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Fuzzy γ–open and Fuzzy γ-closed sets

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Abstract

In this paper the concept of fuzzy γ–open sets and its properties are studied in fuzzy topological spaces .

Introduction :

The concept of fuzzy set and fuzzy set operations were first introduced By L. A. Zadeh in 1965[ ] . After Zadeh 's introduction of fuzzy sets , Chang [ ] defined and studied the notion of fuzzy topological space in 1968 .Since then, much attention has been paid to generalize the basic concepts of classical topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed .

Throughout this paper (X,T) (or simply X) , we shall mean a fuzzy topological spaces (fts , for short) in chang's [6] sense .A fuzzy point [8] with support $x \in X$ and value λ $(0 < \lambda \leq 1)$ at $x \in X$ will be denoted by x_λ , and for fuzzy set A , $x_\lambda \in A$ if and only if $\lambda \leq A(x)$ .For two fuzzy sets A and B , we shall write AqB to mean that A is quasi–coincident (q– coincident, for short) with B , i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$ [13] , and B is said to be a q–neighborhood (q–nbd , for short) of A if there is a fuzzy open set U with $AqU \leq B$ .If A is not q–coincident with B , then we write $A \not q B$. For a fuzzy set A in a fts X , cl(A) , Int(A) , A^c (or 1_X–A)denote the closure , interior , complement of A , respectively . By 0_X and 1_X we mean the constant fuzzy sets taking on the values 0 and 1 , respectively .

Definition 1.1:

A fuzzy set A in a fts X is called :

(1) fuzzy α–open [1] or feebly open [2] or fuzzy strongly semiopen [3] , if $A \leq int(cl(int(A)))$.

(2) fuzzy α–closed [1] or feebly closed [2] or fuzzy strongly semiclosed [3] , if $cl(int(cl(A))) \leq A$.

(3) fuzzy preopen [1] , if $A \leq int(cl(A))$.

(4) fuzzy preclosed [1] , if $cl(int(A)) \leq A$.

(5) fuzzy semiopen [9] , if $A \leq cl(int(A))$.

(6) fuzzy semiclosed [9] , if $int(cl(A)) \leq A$.

Remark 1.2:

Every fuzzy open (resp., fuzzy closed) set is a fuzzy α–open (resp., fuzzy α–closed) ,every fuzzy α–open (resp., fuzzy α–closed)set is a fuzzy semiopen (resp., fuzzy semiclosed) as well as a fuzzy preopen (resp., fuzzy preclosed) set . But the converses may not be true [2 , 7 , 14 , 16] .

Definition 1.3:[12 , 4 , 2]

Let A be a fuzzy set in a fts X , the fuzzy pre–closure (resp., semi–closure , α–closure , pre–interior ,semi–interior ,α–interior) of A denoted by pcl(A) (resp., scl(A) ,αcl(A) , pint(A) , sint(A) , αint(A)) and defined as follows :

$pcl(A) = \wedge\{B : A \le B$, B is fuzzy pre closed $\}$, and

$pint(A) = \vee\{B : B \le A$, B is fuzzy pre open $\}$.

The definitions of scl(A) , αcl(A) , sint(A) and αint(A) are similar by putting fuzzy semiclosed and fuzzy semiopen as well as fuzzy α–closed and fuzzy α–open instead of fuzzy preclosed and fuzzy preopen .

Definition 1.4:[4 , 10, 2]

A fuzzy set B in a fts X is said to be :

(1) semi–q–nhd (simply s–q–nhd)of a fuzzy point x_λ if and only if there exists a fuzzy semiopen subset A in X such that x_λ q $A \le B$.

(2) pre–q–nhd (simply p–q–nhd) of a fuzzy point x_λ if and only if there exists a fuzzy preopen subset A in X such that x_λ q $A \le B$.

(3) α–q–nhd of a fuzzy point x_λ if and only if there exists a fuzzy α–open subset A in X such that x_λ q $A \le B$.

Lemma 1.5:[3]

For a fuzzy set A in a fts X , scl(A) is the union of all fuzzy point x_λ such that every fuzzy semiopen U with x_λ q U is q–coincident with A .

Theorem 1.6:[10]

In a fts X a fuzzy point $x_\lambda \in pcl(A)$ if and only if every pre–q–nhd of x_λ is q–coincident with A .

Theorem 1.7:[2]

Let A be a fuzzy set of a fts X .Then ,αcl(A) is the set of all fuzzy point x_λ such that every fuzzy α–open–q–nhd of x_λ is q–coincident with A .

Lemma 1.8:

Let A be a fuzzy set of a fts X .Then :

(1) $int(A) \le sint(A) \le A \le scl(A) \le cl(A)$.[5]

(2) $int(A) \le \alpha int(A) \le sint(A) \le A \le scl(A) \le \alpha cl(A) \le cl(A)$.[2]

(3) $\alpha cl(1_X - A) = 1_X - \alpha int(A)$.[2]

(4) $\alpha int(1_X - A) = 1_X - \alpha cl(A)$.[2]

(5) $int(A) \le pint(A) \le A \le pcl(A) \le cl(A)$.[10]

(6) $1_X - pint(A) = pcl(1_X - A)$.[10]

(7) $pint(1_X - A) = 1_X - pcl(A)$.[10]

(8) $scl(1_X - A) = 1_X - sint(A)$.[11]

(9) $sint(1_X - A) = 1_X - scl(A)$.[11]

Definition 1.9 :[15]

Let X be a fts . A fuzzy set A of X is called fuzzy γ–open , if $A \le int(cl(A)) \vee cl(int(A))$ .The complement of a fuzzy γ–open set is called fuzzy γ–closed .

The family of all fuzzy γ–open (fuzzy γ–closed) sets of X is denoted by FγO(X) (FγC(X)) .

The following are the properties of fuzzy γ–open sets .

(1) Any union of fuzzy γ–open sets in a fts X is a fuzzy γ–open set

Proof :

Let $\{A_i , i \in J\}$ be a family of fuzzy γ–open sets in a fts X . For each $i \in J$, $A_i \leq int(cl(A_i)) \vee cl(int(A_i))$. Hence ,

$$\bigvee_{i \in J} A_i \leq \bigvee_{i \in J} [\, int(cl(A_i)) \vee cl(int(A_i))]$$
$$= (\bigvee_{i \in J} [\, int(cl(A_i))]) \vee (\bigvee_{i \in J} [\, cl(int(A_i))])$$
$$\leq int(cl(\bigvee_{i \in J} A_i) \vee cl(int(\bigvee_{i \in J} A_i)$$

This proves that $\bigvee_{i \in J} A_i$ is fuzzy γ–open set .

(2) Any intersection of fuzzy γ–closed sets is fuzzy γ–closed set .

Remark 1.10:

Every fuzzy semiopen (resp., fuzzy semiclosed) set is fuzzy γ–open (resp., fuzzy γ–closed) and every fuzzy preopen (resp., fuzzy preclosed) set is fuzzy γ–open (resp., fuzzy γ–closed) . But the converses may not be true .

Example 1.11:

Let H_1 , H_2 be two fuzzy sets in X = {a , b} defined as follows :

H_1 (a) = 0.6 H_1 (b) = 0.7 ,
H_2 (a) = 0.5 H_2 (b) = 0.5 .

Let $\delta = \{0_X , H_1 , 1_X\}$ be a fuzzy topology on X .Then H_2 (resp., H^c_2) is fuzzy γ–open (resp., fuzzy γ–closed) set , but not fuzzy semiopen (resp., fuzzy semiclosed) .

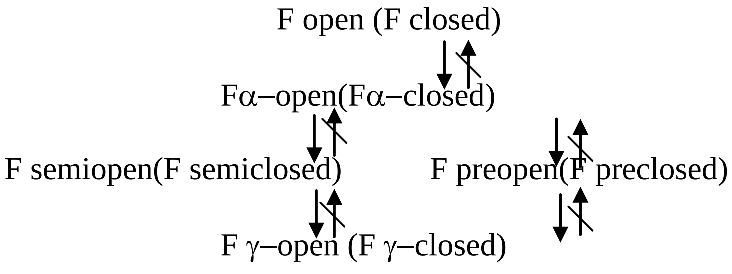
Example 1.12:

Let A , B be two fuzzy sets in X = {a , b} defined as follows :

A (a) = 0.4 A (b) = 0.3 ,
B (a) = 0.4 B (b) = 0.4 .

Let $T = \{0_X , A , 1_X \}$ be a fuzzy topology on X . Then B (resp., B^c) is fuzzy γ–open (resp., fuzzy γ–closed) set ,but not fuzzy preopen (resp., fuzzy preclosed) .

As consequence of Remark 1.2 and Remark 1.10 , the following diagram holds for any fuzzy set in fts X . Let F = fuzzy .



Fuzzy γ–closure and fuzzy γ–interior of a fuzzy set A in a fts X is denoted as γ–cl(A) and γ–int(A) respectively , and defined as follows :

Definition 1.13:

Let A be any fuzzy set in the fts X .Then we define :

γ–cl(A) $= \wedge\{B : B$ is fuzzy γ–closed set and $B \geq A\}$, and
γ–int(A) $= \vee\{B : B$ is fuzzy γ–open set and $B \leq A\}$

It is evident that γ–cl(A) = A if and only if A is fuzzy γ–closed and γ–int(A) = A if and only if A is fuzzy γ–open .

Definition 1.14:

A fuzzy set A in a fts X is called fuzzy γ–nhd of a fuzzy point x_λ if there exists B $\in$ FγO(X) such that $x_\lambda \in B \leq A$.

A fuzzy γ–nhd A is said to be fuzzy γ–open–nhd (respectively, γ–closed–nhd) if and only if A is fuzzy γ–open (respectively, γ–closed)set .

Definition 1.15:

A fuzzy set A in a fts X is called γ–q–nhd of a fuzzy point x_λ if there exists a fuzzy γ–open U in X such that x_λq U $\leq$ A .

Theorem 1.16:

Let x_λ and A be a fuzzy point and a fuzzy set respectively in a fts X . Then $x_\lambda \in \gamma$–cl(A) if and only if every γ–q–nhd of x_λ is q–coincident with A .

Proof :

Suppose $x_\lambda \in \gamma$–cl(A) and if possible , let there exists a γ–q–nhd U of x_λ such that U /q A .Then there exists a fuzzy γ–open set V in X such that x_λ q V $\leq$ U which shows that V /q A and hence A $\leq 1_X$–V . As 1_X–V is fuzzy γ–closed , γ–cl(A) $\leq 1_X$–V . Since $x_\lambda \notin 1_X$–V ,we obtain $x_\lambda \notin \gamma$–cl(A) which is a contradiction .

Conversely , Suppose $x_\lambda \notin \gamma$–cl(A) .Then there exists a fuzzy γ–closed set B such that A $\leq$ B and $x_\lambda \notin$ B .We then have x_λ q (1_X– B)$\in$ FγO(X) and A /q (1_X– B) .

Properties 1.17:

Let A be any fuzzy set in a fts X .Then :

(a) γ–cl(1_X – A) = 1_X –(γ–int(A)) ,

(b) γ–int (1_X – A) = 1_X –(γ–cl(A)) .

Proof :

By definition of fuzzy γ–closure and fuzzy γ–interior ,

γ–cl(A) =$\wedge$\{B : B is fuzzy γ–closed set and B $\geq$ A\} , and

γ–int(A) =$\vee$\{B : B is fuzzy γ–open set and B $\leq$ A\}

Then ,

(a) 1_X –(γ–int(A)) = 1_X –$\vee$\{B : B is fuzzy γ–open set and B $\leq$ A\}

= $\wedge$ \{1_X–B : B is fuzzy γ–open set and B $\leq$ A\}

= $\wedge$ \{V : V is fuzzy γ–closed set and V $\geq 1_X$ –A\}

= γ–cl(1_X – A) .

(b) The proof is similar to (a) .

Lemma 1.18:

Let A , B and C are fuzzy sets in a fts X . A q B$\vee$C if and only if either A q B or A q C .

Proof :

Suppose that A q B$\vee$C , then

A q B$\vee$C if and only if there exists x $\in$X , A(x) + (B$\vee$C)(x) > 1

if and only if A(x) + B(x) > 1 or A(x) + B(x) > 1

if and only if A q B or A q C .

Properties 1.19:

Let A and B be two fuzzy sets in a fts X .Then the following are true :

(1) γ–cl(0_X) = 0_X , γ–cl(1_X) = 1_X .

(2) γ–cl(A) is fuzzy γ–closed in X .

(3) γ–cl(A) ≤ γ–cl(B) if A ≤ B .

(4) if U is a fuzzy γ–open set, then U q A if and only if U q γ–cl(A) .

(5) γ–cl(A) =γ–cl(γ–cl(A)) .

(6) γ–cl(A∨B) = γ–cl(A) ∨ γ–cl(B) .

(7) γ–cl(A∧B) ≤ γ–cl(A) ∧ γ–cl(B) .

Proof : (1) and (2) are obvious .

(3) Let $x_\alpha \notin \gamma$–cl(B) . By Theorem 1.16 ,there is a fuzzy γ–q–nhd V of x_α such that V /q B , so there is a fuzzy γ–open set U in X such that x_α q U ≤ V and U /q B . Since A ≤ B ,then U /q A . Hence $x_\alpha \notin \gamma$–cl(A) by Theorem 1.16 .This shows that γ–cl(A) ≤ γ–cl(B) .

(4) Let U∈FγO(X) and U /q A . Then A ≤ U^c .Since U^c is fuzzy γ–closed , γ–cl(A) ≤ γ–cl(U^c) = U^c by part 2 .Thus U /q γ–cl(A) .

Conversely , let U /q γ–cl(A) . Then γ–cl(A) ≤ U^c , so A ≤ U^c . Thus U /q A .

(5) Since γ–cl(A) ≤ γ–cl(γ–cl(A)) , it suffices to show that γ–cl(γ–cl(A)) ≤ γ–cl(A) . Let $x_\alpha \notin \gamma$–cl(A) . By Theorem 1.16 , there is a fuzzy γ–q–nhd V of x_α such that V /q A , so there is a fuzzy γ–open set U in X such that x_α q U ≤ V and U /q A . By(4) , there is a fuzzy γ–q–nhd U of x_α such that U /q γ–cl(A) . Thus we have $x_\alpha \notin \gamma$–cl(γ–cl(A)) by Theorem 1.16 and hence γ–cl(γ–cl(A)) = γ–cl(A) .

(6) Since A ≤ A∨B and B ≤ A∨B , γ–cl(A) ≤ γ–cl(A∨B) and γ–cl(B) ≤ γ–cl(A∨B) .Then γ–cl(A) ∨ γ–cl(B) ≤ γ–cl(A∨B) .

Conversely , let $x_\alpha \in \gamma$–cl(A∨B) . By Theorem 1.16 , for each fuzzy γ–q–nhd U of x_α , U q (A∨B) . By Lemma 1.18 , either U q A or U q B , so $x_\alpha \in \gamma$–cl(A) or $x_\alpha \in \gamma$–cl(B) .Thus γ–cl(A∨B)≤ γ–cl(A) ∨ γ–cl(B) .This shows that γ–cl(A∨B) = γ–cl(A) ∨ γ–cl(B) .

(7) Since A ∧ B ≤ A and A ∧ B ≤ B , γ–cl(A∧B) ≤ γ–cl(A) and γ–cl(A∧B) ≤ γ–cl(B) .Then γ–cl(A∧B) ≤ γ–cl(A) ∧ γ–cl(B) .

Properties 1.20:

Let A and B be two fuzzy sets in a fts X .Then the following are true :

(1) γ–int(0_X) = 0_X , γ–int(1_X) = 1_X .

(2) γ–int(A) is fuzzy γ–open in X .

(3) γ–int(A) ≤ γ–int(B) if A ≤ B .

(4) γ–int(γ–int(A)) = γ–int(A) .

(5) γ–int(A∧B) = γ–int(A) ∧ γ–int(B) .

(6) γ–int(A∨B) ≥ γ–int (A)∨ γ–int(B) .

Proof : It is obvious .

Theorem 1.21:

Let A be a fuzzy set in a fts X .Then

(a) int(A) ≤ αint(A) ≤ pint(A) ≤ A ≤ pcl(A) ≤ αcl(A) ≤ cl(A) .

(b) int(A) ≤ αint(A) ≤ sint(A) ≤ γ–int(A) ≤ A ≤ γ–cl(A) ≤ scl(A) ≤ αcl(A) ≤ cl(A) .

(c) int(A) ≤ αint(A) ≤ pint(A) ≤ γ–int(A) ≤ A ≤ γ–cl(A) ≤ pcl(A) ≤ αcl(A) ≤ cl(A) .

Proof :

(a) Not that int(A) ≤ pint(A) ≤ A ≤ pcl(A) ≤ cl(A) (see Lemma 1.8.part 5) Then , we must prove that pcl(A) ≤ αcl(A) and αint(A) ≤ pint(A) .

Firstly let us prove pcl(A) ≤ αcl(A) :

Let $x_\lambda \notin$ αcl(A) , so by Theorem 1.16 , there is a fuzzy α−open−q−nhd U of x_λ and U /q A . That is x_λ q U≤ U and U /q A . Since U is fuzzy α−open, then U is fuzzy peropen .Thus $x_\lambda \notin$ pcl(A) by Theorem 1.6 .This shows that pcl(A) ≤ αcl(A) .

Now , let us prove αint(A) ≤ pint(A) . Since A is fuzzy set in X , then A^c is also fuzzy set in X . Then pcl(A^c) ≤ αcl(A^c) by above case . By Lemma 1.8.part 6 and part 3, 1_X −pint(A) ≤ 1_X − αint(A) and this implies that αint(A) ≤ pint(A) .

(b) Not that int(A) ≤ αint(A) ≤ sint(A) ≤ A ≤ scl(A) ≤ αcl(A) ≤ cl(A) (see Lemma 1.8.part 2) . Then we must prove γ−cl(A) ≤ scl(A) and sint(A) ≤ γ−int(A) .

The prove is similar to (a) .

(c) By (a) ,we must prove γ−cl(A) ≤ pcl(A) and pint(A) ≤ γ−int(A) .

The prove is analogous to the proof of (a) .

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