be an open cover for A. Since we can consider C as a cover of $\beta-\omega-$open sets by lemma 1.2 and A is $\beta-\omega-$compact subset of X. Then C contains a finite sub cover , Thus X is compact set.

Example 3.4. Consider the usual topology T for $\mathbb{R}$. The subset $A=\mathbb{Q}\cap[-\sqrt{2},\sqrt{2}]$ is $\omega-$compact but not $b-\omega-$compact. Since $\left\{\left(-\sqrt{2}-\frac{1}{n},2\right]\right\}_{n\in\mathbb{N}}$ is $b-\omega-$open cover for A, but it not have a finite sub cover for A.

Theorem 3.5. Let (X,T) be a topological space

1.If (X,T) is door space, then any $\omega-$compact set is $pre-\omega-$compact.

2. If (X,T) is door space, then any $b-\omega-$compact set is $\beta-\omega-$compact.

3. If (X,T) satisfies the $\omega-$condition, then any compact set is $\omega-$compact.

4. If (X,T) satisfies the $\omega-B_{\alpha}-$condition, then any compact set is $\alpha-\omega-$compact.

5. If (X,T) satisfies the $\omega-B-$condition, then any compact set is $pre-\omega-$compact.

Proof:

1. Let X be a topological door space, and let A be an $\omega-$compact sub set of X , and C be a cover of $pre-\omega-$open subsets of X . Since X is a door space so we can consider C as a cover of $\omega-$open sets. And by the $\omega-$compactness of X, C contains a finite sub cover of $pre-\omega-$open sets. Hence A is $pre-\omega-$compact.

Similarly we can prove (2).

3. Let X be a topological space satisfies the ω −condition, and A be a compact subset of X, to prove A is ω −compact, let C be a cover of ω −open sets for A. Since X satisfies the ω −condition, we can consider C as a cover of open sets and by the compactness of A , C contains a finite subcover of open(also ω −open) sets for A. This implies X is ω −compact.

Similarly we can prove (4) and (5) ◉

Theorem 3.6. An ω −closed (resp. $\alpha-\omega$ −closed , pre −ω −closed, b −ω −closed and $\beta-\omega$ −closed) sub set of ω −compact (resp. $\alpha-\omega$ −compact , pre −ω −compact, b −ω −compact and $\beta-\omega$ −compact) subspace is ω −compact (resp. $\alpha-\omega$ −compact , pre −ω −compact, $b-\omega$ −compact and β −ω −compact).

Proof:

Let Y be an ω-compact subspace of the topological space X, and let F be an ω − closed subset of Y. Let $C=\{G_\lambda,\lambda\in\Lambda\}$ be a cover of ω −open sets for F. Then $C\cup(Y\backslash F)=D$ is a cover of ω −open sets for Y. Since Y is ω −compact there is a finite sub cover $\acute{D}$ of D for Y, and hence without $Y\backslash F$, a cover for F (because F and $Y\backslash F$ are disjoint). So we have shown that a finite sub collection of C cover F. Thus F is ω − compact. Similarly we can prove the other cases ◉

Theorem 3.7. Let $f:X\longrightarrow Y$ be an ω-continuous (resp. $\alpha-\omega-$ continuous, pre $-\omega-$ continuous, $b-\omega-$ continuous, and β $-\omega-$ continuous) map from the ω −compact (resp. $\alpha-\omega$ −compact , pre −ω −compact, $b-\omega$ −compact, and β −ω −compact) space X onto a topological space Y. Then Y is ω −compact (resp. $\alpha-\omega$ −compact , pre −ω −compact , $b-\omega$ −compact and β −ω −compact) space..

Proof:

Let $f: X \longrightarrow Y$ be an ω −continuous map from the ω −compact space X on to Y. Let $\{Y_\lambda, \lambda \in \Lambda\}$ be a cover of ω −open sets for Y, then since f is ω −continuous map so $\{f^{-1}(Y_\lambda), \lambda \in \Lambda\}$ is a cover of ω −open sets for X. Since X is ω −compact so it has a finite subcover $\{f^{-1}(Y_{\lambda_i}), i = 1,2, \ldots, n\}$. Then by the surjection of f we get $\{Y_{\lambda_i}, i = 1,2, \ldots, n\}$ is an ω −open cover for Y. Hence Y is ω −compact. With a simple modification to that prove one can prove the other cases

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