T_i Spaces with Respect to Weak Forms of ω −Open Sets, for $i = 0,1,2$

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Abstract. *In this paper we introduce the associative separation axioms of the weak* ω *−open sets defined in [5], and then give some new theorems about them.*

Key words. *Weak separation axioms, weak* ω −open sets, *weak* T_0*spaces, weak* T_1*spaces, weak* T_2*spaces.*

1. Introduction and Preliminaries

Through out this paper , (X,T) stands for topological space. Let (X,T) be a topological space and A a subset of X. A point x in X is called ***condensation point*** of A if for each U in T with x in U, the set $U \cap A$ is un countable [3]. In 1982 the ω −closed set was first introduced by H. Z. Hdeib in [3], and he defined it as: A is ***ω −closed*** if it contains all its condensation points and the ***ω −open*** set is the complement of the ω −closed set. Equivalently. A sub set W of a space (X,T), is ω −open if and only if for each $x \in W$, there exists $U \in T$ such that $x \in U$and $U \backslash W$ is countable. The collection of all ω −open sets of (X,T)denoted T_ω form topology on X and it is finer than T. Several characterizations of ω −closed sets were provided in [1, 3, 4, 6].

In 2009 in [5] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced and investigated new notions called $\alpha - \omega$ −open, pre $-\omega$ −open, $b - \omega$ −open and $\beta - \omega$ −open sets which are weaker than ω −open set. Let us introduce these notions in the following definition:

Definition 1.1. [5] A subset A of a space X is called
1. ***$\alpha - \omega$ −open*** if $A \subseteq int_\omega\left(cl\left(int_\omega(A)\right)\right)$.
2. ***pre −ω −open*** if $A \subseteq int_\omega\left(cl(A)\right)$.
3. ***b −ω −open*** if $A \subseteq int_\omega\left(cl(A)\right) \cup cl\left(int_\omega(A)\right)$.
4. ***$\beta - \omega$ −open*** if $A \subseteq cl\left(int_\omega\left(cl(A)\right)\right)$.

For a subset A of X, the ***ω − interior*** of the set A defined as the union of all ω − open sets contained in A, and denoted by $int_\omega(A)$. The closure of A will be denoted by $cl(A)$.

Remark 1.2. [5] Any ω −open (resp. $\alpha - \omega$ −open, $pre - \omega$ −open, $b - \omega$ −open and $\beta - \omega$ −open) sets need not be open (resp. α −open, pre −open, b −open and β −open) as can be seen in the following example:

In [5] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced relationships among the weak open sets above by the lemma below:

Lemma 1.3. [5] In any topological space:
1. Any open set is ω −open.
2. Any ω −open set is $\alpha - \omega$ −open.
3. Any $\alpha - \omega$ −open set is pre - ω −open.
4. Any $pre - \omega$ −open set is b - ω −open.
5. Any $b - \omega$ −open set is $\beta - \omega$ −open.
The converse is not true [5].

Remark 1.4. [5] The intersection of two $pre - \omega$ −open, (resp. $b - \omega$ −open and $\beta - \omega$ −open) sets need not be $pre - \omega$ −open, (resp. $b - \omega$ −open and $\beta - \omega$ −open) sets. As can be seen in the following example:

Example 1.5. [5] Let $X = \mathbb{R}$ with the usual topology T. Let $A = Q$ and $B = (R\backslash Q) \cup \{1\}$, then A and B are $pre - \omega$ −open, but $A \cap B = \{1\}$, is not $\beta - \omega$ −open since $cl(int_\omega\,(cl(\{1\}))) = cl(int_\omega\,(\{1\})) = cl(\{\emptyset\}) = \emptyset$.

Lemma 1.6. [5] The intersection of an $\alpha-\omega-$open (resp. $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) subset of any topological space and an open subset is $\alpha-\omega-$open (resp. $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) set.

Theorem 1.7. The union of an $\alpha-\omega-$closed (resp. $pre-\omega-$ closed, $b-\omega-$ closed and $\beta-\omega-$ closed) subset of any topological space and a closed subset is $\alpha-\omega-$ closed (resp. $pre-\omega-$ closed, $b-\omega-$ closed and $\beta-\omega-$ closed) set.

Proof:

Let A be an $\alpha-\omega-$closed subset of a topological space X and B is a closed subset of X. Then A^c is $\alpha-\omega-$open subset of X and B^c is an open subset of X. Then by Lemma 1.6 we have $A^c \cap B^c$ is an is $\alpha-\omega-$open subset of X and $(A^c \cap B^c)^c$ is an $\alpha-\omega-$closed subset of X Therefore $(A^c \cap B^c)^c = A \cup B$ is $\alpha-\omega-$closed subset of X X

Theorem 1.8. [5] If $\{A_\alpha : \alpha \in \Delta\}$ is a collection of $\alpha-\omega-$open (resp. $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) subsets of the topological space (X,T), then $\cup_{\alpha\in\Delta} A_\alpha$ is $\alpha-\omega-$open (resp. $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) set.

Theorem 1.9. If $\{A_\alpha : \alpha \in \Delta\}$ is a collection of $\alpha-\omega-$closed (resp. $pre-\omega-$closed, $b-\omega-$closed and $\beta-\omega-$closed) subsets of the topological space (X,T), then $\cap_{\alpha\in\Delta} A_\alpha$ is $\alpha-\omega-$closed (resp. $pre-\omega-$closed, $b-\omega-$closed and $\beta-\omega-$closed) set.

Proof:

Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of $\alpha-\omega-$closed subsets of X, then ${A_\alpha}^c$ (the complement set of A_α) is $\alpha-\omega-$open set for each $\alpha \in \Delta$. Then by Theorem 1.8 we have $\cup_{\alpha\in\Delta} {A_\alpha}^c$ is $\alpha-\omega-$open set . Therefore $(\cup_{\alpha\in\Delta} {A_\alpha}^c)^c = \cap_{\alpha\in\Delta} A_\alpha$, is $\alpha-\omega-$closed subsets of X. A similar proof for the other cases X

Definition 1.10. [5] A space (X,T) is called a ***door space*** if every subset of X is either open or closed.

Lemma 1.11. [5] If (X,T) is a door space, then every $pre-\omega-$open set is $\omega-$open.

Theorem 1.12. Let A be a $\beta-\omega-$open set in the topological space (X,T), then A is $b-\omega-$open , whenever X is door space.

Proof:

Let A be a $\beta-\omega-$open subset of X. If A is open then by Lemma 1.4 it is $b-\omega-$open. Then if A is closed we get $A \subseteq cl(int_\omega(A)) \subseteq int_\omega(cl(A)) \cup cl(int_\omega(A))$. Thus A is $b-\omega-$open set in X X

Definition 1.13. [5] A subset A of a space X is called

1. An ***ω − t −set***, if $int(A) = int_\omega(cl(A))$.
2. An ***ω − B −set*** if $A = U \cap V$, where U is an open set and V is an $\omega-t-$set.
3. An ***ω − t_α −set***, if $int(A) = int_\omega(cl(int_\omega(A)))$.
4. An ***ω − B_α −set*** if $A = U \cap V$, where U is an open set and V is an $\omega-t_\alpha-$set.
5. An ***ω −set*** if $A = U \cap V$, where U is an open set and $int(V) = int_\omega(V)$.

Definition 1.14. Let (X,T) be topological space. It said to be satisfy

1. The ***ω −condition*** if every ω −open set is $\omega-$set.
2. The ***ω − B_α −condition*** if every $\alpha-\omega-$open set is $\omega-B_\alpha-$set.
3. The ***ω − B −condition*** if every $pre-\omega-$open is $\omega-B-$set.

Now let us introduce the following lemma from [5].

Lemma 1.15. [5] For any subset A of a space X, We have

1. A is open if and only if A is $\omega-$open and $\omega-$set.
2. A is open If and only if A is $\alpha-\omega-$open and $\omega-B_\alpha-$set.
3. A is open if and only if A is $pre-\omega-$open and $\omega-B-$set.

Lemma 1.16. [5] Let (X,T) be a topological space, and let $A \subseteq X$. If A is $b-\omega-$open set such that $int_\omega(A) = \emptyset$, then A is $pre-\omega-$open.

Definition 1.17. Let X be a topological space. We say that a subset A of X is ***ω −compact*** [2] (resp. ***α − ω −compact*, *pre −ω −compact*, *b − ω −compact*** and ***β − ω −compact***) if for each cover of ω −open (resp. α − ω −open , pre −ω −open, b −ω −open and β − ω −open) sets from X contains a finite subcover for A.

Definition 1.18. A function $f:(X,\sigma) \to (Y,\tau)$ is called ***ω -continuous*** (resp**.** ***α − ω -continuous* , *pre −ω −continuous*, *b −ω −continuous and β − ω −continuous***), if for each $x \in X$, and each $\omega -$open (resp. $\alpha - \omega -$open, $pre -\omega -$open, $b -\omega -$open and $\beta - \omega -$open) set V containing $f(x)$, there exists an $\omega -$open (resp. $\alpha - \omega -$open, $pre -\omega -$open, $b -\omega -$open and $\beta - \omega$-open,) set U containing x, such that $f(U) \subset V$.

2. weak T_0 spaces

In this article, let us introduce the weak T_0 spaces with some relations, propositions and theorems.

Definition 2.1. Let X be a topological space. If for each $x \neq y \in X$, either there exists a set U, such that $x \in U$, $y \notin U$,or there exists a set U such that $x \notin U, y \in U$. Then X called
1. $\boldsymbol{\omega - T_0}$ space, whenever U is $\omega -$open set in X.
2. $\boldsymbol{\alpha - \omega - T_0}$ space, whenever U is $\alpha - \omega -$open set in X.
3. $\boldsymbol{pre-\omega - T_0}$ space, whenever U is pre $-\omega -$open set in X.
4. $\boldsymbol{b - \omega - T_0}$ space, whenever U is $b - \omega -$open set in X.
5. $\boldsymbol{\beta - \omega - T_0}$ space, whenever U is $\beta - \omega -$open set in X.

Using Lemma 1.3 we can write the following proposition:

Proposition 2.2. Let (X,T) be a topological space.
1. If (X,T) is T_0, then it is $\omega - T_0$.
2. If (X,T) is $\omega - T_0$, then it is $\alpha - \omega - T_0$.
3. If (X,T) is $\alpha - \omega - T_0$, then it is $pre-\omega - T_0$
4. If (X,T) is $pre-\omega - T_0$, then it is $b - \omega - T_0$.
5. If (X,T) is $b - \omega - T_0$, then it is $\beta - \omega - T_0$.

Remark 2.3. The converse of the above theorem is not true as we see in the following example:

Example 2.4. Let $X = \{1,2,3\}$ with the topology $T = \{\emptyset, X, \{1\}\}$. It is clear that (X,T) is $\omega - T_0$ space but not T_0 space.

Theorem 2.5. Let (X,T) be a door space. Then we have:
1. Every $pre -\omega - T_0$ space is $\omega - T_0$.
2. Every $\beta -\omega - T_0$ space is $b - \omega - T_0$.
Proof:

Directly from Definition 2.1, Lemma 1.11 and Theorem 1.12 X

Theorem 2.6. Let (X,T), be a topological space.
1. If (X,T) is $\omega - T_0$ topological space satisfies the $\omega -$condition, then it is T_0 topological space.
2. If (X,T) is $\alpha - \omega - T_0$topological space satisfies the $\omega - B_\alpha$ −condition, then it is T_0 topological space.
3. If (X,T) is $pre - \omega - T_0$ topological space satisfies the $\omega - B$ −condition, then it is T_0 topological space.
Proof:

Directly from Definition 2.1, Definition 1.14 and Lemma 1.15 X

Proposition 2.7. If (X,T) is $b - \omega - T_0$ topological space with the property that any $b - \omega -$open subset has empty $\omega -$interior. Then it is $pre - \omega - T_0$.
Proof:

Directly from Definition 2.1 and Lemma 1.16 X

One can summarize the theorems above by Figure 1.

3. weak T_1 space

Weak types of $\omega - T_1$ spaces is the subject of this article. Also we introduce some related results.

Definition 3.1. Let X be a topological space. For each $x \neq y \in X$, there exists a set U, such that $x \in U, y \notin U$, and there exists a set V such that $y \in V, x \notin V$, then X is called

1. $\boldsymbol{\omega - T_1}$ ***space*** if U is open and V is ω −open sets in X.
2. $\boldsymbol{\alpha - \omega - T_1}$ ***space*** if U is open and V is $\alpha - \omega$ −open sets in X.
3. $\boldsymbol{\omega^\star - T_1}$ ***space*** [3] if U and V are ω −open sets in X.
4. $\boldsymbol{\alpha - \omega^\star - T_1}$ ***space*** if U is ω-open and V is $\alpha - \omega$ −open sets in X.
5. $\boldsymbol{\alpha - \omega^{\star\star} - T_1}$ ***space*** if U and V are $\alpha - \omega$ −open sets in X.
6. $\boldsymbol{pre - \omega - T_1}$ ***space*** if U is open and V is $pre - \omega$ −open sets in X.
7. $\boldsymbol{pre - \omega^\star - T_1}$ ***space*** if U is ω-open and V is $pre - \omega$ −open sets in X.
8. $\boldsymbol{\alpha - pre - \omega - T_1}$ ***space*** if U is $\alpha - \omega$ − open and V is $pre - \omega$ −open sets in X.
9. $\boldsymbol{pre - \omega^{\star\star} - T_1}$ ***space*** if U and V are $pre - \omega$ −open sets in X.
10. $\boldsymbol{b - \omega - T_1}$ ***space*** if U is open and V is $b - \omega$ −open sets in X.
11. $\boldsymbol{b - \omega^\star - T_1}$ ***space*** if U is ω −open and V is $b - \omega$ −open sets in X.
12. $\boldsymbol{\alpha - b - \omega - T_1}$ ***space*** if U is $\alpha - \omega$ −open and V is $b - \omega$ −open sets in X.
13. $\boldsymbol{pre - b - \omega - T_1}$ ***space*** if U is $pre - \omega$ −open and V is $b - \omega$ −open sets in X.
14. $\boldsymbol{b - \omega^{\star\star} - T_1}$ ***space*** if U and V are $b - \omega$ −open sets in X.
15. $\boldsymbol{\beta - \omega - T_1}$ ***space*** if U is open and V is $\beta - \omega$ −open sets in X.
16. $\boldsymbol{\beta - \omega^\star - T_1}$ ***space*** if U is ω −open and V is $\beta - \omega$ −open sets in X.
17. $\boldsymbol{\alpha - \beta - \omega - T_1}$ ***space*** if U is $\alpha - \omega$ −open and V is $\beta - \omega$ −open sets in X.
18. $\boldsymbol{pre - \beta - \omega - T_1}$ ***space*** if U is $pre - \omega$ −open and V is $\beta - \omega$ −open sets in X.
19. $\boldsymbol{\beta - \omega^{\star\star} - T_1}$ ***space*** if U and V are $\beta - \omega$ −open sets in X.
20. $\boldsymbol{b - \beta - \omega - T_1}$ ***space*** if U is $b - \omega$ −open and V is $\beta - \omega$ −open sets in X

Theorem 3.2. Let X be a topological space,

1. X is $\omega - T_1$ space if and only if for each $x \neq y \in X, \{x\}$ is closed and $\{y\}$ is ω −closed set in X.
2. X is $\alpha - \omega - T_1$ space if and only if for each $x \neq y \in X, \{x\}$ is closed and $\{y\}$ is $\alpha - \omega$ −closed set in X.
3. X is $\omega^\star - T_1$ space if and only if for each $x \in X, \{x\}$ is ω −closed set in X.
4. X is $\alpha - \omega^\star - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is ω −closed and $\{y\}$ is $\alpha - \omega$ −closed set in X.
5. X is $\alpha - \omega^{\star\star} - T_1$ space if and only if for each $x \in X, \{x\}$ is $\alpha - \omega$ −closed set in X.
6. X is $pre - \omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is closed and $\{y\}$ is $pre - \omega$ −closed set in X.
7. X is $pre - \omega^\star - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is ω −closed and $\{y\}$ is $pre - \omega$ −closed set in X.
8. X is $\alpha - pre - \omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is $\alpha - \omega$ −closed and $\{y\}$ is $pre - \omega$ −closed set in X.
9. X is $pre - \omega^{\star\star} - T_1$ space if and only if for each $x \in X, \{x\}$ is $pre - \omega$ −closed set in X.
10. X is $b - \omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ closed and $\{y\}$ is $b - \omega$ −closed set in X.
11. X is $b - \omega^\star - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is ω −closed and $\{y\}$ is $b - \omega$ −closed set in X.
12. X is $\alpha - b - \omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is $\alpha - \omega$ −closed and $\{y\}$ is $b - \omega$ −closed set in X.
13. X is $pre - b - \omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is pre −closed and $\{y\}$ is $b - \omega$ −closed set in X.
14. X is $b - \omega^{\star\star} - T_1$ space if and only if for each $x \in X, \{x\}$ is $b - \omega$ −closed set in X.
15. X is $\beta - \omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$is closed and $\{y\}$ is $\beta - \omega$ −closed set in X.
16. X is $\beta - \omega^\star - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is ω −closed and $\{y\}$ is $\beta - \omega$ −closed set in X.
17. X is $\alpha - \beta - \omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is $\alpha - \omega$ −closed and $\{y\}$ is $\beta - \omega$ −closed set in X.
18. X is $pre - \beta - \omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is $pre - \omega$ −closed and $\{y\}$ is $\beta - \omega$ −closed set in X.
19. X is $\beta - \omega^{\star\star} - T_1$ space if and only if for each $x \in X, \{x\}$ is $\beta - \omega$ −closed set in X.
20. X is $b - \beta - \omega - T_1$ space if and only if for each $x \neq y \in X$, $\{x\}$ is $b - \omega$ −closed and $\{y\}$ is $\beta - \omega$ −closed set in X.

Proof of (4):

Let $x \in X$. If $y \in X$, such that $y \neq x$, then there exist an ω −open set U_y containing y but not x, and $\alpha - \omega$ −open set U_x containing x but not y . Hence $y \in U_y \subset \{x\}^c$. Therefore $\{x\}^c = \cup_{y \in \{x\}^c} U_y$, which is ω −open set and $\{x\}$ is ω −closed set in X. Also $\{y\}$ is $\alpha - \omega$ − closed set in X . In fact $x \in U_x \subset \{y\}^c$, which implies $\{y\}^c = \cup_{x \in \{y\}^c} U_x$. Then because U_x is $\alpha - \omega$ −open set, for each $x \in \{y\}^c$, so $\{y\}^c$ is $\alpha - \omega$ −open set, and $\{y\}$ is $\alpha - \omega$ −closed set

Now for the converse, let $x \neq y \in X$, $U_x = X \backslash \{y\}$ is $\alpha - \omega$ −open set containing x but not y, and $U_y = X \backslash \{x\}$ is ω −open set, containing y but not x. Thus X is $\alpha - \omega^\star - T_1$ space.

A similar proof for 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, and 20 X

Theorem 3.3. For any topological space.
1. Any T_1 is $\omega - T_1$ space.
2. Any $\omega - T_1$ is $\alpha - \omega - T_1$ space.
3. Any $\omega - T_1$ is $\omega^\star - T_1$ space.
4. Any $\omega^\star - T_1$ is $\alpha - \omega^\star - T_1$ space.
5. Any $\alpha - \omega^\star - T_1$ is $pre - \omega^\star - T_1$ space.
6. Any $\alpha - \omega - T_1$ is $\alpha - \omega^\star - T_1$ space.
7. Any $\alpha - \omega - T_1$ is $pre - \omega - T_1$ space.
8. Any $pre - \omega - T_1$ is $\beta - \omega - T_1$ space.
9. Any $\alpha - \omega^\star - T_1$ is $\alpha - \omega^{\star\star} - T_1$ space.
10. Any $\alpha - \omega^{\star\star} - T_1$ is $\alpha - pre - \omega - T_1$ space.
11. Any $\alpha - pre - \omega - T_1$ is $pre - \omega^{\star\star} - T_1$ space.
12. Any $pre - \omega^{\star\star} - T_1$ is $pre - b - \omega - T_1$ space.
13. Any $pre - b - \omega - T_1$ is $b - \omega^{\star\star} - T_1$ space.
14. Any $b - \omega^{\star\star} - T_1$ is $b - \beta - \omega - T_1$ space.
15. Any $b - \beta - \omega - T_1$ is $\beta - \omega^{\star\star} - T_1$ space.
16. Any $pre - \omega^\star - T_1$ is $\alpha - pre - \omega - T_1$ space.
17. Any $pre - \omega^\star - T_1$ is $b - \omega^\star - T_1$ space.
18. Any $\alpha - pre - \omega - T_1$ is $\alpha - b - \omega - T_1$ space.
19. Any $b - \omega^\star - T_1$ is $\alpha - b - \omega - T_1$ space.
20. Any $pre - \omega - T_1$ is $b - \omega - T_1$ space.
21. Any $\beta - \omega - T_1$ is $\beta - \omega^\star - T_1$ space.
22. Any $\beta - \omega^\star - T_1$ is $\alpha - \beta - \omega - T_1$ space.
23. Any $\alpha - \beta - \omega - T_1$ is $pre - \beta - \omega - T_1$ space.
24. Any $pre - \omega - T_1$ is $pre - \omega^\star - T_1$ space.
25. Any $b - \omega - T_1$ is $\beta - \omega - T_1$ space.
26. Any $b - \omega^\star - T_1$ is $\beta - \omega^\star - T_1$ space.
27. Any $\alpha - b - \omega - T_1$ is $\alpha - \beta - \omega - T_1$ space.
28. Any $\alpha - b - \omega - T_1$ is $pre - b - \omega - T_1$ space.
29. Any $pre - b - \omega - T_1$ is $pre - \beta - \omega - T_1$ space.
30. Any $pre - \beta - \omega - T_1$ is $b - \beta - \omega - T_1$ space.
31. Any $b - \omega - T_1$ is $b - \omega^\star - T_1$ space.
Proof:
Easy. By using Lemma 1.3 X

Remark 3.4. The converse of the theorem above is not satisfied in general. As we see in the following examples.

Example 3.5. Let $X = \{1,2,3\}$ with the topology $T = \{\emptyset, X, \{1\}, \{3\}, \{1,3\}\}$. (X, T) is $\omega - T_1$ space, but not T_1.
To have equivalence between the weak T_1s spaces, we shall introduce the following theorems:

Theorem 3.6. Let (X, T) be a door space. Then we have:
1. Every $pre - \omega - T_1$ space is $\omega - T_1$.
2. Every $pre - \omega^\star - T_1$ space is $\omega^\star - T_1$.
3. Every $\alpha - pre - \omega - T_1$ space is $\alpha - \omega^\star - T_1$,
4. Every $pre - b - \omega - T_1$ space is b $- \omega^\star - T_1$.
5. Every $pre - \beta - \omega - T_1$ space is $\beta - \omega^\star - T_1$.
6. Every $pre - \omega^{\star\star} - T_1$ space is $\omega^\star - T_1$.
7. Every $b - \beta - \omega - T_1$ space is $b - \omega^{\star\star} - T_1$.
8. Every $\beta - \omega^{\star\star} - T_1$ space is $b - \omega^{\star\star} - T_1$.
9. Every $pre - \beta - \omega - T_1$ space is $pre - b - \omega - T_1$.
10. Every $\alpha - \beta - \omega - T_1$ space is $\alpha - b - \omega - T_1$.
11. Every $\beta - \omega^\star - T_1$ space is $b - \omega^\star - T_1$.
12. Every $\beta - \omega - T_1$ space is $b - \omega - T_1$.
Proof:
Directly from Lemma 1.11 and Theorem 1.12 X

Using Definition 1.14 and Lemma 1.15 we can prove the following important theorem

Theorem 3.7. For any topological space (X, T).
A. Let (X, T), satisfies the ω −condition.
1. If (X, T) is $\omega - T_1$, then it is T_1.
2. If (X, T) is $\omega^\star - T_1$, then it is T_1.
3. If (X, T) is $\alpha - \omega^\star - T_1$, then it is $\alpha - \omega - T_1$.
4. If (X, T) is $pre - \omega^\star - T_1$, then it is $pre - \omega - T_1$.
5. If (X, T) is $b - \omega^\star - T_1$, then it is $b - \omega - T_1$.
6. If (X, T) is $\beta - \omega^\star - T_1$, then it is $\beta - \omega - T_1$.
B. Let (X, T), satisfies the $\omega - B_\alpha$ −condition.
1. If (X, T) is is $\alpha - \omega - T_1$, then it is T_1.
2. If (X, T) is $\alpha - \omega^\star - T_1$, then it is $\omega - T_1$.
3. If (X, T) is $\alpha - \omega^{\star\star} - T_1$, then it is T_1.
4. If (X, T) is $\alpha - pre - \omega - T_1$, then it is $pre - \omega - T_1$.
5. If (X, T) is $\alpha - b - \omega - T_1$, then it is $b - \omega - T_1$.
6. If (X, T) is $\alpha - \beta - \omega - T_1$, then it is $\beta - \omega - T_1$.
C. Let (X, T), satisfies the $\omega - B$ −condition.
1. If (X, T) is $pre - \omega - T_1$, then it is T_1.
2. If (X, T) is $pre - \omega^\star - T_1$, then it is $\omega - T_1$.
3. If (X, T) is $pre - \omega^{\star\star} - T_1$, then it is T_1.
4. If (X, T) is $pre - b - \omega - T_1$, then it is $b - \omega - T_1$.
5. If (X, T) is $pre - \beta - \omega - T_1$, then it is $\beta - \omega - T_1$.
6. If (X, T) is $\alpha - pre - \omega - T_1$, then it is $\alpha - \omega - T_1$.

Proposition 3.8. Let (X, T) be a topological space with the property that any $b - \omega -$open subset has empty ω −interior.
1. If (X, T) is $b - \omega - T_1$, then it is $pre - \omega - T_1$.
2. If (X, T) is $b - \omega^\star - T_1$, then it is $pre - \omega^\star - T_1$.
3. If (X, T) is $\alpha - b - \omega - T_1$, then it is $\alpha - pre - \omega - T_1$.
4. If (X, T) is $pre - b - \omega - T_1$, then it is $pre - \omega^{\star\star} - T_1$.
5. If (X, T) is $b - \omega^{\star\star} - T_1$, then it is $pre - \omega^{\star\star} - T_1$.
6. If (X, T) is $b - \beta - \omega - T_1$, then it is $pre - \beta - \omega - T_1$.
Proof:

Directly from Lemma 1.16 X

One can summarize the relationships among weak T_1s spaces by Figure 2.

Definition 3.9. A topological space (X, T) is ***ω −symmetric*** if for x and y in the space X, $x \in cl_\omega(\{y\})$ implies $y \in cl_\omega(\{x\})$.

Proposition 3.10. Let X be a door, ω −symetric topological space . Then for each $x \in X$, the set $\{x\}$ is ω −closed.
Proof:

Let $x \neq y \in X$, since X is a door space so $\{y\}$ is open or closed set in X. When $\{y\}$ is open, so it is ω −open, let $V_y = \{y\}$. Whenever $\{y\}$ is ω −closed , $x \notin \{y\} = cl_\omega(\{y\})$. Since X is ω −symetric we get $y \notin cl_\omega(\{x\})$. Put $V_y = X \backslash cl_\omega(\{x\})$, then $x \notin V_y$ and $y \in V_y$, and V_y is ω −open set in X. Hence we get for each $y \in X \backslash \{x\}$ there is an ω −open set V_y such that $x \notin V_y$ and $y \in V_y$. Therefore $X \backslash \{x\} = \bigcup_{y \in X \backslash \{x\}} V_y$ is ω −open, and $\{x\}$ is ω −closed X

Proposition 3.11. Let (X, T) be an $\omega - T_1$ (resp. $\omega^\star - T_1$, $\alpha - \omega - T_1$, $\alpha - \omega^\star - T_1$, $b - \omega - T_1$, $b - \omega^\star - T_1$, $pre - \omega - T_1$, $pre - \omega^\star - T_1$, $\beta - \omega - T_1$, $\beta - \omega^\star - T_1$) topological space, then it is ω −symetric space.
Proof:

Assume $y \notin cl_\omega(\{x\})$, so $x \neq y$, then since X is $\omega - T_1$ there is an open set U containing x but not y, so $x \notin cl_\omega(\{y\})$. This completes the proof X

Theorem 3.12. The topological door space is $\omega -$ symmetric if and only if it is $\omega^\star - T_1$.
Proof:

Let (X, T) be a door $\omega-$ symmetric space. Then using Proposition 3.10 for each $x \in X$, $\{x\}$ is $\omega-$closed set in X. Then by (3) of Theorem 3.2, we get that (X, T) is $\omega^{\star}-T_1$. On the other hand, assume (X, T) is $\omega^{\star}-T_1$, then directly by Proposition 3.11 (X, T) is $\omega-$ symmetric space

X

4. Weak $\omega - T_2$ spaces

In this article we will define weak types of $\omega-\mathrm{T}_2$ spaces and introduce some results about it.

Definition 4.1. Let X be a topological space. And for each $x \neq y \in X$, there exist two disjoint sets U and V with $x \in U$ and $y \in V$, then X is called:

1. $\boldsymbol{\omega-T_2}$ ***space*** if U is open and V is $\omega-$open sets in X.
2. $\boldsymbol{\alpha-\omega-T_2}$ ***space*** if U is open and V is $\alpha-\omega-$open sets in X.
3. $\boldsymbol{\omega^{\star}-T_2}$ ***space*** if U and V are $\omega-$open sets in X.
4. $\boldsymbol{\alpha-\omega^{\star}-T_2}$ ***space*** if U is $\omega-$open and V is $\alpha-\omega-$open sets in X.
5. $\boldsymbol{\alpha-\omega^{\star\star}-T_2}$ ***space*** if U and V are $\alpha-\omega-$open sets in X.
6. $\boldsymbol{pre-\omega-T_2}$ ***space*** if U is open and V is $pre-\omega-$open sets in X.
7. $\boldsymbol{pre-\omega^{\star}-T_2}$ ***space*** if U is $\omega-$open and V is $pre-\omega-$open sets in X.
8. $\boldsymbol{\alpha-pre-\omega-T_2}$ ***space*** if U is $\alpha-$open and V is $pre-\omega-$open sets in X.
9. $\boldsymbol{pre-\omega^{\star\star}-T_2}$ ***space*** if U and V are $pre-\omega-$open sets in X.
10. $\boldsymbol{b-\omega-T_2}$ ***space*** if U is open and V is $b-\omega-$open sets in X.
11. $\boldsymbol{b-\omega^{\star}-T_2}$ ***space*** if U is $\omega-$open and V is $b-\omega-$open sets in X.
12. $\boldsymbol{\alpha-b-\omega-T_2}$ ***space*** if U is $\alpha-\omega-$open and V is $b-\omega-$open sets in X.
13. $\boldsymbol{pre-b-\omega-T_2}$ ***space*** if U is $pre-\omega-$open and V is $b-\omega-$open sets in X.
14. $\boldsymbol{b-\omega^{\star\star}-T_2}$ ***space*** if U and V are $b-\omega-$open sets in X.
15. $\boldsymbol{\beta-\omega-T_2}$ ***space*** if U is open and V is $\beta-\omega-$open sets in X.
16. $\boldsymbol{\beta-\omega^{\star}-T_2}$ ***space*** if U is $\omega-$open and V is $\beta-\omega-$open sets in X.
17. $\boldsymbol{\alpha-\beta-\omega-T_2}$ ***space*** if U is $\alpha-\omega-$open and V is $\beta-\omega-$open sets in X.
18. $\boldsymbol{pre-\beta-\omega-T_2}$ ***space*** if U is $pre-\omega-$open and V is $\beta-\omega-$open sets in X.
19. $\boldsymbol{\beta-\omega^{\star\star}-T_2}$ ***space*** if U and V are $\beta-\omega-$open sets in X.
20. $\boldsymbol{b-\beta-\omega-T_2}$ ***space*** if U is $b-\omega-$open and V is $\beta-\omega-$open sets in X.

Remark 4.2. We can restate Theorem 3.3 for the weak T_2s spaces.

Theorem 4.3. For any door topological space we have:

1. Every $pre-\omega-T_2$ space is $\omega-T_2$.
2. Every $pre-\omega^{\star}-T_2$ space is $\omega^{\star}-T_2$.
3. Every $\alpha-pre-\omega-T_2$ space is $\alpha-\omega^{\star}-T_2$.
4. Every $pre-b-\omega-T_2$ space is b $-\omega^{\star}-T_2$.
5. Every $pre-\beta-\omega-T_2$ space is $\beta-\omega^{\star}-T_2$.
6. Every $pre-\omega^{\star\star}-T_2$ space is $\omega^{\star}-T_2$.
7. Every $b-\beta-\omega-T_2$ space is $b-\omega^{\star\star}-T_2$.
8. Every $\beta-\omega^{\star\star}-T_2$ space is $b-\omega^{\star\star}-T_2$.
9. Every $pre-\beta-\omega-T_2$ space is $pre-b-\omega-T_2$.
10. Every $\alpha-\beta-\omega-T_2$ space is $\alpha-b-\omega-T_2$.
11. Every $\beta-\omega^{\star}-T_2$ space is $b-\omega^{\star}-T_2$.
12. Every $\beta-\omega-T_2$ space is $b-\omega-T_2$.

Proof:

Directly from Lemma 1.11 and Theorem 1.12 X

Theorem 4.4. For any topological space (X, T).

A. Let (X, T), satisfies the $\omega-$condition.
1. If (X, T) is $\omega-T_2$, then it is T_2.
2. If (X, T) is $\omega^{\star}-T_2$, then it is T_2.
3. If (X, T) is $\alpha-\omega^{\star}-T_2$, then it is $\alpha-\omega-T_2$.
4. If (X, T) is $pre-\omega^{\star}-T_2$, then it is $pre-\omega-T_2$.
5. If (X, T) is $b-\omega^{\star}-T_2$, then it is $b-\omega-T_2$.
6. If (X, T) is $\beta-\omega^{\star}-T_2$, then it is $\beta-\omega-T_2$.
B. Let (X, T), satisfies the $\omega-B_{\alpha}-$condition.
1. If (X, T) is $\alpha-\omega-T_2$, then it is T_2.
2. If (X, T) is $\alpha-\omega^{\star}-T_2$, then it is $\omega-T_2$.

3. If (X,T) is $\alpha - \omega^{\star\star} - T_2$, then it is T_2.
4. If (X,T) is $\alpha - pre - \omega - T_2$, then it is $pre - \omega - T_2$.
5. If (X,T) is $\alpha - b - \omega - T_2$, then it is $b - \omega - T_2$.
6. If (X,T) is $\alpha - \beta - \omega - T_2$, then it is $\beta - \omega - T_2$.
C. Let (X,T), satisfies the $\omega - B$ −condition.
1. If (X,T) is $pre - \omega - T_2$, then it is T_2.
2. If (X,T) is $pre - \omega^{\star} - T_2$, then it is $\omega - T_2$.
3. If (X,T) is $pre - \omega^{\star\star} - T_2$, then it is T_2.
4. If (X,T) is $pre - b - \omega - T_2$, then it is $b - \omega - T_2$.
5. If (X,T) is $pre - \beta - \omega - T_2$, then it is $\beta - \omega - T_2$.
6. If (X,T) is $\alpha - pre - \omega - T_2$, then it is $\alpha - \omega - T_2$.
Proof:

Using Definition 4.1 Definition 1.14 and Lemma 1.15 X

Proposition 4.5. Let (X,T) be a topological space with the property that any $b - \omega -$open subset has empty $\omega -$interior.
1. If (X,T) is $b - \omega - T_2$, , then it is $pre - \omega - T_2$.
2. If (X,T) is $b - \omega^{\star} - T_2$, , then it is $pre - \omega^{\star} - T_2$.
3. If (X,T) is $\alpha - b - \omega - T_2$, , then it is $\alpha - pre - \omega - T_2$.
4. If (X,T) is $pre - b - \omega - T_2$, , then it is $pre - \omega^{\star\star} - T_2$.
5. If (X,T) is $b - \omega^{\star\star} - T_2$, , then it is $pre - \omega^{\star\star} - T_2$.
6. If (X,T) is $b - \beta - \omega - T_2$, , then it is $pre - \beta - \omega - T_2$.
Proof:

Directly from Lemma 1.16 X

One can summarize the relationships among weak T_2s spaces by a figure coincide with Figure 2.

Theorem 4.6. Let (X,τ) and (Y,σ) be two topological spaces, and $f:(X,\tau) \longrightarrow (Y,\sigma)$ be injective map.
1. If f is ω −continuous, and Y is $\omega^{\star} - T_2$, then X is also $\omega^{\star} - T_2$.
2. If f is $\alpha - \omega$ −continuous, and Y is $\alpha - \omega^{\star\star} - T_2$, then X is also $\alpha - \omega^{\star\star} - T_2$.
3. If f is $pre - \omega$ −continuous, and Y is $pre - \omega^{\star\star} - T_2$, then X is also $pre - \omega^{\star\star} - T_2$.
4. If f is $b - \omega$ −continuous, and Y $b - \omega^{\star\star} - T_2$, then X is also $b - \omega^{\star\star} - T_2$.
5. If f is $\beta - \omega$ −continuous, and Y is $\beta - \omega^{\star\star} - T_2$, then X is also $\beta - \omega^{\star\star} - T_2$.
Proof of (2):

Let us prove one case and the others are similar. Let Y be $\alpha - \omega^{\star\star} - T_2$ space, to prove X is $\alpha - \omega^{\star\star} - T_2$, let $x, y \in X$ with $x \neq y$, Since f is injective, so $f(x) \neq f(y)$. And since Y is $\alpha - \omega^{\star\star} - T_2$, there exist $\alpha - \omega$ −open sets V and U such that $f(x) \in U$ and $f(y) \in V$ with $U \cap V = \emptyset$. Let $G = f^{-1}(U)$ and $H = f^{-1}(V)$. Since f is $\alpha - \omega$ −continuous, so G and H are $\alpha - \omega$ −open sets in X, with $x \in G$, and $y \in H$. Also $G \cap H = f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$. Hence X is $\alpha - \omega - T_2$ space X

Theorem 4.7. Let (X,τ) and (Y,σ) be two topological spaces, and $f:(X,\tau) \longrightarrow (Y,\sigma)$ be injective map.
1. If X satisfies ω −condition, f is ω −continuous, and Y is $\omega - T_2$, then X is also $\omega - T_2$.
2. If X satisfies $\omega - B_\alpha$ −condition, f is $\alpha - \omega$ −continuous, and Y is $\alpha - \omega - T_2$, then X is also $\alpha - \omega - T_2$.
3. If X satisfies $\omega - B_\alpha$ −condition, f is $\alpha - \omega$ −continuous, and Y is $\alpha - \omega^{\star} - T_2$, then X is also $\alpha - \omega^{\star} - T_2$.
4. If X satisfies , $\omega - B$ −condition, f is $pre - \omega$ −continuous, and Y is $pre - \omega - T_2$, then X is also $pre - \omega - T_2$.
5. If X is a door space or satisfies $\omega - B$ −condition, f is $pre - \omega$ −continuous, and Y is $pre - \omega^{\star} - T_2$, then X is also $pre - \omega^{\star} - T_2$.
6. If X is a door space or satisfies $\omega - B$ −condition, f is $pre - \omega$ −continuous, and Y is $\alpha - pre - \omega - T_2$, then X is also $\alpha - pre - \omega - T_2$.
7. If X is a door space, f is $\beta - \omega$ −continuous, and Y is $b - \beta - \omega - T_2$, then X is also $b - \beta - \omega - T_2$.
Proof of (7):

Let Y be $b - \beta - \omega - T_2$, and let $x, y \in X$ with $x \neq y$, Since f is injective, so $f(x) \neq f(y)$. And since Y is $b - \beta - \omega - T_2$there exist $b - \omega$ −open set U and $\beta - \omega$ −open set V such that $f(x) \in U$ and $f(y) \in V$ with $U \cap V = \emptyset$. Let $G = f^{-1}(U)$ and $H = f^{-1}(V)$. $\beta - \omega$ −continuity implies G and H are $\beta - \omega$ −open sets in X. When X is a door space we can consider one of the two $\beta - \omega$ −open sets as a $b - \omega$ −open with $x \in G$, and $y \in H$. Also $G \cap H = f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$. Hence X is $b - \beta - \omega - T_2$ space X

Proposition 4.8. Let (X,T) be a topological space.

1. If X is an $\omega - T_2$ space, $x \notin Y$ and Y is $\omega -$compact subset of X. Then there exist disjoint sets U is open and V is ω −open in X such that U containing x and V containing Y.
2. If X is an $\omega^\star - T_2$ space, $x \notin Y$and Y is $\omega -$compact subset of X. Then there exist disjoint sets U and V are ω −open in X such that U containing x and V containing Y.
3. If X is an $\alpha - \omega - T_2$ space, $x \notin Y$and Y is $\alpha - \omega -$compact subset of X. Then there exist disjoint sets U is open and V is $\alpha - \omega -$open in X such that U containing x and V containing Y.
4. If X is an $\alpha - \omega^\star - T_2$, space, $x \notin Y$and Y is $\alpha - \omega -$compact subset of X. Then there exist disjoint sets U is ω −open and V is $\alpha - \omega -$open in X such that U containing x and V containing Y.
5. If X is a $pre - \omega - T_2$space, $x \notin Y$ and Y is $pre - \omega -$compact subset of X. Then there exist disjoint sets U is open and V is $pre - \omega -$open in X such that U containing x and V containing Y.
6. If X is a $pre - \omega^\star - T_2$space, $x \notin Y$and Y is $pre - \omega -$compact subset of X. Then there exist disjoint sets U is $\omega -$open and V is $pre - \omega -$open in X such that U containing x and V containing Y.
7. If X is a $b-\omega - T_2$space, $x \notin Y$and Y is $b - \omega -$compact subset of X. Then there exist disjoint sets U is open and V is $b - \omega -$open in X such that U containing x and V containing Y.
8. If X is a $b -\omega^\star - T_2$space, $x \notin Y$and Y is $b -\omega -$compact subset of X. Then there exist disjoint sets U is $\omega -$open and V is $b - \omega -$open in X such that U containing x and V containing Y.
9. If X is a $\beta - \omega - T_2$space, $x \notin Y$and Y is $\beta - \omega -$compact subset of X. Then there exist disjoint sets U is open and V is $\beta - \omega -$open in X such that U containing x and V containing Y.
10. If X is a $\beta - \omega^\star - T_2$ space, $x \notin Y$and Y is $\beta - \omega -$compact subset of X. Then there exist disjoint sets U is $\omega -$open and V is $\beta - \omega -$open in X such that U containing x and V containing Y.

Proof of (3):

Let $x \notin Y$. Assume $y \in Y$, since X is an $\alpha - \omega - T_2$ space , so there exist two disjoint sets U_y open and V_y $\alpha - \omega -$open in X with $x \in U_y$, and $y \in V_y$, so $Y \subset \cup_{y \in Y} V_y$. Since Y is an $\alpha - \omega -$compact so there exist $y_1, y_2, \ldots, y_n$, such that $Y \subset \cup_{i=1}^{n} V_{y_i}$. Let $V = \cup_{i=1}^{n} V_{y_i}$, V is $\alpha - \omega -$open set containing Y, and $U = \cap_{i=1}^{n} U_{y_i}$ is open set containing x. U and V are disjoint because if there is $z \in U \cap V$, then $z \in V_{y_i}$ for some i and $z \in U_{y_i}$ for each i. This contradicts U_{y_i} and V_{y_i} are disjoint. Similarly we can prove the other cases X

As a consequence of the proof of the theorem above one can get the following corollary.

Corollary 4.9. Let (X, T) be a topological space. If X is an $\omega - T_2$ space, $x \in X$ and Y is compact set not containing x. Then there exist disjoint sets U open containing Y and V $\omega -$open containing x.

Theorem 4.10. For any topological space.
1. Every $\omega -$compact subset of $\omega - T_2$ space is closed.
2. Every $\alpha - \omega -$compact subset of $\alpha - \omega - T_2$ space is closed.
3. Every $\omega -$compact subset of $\omega^\star - T_2$ space is $\omega -$closed.
4. Every $\alpha - \omega -$compact subset of $\alpha - \omega^\star - T_2$ space is $\omega -$ closed.
5. Every $pre - \omega -$compact subset of $pre - \omega - T_2$ space is closed.
6. Every $pre-\omega$-compact subset of $pre - \omega^\star - T_2$ space is $\omega -$ closed.
7. Every $b - \omega -$compact subset of $b - \omega - T_2$ space is closed.
8. Every $b-\omega -$compact subset of $b - \omega^\star - T_2$ space is $\omega -$ closed.
9. Every $\beta - \omega -$compact subset of $\beta - \omega - T_2$ space is closed.
10. Every $\beta - \omega -$compact subset of $\beta - \omega^\star - T_2$ space is $\omega -$closed.

Proof of (2):

Let Y be an $\alpha - \omega -$compact subset of the $\alpha - \omega - T_2$ space X. To prove Y is closed, we shall prove $X \backslash Y$ is open. Let $x_0 \in X \backslash Y$, but X is $\alpha - \omega - T_2$, so for each $y \in Y$ there are disjoint sets U_y and V_y such that U_y is open set containing x_0 and V_y is $\alpha - \omega -$open set containing y. The collection $\{V_y, y \in Y\}$ is a cover for Y consists of $\alpha - \omega -$open sets in X. Since Y is $\alpha - \omega -$compact so we can find a finite subcover V for Y, $V = \cup_{i=1}^{n} V_{y_i}$. Let $U = \cap_{i=1}^{n} U_{y_i}$. Note that U is open set and V is $\alpha - \omega -$open set in X, also they are disjoint. If $z \in V$ then there is i, such that $z \in V_{y_i}$ and $z \notin U$, therefore U is an open set containing x_0 disjoint from Y. Hence $X \backslash Y$ is open and Y is closed . Similarly we can prove the other statements X

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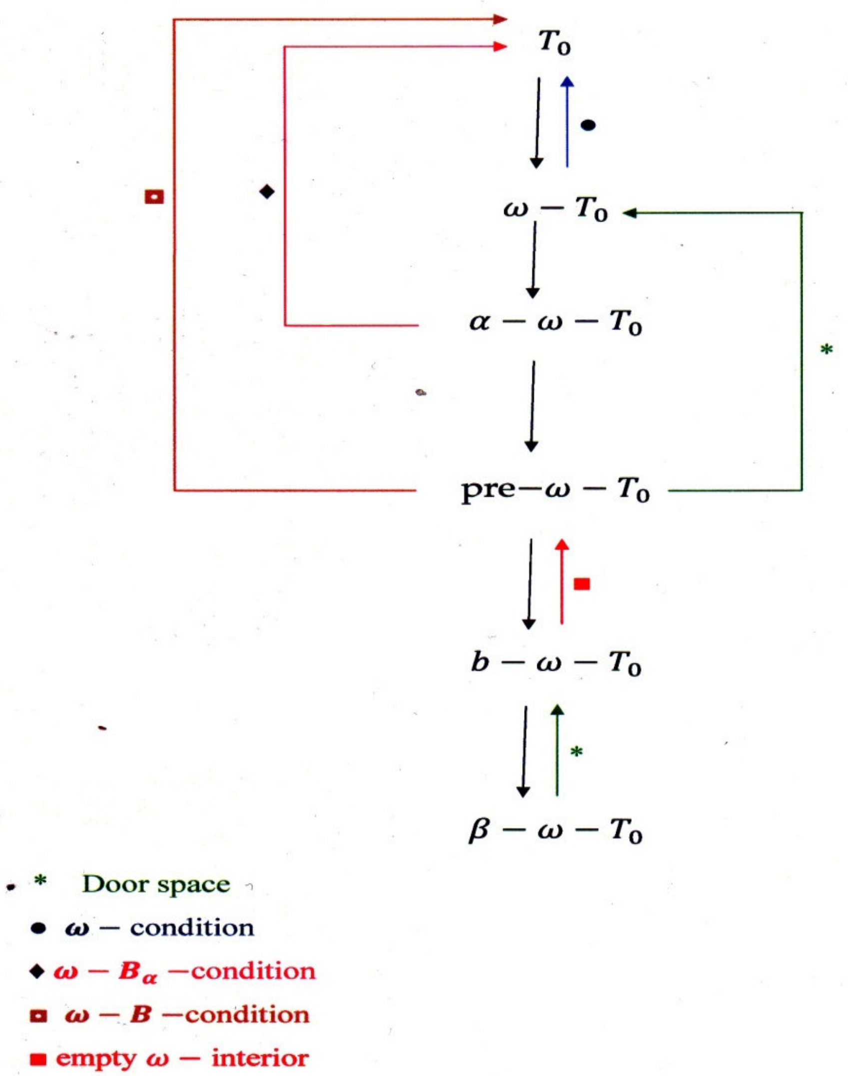

Figure 1

Weak T_0 spaces

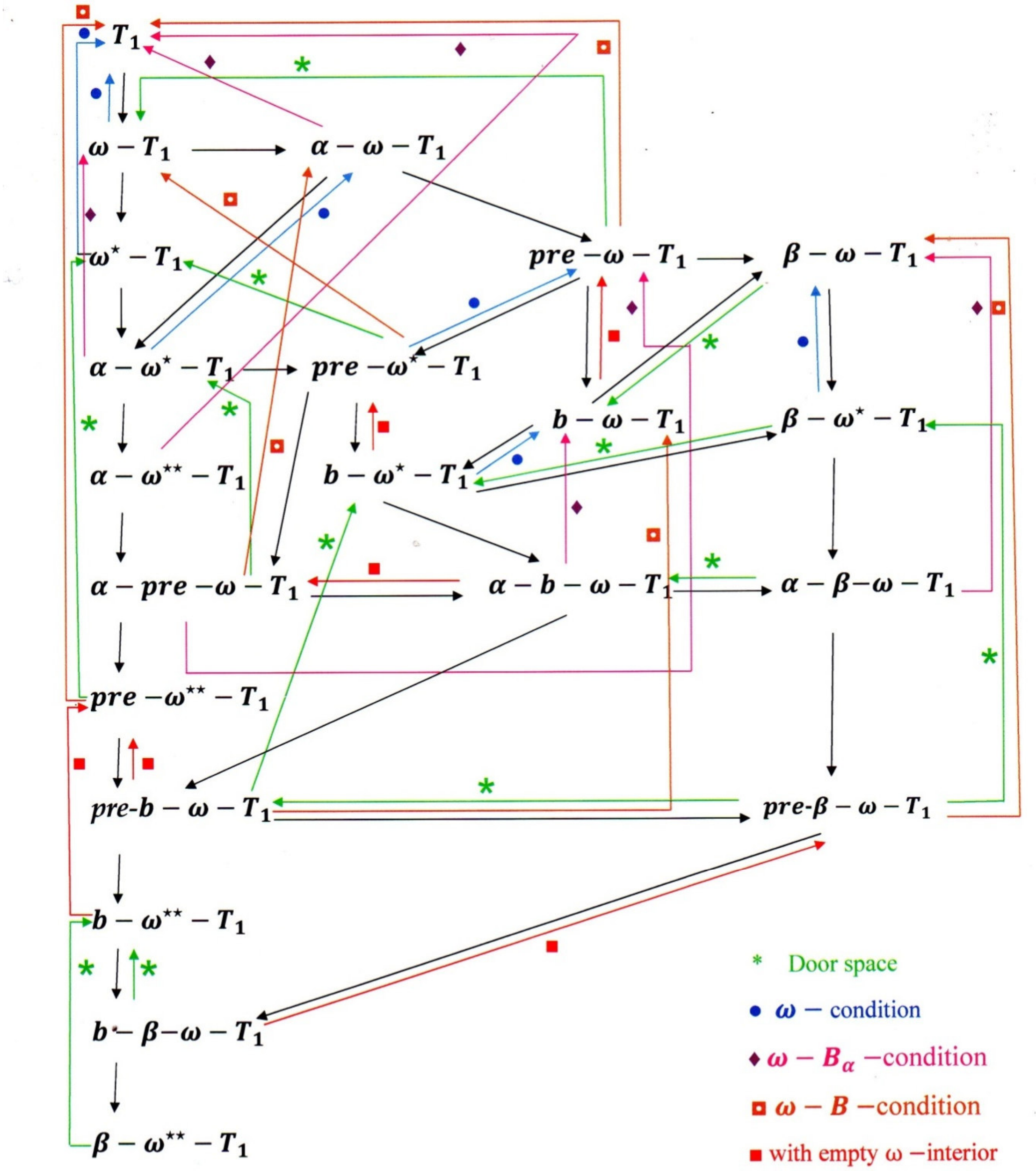

Figure 2

Weak T_1 spaces