

New Separation Axioms Using the idea of "Gem-Set" in Topological Space

Luay A. Al-Swidi¹ and Ahmed B. AL-Nafee^{2*}

1. 1 Mathematics Department, College of Education For Pure sciences University of Babylon.

2. 2 Mathematics Department, College of Education For Pure sciences University of Babylon.

*E-meal:Ahm_math_88@yahoo.com.

Abstract

In this paper, we create a new set of topological space namely "Gem-Set" and immersed it with a new separation axioms in topological space and investigate the relationship between them

Keywords: "Gem-Set", separation axioms.

1. Introduction and Preliminaries.

The idea of "Gem-Set" is defined as: for a topological space (X, T) , and $A \subseteq X$, we defined A^{*x} with respect to space (X, T) as follows: $A^{*x} = \{y \in X : G \cap A \notin I_x, \text{ for every } G \in T(y)\}$ where $T(y) = \{G \in T : y \in G\}$, I_x is an ideal on a topological space (X, T) at point x is defined by $I_x = \{U \subseteq X : x \in U^c\}$, where U is non-empty set of X .

Within this paper "Gem-Set" is studied with some its properties, a set of new separation axioms in topological spaces, namely " I^* - T_0 -space", " I^* - T_1 -space", " I^* - T_2 -space", " I^{**} - T_0 -space", " I^{**} - T_1 -space", " I^{**} - T_2 -space" and the axioms R_i , $i = 0, 1, 2, 3$ are proposed by using the idea of "Gem-Set", the relationship between them is studied. Also two mappings " I^* -map" and " I^{**} -map" are defined to carry properties of "Gem-Set" from a space to other space.

Throughout this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned. Let A be a subset of a space X . The closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

Definition 1.1.

A topological space (X, T) is called

- R_0 -space [1,2,4] if and only if for each open set G and $x \in G$ implies $cl(\{x\}) \subseteq G$.
- R_1 -space [1,2,4] if and only if for each two distinct point x, y of X with $cl(\{x\}) \neq cl(\{y\})$, then there exist disjoint open sets U, V such that $cl(\{x\}) \subseteq U$ and $cl(\{y\}) \subseteq V$
- R_2 -space [2] if it is property regular space.
- R_3 -space [3] if and only if (X, T) is a normal and R_1 -space.

Remark 1.2[3]

Each separation axiom is defined as the conjunction of two weaker axiom: T_k -space = R_{k-1} -space and T_{k-1} -space = R_{k-1} -space and T_0 -space, $k=1, 2, 3, 4$

Remark 1.3[3]

Every R_i -space is an R_{i-1} -space $i = 0, 1, 2, 3$.

2. "Gem-Set" in Topological Space

Definition 2.1

For a topological space (X, T) , $x \in X$, $Y \subseteq X$, we define an ideal ${}^Y I_x$ with respect to subspace (Y, T_Y) as follows: ${}^Y I_x = \{G \subseteq Y : x \in (X-G)\}$.

Remark 2.2

For a topological space (X, T) , $Y \subseteq X$, for each $G \neq \emptyset$. Then ${}^Y I_x = \{G \subseteq Y : x \in (X-G), \text{ for each } x \in Y\} = \{G \subseteq Y : x \in (Y-G) \text{ for each } x \in Y\}$.

Proposition 2.3

For a topological space (X, T) , $Y \subseteq X$, for each $G \neq \emptyset$. Then ${}^Y I_x = \{G \subseteq Y : x \in (X-G)\} = \{G \cap Y : \text{for each } \emptyset \neq G \in I_x\}$,

Definition 2.4

For a topological space (X, T) , $Y \subseteq X$, and $A \subseteq Y$, we defined ${}_Y A^{*x}$ with respect to subspace (Y, T_Y) as follows: for $A \subseteq Y$, ${}_Y A^{*x} = \{y \in Y : G \cap A \notin I_x, \text{ for every } G \in T_Y(y)\}$ where $T_Y(y) = \{G \in T_Y : y \in G\}$.

Note 2.5

For a topological space (X, T) contains singleton point (say x), then $I_x = \emptyset$.

Proposition 2.6

Let (X, T) be a topological space, and let A, B be subsets of X , $x \in X$. Then

- $\phi^{*x} = \phi$
- $X^{*x} = X$, whenever $I_x = \emptyset$.
- $A \subseteq B$ implies $A^{*x} \subseteq B^{*x}$.
- For another ideal $I_y \supseteq I_x$ on X , $A^{*y} \subseteq A^{*x}$.
- If $x \in X$. Then $x \in A$ if and only if $x \in A^{*x}$.
- If $x \in A$, then $(A^{*x})^{*x} = A^{*x}$.
- If $x \in A, y \in B$ such that $x \neq y$, then $A^{*x} \cap B^{*y} = \emptyset$.
- If $x, y \in X$ such that $x \neq y$, then $y \in \{x\}^c$ implies $x \notin \{x\}^{*y}$ and $y \notin \{y\}^{*x}$
- $A^{*x} \cup B^{*x} = (A \cup B)^{*x}$
- $(A \cap B)^{*x} \subseteq A^{*x} \cap B^{*x}$.
- $A^{*x} \subseteq \text{cl}(A)$.

Proof : Straight forward .

Proposition 2.7

Let a topological space (X, T) then for open set V , $V \cap A^{*x} = V \cap (V \cap A)^{*x} \subseteq (V \cap A)^{*x}$, for any $x \in X$.

Proof: Straight forward

Definition 2.8 Let (X, T) be a topological space and $A \subseteq X$. We define ${}^{*x}\text{pr}(A)$, as following: ${}^{*x}\text{pr}(A) = A^{*x} \cup A$, for each $x \in X$.

Theorem 2.9 Let E and F be such sets of (X, T) , $x \in X$. Then

- ${}^{*x}\text{pr}(\phi) = \phi$
- ${}^{*x}\text{pr}(X) = X$
- If $E \subseteq F$, then ${}^{*x}\text{pr}(E) \subseteq {}^{*x}\text{pr}(F)$.
- ${}^{*x}\text{pr}(E \cup F) = {}^{*x}\text{pr}(E) \cup {}^{*x}\text{pr}(F)$.
- ${}^{*x}\text{pr}(E \cap F) \subseteq {}^{*x}\text{pr}(E) \cap {}^{*x}\text{pr}(F)$.

Proof Straight forward.

Proposition 2.10

Let (X, T) be a topological space and $A \subseteq X$. If A is a closed set, then ${}^{*x}\text{pr}(A) = A^{*x} = A = \text{cl}(A)$, for each $x \in X$.

Definition 2.11

A subset A of a topological space (X, T) is called *prefected set* if $A^{*x} \subseteq A$, for each $x \in X$.

Definition 2.12

A subset A of a topological space (X, T) is called *coprefected set* if A^c is a *prefected set*.

Lemma 2.13

Let (X, T) be a topological space, then every a closed set is *prefected set*

Proof

Let A be a subset closed of X , then $\text{cl}(A) = A$. But $A^{*x} \subseteq \text{cl}(A)$, for each $x \in X$ [By proposition 2.6], so that $A^{*x} \subseteq \text{cl}(A) = A$, thus $A^{*x} \subseteq A$, for each $x \in X$. Hence A is *prefected set*.

3. $I^*-T_i, I^{}-T_i, i=0,1,2$ and $R_i, i=0,1,2$ and 3**

Definition 3.1.

- A topological space (X, T) is called
- I^*-T_0 -space if and only if for each pair of distinct points x, y of X , there exist non-empty subsets A, B of X such that $y \notin A^{*x}$ or $x \notin B^{*y}$.
 - I^*-T_1 -space if and only if for each pair of distinct points x, y of X , there exist non-empty subsets A, B of X such that $y \notin A^{*x}$ and $x \notin B^{*y}$.
 - I^*-T_2 -space if and only if for each pair of distinct points x, y of X , there exist subsets A, B of X such that $A^{*x} \cap B^{*y} = \emptyset$, with $y \notin A^{*x}$ and $x \notin B^{*y}$.
 - $I^{**}-T_0$ -space if and only if for each pair of distinct points x, y of X , there exists non-empty subset A of X such that $y \notin A^{*x}$ or $x \notin A^{*y}$.
 - $I^{**}-T_1$ -space if and only if for each pair of distinct points x, y of X , there exists non-empty subset A of X such that $y \notin A^{*x}$ and $x \notin A^{*y}$.
 - $I^{**}-T_2$ -space if and only if for each pair of distinct points x, y of X , there exists subset A of X such that $A^{*x} \cap A^{*y} = \emptyset$, with $y \notin A^{*x}$ and $x \notin A^{*y}$.

Definition 3.2

If (X, T) is a topological space and $Y \subseteq X$, we say that Y is an I^*-T_0 -subspace (I^*-T_1 -subspace) of X iff for each pair of distinct points y^1, y^2 of X , there exist non-empty subsets A, B of Y such that $y^2 \notin A^{*y^1}$ or (and) $y^1 \notin B^{*y^2}$.

Definition 3.3

Let (X, T) be a topological space, for each $x \in X$, a non-empty subset A of X , is called a strongly set if and only if $(A^{*x}$ is open set and $x \in A$).

Definition 3.4

A topological space (X, T) is said to be a strongly- T_1 -space (briefly $s-T_1$ -space) if and only if, for each non-empty subset A of X is a strongly set.

Theorem 3.5

For a topological space (X, T) , then the following properties hold:

1. Every T_0 -space is a I^*-T_0 -space.
2. Every T_1 -space is a I^*-T_1 -space.
3. Every T_2 -space is a I^*-T_2 -space.
4. Every T_0 -space is a $I^{**}-T_0$ -space.
5. Every T_1 -space is a $I^{**}-T_1$ -space.
6. Every T_2 -space is a $I^{**}-T_2$ -space.

Proof:(1)

Let $x, y \in X$ such that $x \neq y$ and let (X, T) is T_0 -space. Then there exist an open set U such that, $x \in U, y \notin U$ or there exist an open set V such that, $y \in V, x \notin V$ and so, $U \cap \{y\} = \emptyset \in I_y$ or $V \cap \{x\} = \emptyset \in I_x$. Put $A = \{x\}, B = \{y\}$. It follows that $x \notin B^{*y}$ or $y \notin A^{*x}$. Hence let (X, T) be a I^*-T_0 -space.

Proof:(2)

Let $x, y \in X$ such that $x \neq y$ and let (X, T) is T_1 -space. Then there exist an open set U such that, $x \in U, y \notin U$, and there exist an open set V such that, $y \in V, x \notin V$ and so, $U \cap \{y\} = \emptyset \in I_y$ and $V \cap \{x\} = \emptyset \in I_x$. Put $A = \{x\}, B = \{y\}$. It follows that $x \notin B^{*y}$ and $y \notin A^{*x}$. Hence let (X, T) be a I^*-T_1 -space.

Proof:(3)

Let (X, T) be T_2 -space. Then for each $x \neq y \in X$ there exist open sets U, V such that $x \in U$, and $y \in V$ and $U \cap V = \emptyset$. But $U^{*x} \cap V^{*y} = \emptyset$ [Proposition 3.1.10]. Put $A = U, B = V$. It follows that there exist subsets A, B of X such that $A^{*x} \cap B^{*y} = \emptyset$, with $y \notin A^{*x}$ and $x \notin B^{*y}$. Thus (X, T) is I^*-T_2 -space.

Proof:(4) By the same proof of part(1).

Proof:(5) Assume that (X, T) is an T_1 -space and let $x, y \in X$ such that $x \neq y$. By assumption, Then there exist an open set U such that, $x \in U, y \notin U$, and there exist an open set V such that, $y \in V, x \notin V$. So that $U \cap \{y\} = \emptyset \in I_y$ and $V \cap \{x\} = \emptyset \in I_x$. Put $A = \{y\}$ and so $x \notin A^{*y}$ and $y \notin A^{*x}$. Hence (X, T) is $I^{**}-T_1$ -space.

Proof:(6) By the same way of proof of part(3).

Remark 3.6

The converse of theorem need not be true as seen from the following examples.

Example 3.7

Let (X, T) be a topological space such that $X = \{x, y, z\}, T = \{\emptyset, X, \{y, z\}, \{x, z\}, \{z\}\}$ and $I_x = \{\emptyset, \{y\}, \{z\}, \{y, z\}\}, I_y = \{\emptyset, \{x\}, \{x, z\}, \{z\}\}$. Set $A = \{x\}$. $A^{*x} = \{x\}$, so that $y \notin A^{*x}$. Hence (X, T) is an I^*-T_0 -space ($I^{**}-T_0$ -space), but not T_0 -space.

Example 3.8

Let (X,T) be a topological space such that $X=\{x,y,z\}$ $T=\{\emptyset,X,\{y,z\},\{x,z\},\{z\}\}$, and $I_x=\{\emptyset,\{y\},\{z\},\{y,z\}\}$, $I_y=\{\emptyset,\{x\},\{x,z\},\{z\}\}$.Set $A=\{x\}$. $B=\{y\}$, $A^{*x}=\{x\}$, and $B^{*y}=\{y\}$, so that $y \notin A^{*x}$ and $x \notin B^{*y}$. Hence (X,T) is an I^*-T_1 -space ($I^{**}-T_1$ -space), but not T_1 -space.

Example 3.9

Let (X,T) be a topological space such that $X=\{x,y,z\}$ $T= \{\emptyset,X,\{y,z\},\{x,z\},\{z\}\}$, and $I_x=\{\emptyset,\{y\},\{z\},\{y,z\}\}$, $I_y=\{\emptyset,\{x\},\{x,z\},\{z\}\}$.Set $A=\{x\}$. $B=\{y\}$, $A^{*x}=\{x\}$, and $A^{*y}=\{y\}$, so that $A^{*x} \cap B^{*y}=\emptyset$. Hence (X,T) is an I^*-T_2 -space ($I^{**}-T_2$ -space), but not T_2 -space.

Remark 3.10

The converse of theorem 3.5, need not be true. But it is true generally, if (X,T) is a $s-T_1$ -space

Theorem 3.11

If (X,T) is an I^*-T_0 -space and $Y \subseteq X$, then Y is I^*-T_0 -subspace

Proof

Let (X,T) is an I^*-T_0 -space and Y is a subspace of X . Let y^1 and y^2 be two distinct points of Y . Since $Y \subseteq X$ and y^1, y^2 are distinct points of X . Again, since X is an I^*-T_0 -space, there exist non-empty subset A,B of X such that $y^2 \notin B^{*y^1}$ or $y^1 \notin A^{*y^2}$. Suppose, $y^1 \notin A^{*y^2}$, so that there exists an T -open set U such that, $y^1 \in U$, $U \cap A \in I_{y^2}$. Put $U' = U \cap Y$ is T_Y -open and $A' = A \cap Y$, so that U' containing y^1 and $U' \cap A' \subseteq U \cap A \in I_{y^2}$. It is follows that $y^1 \notin A'^{*y^2}$. So by definition, we have that Y is I^*-T_0 -subspace

Theorem 3.12

If (X,T) is an I^*-T_1 -space and $Y \subseteq X$, then Y is I^*-T_1 -subspace

Proof

Let (X,T) is an I^*-T_1 -space and Y is a subspace of X . Let y^1 and y^2 be two distinct points of Y . Since $Y \subseteq X$ and y^1, y^2 are distinct points of X . Again, since X is an I^*-T_1 -space, there exist a subset A,B of X such that $y^2 \notin B^{*y^1}$ and $y^1 \notin A^{*y^2}$, so that there exist an T -open set U such that, $y^1 \in U$, $U \cap A \in I_{y^2}$, and there exist an T -open set V such that, $y^2 \in V$, $V \cap B \in I_{y^1}$. Put $U' = U \cap Y$ and $V' = V \cap Y$ are T_Y -open, $A' = A \cap Y$, $B' = B \cap Y$, so that U' containing y^1 , V' containing y^2 , thus $U' \cap A' \subseteq U \cap A \in I_{y^2}$ and $V' \cap B' \subseteq V \cap B \in I_{y^1}$. It is follows that $y^2 \notin B'^{*y^1}$ and $y^1 \notin A'^{*y^2}$. So by definition, we have that Y is I^*-T_1 -subspace

Theorem 3.13

A topological space (X,T) is an R_0 -space if and only if for each $x \in X$ and U open set such that $x \in U$, then $cl(\{x\}^{*x}) \subseteq U$.

Proof

Let $x \in X$ and U open set such that $x \in U$. By assumption, then $cl(\{x\}) \subseteq U$. But $\{x\}^{*x} \subseteq cl(\{x\})$ [By Proposition 2.8]. Therefore $cl(\{x\}^{*x}) \subseteq cl(cl(\{x\}))$ implies $cl(\{x\}^{*x}) \subseteq cl(\{x\})$. Thus $cl(\{x\}^{*x}) \subseteq U$. Conversely, to prove (X,T) is R_0 -space, let $U \in T$ and $x \in U$. Since, $\{x\} \subseteq \{x\}^{*x}$. Then $cl(\{x\}) \subseteq cl(\{x\}^{*x}) \subseteq U$. Thus $cl(\{x\}) \subseteq U$. Therefore (X,T) is R_0 -space.

Theorem 3.14

A topological space (X,T) is R_1 -space if and only if, for each $x,y \in X$ and $A \in X$, such that $x \neq y$ and $cl(\{x\}) \neq cl(\{y\})$, then, there exist disjoint open sets U,V such that $cl(\{x\}^{*x}) \subseteq U$ and $cl(\{y\}^{*y}) \subseteq V$.

Proof

Let $x,y \in X$ and $A \in X$, with $x \neq y$, and $cl(\{x\}) \neq cl(\{y\})$. By assumption, then there exist disjoint open sets U,V such that $cl(\{x\}) \subseteq U$ and $cl(\{y\}) \subseteq V$. But $\{x\}^{*x} \subseteq cl(\{x\})$ and $\{y\}^{*y} \subseteq cl(\{y\})$ [By Proposition 2.8]. Therefore $cl(\{x\}^{*x}) \subseteq cl(cl(\{x\}))$ and $cl(\{y\}^{*y}) \subseteq cl(cl(\{y\}))$. This implies $cl(\{x\}^{*x}) \subseteq cl(\{x\})$ and $cl(\{y\}^{*y}) \subseteq cl(\{y\})$. Thus $cl(\{x\}^{*x}) \subseteq U$ and $cl(\{y\}^{*y}) \subseteq V$.

Conversely, let $x,y \in X$ such that $x \neq y$ and $cl(\{x\}) \neq cl(\{y\})$. By assumption, then there exist disjoint open sets U,V such that $cl(\{x\}^{*x}) \subseteq U$ and $cl(\{y\}^{*y}) \subseteq V$. Now, since, $\{x\} \subseteq \{x\}^{*x}$ and $\{y\} \subseteq \{y\}^{*y}$. Then, $cl(\{x\}) \subseteq cl(\{x\}^{*x}) \subseteq U$ and $cl(\{y\}) \subseteq cl(\{y\}^{*y}) \subseteq V$. Thus $cl(\{x\}) \subseteq U$ and $cl(\{y\}) \subseteq V$. Therefore (X,T) is R_1 -space.

Theorem 3.15

A $s-T_1$ -space (X,T) is regular space iff for each F closed set and $x \notin F$, then $\{x\}^{*x} \cap F^{*y} = \emptyset$.

Proof

Let F be closed set and $x \notin F$, thus $y \in F$, so that $\{x\}^{*x} \cap F^{*y} = \emptyset$ [Proposition 2.6].

Conversely, let F be closed set and $x \notin F$ and $y \in F$ implies $\{x\}^{*x} \cap F^{*y} = \emptyset$. Since (X,T) is a $s-T_1$ -space and by definition 3.1, we get that $\{x\}, F$ are a strongly sets, so $\{x\}^{*x}, F^{*x}$ an open subsets of X , with $x \in \{x\}^{*x}$ and $F \subseteq F^{*y}$. Thus (X,T) is regular space.

Theorem 3.16

A $s-T_1$ -space (X, T) is normal space iff for each disjoint closed sets F, H , then $F^{*x} \cap H^{*x} = \emptyset$.

Proof

By the same way of proof of above theorem.

Theorem 3.17

For a topological space (X, T) , then the following properties hold:

1. (X, T) is I^*-T_0 -space iff $I^{**}-T_0$ -space.
2. Every I^*-T_1 -space is a $I^{**}-T_1$ -space.
3. Every I^*-T_2 -space is a $I^{**}-T_2$ -space.

Proof :Straight forward.

Remark 3.18

The converse of part(2),(3), need not be true as seen from the following examples.

Example 3.19

Let (X, T) be a topological space such that $X = \{x, y, z\}$, $T = \{\emptyset, X, \{z\}\}$, and $I_x = \{\emptyset, \{y\}, \{y, z\}, \{z\}\}$, $I_y = \{\emptyset, \{x\}, \{x, z\}, \{z\}\}$. Set $A = \{z\}$, $B = \{x, y\}$, then $A^{*x} = \{\emptyset\}$, $A^{*y} = \{\emptyset\}$, $B^{*y} = \{x, y\}$, that means $y \notin A^{*x}$ and $x \notin A^{*y}$ but $y \in A^{*x}$ and $x \in B^{*y}$. Hence (X, T) is an $I^{**}-T_1$ -space but not I^*-T_1 -space

Example 3.20

Let (X, T) be a topological space such that $X = \{x, y, z, w\}$, $T = \{\emptyset, X, \{x, y\}, \{x, y, z\}, \{z\}\}$, and $I_x = \{\emptyset, \{y\}, \{x\}, \{w\}, \{x, y\}, \{x, w\}, \{x, y, w\}, \{w, y\}\}$, $I_y = \{\emptyset, \{x\}, \{z\}, \{w\}, \{x, z\}, \{x, w\}, \{z, w\}, \{x, z, w\}\}$. Set $A = \{y, z\}$, $B = \{a, c\}$ then $A^{*y} = \{x, y, w\}$ and $A^{*z} = B^{*z} = \{z, w\}$, so that $y \notin A^{*y}$ and $x \notin A^{*z}$. But $A^{*y} \cap B^{*z} \neq \emptyset$. Hence (X, T) is an $I^{**}-T_1$ -space, but not I^*-T_2 -space.

Theorem 3.21

A $s-T_1$ -space (X, T) is a T_1 -space if and only if it is R_{i-1} -space and I^*-T_j -space, $i = 1, 2, 3, 4, j = 0, 1, 2$

Proof

By theorem 3.1, remark 3.10 and remark 1.2.

Theorem 3.22

A $s-T_1$ -space (X, T) is a T_1 -space if and only if it is R_{i-1} -space and $I^{**}-T_j$ -space, $i = 1, 2, 3, 4, j = 0, 1, 2$

Proof

By theorem 3.1, theorem 3.17, remark 3.10 and remark 1.2.

4. I^* -map and I^{} -map**

Definition 4.1

A mapping $f: (X, T) \rightarrow (Y, \sigma)$ is called I^* -map. If and only if, for every subset A of X , $x \in X$, $f(A^{*x}) = (f(A))^{*f(x)}$.

Definition 4.2

A mapping $f: (X, T) \rightarrow (Y, \sigma)$ is called I^{**} -map. If and only if, for every subset A of Y , $y \in Y$, $f^{-1}(A^{*y}) = (f^{-1}(A))^{*f^{-1}(y)}$.

Theorem 4.3

If $f: (X, T) \rightarrow (Y, \sigma)$ is one-one I^* -map of an I^*-T_0 -space X onto a space Y , then Y is an I^*-T_0 -space.

Proof

Let (X, T) be I^*-T_0 -space and $f: X \rightarrow Y$ be onto, one-one and I^* -map. We want to prove that Y is I^*-T_0 -space. Let y^1 and y^2 be two distinct points of Y . Since f is one-one and onto, there exists distinct points x_1, x_2 of X such that $f(x_1) = y^1$ and $f(x_2) = y^2$. Since (X, T) is I^*-T_0 -space, there exist non-empty subsets A, B of X such that $x_2 \notin A^{*x_1}$ or $x_1 \notin B^{*x_2}$, so that $f(x_2) \notin (f(A))^{*f(x_1)} = (f(A))^{*f(x_1)}$ or $f(x_1) \notin (f(B))^{*f(x_2)} = (f(B))^{*f(x_2)}$. Thus $y^2 \notin (f(A))^{*f(x_1)=y^1}$ and $y^1 \notin (f(B))^{*f(x_2)=y^2}$. Therefore we get that Y is I^*-T_0 -space.

Theorem 4.4

If $f: (X, T) \rightarrow (Y, \sigma)$ is one-one I^* -map of an I^*-T_1 -space X onto a space Y , then Y is an I^*-T_1 -space.

Proof

By the same way of proof of above theorem.

Theorem 4.5

If $f: (X, T) \rightarrow (Y, \sigma)$ is I^{**} -map injection of a space X into I^*-T_0 -space Y , then X is an I^*-T_0 -space.

Proof

Let (Y, σ) be I^*-T_0 -space and $f: X \rightarrow Y$ be I^{**} -map injection. We want to prove that X is I^*-T_0 -space. Let x^1 and x^2 be two distinct points of X . Since f is injection, then $f(x^1) \neq f(x^2)$. Since (Y, σ) is I^*-T_0 -space, there exist non-empty subsets C, D of Y such that $f(x^1) \notin C^{*f(x^2)}$ or $f(x^2) \notin D^{*f(x^1)}$, so that $f^{-1}(f(x^1)) \notin f^{-1}(C^{*f(x^2)}) = (f^{-1}(C))^{*f^{-1}(f(x^2))}$ or $f^{-1}(f(x^2)) \notin f^{-1}(D^{*f(x^1)}) = (f^{-1}(D))^{*f^{-1}(f(x^1))}$. This implies $x^1 \notin (f^{-1}(C))^{*x^2}$ or $x^2 \notin (f^{-1}(D))^{*x^1}$. Therefore we get that X is I^*-T_0 -space.

Theorem 4.6

If $f: (X, T) \rightarrow (Y, \sigma)$ is I^{**} -map injection of a space X into I^*-T_1 -space Y , then X is an I^*-T_1 -space.

Proof

By the same way of proof of above theorem.

Theorem 4.7

If $f: (X, T) \rightarrow (Y, \sigma)$ is one-one, I^* -map of an I^*-T_2 -space X onto a space Y , then Y is an I^*-T_2 -space.

Proof

Let (X, T) be I^*-T_2 -space and $f: X \rightarrow Y$ be one-one onto I^* -map. We want to prove that $f(X) = Y$ is I^*-T_2 -space. Let y^1 and y^2 be two distinct points of Y . Since f is onto I^* -map, there exists distinct points x_1, x_2 of X such that $f(x_1) = y^1$ and $f(x_2) = y^2$. Since (X, T) is I^*-T_2 -space, there exist non-empty subsets A, B of X such that $A^{*x_1} \cap B^{*x_2} = \emptyset$, with $x_2 \notin A^{*x_1}$ and $x_1 \notin B^{*x_2}$. But f is onto I^* -map, so that $f(A^{*x_1}) \cap f(B^{*x_2}) = f(A)^{*f(x_1)} \cap f(B)^{*f(x_2)} = f(A)^{*y^1} \cap f(B)^{*y^2} = \emptyset$, with $f(x_2) \notin f(A)^{*f(x_1)}$ and $f(x_1) \notin f(B)^{*f(x_2)}$. Thus there exist non-empty subsets $f(A), f(B)$ of Y such that $f(A)^{*y^1} \cap f(B)^{*y^2} = \emptyset$, with $y^2 \notin f(A)^{*y^1}$ and $y^1 \notin f(B)^{*y^2}$. Therefore by definition we get that Y is I^*-T_2 -space.

Theorem 4.8

If $f: (X, T) \rightarrow (Y, \sigma)$ is I^{**} -map injection of a space X into I^*-T_2 -space Y , then X is an I^*-T_2 -space.

Proof

Let (Y, σ) be I^*-T_2 -space and $f: X \rightarrow Y$ be I^{**} -map continuous injection. We want to prove that X is I^*-T_2 -space. Let x^1 and x^2 be two distinct points of X . Since f is injection, then $f(x^1) \neq f(x^2)$. Since (Y, σ) is I^*-T_2 -space, there exist non-empty subsets C, D of Y such such $C^{*f(x^1)} \cap D^{*f(x^2)} = \emptyset$, with $f(x_2) \notin f(C)^{*f(x^1)}$ and $f(x_1) \notin f(D)^{*f(x^2)}$. But f is I^{**} -map injection, so that $(f^{-1}(C))^{*(x^1)} \cap (f^{-1}(D))^{*(x^2)} = f^{-1}(C^{*f(x^1)}) \cap f^{-1}(D^{*f(x^2)}) = f^{-1}(C^{*f(x^1)} \cap D^{*f(x^2)}) = f^{-1}(\emptyset) = \emptyset$. Thus there exist non-empty subsets $f^{-1}(C), f^{-1}(D)$ of X such that $(f^{-1}(C))^{*(x^2)} \cap (f^{-1}(D))^{*(x^1)} = \emptyset$, with $x^2 \notin (f^{-1}(C))^{*x^1}$ and $x^1 \notin (f^{-1}(D))^{*x^2} = \emptyset$, for each x^1 and x^2 be two distinct points of X . Therefore by definition we get that X is I^*-T_2 -space.

Corollary 4.9

If $f: (X, T) \rightarrow (Y, \sigma)$ is I^{**} -map injection of a $s-T_1$ -space X into I^*-T_2 -space Y , then X is T_0 -space.

Corollary 4.10

If $f: (X, T) \rightarrow (Y, \sigma)$ is I^{**} -map injection of a $s-T_1$ -space X into I^*-T_2 -space Y , then X is an T_1 -space.

Theorem 4.11

If $f: (X, T) \rightarrow (Y, \sigma)$ is continuous, injection function of a space X into T_2 -space Y , then X is an I^*-T_1 -space.

Proof

Let (Y, σ) be T_2 -space and $f: X \rightarrow Y$ be continuous, injection function. We want to prove that X is I^*-T_1 -space. Let x^1 and x^2 be two distinct points of X . Since f is injection, then $f(x^1) \neq f(x^2)$. Since (Y, σ) is T_2 -space, then there exist V_1 and $V_2 \in T_Y$ such that $f(x^1) \in V_1, f(x^2) \in V_2$ and $V_1 \cap V_2 = \emptyset$. This implies $x^1 \in f^{-1}(V_1)$ and $x^2 \in f^{-1}(V_2)$. So that $f^{-1}(V_1) \cap \{x^2\} = \emptyset \in I_{x^2}$ and $f^{-1}(V_2) \cap \{x^1\} = \emptyset \in I_{x^1}$. Put $A = \{x^2\}, B = \{x^1\}$. It follows that $x^1 \notin A^{*x^2}$ and $x^2 \notin A^{*x^1}$. Therefore by definition we get that X is I^*-T_1 -space.

Corollary 4.12

If $f: (X, T) \rightarrow (Y, \sigma)$ is injection function of a space X into T_2 -space Y , then X is an I^*-T_0 -space.

Proof It is clear [Since every I^*-T_1 -space is I^*-T_0 -space].

REFERENCE

[1] N. A. Shanin, On separation in topological spaces, Dokl. Akad. Nauk SSSR, 38 (1943), 110–113.
 [2] A.S. Davis, "Indexed Systems of Neighborhoods for General Topological Spaces," The Amer. Math. Monthly, 68, 886 – 893 (1961).
 [3] L. A. AL-Swidi and Mohammed, B.(2012). "Separation axioms via kernel set in

- topological spaces" Archive Des sciences , Vol .65, No.7 ,pp 41 -48.
[4] Bishwambhar Roy and M.N.Mukherjee. A unified theory for R_0 , R_1 and certain other separation properties and their variant forms ,Bol .Soc.paran. Mat.(3s)v.28.2(2010):15-24

This academic article was published by The International Institute for Science, Technology and Education (IISTE). The IISTE is a pioneer in the Open Access Publishing service based in the U.S. and Europe. The aim of the institute is Accelerating Global Knowledge Sharing.

More information about the publisher can be found in the IISTE's homepage:

<http://www.iiste.org>

CALL FOR PAPERS

The IISTE is currently hosting more than 30 peer-reviewed academic journals and collaborating with academic institutions around the world. There's no deadline for submission. **Prospective authors of IISTE journals can find the submission instruction on the following page:** <http://www.iiste.org/Journals/>

The IISTE editorial team promises to review and publish all the qualified submissions in a **fast** manner. All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Printed version of the journals is also available upon request of readers and authors.

IISTE Knowledge Sharing Partners

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digital Library, NewJour, Google Scholar

