Turing Point of proper Ideal

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Abstract:

In this paper we introduce and study the concepts of a new class of points, namely turing points of proper ideal and some of their properties are analyzed.

Keywords: Turing point, proper ideal, net and compact.

1. Introduction:

Ideal in topological spaces have been considered since 1930. This topic won its importance by the paper of Vaidyanathaswamy [7] and Kuratwoski [6]. Applications to various fields were further investigated by Janković and Hamlett [3]; Dontchev et al. [2]; Mukherjee et al. [4]; Arenas et al. [5]; Navaneethakrishnan et al. [1]; Nasef and Mahmoud [8], etc. In this paper We used only the proper ideal to work , ideal bace, turing point, finer than, and maximal ideal to prove continuous, compactness, T_2 space and ideal net. Let (X, T) be a topological space, by N_x we will denote the open neighborhood system at a point $x \in X$.

A set Δ is said to be directed if and only if there is a relation $\succcurlyeq$ on Δ satisfying, (i) $\alpha \succcurlyeq \alpha$ for each $\alpha \in \Delta$, (ii) if $\alpha_1 \succcurlyeq \alpha_2$ and $\alpha_2 \succcurlyeq \alpha_3$ then $\alpha_1 \succcurlyeq \alpha_3$, (iii) if $\alpha_1, \alpha_2 \in \Delta$, then there is $\alpha_3 \in \Delta$ such that $\alpha_3 \succcurlyeq \alpha_1$ and $\alpha_3 \succcurlyeq \alpha_2$ [9]. A net in a set X is a function x on a directed set into X [9]. By $\{x_\alpha\}_{\alpha\in\Delta}$ we will denote the net x. Let $A \subseteq X$, a net x on X is said to be eventually in A iff there exists $\alpha_0 \in \Delta$ such that $\alpha \succcurlyeq \alpha_0$ implies that $x_\alpha \in A$. A net $\{x_\alpha\}_{\alpha\in\Delta}$ convergent to $x_0 \in X$ iff its eventually for each $N \in N_{x_0}$ and denoted by $\{x_\alpha\}_{\alpha\in\Delta} \to x_0$. A topological space X is compact if and only if every class $\{F_i\}$ of closed subsets of X which satisfies the finite intersection has, itself, a non-empty intersection [9].

Definition 2.1[7]:

Let X be a set. A family I of subsets of X is an ideal on X if

1. $A, B \in I$ implies $A \cup B \in I$.
2. $A \in I$ and $B \subseteq A$ implies $B \in I$.

Definition 2.2[4]:

Let X be a set. A family I of subsets of X is a proper ideal on X if I is an ideal on X and $X \notin I$.

Theorem 2.3:

Let $\{I_\lambda : \lambda \in \Delta\}$ be any femily ideals on X. Then $I = \cap \{I_\lambda : \lambda \in \Delta\}$ is also ideal on X.

Remark 2.4:

1. The union of two ideals on a set X not necessary is ideals, for example. Let $X = \{x, y\}$, then $I_0 = \{\emptyset, \{x\}\}$ and $I_1 = \{\emptyset, \{y\}\}$ are ideals on X but $I_0 \cup I_1 = \{\emptyset, \{x\}, \{y\}\}$ is not ideal on X.
2. The intersection of all ideals on X is the ideal $\{\emptyset\}$.

Example 2.5:

Let X be infinite set . Then $I = \{A\colon A \text{ is finite subset of } X\}$ is an ideal on X is called finite ideal.

Theorem 2.6:

Let $f\colon X \to Y$ be a function. Then If I is an ideal on X, then the family $f(I) = \{f(A)\colon A \in I\}$ is an ideal on Y.

Definition 2.7:

An ideal base on X is a family I_0 of subsets of X satisfies (i) $X \notin I_0$ (ii) if $A \in I_0$ and $B \in I_0$, then there exists $C \in I_0$ such that $A \cup B \subseteq C$. Observe that if $X \neq A \cup B \in I_0$ for each A and B in I_0, then I_0 is an ideal base on X.

Example 2.8:

Let $X = \{a, b, c\}$, then $I_0 = \{\{b\}, \{c\}, \{b, c\}\}$ is an ideal base on X.

Example 2.9:

Let I be an ideal on X and $A \subseteq X$ such that $A \cup B \neq X$ for each $B \in I$. Then, $I_0 = \{A \cup B : B \in I\}$ is an ideal base on X.

Proposition 2.10:

Let I_0 be an ideal base on X, then $I = \{A \subseteq X : A \subseteq B \text{ for some } B \in I_0\}$ is an ideal on X generated by I_0.

Proposition 2.11:

Let $X \neq \emptyset$ and $Y \subseteq X$. If I_0 is an ideal base on Y, then its ideal base on X.

Proof.

Directly by using definition 2.7.

Remark 2.12:

The converse is not true for example if $X = \{x, y, z\}$ and $Y = \{z\}$ then $I_0 = \{\{x\}, \{z\}, \{x, z\}\}$ is an ideal base on X but is not ideal base on Y.

Theorem 2.13:

Let $f\colon X \to Y$ be a function. If I_0 is an ideal base on X, then the family $f(I_0) = \{f(A)\colon A \in I_0\}$ is an ideal base on Y.

Definition 2.14:

Let I be an ideal on X and $x \in X$. We say that x is turing point of I if for each $N \in N_x$ implies $N^c \in I$.

Example 2.15:

Let X be a set and $x \in X$. Then $I = \{A \subseteq X\colon x \in A^c\}$, is an ideal on X and x is a turing point of I.

Theorem 2.16:

Let $f\colon X \to Y$. Then f is continuous at $x \in X$ if and only if whenever x is a turing point of an ideal I then $f(x)$ is a turing point of an ideal $f(I)$.

Proof.

Suppose f is continuous at x and x is turing point of an ideal I. Let $V \in N_{f(x)}$ in Y. Since f is continuous then for some $U \in N_x$ in X, $f(U) \subset V$. So $V^c \subset f(U^c)$. Since x is a turing point of an ideal I, then $U^c \in I$. Then $f(U^c) \in f(I)$. But $V^c \subset f(U^c)$, then $V^c \in f(I)$. So that $f(x)$ is a turing point of an ideal $f(I)$.

Conversely, suppose whenever x is a turing point of an ideal I then $f(x)$ is a turing point of an ideal $f(I)$. Define $I = \{N^c \subseteq X: N \in N_x\}$. Then each $M \in N_{f(x)}$, $M^c \in f(I)$, so for some $N \in N_x$, $M^c \subset f(N^c)$. So $f(N) \subset M$. Thus f is continuous at x.

Definition 2.17:

Let I and J be two ideals on the same set X. Then I is said to be finer than J if and only if $J \subseteq I$.

Example 2.18:

Let $X = \{1,2,3\}$. Then $I_1 = \{\emptyset, \{1\}\}$ and $I_2 = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ are an ideals on X and I_2 finer than I_1.

Example 2.19:

If X is any set. Then any ideal on X is finer than $\{\emptyset\}$.

Theorem 2.20:

Let X be any set and let I be an ideal on X such that $\cup \{B: B \in I\} = X$. Show that I is finer than the finite ideal on X.

Proof:

Let J be the finite ideal on X. To show that $J \subseteq I$. Suppose if possible $J \nsubseteq I$. Then there exists $A \in J$ such that $A \notin I$. Then A is finite subset of X. Let $A = \{x_1, x_2, \dots, x_n\}$. Now $\cup \{B: B \in I\} = X$. Then $x_i \in B_i$ for some $B_i \in I (i = 1, \dots, n)$. Since I is an ideal, $D = \cup \{B_i: i = 1,2, \dots, n\} \in I$. Thuse D is contain any element of $\{x_1, x_2, \dots, x_n\}$. Hence $A \subseteq D$. Since I is an ideal, $A \subseteq D$ implies $A \in I$. But this is a contradiction with $A \notin I$. Hence I must be finer than the finite ideal.

Definition 2.21:

Let I be an ideal on X. Then I is said to be maximal ideal on X if and only if I is not contained in any other ideal on X. i.e I is a maximal ideal on X if and only if for every ideal J on X such that $I \subseteq J$, then $I = J$.

Theorem 2.22:

Let X be a set. Every ideal on X is contained in a maximal ideal on X.

Proof.

Let I be any ideal on X and let W be the class of all ideals on X containing I. Then W is non-empty since $I \in W$. Also W is partially orderd by the inclusion relation $\subseteq$. Now let K be a linearly ordered subset of W. Then by definition of linear ordering for any two members I_1, I_2 of K, we have either $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. Let $S = \cup \{I_\gamma : I_\gamma \in K\}$. To show that S is an ideal on X.

1. Since each I_γ is an ideal, we have $X \notin I_\gamma$ for each an ideal $I_\gamma \in S$ and so $X \notin S$.
2. Let $A \in S$ and $B \subseteq A$. Then $A \in I_\gamma$ for some $I_\gamma \in S$ and since I_γ is an ideal, $B \in I_\gamma$. It follows that $B \in S$.
3. Let $A \in S$ and $B \in S$. $A \in I_\gamma$ and $B \in I_\delta$ for some $I_\gamma, I_\delta \in S$. Since S is linearly ordered, we have either $I_\gamma \subseteq I_\delta$ or $I_\delta \subseteq I_\gamma$. Hence both A and B belong either to I_γ or to I_δ and so $A \cup B$ belongs either to I_γ or to I_δ. It follows that $A \cup B \in S$.

Further S is finer than every member of K and so S is upper bound of K. Thus we have shown that W is a non-empty partially ordered set in which every linearly ordered subset has an upper bound. Hence by Zorn's lemma W contains a maximal element J. This maximal element J is by definition, maximal ideal on X containing I.

Example 2.23:

Let $X = \{1,2,3\}$. Then $I_1 = \{\emptyset\}$, $I_2 = \{\emptyset, \{1\}\}$, $I_3 = \{\emptyset, \{2\}\}$, $I_4 = \{\emptyset, \{3\}\}$,

$I_5 = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$,

$I_6 = \{\emptyset, \{1\}, \{3\}, \{1,3\}\}$ and

$I_7 = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}$ are ideals on X. Clear that $I_1 \subset I_2 \subset I_5$, $I_1 \subset I_3 \subset I_7$ and $I_1 \subset I_4 \subset I_6$. So that I_5, I_6 and I_7 are maximal ideals on X.

Proposition 2.24:

Let I be an ideal on X. I is a maximal ideal on X if and only if for each $A \subseteq X$, then either $A \in I$ or $A^c \in I$.

Proof.

If $A^c \notin I$, then $A \cup B \neq X$ for each $B \in I$, because if there exists $B \in I$ such that $A \cup B = X$, then $A^c \subseteq B$, then $A^c \in I$ construction. Let J be an ideal generated by ideal base $\{A \cup B : B \in I\}$. Since $A \subseteq A \cup B$ for each $B \in I$, then $A \in J$ ... (1). To show that $I \subseteq J$. Let $K \in I$, then $K \in J$, because $K \subseteq A \cup K$. But I is a maximal ideal, then $I = J$. By (1) $A \in I$.

Conversely. Let J be an ideal and $I \subseteq J$. To prove that $J \subseteq I$. Suposse there exists $B \in J$ such that $B \notin I$, then by hypothesis $B^c \in I$. But $I \subseteq J$. So $B^c \in J$. Then $B \cup B^c = X \in J$. But the contradictio with J is ideal. Then $J \subseteq I$. Then $I = J$. So that I is maximal ideal.

Theorem 2.25:

Let I be an ideal on X. An ideal I on X is maximal ideal if and only if I contains all those subsets A of X which $A \cup B \neq X$ for each $B \in I$.

Proof.

Let I be a maximal ideal and let $A \subseteq X$ such that $A \cup B \neq X$ for each $B \in I$.Define

$J = \{D : D \subseteq A \cup B$ for each $B \in I\}$

Observe that $I \subseteq J$and J is ideal. Since I is maximal ideal, then $I = J$. Since $A \subseteq A \cup B$ for each $B \in I$ so that $A \in J$ so also $A \in I$.

Conversely. Let I be an ideal satisfying the condition. Let J be an ideal on X such that $I \subseteq J$. Let $A \in J$, then $A \cup B \neq X$ for each $B \in J$. Since $I \subseteq J$, then $A \cup B \neq X$ for each $B \in I$. then $A \in I$. So $J \subseteq I$, then $I = J$. Therefore I is maximal ideal.

Theorem 2.26:

Let I be an ideal on X. I is maximal ideal if and only if for any two subsets A and B of X such that $A \cap B \in I$, we have either $A \in I$ or $B \in I$.

Proof.

Let $A \cap B \in I$. If possible, that $A \notin I$ and $B \notin I$. Then $A \neq X$ and $B \neq X$ (if possible $A = X$, then $A \cap B = X \cap B = B \in I$ contradiction with $B \notin I$ and the similer if $B = X$. Define $J = \{C : C \cap A \in I\}$. Then J is an ideal and $B \in J$. To prove that $I \subseteq J$. Let $D \in I$. Since $A \cap D \subseteq D$, then $D \cap A \in I$. So $D \in J$. Therefore $I \subseteq J$. Since $B \in J$ and $B \notin I$, then $I \neq J$. But this contradicts the hypothesis I is a maximal ideal. Hence either $A \in I$ or $B \in I$.

Conversely. Let the condition hold and let A be any subset of X. Since I is an ideal then $\emptyset \in I$. But $\emptyset = A \cap A^c$. Hence by hypothesis either $A \in I$ or $A^c \in I$. Hence by proposition 2.24, I is maximal ideal on X.

Corollary 2.27:

If I is maximal ideal on X and $A_1 \cap A_2 \cap \ldots \cap A_n \in I$, then at least one $A_i \in I(i = 1,2,\ldots,n)$.

Corollary 2.28:

If I is maximal ideal on X and $A_1, A_2, \ldots, A_n$ are subsets of X such that $\cup \{A_i^c : i = 1,2,\ldots,n\}$ covers X, then some $A_i \in I(i = 1,2,\ldots,n)$.

Proof.

Since $\cup \{A_i^c : i = 1,2,\ldots,n\} = X$, then $\cap \{A_i : i = 1,2,\ldots,n\} = \emptyset$. Then

$\cap \{A_i : i = 1,2,\ldots,n\} \in I$. So by corollary 2.27, some $A_i \in I(i = 1,2,\ldots,n)$.

Proposition 2.29:

Let G be a collection of subsets of X such that for all $n \in \mathbb{N}$ and $A_1, A_2, \ldots, A_n \in G$ we have $\cup_{i=1}^{n} A_i \neq X$. Then $I = \{A \subseteq X : \text{there exists } A_1, A_2, \ldots, A_n \in G, A \subseteq \cup_{i=1}^{n} A_i\}$. I is an ideal containing G. Inded I is the ideal generated by G.

Proof.

First to show that $G \subseteq I$. Let $A \in G$. Then for all $n \in \mathbb{N}$ such that $A_1, A_2, \ldots, A_n \in G$, then $A \subseteq \cup_{i=1}^{n} A_i \cup A$. So $A \in I$. So $G \subseteq I$. Clear I is an ideal on X. So that I is an ideal on X and G is called a subbase of I.

Theorem 2.30:

Let G be a collection of subsets of X with the finite intersection property. Then

$G^c = \{A^c : A \in G\}$ forms a subbase for an ideal.

Proof.

Let $n \in \mathbb{N}$ and $A_1^c, A_2^c, \ldots, A_n^c \in G^c$. If $\cup_{i=1}^{n} A_i^c = X$, then $\cap_{i=1}^{n} A_i = \emptyset$ such that $A_i \in G$ for all $i = 1,2,\ldots,n$. But the contradiction with G has intersection property. So $\cup_{i=1}^{n} A_i^c \neq X$. Therefore by proposition 2.29, G^c forms a subbase for an ideal.

Theorem 2.31:

Let X be a topological space. X is compact if and only if every ideal I can be found that an ideal J finer than I and x is turing point of ideal J for some $x \in X$.

Proof.

Suppose X is compact. Let I be an ideal. Then $K = \{cl(A): A^c \in I\}$ is a closed sets with the finite intersection property (other wise if $cl(A_1) \cap cl(A_2) \cap \ldots \cap cl(A_n) = \emptyset$, then $A_1 \cap A_2 \ldots \cap A_n = \emptyset$, then $A_1^c \cup A_2^c \ldots \cup A_n^c = X \in I$, but this contradiction with $X \notin I$). Let $x \in \cap K$. Then for each $A^c \in I$ and each $V \in N_x$, we have $A \cap V \neq \emptyset$, as $x \in cl(A)$. So $A^c \cup V^c \neq X$, as $x \in cl(A)$. Thus $I \cup \{N^c : N \in N_x\}$ forms sub base for an ideal J, $I \subseteq J$ and x is turing point of an ideal J.

Conversely. Let G be a collection of closed sets with finite intersection property. Then G^c forms subbase for an ideal I. Can be found that an ideal J finer than I and x is turing point of J for some $x \in X$. Thus for each $U \in N_x$ and each $V^c \in G^c$ we must have $U^c \cup V^c \neq X$. So that $U \cap V \neq \emptyset$. Since V is closed, we must have $x \in V$. So $x \in \cap G$. Thus $\cap G \neq \emptyset$. So X is compact.

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Corollary 2.32:

Let X be a topological space. X is compact if and only if every maximal ideal Ion X there exists $x \in X$ is turing point of I.

Theorem 2.33:

Let (X, T) be a topological space. X is T_2 if and only if every ideal I on X and $x, y \in X$ are turing point of I, then $x = y$.

Proof.

Let $x, y \in X$ such that $x \neq y$ and let I be an ideal such that x, y are turing point of I. Then every $N \in N_x$ and every $M \in N_y$, $N^c \in I$ and $M^c \in I$. Then $N^c \cup M^c \in I$. But I is proper ideal then $N^c \cup M^c \neq X$. Then $N \cap M \neq \emptyset$. Thus X is not T_2.

Conversely. Let X be not T_2 then there exists two points $x, y \in X$, with $x \neq y$, such that for any open sets $U, V \subseteq X$, with $U \in N_x$ and $V \in N_y$ we have $U \cap V \neq \emptyset$. So that $U^c \cup V^c \neq X$ for each $U \in N_x$ and for each $V \in N_y$. Thus the collection

$$I_0 = \{U^c \cup V^c : U \in N_x, V \in N_y\}$$

is an ideal base on X (since for any $U_1^c \cup V_1^c$, $U_2^c \cup V_2^c \in I_0$, we have $(U_1^c \cup V_1^c) \cup (U_2^c \cup V_2^c) = (U_1^c \cup U_2^c) \cup (V_1^c \cup V_2^c) = (U_1 \cap U_2)^c \cup (V_1 \cap V_2)^c \in I_0$). Clearly every $U \in N_x$, $U^c = U^c \cup X^c$, is in I_0, and similarly, every $V \in N_y$, $V^c = V^c \cup X^c$, is in I_0. So that x, y are turing point of I where I is an ideal generated by I_0.

Proposition 2.34:

Let $\{x_\alpha\}_{\alpha \in \Delta}$ be a net on X and let

$I = \{A \subseteq X : \{x_\alpha\}_{\alpha \in \Delta}$ is eventually in $A^c\}$. I is an ideal on X.

Remark 2.35:

An ideal in proposition above its an ideal generating by the net $\{x_\alpha\}_{\alpha \in \Delta}$ and denoted by Ideal(x).

Proposition 2.36:

Let I be an ideal generating by the net $\{x_\alpha\}_{\alpha \in \Delta}$ in a topological space (X, T) and $x \in X$. Then x is turing point of I if and only if $x_\alpha \to x$.

Proof.

Let x be a turing point of I. Then $N^c \in I$, for each $N \in N_x$. So $\{x_\alpha\}_{\alpha \in \Delta}$ eventually in N. Thus $x_\alpha \to x$.

Conversely. Let $x_\alpha \to x$. Then $\{x_\alpha\}_{\alpha \in \Delta}$ is eventually in N for each $N \in N_x$. So $N^c \in I$ for each $N \in N_x$. Thus x is turing point of I.

Definition 2.37:

Let I_0 be an ideal bace on X and $x \in X$. We say that x is turing point of I_0 if and only if x is turing point of I where I is an ideal generated by I_0.

Proposition 2.38:

Let I_0 be an ideal bace on X and $x \in X$. Then x is turing point of I_0 if and only if for each $N \in N_x$ then $N^c \subseteq A$ for some $A \in I_0$.

Proof.

Let x is turing point of I_0. Then x is turing point of I where I is an ideal generated by I_0. Thus for each $N \in N_x$, $N^c \in I$. So $N^c \subseteq A$ for some $A \in I_0$.

Conversely. Let I be an ideal generated by I_0 and let for each $N \in N_x$, $N^c \subseteq A$ for some $A \in I_0$. Since $I_0 \subseteq I$ and $A \in I_0$, then $A \in I$. But $N^c \subseteq A$, so $N^c \in I$ for each $N \in N_x$, then x is turing point of I. Therefore x is turing point of I_0.

Definition 2.39:

Let I_0 be an ideal bace on X and let $\Delta = \{(x, A): x \in A \text{ and } A^c \in I_0\}$. Then $(\Delta, \succcurlyeq)$ is a directed set where $(x_1, A_1) \succcurlyeq (x_2, A_2)$ if and only if $A_1 \subseteq A_2$ {because $(x_1, A_1) \succcurlyeq (x_2, A_2)$ if and only if $A_1 \subseteq A_2$, then $A_2^c \subseteq A_1^c$ where $A_1^c, A_2^c \in I_0$, then there exists $A_3^c \in I_0$ such that $A_1^c \cup A_2^c \subseteq A_3^c$, then $A_3 \subseteq A_1 \cap A_2$, then $A_3 \subseteq A_1$ and $A_3 \subseteq A_2$. So that $A_3 \succcurlyeq A_1$ and $A_3 \succcurlyeq A_2$}. Define a net $x : \Delta \to X$ such that $x(\alpha) = x_\alpha = x$ where $\alpha = (x, A) \in \Delta$. So that $\{x_\alpha\}_{\alpha \in \Delta}$ is a net generating by I_0, and denoted by Net(I_0).

Proposition 2.40:

Let I_0 be an ideal bace on a nonemrty set X and $x \in X$. If $\{x_\alpha\}_{\alpha \in \Delta}$ is Net(I_0), then x is turing point of I_0 if and only if $x_\alpha \to x$.

Proof.

Let x be a turing point of I_0 and $N \in N_x$. By proposition 2.38, we have $A_0 \subseteq N$ for some $A_0^c \in I_0$. Since $A_0^c \in I_0$, then $A_0 \neq \emptyset$ (because $X \notin I_0$). So there exists $x_0 \in A_0$, take $\alpha_0 = (x_0, A_0)$. Then $x_{\alpha_0} \in N$. So that $x_\alpha \in N$ for each $\alpha \succcurlyeq \alpha_0$. Therefore $x_\alpha \to x$.

Conversely. Let $x_\alpha \to x$ and $N \in N_x$ then there exists $\alpha_0 \in \Delta$ such that $x_\alpha \in N$ for each $\alpha \succcurlyeq \alpha_0$. Thus there exists $A_0^c \in I_0$ and $x_0 \in A_0$ such that $\alpha_0 = (x_0, A_0)$. To show that $A_0 \subseteq N$. Let $x \in A_0$, then $\alpha = (x, A_0) \succcurlyeq (x_0, A_0) = \alpha_0$. So that $x_\alpha \in N$, then $x \in N$, then $A_0 \subseteq N$. By proposition 2.38, we have x is turing point of I_0.

Proposition 2.41:

Let $X \neq \emptyset$ and I_0 be an ideal base on X. Let $\{x_\alpha\}_{\alpha \in \Delta}$ be a Net(I_0). If J is an Ideal(x), then $I = J$ where I is an ideal generating by I_0.

Proof.

Let $A \in I$, then there exists $A_0 \in I_0$ such that $A \subseteq A_0$, then $A_0^c \subseteq A^c$. Let

$$J = \{B \subseteq X : \{x_\alpha\}_{\alpha \in \Delta} \text{ is eventually in } B^c\}.$$

Since $A_0^c \neq \emptyset$ (because $X \notin I_0$), then there exists $x_0 \in A_0^c$, so if $\alpha = (x, A_1^c) \succcurlyeq (x_0, A_0^c) = \alpha_0$ then $x_\alpha = x \in A_1^c \subseteq A_0^c \subseteq A^c$. So that $\{x_\alpha\}_{\alpha \in \Delta}$ is eventually in A^c. So $A \in J$. Therefore $I \subseteq J$.

Conversely. Let $A^c \in J$, then $\{x_\alpha\}_{\alpha \in \Delta}$ is eventually in A, then there exists $\alpha_0 = (x_0, A_0)$ such that $x_\alpha \in A$ for each $\alpha \succcurlyeq \alpha_0$. Since $\alpha = (x, A_0) \succcurlyeq (x_0, A_0) = \alpha_0$ for each $x \in A_0$ then $x_\alpha = x \in A$. So $A_0 \subseteq A$. So that there exists $A_0^c \in I_0$ such that $A_0 \subseteq A$, then $A^c \subseteq A_0^c$. So $A^c \in I$. Therefore $J \subseteq I$.

Definition 2.42:

Let I be an ideal on X and let $\Delta = \{(x, A): x \in A \text{ and } A^c \in I\}$. Then $(\Delta, \succcurlyeq)$ is a directed set where $(x_1, A_1) \succcurlyeq (x_2, A_2)$ if and only if $A_1 \subseteq A_2$ {because $(x_1, A_1) \succcurlyeq (x_2, A_2)$ if and only if $A_1 \subseteq A_2$, then $A_2^c \subseteq A_1^c$ where $A_1^c, A_2^c \in I$, then $A_1^c \cup A_2^c \in I$, then $A_1^c \subseteq A_1^c \cup A_2^c$ and $A_2^c \subseteq A_1^c \cup A_2^c$, then $A_1 \cap A_2 \subseteq A_1$ and $A_1 \cap A_2 \subseteq A_2$. So that $A_1 \cap A_2 \succcurlyeq A_1$ and $A_1 \cap A_2 \succcurlyeq A_2$}. Define a net $x : \Delta \to X$ such that $x(\alpha) = x_\alpha = x$ where $\alpha = (x, A) \in \Delta$. So that $\{x_\alpha\}_{\alpha \in \Delta}$ is a net generating by I, and denoted by Net(I).

Proposition 2.43:

Let $X \neq \emptyset$ and let I be any ideal on X, $x: \Delta \to X$ be any net. Then

$$I = Ideal\big(Net(I)\big).$$

Proof.

Let $A \in I$, and let

$Ideal(Net(I)) = J = \{B \subseteq X : \{x_\alpha\}_{\alpha\in\Delta}$ is eventually in $B^c\}$.

Since $A^c \neq \emptyset$ (because $X \notin I$), then there exists $x_0 \in A^c$, so if $\alpha = (x, A_1^c) \geqslant (x_0, A^c) = \alpha_0$ then $x_\alpha = x \in A_1^c \subseteq A^c$. So that $\{x_\alpha\}_{\alpha\in\Delta}$ is eventually in A^c. So $A \in J$. Therefore $I \subseteq J$.

Conversely. Let $A^c \in J$, then $\{x_\alpha\}_{\alpha\in\Delta}$ is eventually in A, then there exists $\alpha_0 = (x_0, A_0)$ such that $x_\alpha \in A$ for each $\alpha \geqslant \alpha_0$. Since $\alpha = (x, A_0) \geqslant (x_0, A_0) = \alpha_0$ for each $x \in A_0$ then $x_\alpha = x \in A$. So $A_0 \subseteq A$. So that $A^c \subseteq A_0^c$. Since $A_0^c \in I$, then $A^c \in I$.

Theorem 2.44:

Let $X \neq \emptyset$ and let I be an ideal on X such that $Net(I)$ is a net generated by ideal I. $x \in X$ is turing point of I if and only if $Net(I)$ converge to x.

Proof.

Let x be a turing point of I. Then $N^c \in I$ for each $N \in N_x$. Since $Net(I)$ is a net generating by I, then $x_\alpha \in N$ for each $\alpha \geqslant \alpha_0$. So that $Net(I)$ converge to x.

Conversely. Let $Net(I)$ converge to x. Then $Net(I)$ is eventually in N for each $N \in N_x$. But $I = Ideal(Net(I))$ (theorem 2.43), then $N^c \in I$ for each $N \in N_x$. Therefore x is turing point of I.

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