Compactness and Proper Ideal

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Abstract:

In this paper we introduce relations between proper ideal and compactness.

1. Introduction:

Ideal in topological spaces have been considered since 1930. This topic won its importance by the paper of Vaidyanathaswamy [11] and Kuratwoski [7].

Applications to various fields were further investigated by Newcomb [10], Rancin [5], Hamlett and Jankovic [12], David A. Rose and T. R. Hamlett [3], Julian Dontchev and Maximilian Ganster [6], A.A . Nasef and T . Noiri [1], Abd El-Monsef et al. [8], Arafa A. Nasef [2], M. K. Gupta and T. Noiri [11], and M. K. Gupta and Rajneesh [10].

A family I of subsets of X is an ideal on X if (i)$A, B \in I$ implies $A \cup B \in I$ (ii) $A \in I$ and $B \subseteq A$ implies $B \in I$. If (iii) $X \notin I$ then I is called proper ideal. In this paper we used only the proper ideal i.e. in this paper any ideal is proper idea. Let (X, T) be a topological space, by N_x we will denote the open neighborhood system at a point $x \in X$.

Definition 2.1[13]:

Let I be an ideal on a topological space X and $x \in X$. We say that x is turing point of I if for each $N \in N_x$ implies $N^c \in I$.

Example 2.2[13]:

Let X be a set and $x \in X$. Then $I = \{A \subseteq X: x \in A^c\}$, is an ideal on X and x is a turing point of I.

Definition 2.3:

Let I be an ideal on a topological space X and $x \in X$. We say that x is a bench point of I if for each $N \in N_x$ and for each $A \in I$ we have $A \cap N^c \neq A$.

Theorem 2.4:

Let I be an ideal on a topological space X and $x \in X$. if x is turing point then x is bench point.

Proof:

Let x is a turing point of I. If possible x is not bench point then there is $N \in N_x$ and $A \in I$ such that $A \cup N^c = X$ and we have $N \in I$, but $N^c \in I$ so $N \cup N^c = X \in I$ contradiction with $X \notin I$.

Remark 2.5:

The converse not true in general for example: let (X, T) be the usuall topological space, then $I = \{A \subseteq \mathbb{R}: (0,2) \subseteq A^c\}$ is an ideal on X and 1 is bench point of I but 1 is not turing point of I because $(0, \frac{3}{2}) \in N_1$, but $(0, \frac{3}{2})^c \notin I$.

Definition 2.6[13]:

Let I be a ideal on X. Then I is said to be maximal ideal on X if and only if I is not contained in any other ideal on X. i.e I is a maximal ideal on X if and only if for every ideal J on X such that $I \subseteq J$, then $I = J$.

Theorem 2.7:

Let X be a topological space. X is compact if and only if every maximal ideal Ion X there exists $x \in X$ is turing point of I.

Definition 2.8:

Let (X, T) be a topological space. An ideal I on X is called C-ideal if and only if for each $A \in T$ and every point $x \in A$ there is $B \in T$ such that $x \in B \subseteq A$ and either $B \in I$ or $B^c \in I$.

Theorem 2.9:

Let (X, T) be a topological space. An ideal I on X is called C-ideal if and only if F is a closed set and $x \in F^c$ imply there is a closed set K such that $K \subseteq F,\ x \in K^c$ and either $K \in I$ or $K^c \in I$.

Proposition 2.10:

Let (X, T) be a topological space. Then

1. An ideal I on X is called C-ideal if and only if A is open amd $x \in A$ imply there is B open such that $x \in B$ and either $(B \cap A)^c \in I$ or $B \in I$.
2. An ideal I on X is called C-ideal if and only if A is open amd $x \in A$ imply there is B open such that $x \in B$ and either $(B \cap A)^c \in I$ or $(B \cap A) \in I$.

Theorem 2.11:

The topological space (X, T) is compact if and only if every C-ideal on X has a turing point.

Proof:

Let the space be a compact and let I be C-ideal on X. If possible, let I has not turing point. Then for each $x \in X$ implies there is an open set A containing x such that $A^c \notin I$ …(*)

Since I is C-ideal, then there is an open set B_x such that $x \in B_x \subseteq A$ and either $B_x \in I$ or $B^c \in I$. If $B_x^c \in I$ and $A^c \subseteq B_x^c$ implies $A^c \in I$ contradiction (*). So that $B_x \in I$. Let $\mathcal{B} = \{B_x : x \in X\}$. Then $\mathcal{B}$ is an open cover of X. Since X is compact, then there is a finite number $B_1, \dots, B_n$ of members of $\mathcal{B}$ such that $X = \cup_{i=1}^{n} B_i$. But $B_i \in I, i = 1, \dots, n$ and so $X = \cup_{i=1}^{n} B_i \in I$ and this contradiction because $X \notin I$. Therefore I has a turing point.

Conversely. Assume every C-ideal has a turing point and let $\mathfrak{F}$ be a family of closed sets with the finite intersection property. Let $\mathfrak{A}$ the family of all finite intersection of members of $\mathfrak{F}$, then by theorem , $\mathfrak{A}^c = \{A^c : A \in \mathfrak{A}\}$ is a subbase of an ideal I and $\mathfrak{A}^c \subseteq I$. Suppose $\mathfrak{A}$ has empty intersection.

To show that I is a C-ideal, let F be a closed set and let $x \in F^c$. Now there is $A \in \mathfrak{A}$ such that $x \in A^c$ and so $K = A \cup F$ is a closed set such that $x \in K^c, F \subseteq K$. Hence $K^c \in I$ because $F \in \mathfrak{A}, F \subseteq K$, then $F^c \in \mathfrak{A}^c \subseteq I, K^c \subseteq F^c$. So that I is a C-ideal and so has a turing point y. Hence y is in each member of $\mathfrak{A}$ (because if there is $U \in \mathfrak{A}$ and $y \notin U$, then $y \in U^c$. i.e. $U^c \in N_y$. Since y is turing point of I, then $U \in I$. Since $U^c \in \mathfrak{A}^c$, then $U^c \in I$. So that $U \cup U^c = X \in I$ contradiction). So that $y \in \cap \mathfrak{A}$ and we have X is compact.

Theorem 2.12:

Let (X, T) be a topological space, $x \in X$ and I be an ideal on X. Then this equivalent:

1. I is C-ideal
2. x is bench point of I if and only if x is turing point of I.

Proof:

(1. implies 2.). Let I be a C-ideal and let x be a bench point of I. To show x is turing point of I, let $N \in N_x$, then there is an open set B such that $x \in B \subseteq N$ and either $B \in I$ or $B^c \in I$ because I is a C-ideal. Hence $B \notin I$, because if possible $B \in I$, $B \in N_x$ and x is bench point of I then $B \cup B^c \neq X$ contradiction. So that $B^c \in I$. Since $N^c \subseteq B^c$, then $N^c \in I$. So x is turing point of I.

Conversely. Clear from theorem2.4.

(2. implies 1.). Assume 2., hold. Let A is an open set and $x \in A$ then x is either a bench point or not a bench point of I. If x is bench point then x is turing point, so $A^c \in I$ and $x \in A \subseteq A$. If x is not bench point, then there is $U \in N_x$ and $B \in I$ such that $B \cup U^c = X$, so $U \subseteq B$ and we have $U \in I$. Then $M = A \cap U$ is open, $x \in M \subseteq A$. Hence $M \in I$ because $M \subseteq U$ and $U \in I$. So that I is a C-ideal.

Corollary 2.13:

A topological space (X, T) is compact if and only if every C-ideal has a bench point.

Lemma 2.14:

If an ideal I on a compact space has one and only one bench point x then x is a turing point of I.

Proof:

Let I be an ideal on a compact space X and x is a bench point of I. Let $N \in N_x$. If $y \in N^c$ then y is not bench point of I because only x is bench point of I. Then there is $B_y \in N_y$ and $A \in I$ such that $B_y^c \cup A = X$, so $B_y \subseteq A$ and we have $B_y \in I$. Now $\mathcal{B} = \{B_y : y \in N^c\}$ is an open cover of N^c and N^c is compact because any closed subset of compact space is compact. Hence a finite subfamily $B_1, \ldots, B_n$ of $\mathcal{B}$ covers N^c i.e. $N^c \subseteq \cup_{i=1}^{n} B_i$. But $B_i \in I, i = 1, \ldots, n$, so $\cup_{i=1}^{n} B_i \in I$ and we have $N^c \in I$. Therefore x is a turing point of I.

Theorem 2.15:

Let (X, T) be a topological space, A a subset of X and T_A be a relative topology. A is compact if and only if every C-ideal on A has a turing point in a topological space in (A, T_A).

Theorem 2.16:

Let (X, T) be a topological space, A a closed subset of X and T_A the relative topology. If I is a C-ideal on (A, T_A), then I is a C-ideal on (X, T).

Definition 2.17:

Let $X \neq \emptyset$ and let $P(X)$ denote the family of all subset of X. An ideal I is called $P(X)$-ideal if and only if $A \in P(X)$ and $x \in A$ imply there is $B \in P(X)$ such that $x \in B \subseteq A$ and $B \in I$ or $B^c \in I$.

Observe that an ideal is $P(X)$-ideal if and only if $A \in P(X)$ and $x \in A^c$ imply there is $B \in P(X)$ such that $x \in B^c, A \subseteq B$ and $B \in I$ or $B^c \in I$.

Remark 2.18:

1. Not every C-ideal is $P(X)$-ideal for example: let $X = \{1, 2, 3\}$ and $T = \{\emptyset, \{1,2\}, X\}$. Then $I = \{\emptyset, \{3\}\}$ is C-ideal but is not $P(X)$-ideal.
2. Not every $P(X)$-ideal is C-ideal for example: let $X = \mathbb{R}$ and $T = \{\emptyset, \mathbb{Q}, \mathbb{R}\}$. Then $I = \{A \subseteq \mathbb{R}: A \text{ is finite}\}$ is $p(x)$-ideal because for each $A \in p(x)$ and $x \in A$, then $x \in \{x\} \subseteq A$ and $\{x\} \in I$. but I is not C-ideal because $\mathbb{Q} \in T$ and any $x \in \mathbb{Q}$ only $\mathbb{Q} \in T$ such that $x \in \mathbb{Q} \subseteq \mathbb{Q}$ and $\mathbb{Q}, \mathbb{Q}^c \notin I$.

Theorem 2.19:

Let (X, T) be a topological space. Then

1.Every ideal I on X has not bench point is $p(x)$-ideal.
2.Every ideal I on X has not bench point is C-ideal.

Proof:

1.Let I be an ideal and has not bench point. Let $A \in p(x)$ and $x \in A$. Since x is not bench point of I, then there exists $U \in N_x, B \in I$ such that $U^c \cup B = X$, so $U \subseteq B$ and we have $U \in I$. Let $M = A \cap U$, then $x \in M \subseteq A$. Hence $M \in I$ because $M \subseteq U$. So that I is $p(x)$-ideal.
2.A similar proof 1.

Theorem 2.20:

Every $p(x)$-ideal has a bench point if and only if every C-ideal has a bench point.

Proof:

Suppose every $p(x)$-ideal has a bench point. If possible I is a C-ideal and has not bench point then I is $p(x)$-ideal and so I has a bench point.

Conversely. By theorem2.4.

Theorem 2.21:

Let (X,T) be a topological space. X is compact if and only if every $P(X)$-ideal has a bench point.

Definition 2.22:

Let (X,T) be a topological space and let $\mathfrak{B}$ ba a bace for the toplogy T. Let us call an ideal I a $\mathfrak{B}$-ideal if and only if $A \in \mathfrak{B}$ and $x \in A$ imply there is $B \in \mathfrak{B}$ such that $x \in B \subseteq A$ and $B \in I$ or $B^c \in I$.

Theorem 2.23:

Let (X,T) be a topological space and let I be an ideal on X. I is C-ideal if and only if I is $\mathfrak{B}$-ideal.

Proof:

Let I be a C-ideal on X. Let $A \in \mathfrak{B}$ and $x \in A$, then $A \in T$. Since I is C-ideal, then there is $B \in T$ such that $x \in B \subseteq A$ and either $B \in I$ or $B^c \in I$. If $B^c \in I$, then $A^c \in I$ because $A^c \subseteq B^c$. Suppose $B \in I$, then $B = \cup_i B_i$ for some $B_i \in \mathfrak{B}$. Since $x \in B$, then $x \in B_i$ for some B_i. Since $B_i \subseteq \cup_i B_i = B$ and $B \in I$, then $B_i \in I$. Therefore I is $\mathfrak{B}$-ideal.

Conversely. Let I be a $\mathfrak{B}$-ideal on X. Let $A \in T$ and $x \in A$, then $A = \cup_i B_i$ for some $B_i \in \mathfrak{B}$, so there is $B_i \subseteq A$ such that $x \in B_i$ and $B_i \in \mathfrak{B}$. Since I is $\mathfrak{B}$-ideal then there is $B \in \mathfrak{B}$ such that $x \in B \subseteq B_i$ and either $B \in I$ or $B^c \in I$. So that there is $B \in T$ such that $x \in B \subseteq A$ and either $B \in I$ or $B^c \in I$. Therefore I is C-ideal.

Theorem 2.24:

Let (X,T) be a topological space. X is compact if and only if every $\mathfrak{B}$-ideal has a turing point if and only if every $\mathfrak{B}$-ideal has a bench point.

Definition 2.25:

Let (X,T) be a topological space and let $\mathcal{S}$ be a subbace for the topology T. An ideal I is called $\mathcal{S}$-ideal if and only if $S_1 \in \mathcal{S}$ and $x \in S_1$ imply there is $S_2 \in \mathcal{S}$ such that $x \in S_2$ and either $(S_1 \cap S_2)^c \in I$ or $S_2 \in I$.

Observe that any maximal ideal is $\mathcal{S}$-ideal.

Definition 2.26:

Let (X,T) be a topological space, $x \in X$, I ideal on X and let $\mathcal{S}$ be a subbace for the topology T. We say that

1.x is $\mathcal{S}$-turing point of I if for each $S \in \mathcal{S}$ and $x \in S$ implies $S^c \in I$.
2.x is a $\mathcal{S}$-bench point of I if and only if $S^c \cup A \neq X$ for each $x \in S \in \mathcal{S}$.

Theorem 2.27:

Let (X,T) be a topological space, $\mathcal{S}$ be a subbace for the topology T, I be an ideal on X and $x \in X$. x is turing point of I if and only if x is a $\mathcal{S}$-turing point of I.

Proof:

Suppose x is $\mathcal{S}$-turing point of I. Let $N \in T$ and $x \in N$, so $x \in N = \cup_i (S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_{n_i}})$ where $S_{i_k} \in \mathcal{S}$.Then $x \in S = (S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_{n_i}}) \subseteq N$ for some i, then $x \in S_{i_1}, S_{i_2}, \dots, S_{i_{n_i}}$ and we have by hypothesis we have $(S_{i_1})^c, (S_{i_2})^c, \dots, (S_{i_{n_i}})^c \in I$then $S^c = (S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_{n_i}})^c = (S_{i_1})^c \cup (S_{i_2})^c \cup \dots \cup (S_{i_{n_i}})^c \in I$ Since $N^c \subseteq S^c$, then $N^c \in I$. Therefore x is turing point of I.

Conversely. Suppose x is a turing point of I. Let $S \in \mathcal{S}$ such that $x \in S$, then $S \in T$. Since x is a turing point of I, then $S^c \in I$. So that x is a $\mathcal{S}$-turing point of I.

Theorem 2.28:

Let (X,T) be a topological space, $\mathcal{S}$ be a subbace for the topology T, I be an ideal on X and $x \in X$. If x is bench point of I, then x is a $\mathcal{S}$-bench point of I.

Proof:

Let x be a bench point of I. Let $S \in \mathcal{S}$, $x \in S$, then $S \in T$. Since x is bench point of I, then $S^c \cup A \neq X$ for each $A \in I$. Therefore x is a $\mathcal{S}$-bench point of I.

Remark 2.29:

The converse for theorem above is not true for example: Let $X = \{a, b, c, d, e\}$, then $\mathcal{S} = \{\{a, b, c\}, \{c, d\}, \{d, e\}\}$ is a subbace of a topology $T = \{X, \{a, b, c\}, \{c, d\}, \{d, e\}, \{c\}, \{d\}, \emptyset, \{a, b, c, d\}, \{c, d, e\}\}$. Let $I = \{\emptyset, \{c\}\}$, then c is $\mathcal{S}$-bench point of I but x is not bench point of I because $c \in \{c\} \in T$ and $\{c\} \in I$, so that $\{c\} \cup \{c\}^c = X$.

Theorem 2.30:

Let (X, T) be a topological space, $x \in X$ and I be an ideal on X. Then this equivalent:

1.I is $\mathcal{S}$-ideal
2.x is $\mathcal{S}$-bench point of I if and only if x is $\mathcal{S}$-turing point of I.

Proof:

(1. implies 2.). Let I be a $\mathcal{S}$-ideal and let x be a $\mathcal{S}$-bench point of I. To show x is $\mathcal{S}$-turing point of I, let $S \in \mathcal{S}$ and $x \in S$, then there is $S_1 \in \mathcal{S}$ such that $x \in S_1$ and either $S_1 \in I$ or $(S \cap S_1)^c \in I$ because I is a $\mathcal{S}$-ideal. Hence $S_1 \notin I$, because if possible $S_1 \in I$, and x is $\mathcal{S}$-bench point of I then $S_1 \cup S_1^c \neq X$ contradiction. So that $(S \cap S_1)^c \in I$. Since $S^c \subseteq (S \cap S_1)^c$, then $S^c \in I$. So x is $\mathcal{S}$-turing point of I.

Conversely. Let x be a $\mathcal{S}$-turing point of I, then $S^c \in I$ for each $S \in \mathcal{S}$ such that $x \in S$. To prove that $S^c \cup B \neq X$ for each $S \in \mathfrak{A}$ such that $x \in S$ and for each $B \in I$. if possible $S^c \cup B = X$, then $S \subseteq B$, then $S \in I$. But $S^c \in I$, so $S \cap S^c = X \in I$ contradiction. Therefore x is $\mathcal{S}$-bench point of I.

(2. implies 1.). Assume 2., hold. Let $S \in \mathcal{S}$ and $x \in S$ then x is either a $\mathcal{S}$-bench point or not a $\mathcal{S}$-bench point of I. If x is $\mathcal{S}$-bench point then x is $\mathcal{S}$-turing point, so $S^c = (S \cap S)^c \in I$. If x is not $\mathcal{S}$-bench point, then there is $S_1 \in \mathcal{S}$, $x \in S_1$ and there is $B \in I$ such that $B \cup S_1^c = X$, so $S_1 \subseteq B$ and we have $S_1 \in I$. Therefore I is $\mathcal{S}$-ideal.

Corollary 2.31:

An ideal without any $\mathcal{S}$-bench points is a $\mathcal{S}$-ideal. hence if an ideal is not $\mathcal{S}$-ideal then it has a $\mathcal{S}$-bench point.

Theorem 2.32:

Let (X, T) be a topological space, $x \in X$ and I be a $\mathcal{S}$-ideal on X. Then x is bench point of I if and only if x is turing point of I.

Proof:

Direct by theorem 2.28, theorem 2.30, and theorem 2.27.

Theorem 2.33:

Let (X, T) be a topological space I be an ideal on X. If I is $\mathcal{S}$-ideal, then I is C-ideal.

Proof:

Direct from theorem 2.32, and theorem 2.12.

Theorem 2.34:

A space is compact if and only if every $\mathcal{S}$- ideal has a $\mathcal{S}$-turing point.

Proof:

Let X is compact .Let I is $\mathcal{S}$- ideal, then by theorem 2.33, we have I is C-ideal. Since X is compact then by theorem 2.11, we have I has a turing point. So by theorem 2.30 we have I has a $\mathcal{S}$-turing point.

Conversely. Suppose every $\mathcal{S}$- ideal has a $\mathcal{S}$-turing point. Since every maximal ideal is $\mathcal{S}$-ideal, then every maximal ideal has a $\mathcal{S}$-turing point. By theorem2.30, we have every maximal ideal has a turing point and by theorem2.7 , X is compact.

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