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New Classes of Separation Axiom via Special Case of Local Function

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Abstract

In this paper, are investigated some new weak separation axioms connected with "Gem set". Besides are both introduced and studied some of their properties and the relation between them and other weak separation axioms is highlighted .
Keywords: Gem set, New separation axioms, $S_{i\_}^{*}$ $space$, $S_{i\_}^{**}$ $space$, $S_{i\_}^{***}$ $space$, $S_{i\_}^{****}$ space where i=0,1,2.

1. Introduction and Preliminaries

Shanin, 1943 [10] presented the notion of $R_\circ$ topological spaces, and Davis [11] did that of R_1 topological spaces, studying certain characteristics of the weak separation axioms as well as found some properties of $R_\circ$ topological spaces. The idea of ideals in topological spaces has been studied by Kuratowski [4] and Vaidyanathasamy [8]. An ideal I on a topological space (X,T) is a nonempty collection of subsets of X which satisfies the following properties: (1) $A \in I$ and $B \subseteq A$ implies $B \in I$. (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$ and called the (X,T,I) ideal topological space. Ideal plays an important role in topics related to topological domains as in separation axioms. Arenas, Dontchev and Puertas, 2008 [1] unified some weak separation properties via topological ideals. Noiri and Keskin 2011 [12] introduced the notions of Λ_I-sets in ideal topological spaces and benefited from them in the definition of new types of weak separation axioms. In

2013 Balaji and Rajesh [9] used *b-I*-open sets to define some weak separation axioms and study some of their basic properties. The implications of these axioms among themselves and with the known axioms are investigated. Al-Swidi and AL-Nefee, 2013 [6] introduced new separation axioms by using "Gem set " in an ideal topological space; namely I^*-T_i- space and I^{**} -T_i- space for each i=0,1,2. Attention: there are some mistakes appear in our first article [7] the mistakes appear are(Theorem 3.14, Corollary 3.16)and the true of this mistakes are ([Let *f* be open and injection map from (X,T) onto S_2_space (Y,ρ). Then (X,T) is S_2_space, if *f* is contiouns function].[omit]) respectably. This correct article is our second one. In this paper, we introduce new definition of weak separation axiom in an ideal topological space, study relationship between them and investigated the properties and theory related to them.

Theorem 1.1:[2]Let (X,T) be a topological space. Then (X,T) is $T_1 -$ space if and only if every singleton subset of X is closed.

Definition 1.2:[2]The intersection of all open subsets a topological space (X,T) containing A is called the kernel of A (briefly Ker(A)), this means that Ker (A)=∩{G∈T:A⊆G}.

Theorem1.3:[12]A topological space (X,T) is T_1-space if and only if for each x∈X then Ker{x}={x}.

Definition1.4:[10]A topological space (X,T) is said to be $R_\circ$-space if and only if for each open set G and x∈ G implies cl{x}⊆ G.

Remark 1.5:[5]Let (X,T) be a topological space and x∈ X, we denote by I_x to an ideal {G⊆ X: x∈ G^c}, where X is a nonempty set.

Definition 1.6:[6]A subset B of a topological space (X,T). Then they are defined B^{*x} with respect to space (X,T) as follows: B^{*x}={y∈ X: G∩B ∉ I_x, for every G∈ T(y)}, where T(y)={G∈ T: y ∈ G}. A set B^{*x} is called "Gem set".

Proposition 1.7:[6]Let (X,T)be topological space, and A subset of X, x∈ X.Then if x∈ A if and only if x∈ A^{*x}.

Definition1.8:[6]Let (X,T) be a topological space, A⊆ X we define $Pr^{*x}(A) = A^{*x} \cup A,$ for each x ∈ X.

Proposition 1.9: [6] Let A be such sets of (X, T), x∈X. Then pr^{*x}(A)⊆cl (A).

Definition 1.10:[6]A subset A of a topological space (X,T) is called perfected set if A^{*x}⊆ A, for each x∈ X**.**

Definition 1.11:A topological space (X,T) is said perfected space if every sub set of X is perfected set.

Proposition 1.12:[6]Let (X,T) be a topological space and A⊂ X. If A is a perfected set. Then $pr^{*x}(A) = A$, for each x∈ X.

Proposition 1.13:[6]Let (X,T) be a topological space then every closed set is a perfected set.

Proposition 1.14:[6]Let (X,T) be a topological space, then for A⊆X and x ∈ X. If A is a closed set then $pr^{*x}(A)$ =A= cl(A).

Remark 1.15: $pr^{*x}(A)$ =A=A^{*x}= cl(A) is true only A is a singleton set and closed. Let A ={x} for each x∈ X **(** by Proposition 1.14) then $pr^{*x}(A)$ =A= cl(A) and since A is closed then A is a perfected set, then $A^{*x} = \{x\}^{*x} \subseteq \{x\} = A$ and A={x}⊆ $\{x\}^{*x} = A^{*x}$ then A={x}= $\{x\}^{*x} = A^{*x}$.Hence $pr^{*x}(A)$ =A= A^{*x}= cl(A).

Definition 1.16:[6] Let (X,T) be a topological space, for each x∈ X, a nonempty subset A of X, is called a strong set if and only if A^{*x} is open set and $x \in A$ **.**

Definition 1.17: A topological space (X,T) is said strong space if every sub set of X is strong set**.**

Proposition 1.18: Let (X,T) be topological space a perfected subset A of X is open. If X is strong space.

Proof:-Let $x \in A$ since X is strong space then A is strong set it follows that A^{*x} is open by proposition 1.7 then x∈ A^{*x} and A is perfected set implies that $A^{*x} \subseteq A$. Hence A is open since neighborhood of each of its point .

Definition 1.19: A subset A of a topological space (X,T) is said Thin set if A⊆ A^{*x}.

Definition1.20: A topological space (X,T) is said Thin space if every sub set of X is Thin set**.**

Remark 1.21:[7]

1. If a function f:(X, T) → (Y, ρ) is one to one mapping, then $f^{-1}(I_y) = I_{f^{-1}(y)}$ for each y ∈Y.
2. If a function f :(X, T) → (Y, ρ) is bijection, then $f(I_x) = I_{f(x)}$ for each y ∈Y.

Corollary1.22:[7]If a function f:(X, T) → (Y, ρ) is continuous, open and bijection then $(f^{-1}(B))^{*f^{-1}(y)} = f^{-1}(B^{*y})$ for each y∈Y.

Definition1.23:[6] A mapping f:X→ Y is called I^*-map. If and only if, for every subset A of X, x∈ X $f((A^{*x}) = f(A)^{*f(x)}$.

Definition 1.24:[6] A topological space (X,T) is called **.**

1. I^* _T∘_space if and only if for each pair of distinct point x ,y of X there exists nonempty subset A ,B of X such that y∉A^{*x}or x∉B^{*y}.
2. I^*_T_1_space if and only if for each pair of distinct point x, y of X there exists nonempty subset A ,B of X such that y∉$A^{*x}$and x∉B^{*y} .

3. I^{*} _T_2_space if and only if for each pair of distinct point x, y of X there exists nonempty subset A,B of X such that $y \notin A^{*x}$ and $x \notin B^{*y}$ with $B^{*y} \cap A^{*x} = \emptyset$.
4. I^{**} _$T_\circ$_space if and only if for each pair of distinct point x ,y of X there exists nonempty subset A of X such that $y \notin A^{*x}$ or $x \notin A^{*y}$.
5. I^{**}_T_1_space if and only if for each pair of distinct point x ,y of X there exists nonempty subset A of X such that $x \notin A^{*y}$ and $y \notin A^{*x}$.
6. I^{**} _T_2_space if and only if for each pair of distinct point x ,y of X there exists nonempty subset A of X such that $x \notin A^{*y}$ and $y \notin A^{*x}$ with $A^{*y} \cap A^{*x} = \emptyset$.

Definition1.25:[6]Let (X,T) be a topological space, A⊆X and x∈X,we define A_x^{θ}={y∈X:there exist an open U containing y such that U-A∈ I_x}.

Theorem1.26:[6]Let (X,T) be a topological space, then for A⊆X and x ∈ X , $(((A^c)^{*x})^c)=A_x^{\theta}$.

2- Weak separation Axioms via benefit of Gem set.

In this section, we define some new weak separation axioms with Gem set, and study the relation between them.

Definition 3.1:A topological space (X, T) is said to be.

1. $S_\circ^{*}$_ space if and only if for each strong set G and x∈ G implies $pr^{*x}\{x\} \subseteq G^{*x}$.
2. $S_\circ^{**}$_ space if and only if for each open set G and x∈ G implies $pr^{*x}\{x\} \subseteq G$.
3. $S_\circ^{***}$_ space if and only if for each a sub set G of X and x∈ G implies $cl\{x\} \subseteq G^{*x}$.
4. $S_\circ^{****}$_space if and only if for each a perfected set G and x∈ G implies $pr^{*x}\{x\} \subseteq cl\{G\}$.

Theorem 3.2: Every $S_\circ^{***}$_space is $S_\circ^{*}$_space.

Proof : Let x∈ X and G be strong set of X since X be an $S_\circ^{***}$_ space then $cl\{x\} \subseteq G^{*x}$ by Proposition 1.9 $pr^{*x}\{x\} \subseteq cl\{x\} \subseteq G^{*x}$, then $pr^{*x}\{x\} \subseteq G^{*x}$. It follows that X is $S_\circ^{*}$_ space.

Theorem 3.3:Every $S_\circ^{***}$_space is $S_\circ^{****}$_space.

Proof : Let $x \in X$ and G be perfected set of $S_\circ^{***}$_ space X such that x∈ G, then $cl\,\{x\} \subseteq G^{*x}$ so, by Proposition 1.9 so we have that $pr^{*x}\{x\} \subseteq cl\{x\} \subseteq G^{*x} \subseteq cl(G)$, then $pr^{*x}\{x\} \subseteq cl\{G\}$, it follows that X is $S_\circ^{****}$_ space.

Proposition 3.4: If (X,T) is perfected space and strong space, then every $S_{\circ}_^{***}$space is $S_{\circ}_^{**}$space.

Proof: Let x∈ X and G be an open subset of X such that x∈ G. But X is perfected space then G is perfected set implies that $G^{*x} \subseteq G$. Since X is $S_{\circ}\_^{***}$ space then cl{x}⊆ G^{*x} by Proposition 1.9 then $pr^{*x}\{x\} \subseteq cl\{x\}$, it follows that $pr^{*x}\{x\} \subseteq G$. Hence X is $S_{\circ}\_^{**}$ space.

proposition 3.5: If (X,T) is strong space, then every $S_{\circ}_^{**}$space is $S_{\circ}_^{****}$space.

Proof: Let x∈ X and G be perfected set of X such that x∈ G. Since X is strong space by Proposition 1.18 then G is open. But X is $S_{\circ}\_^{**}$ space then $pr^{*x}\{x\} \subseteq G^{*x}$, we have that $pr^{*x}\{x\} \subseteq G^{*x} \subseteq cl\{G\}$, then $pr^{*x}\{x\} \subseteq cl\{G\}$, it follows that X is $S_{\circ}\_^{****}$ space.

Proposition 3. 6: Every $R_{\circ} -$ space is $S_{\circ}\_^{**}$ space

Proof: Let x∈ X and G be an open set such that x∈ G .Since X is $R_{\circ}$- space then x∈ G and cl{x}⊆ G, by Proposition 1.9 so we have that $pr^{*x}\{x\} \subseteq cl\{x\} \subseteq G$ then $pr^{*x}\{x\}$ ⊆G. Hence X is $S_{\circ}\_^{**}$ space.

Proposition 3. 7: If (X,T) is strong space, then every $R_{\circ} -$ space is $S_{\circ}_^{****}$space .

Proof : Let x∈ X and G be perfected set of X such that x∈ G. Since X is strong space by proposition 1.18 then G is open set. But X is $R_{\circ}$ space, then $cl\,\{x\} \subseteq G$ so by proposition 1.9 so we have that $pr^{*x}\{x\} \subseteq cl\{x\} \subseteq G \subseteq cl(G)$, then $pr^{*x}\{x\} \subseteq cl\{G$, it follows that X is $S_{\circ}\_^{****}$ space.

proposition 3.8: If (X,T) is perfected space and strong space, then every $S_{\circ}_^{***}$space is $R_{\circ} -$ space .

Proof: Let G open set and x∈ X such that x∈ G. Since X is $S_{\circ}\_^{***}$ space then $cl\{x\} \subseteq G^{*x}$. But X is perfected space then G is perfected set implies that $G^{*x} \subseteq G$, it follows that cl{x}⊆ G. Then X is $R_{\circ} - space$.

Theorem 3.9: A topological space (X,T) is $S_{\circ}\_^{***}$ $space$ if and only if for each x ∈ X and U a subset of X with $cl(\{x\})^{*x} \subseteq U^{*x}$, where $x \in U$.

Proof : Let x ∈ X and U subset containing x. Since X is $S_{\circ}\_^{***}$ space then cl{x}⊆ U^{*x}. But $\{x\}^{*x} \subseteq cl\,\{x\}$ so $cl(\{x\}^{*x}) \subseteq clcl(\{x\})$ it follows $cl(\{x\}^{*x}) \subseteq cl(\{x\}) \subseteq U^{*x}$.

Conversely let x∈ U, but {x}⊆ $\{x\}^{*x}$ then cl{x}⊆ $cl\{x\}^{*x} \subseteq U^{*x}$. Therefor (X,T) is $S_{\circ}\_^{**}$ $space$.

Theorem 3. 10: Let f be bijection, open map and I^{*}-map from (X,T) space into $S_{\circ}^{*}$- space (Y,ρ). Then (X,T) is $S_{\circ}^{*}$-space, if f is continuous map.

Proof: Let x∈ X and A be a strong set in X, implies that A^{*x} is open set and x∈ A since f is open map then $f(A^{*x})$ is open set in Y and f(x)∈ $f(A)$, since f is I^{*}-

map then $f(A^{*x}) = f(A)^{*f(x)}$. But X is $S_\circ^*$-space then $pr^{*f(x)}\{f(x)\} \subseteq A^{*f(x)}$. Now $f^{-1}(pr^{*f(x)}\{f(x)\}) \subseteq f^{-1}(f(A)^{*f(x)})$, by corollary 1.22 then $pr^{*x}\{x\} \subseteq A^{*x}$. Hence X is $S_\circ^*$-space.

Theorem 3.11: Let f be bijection, open and I^*-map from $S_\circ^*$- space (X,T) space into (Y,ρ)space. Then (Y,ρ) is $S_\circ^*$-space, if f is continuous map.

Proof: Let y$\in Y$ and B is strong set in Y, implies that B^{*y} is open set and y$\in B$ by continuity of f then $f^{-1}(B^{*y})$ is open set in X and $f^{-1}(y) \in f^{-1}(B)$ and by corollary 1.22 then $f^{-1}(B^{*y}) = f^{-1}(B)^{*f^{-1}(y)}$. But X is $S_\circ^*$-space then $pr^{*f^{-1}(y)}\{f^{-1}(y)\} \subseteq f^{-1}(B)^{*f^{-1}(y)}$. Now$f\left(pr^{*f^{-1}(y)}\{f^{-1}(y)\}\right) \subseteq f(f^{-1}(B)^{*f^{-1}(y)}$ since f is I^*-map then $pr^{*y}\{y\} \subseteq B^{*y}$. Hence Y is $S_\circ^*$-space.

Definition 3. 12 : A topological space (X, T) is said to be .

1. $S_{1\_}^{*}$ space if and only if for each distinct point x ,y of X with $pr^{*x}\{x\} \neq pr^{*y}\{y\}$ then there exist strong sets U,V such that $pr^{*x}\{x\} \subseteq U^{*x}$ and $pr^{*y}\{y\} \subseteq V^{*y}$.
2. $S_{1\_}^{**}$ space if and only if for each distinct point x ,y of X with $pr^{*x}\{x\} \neq pr^{*y}\{y\}$ then there exist open sets U,V of X such that $pr^{*x}\{x\} \subseteq U$and $pr^{*y}\{y\} \subseteq V$.
3. $S_{1\_}^{***}$ space if and only if for each distinct point x ,y of X with $cl\{x\} \neq cl\{y\}$ then there exist subsets U,V of X such that $cl\{x\} \subseteq U^{*x}$and $cl\{y\} \subseteq V^{*y}$.
4. $S_{1\_}^{****}$ space if and only if for each distinct point x ,y of X with $pr^{*x}\{x\} \neq pr^{*y}\{y\}$ then there exist perfected U,V of X such that $pr^{*x}\{x\} \subseteq cl\{U\}$and $pr^{*y}\{y\} \subseteq cl\{V\}$.

Theorem 3.13: Every $S_{1\_}^{*}$ space is $S_{1\_}^{**}$ space **.**

Proof: Let x, y $\in X$ with $pr^{*x}\{x\} \neq pr^{*y}\{y\}$. Since X is $S_{1\_}^{*}$ space then there exist strong sets U,V such that $pr^{*x}\{x\} \subseteq U^{*x}$ and $pr^{*y}\{y\} \subseteq V^{*y}$ since U,V strong sets then U^{*x}, V^{*y} open sets it follows that X is $S_{1\_}^{**}$ space.

Proposition 3.14 : If (X,T) is strong space, then every $S_{1_}^{****}$space is $S_{1_}^{*}$space.

Proof : Let x, y $\in X$ such that $pr^{*x}\{x\} \neq pr^{*y}\{y\}$ and X is $S_{1\_}^{***}$ $space$ then there exist subsets U,V of X such that cl{x}$\subseteq U^{*x}$, cl{y}$\subseteq V^{*y}$ and by proposition1.9, then $pr^{*x}\{x\} \subseteq cl\{x\} \subseteq U^{*x}$ and $pr^{*y}\{y\} \subseteq cl\{y\} \subseteq V^{*y}$ so $pr^{*x}\{x\} \subseteq U^{*x}$ and $pr^{*y}\{y\} \subseteq V^{*y}$. But X is strong space then U, V are strong sets. Hence X is $S_{1\_}^{*}$ $space$.

Proposition 3.15 : If (X,T) is perfected space ,then every $S_{1_}^{*}$space is $S_{1_}^{****}$space.

Proof : Let $x, y \in X$ such that $pr^{*x}\{x\} \neq pr^{*y}\{y\}$ and X is $S_{1\_}^{*}$ space then there exist strong sets U,V such that $pr^{*x}\{x\} \subseteq U^{*x}$ and $pr^{*y}\{y\} \subseteq V^{*y}$, and by proportion 1.9 $U^{*x} \subseteq cl(U), V^{*y} \subseteq cl(V)$ then $pr^{*x}\{x\} \subseteq cl(U)$ and $pr^{*y}\{y\} \subseteq cl(V)$. Since X is perfected space then U,V are perfected sets. Hence X is ${S_1}_{-}^{****}$ space.

Proposition 3.16: If (X,T) is thin and perfected space then every ${S_1}_{-}^{**}$ −space is ${S_1}_{-}^{*}$ space.

Proof : Let $x, y \in X$ such that $pr^{*x}\{x\} \neq pr^{*x}\{y\}$. Since X is ${S_1}_{-}^{**}$ space then there exist two open sets U,V such that $pr^{*x}\{x\} \subseteq U$ and $pr^{*y}\{y\} \subseteq V$ but X is thin space then U,V are thin sets so $pr^{*x}\{x\} \subseteq U \subseteq U^{*x}$ and $pr^{*y}\{y\} \subseteq V \subseteq V^{*y}$, it follows $pr^{*x}\{x\} \subseteq U^{*x}$ and $pr^{*y}\{y\} \subseteq V^{*y}$. Since $x \in U \subseteq U^{*x}$ and $y \in V \subseteq V^{*y}$ with U,V are open sets then U^{*x} and $V^{*y}$are open sets because a neighborhood of each of its points. Hence (X,T) is ${S_1}_{-}^{*}$ −space.

Theorem 3.17:A topological space (X,T) is ${S_1}_{-}^{***}$ *space* if and only if for each x ,y ∈ X with cl{x} ≠ cl{y} there exist U,V a subset of X such that $cl(\{x\})^{*x} \subseteq U^{*x}$ and $cl(\{y\})^{*y} \subseteq V^{*y}$.

Proof : Let x ,y$\in X$ *and* $cl\{x\} \neq cl\{y\}$ By assumption there exists U ,V with $cl\{x\} \subseteq U^{*x}$ and $cl(\{y\} \subseteq V^{*y}$, $\{x\}^{*x} \subseteq cl\,\{x\}$ and $\{y\}^{*y} \subseteq cl\,\{y\}$ therefor $cl(\{x\}^{*x}) \subseteq cl\,cl\,\{x\}$ and $cl(\{y\}^{*y}) \subseteq cl\,cl\,\{y\}$ then $cl(\{x\}^{*x}) \subseteq cl\,\{x\} \subseteq U^{*x}$ and $cl(\{y\}^{*y}) \subseteq cl\,\{y\} \subseteq V^{*y}$, it follows $cl(\{x\}^{*x}) \subseteq U^{*x}$ and $cl(\{y\}^{*y}) \subseteq V^{*y}$.

Conversely Let x ,y such that $cl\{x\} \neq cl\{y\}$ By assumption then $cl(\{x\}^{*x}) \subseteq U^{*x}$ and $cl(\{y\}^{*y}) \subseteq V^{*y}$ but $\{x\} \subseteq \{x\}^{*x}$ and $\{y\} \subseteq \{y\}^{*y}$ hence $cl\,\{x\} \subseteq cl(\{x\}^{*x}) \subseteq U^{*x}$ and $cl\,\{y\} \subseteq cl(\{y\}^{*y}) \subseteq V^{*y}$. Then (X,T) is ${S_1}_{-}^{***}$ *space*

Theorem 3.18: For a topological space (X,T) then the following statements are hold:

1. If (X,T) is $T_1 - space$ then X is ${S_1}_{-}^{***}$ *space*.
2. If (X,T) is $T_1 - space$ then X is ${S_1}_{-}^{**}$ *space* .

Proof : (1) Let x ≠y ∈ X with cl{x} ≠ cl{y}, since X is $T_1 - space$ then there exist two open subset U,V of X such that x ∈ U ,y ∈ V so $\{x\}^{*x} \subseteq U^{*x}$ and $\{y\}^{*y} \subseteq V^{*y}$ but X is $T_1 -$ space then {x},{y} are closed sets and by Remark 1.15 we have $pr^{*x}\{x\} = \{x\}^{*x} = \{x\} = cl\{x\}$ and $pr^{*y}\{y\} = \{y\}^{*y} = \{y\} = cl\{y\}$. Hence $cl\{x\} \subseteq U^{*x}$ and $cl\{y\} \subseteq V^{*y}$, which implies that X is ${S_1}_{-}^{***}$ space.

Proof : (2) Let x ≠y $\in X$ with $cl\{x\} \neq cl\{y\}$ and (X,T) is $T_1 - space$ then there exist two open subset U,V of X such that x $\in U$,y $\in V$ also we have {x}=cl{x},{y}=cl{y}and by Remark 1.15 we have $pr^{*x}\{x\} = \{x\}^{*x} = \{x\} = cl\{x\}$ and $pr^{*y}\{y\} = \{y\}^{*y} = \{y\} = cl\{y\}$. Hence $pr^{*x}\{x\} \subseteq U$ and $pr^{*y}\{y\} \subseteq V$. Therefore X is ${S_1}_{-}^{**}$ *space*.

Propoistion3.19: If (X,T) is strong space , then every $T_1 - \text{space}$ is $S_{1\_}^{*}$ space .

Proof : Let x ,y $\in X$ with $pr^{*x}\{x\} \neq pr^{*y}\{y\}$ and (X,T) is $T_1 - space$ then there exist two open subset U,V of X such that x $\in U$,y $\in V$ also since X is $T_1 - \text{space}$ then {x}=cl{x},{y}=cl{y} and by Remark 1.15 we have $pr^{*x}\{x\} = cl\{x\} = \{x\}^{*x}$ and $pr^{*y}\{y\} = cl\{y\} = \{y\}^{*x}$, so $pr^{*x}\{x\} = cl\{x\} = \{x\}^{*x} \subseteq U^{*x}$ and $pr^{*y}\{y\} = cl\{y\} = \{y\}^{*x} \subseteq V^{*y}$ then $pr^{*x}\{x\} \subseteq U^{*x}$ and $pr^{*y}\{y\} \subseteq V^{*y}$ since X is strong space then U,V are strong sets. Hence (X,T) is $S_{1\_}^{*}$ space.

Theorem 3.20: Let f be bijection ,open map and I^*-map from (X,T) space into S_1^{**}- space (Y,ρ). Then (X,T) is S_1^{**}-space, if f is continuous map.

Proof : Let $x_1, x_2 \in X$ such that $pr^{*x_1}(\{x_1\}) \neq pr^{*x_2}(\{x_2\})$, since f is I^*-map then$f(pr^{*x_1}(\{x_1\}))= pr^{*f(x_1)}(\{f(x_1)\})$ and $f(pr^{*x_2}\{(x_2)\})= pr^{*f(x_2)}(\{f(x_2)\})$ with $pr^{*f(x_1)}(\{f(x)\}) \neq pr^{*f(x_2)}(\{f(x_2)\})$. But Y is S_1^{**}- space then there exist open subsets U,V of Y such that $pr^{*f(x_1)}(\{f(x_1)\}) \subseteq$ U and $pr^{*f(x_2)}(\{f(x_2)\}) \subseteq$ V. Now $f^{-1}(pr^{*f(x_1)}(\{f(x_1)\})) \subseteq f^{-1}$(U) and $f^{-1}(pr^{*f(x_2)}(\{f(x_2)\})) \subseteq f^{-1}$(V) since f is continuous then $f^{-1}(U)$ and f^{-1}(V) are open sets in X, by corollary 1.22 then $pr^{*x_1}(\{x_1\}) \subseteq f^{-1}(U)$ and $pr^{*x_2}(\{x_2\}) \subseteq f^{-1}(V)$. Hence X is S_1^{**}-space.

Theorem 3.21: Let f be bijection, open and I^*-map from S_1^{**}- space (X,T) space into (Y,ρ). Then (Y,ρ) is S_1^{**}-space, if f is continuous map.

Proof : Let $y_1, y_2 \in Y$ such that $pr^{*y_1}(\{y_1\}) \neq pr^{*y_2}(\{y_2\})$ by corollary 1.22 then $f^{-1}(pr^{*y_1}(\{y_1\}))= pr^{*f^{-1}(y_1)}(\{f^{-1}(y_1)\})$ and $f^{-1}(pr^{*y_2}(\{y_2\}))=pr^{*f^{-1}(y_2)}(\{f^{-1}(y_2)\})$with $pr^{*f^{-1}(y_1)}(\{f^{-1}(y_1)\}) \neq pr^{*f^{-1}(y_2)}(\{f^{-1}(y_2)\}$. But X is S_1^{**}-space then there exist open subsets U,V of X such that $pr^{*f^{-1}(y_1)}(\{f^{-1}(y_1)\}) \subseteq$ U and $pr^{*f^{-1}(y_2)}(\{f^{-1}(y_2)\}) \subseteq$ V. Now $f(pr^{*f^{-1}(y_1)}(\{f^{-1}(y_1)\})) \subseteq f$(U) and $f(pr^{*f^{-1}(y_2)}(\{f^{-1}(y_2)\})) \subseteq f$(V) since f is open map then $f(U)$ and f(V) are open set in Y and $pr^{*y_1}(\{y_1\}) \subseteq f(U)$ and $pr^{*y_2}(\{y_2\}) \subseteq f(V)$. HenceY is S_1^{**}-space.

Definition 3.22: A topological space (X, T) is said

1. $S_{2\_}^{*}$ space if and only if for each distinct point x ,y of X with $pr^{*x}(Ker\{x\}) \neq pr^{*y}(Ker\{y\})$ then there exist strong sets U,V such that $pr^{*x}(Ker\{x\}) \subseteq U^{*x}$ and $pr^{*y}(Ker\{y\}) \subseteq V^{*y}$.
2. $S_{2\_}^{**}$ space if and only if for each distinct point x ,y of X with $cl(Ker\{y\}) \neq cl(Ker\{x\})$ then there exist open sets U,V of X such that $cl(Ker\{y\}) \subseteq U^{*x}$and $cl(Ker\{x\}) \subseteq V^{*y}$.

3. $S_{2\_}^{***}$ space if and only if for each distinct point x ,y of X with Ker{y} ≠ Ker{x} then there exist subsets U,V of X such that Ker{x} ⊆ $U^{*x}$and Ker{y}) ⊆ V^{*y}.
4. $S_{2\_}^{****}$ space if and only if for each distinct point x ,y of X with Ker{y} ≠ Ker{x} then there exist perfected U,V of X such that Ker{x} ⊆ cl(U) and Ker{y} ⊆ cl(V).

Proposition 3.23: Let (X,T) be topological space if X is strong space and perfected space. Then the following statement are holds:

1. Every $S_{2\_}^{***} -space$ is $S_{2\_}^{****} -space.$
2. Every $S_{2\_}^{***} -space$ is $S_{2\_}^{*}\ space$.
3. Every $S_{2\_}^{*} -space$ is $S_{2\_}^{****}\ space$

Proof :(1) Let x ,y ∈X such that (Ker{x})≠ (Ker{y}). Since X is $S_{2\_}^{***}$ space then there exist tow sub set U,V such that Ker{x}⊆ U^{*x} and Ker{y}⊆ V^{*y} which implies that Ker{x}⊆ $U^{*x} \subseteq cl(U)$ and Ker{y}⊆ $V^{*y} \subseteq cl(V)$, but X is perfected space then U,V are perfected sets. Hence X is $S_{2\_}^{****}\ space.$

Proof : (2) Let x ,y ∈X such that pr^{*x}(Ker{x})≠ pr^{*y} (Ker{y}). Since X is perfected space by proposition 1.12 then pr^{*y} (Ker{y})=Ker{y} and pr^{*x} (Ker{x})=Ker{x}, it follows that Ker{x})≠ (Ker{y}). But X is $S_{2\_}^{***}\ space$ then there exist tow sub sets U,V such that Ker{x}⊆ U^{*x} and Ker{y}⊆ V^{*y}so pr^{*x}(Ker{x}) =Ker{x}⊆ U^{*x} and pr^{*y}(Ker{y})= Ker{y}⊆ V^{*y} which implies that pr^{*x}(Ker{x})⊆ U^{*x} and pr^{*y}(Ker{y}) ⊆ V^{*y}. Since X is strong space then U,V are strong sets. Hence X is $S_{2\_}^{*}$ space.

Proof: (3) Let x ,y ∈X such that Ker{y} ≠ Ker{x} since X is perfected then pr^{*x} (Ker{x})=Ker{x} and pr^{*y} (Ker{y})=Ker{y}then pr^{*x}(Ker{x})≠ pr^{*y} (Ker{y}). Since X is $S_{2\_}^{*}\ space$ then there exist tow strong sets U,V such that pr^{*x} (Ker{x})⊆ U^{*x} and pr^{*y} (Ker{y})⊆ V^{*y}. But Ker{x}= pr^{*x}(Ker{x}) ⊆ $U^{*x} \subseteq cl(U)$ and Ker{y}= pr^{*x}(Ker{x}) ⊆ $V^{*y} \subseteq cl(V)$ then Ker{x}⊆ $cl(U)$ and Ker{y}⊆ $cl(V)$. Since X is perfected then U,V are perfected set. Hence X is $S_{2\_}^{****}\ space.$

Proposition 3.24: If X is a perfected space and $T_1 -$ space then every $S_{2\_}^{*}\ space$ is $S_{2_}^{**}$space

Proof : Let x ,y ∈X such that cl(Ker{x})≠ cl (Ker{y}). Since X is $T_1 -$ space by theorem 1.3 then Ker{x}={x} and Ker{y}={y} it follows that cl{x}≠ $cl\{y\}$. Also since X is $T_1 -$ space {x}, {y}are closed sets by remark 1.15 then pr^{*y}({y})={y}=cl{y}=$\{y\}^{*y}$and pr^{*x}({x})={x}=cl{x}=$\{x\}^{*x}$then pr^{*x}(Ker{x})≠ pr^{*y} (Ker{y}). But X is $S_{2\_}^{*}\ space$ two strong sets U,V such that

pr^{*x}(Ker{x}) ⊆ U^{*x} and pr^{*y}(Ker{y}) ⊆ V^{*y} by remark 1.15 then that cl(Ker{y}) ⊆ $U^{*x}$and cl(Ker{x}) ⊆ V^{*y}. Since X is perfected space then U,V are perfected sets and U,V are strong set by proposition 1.18 then U,V are open set .Hence (X,T) is $S_{2\_}^{**}$ *space*.

Proposition 3.26: If X is a $T_1 -$ space then every $S_{2\_}^{*}$ *space* is $S_{1\_}^{*}$ *space*.

Proof: Let x, y ∈X such that pr^{*x}({x})≠ pr^{*y} ({y}). By theorem1.3 then Ker{x}={x} and Ker{y}={y}, it follows that pr^{*x}(Ker{x})≠ pr^{*y} (Ker{y}). But X is $S_{2\_}^{*}$ *space* then there exist two strong sets U,V such that pr^{*x}(Ker{x}) ⊆ U^{*x} and pr^{*y}(Ker{y}) ⊆ V^{*y} by the same theorem then pr^{*x}({x}) ⊆ U^{*x} and pr^{*y}({y}) ⊆ V^{*y}. Hence X is $S_{1\_}^{*}$ *space*.

3. Weak separation axioms via A_x^θ

In this section introduces a set of new separation axioms in topological space, namely $M_{i_}$space**, i=**0,1,2 under the idea of A_x^θ ,we investigate the relationship between them**.**

Proposition 3.1:Let (X,T) be a topological space, and A subset of X, x∈ X. Then, the following properties are held.

i. x∈ A, if and only if x ∈ A_x^θ.
ii. For any point x, y∈ X, with $I_x \subseteq I_y$ then $A_x^\theta \subseteq A_y^\theta$.
iii. If A⊂ B, then $A_x^\theta \subset B_x^\theta$.
iv. $(A \cap B)_x^\theta = A_x^\theta \cap B_x^\theta$.

Proof : (i) Let x ∈ A_x^θ. To prove x∈ A, suppose x ∉A since x∈ A_x^θ then there exists an open set U such that x∈U and U-A∈ I_x then U∩ $A^c \in I_x$ since x∉A then x∈ A^c, it follows U∩ $A^c \notin I_x$ then x∉ A_x^θ .This is contradiction then x ∈ A.

Conversely, to prove x∈ A_x^θ .Let x∈ A then x ∈G∩A for each open set G such that x∈G ,x∈G∩A then x∉ G∩ A^c =G-A ∈ I_x. Hence, x∈ A_x^θ.

Proof : (ii) Let z∈ A_x^θ then there exist a open set U such that z∈U and U-A∈ $I_x \subseteq I_y$ So U-A⊆ I_y it follows z∈ A_y^θ then , $A_x^\theta \subseteq A_y^\theta$.

Proof : (iii) Let A⊂B and assume that a∈ A_x^θ so there exist $U \in$ T(a) such that U-A ∈ I_x. But A⊂ B, it follows that U-B⊂ $U - A$ which implies that U-B∈ I_x. Therefor, $A_x^\theta \subset B_x^\theta$.

Proof :(iv) Since $(A \cap B)_x^\theta$ =X-(X- $(A \cup B)^{*x}$) by theorem 1.26 then X-[(X − A) ∪ (X − B)$]^{*x}$ us X-[$(X - A)^{*x} \cup (X - B)^{*x}$]=[X-$(X - A)^{*x}$] ∩[X-$(X - B)^{*x}$] = $A_x^\theta \cap B_x^\theta$.

Definition 3.2: A topological space (X,T) is said to be.

1. $M_\circ$_space if and only if for each pair of distinct point x ,y of X there exist nonempty subset A of X such that y∈ $A^{\theta}{}_{x}$ or x∈ $A_y{}^{\theta}$.
2. M_1_space if and only if for each pair of distinct point x ,y of X there exist nonempty subset A of X such that y∈ $A^{\theta}{}_{x}$ and x∈ $A_y{}^{\theta}$.
3. M_2_space if and only if for each pair of distinct point x ,y of X there exist nonempty subset A of X such that y∈ $A^{\theta}{}_{x}$ and x∈ $A_y{}^{\theta}$ with $A^{\theta}{}_{x} \cap A_y{}^{\theta}=\emptyset$.

Theorem 3.3: For topological space (X,T), then the following properties hold:

1. Every M_1_space is $M_\circ$_space.
2. Every M_2_space is $M_\circ$_space..
3. Every M_2space is M_1_space .

Proof : Straight Forward

Theorem 3.4 :Every subspace of M_i_space is M_i_space. for each i=0,1,2

Proof: Assume i=2 Let X be M_2-space and $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since Y⊆ X then $y_1, y_2 \in X$, but X is M_2-space so there exist A ⊆ X such that $y_1 \in {}_XA^{\theta}_{y_2}$ and $y_2 \in {}_XA^{\theta}_{y_1}$, with ${}_XA^{\theta}_{y_2} \cap {}_XA^{\theta}_{y_1} = \emptyset$, which implies that $y_1 \in {}_XA^{\theta}_{y_2} \cap Y$ and $y_2 \in {}_XA^{\theta}_{y_1} \cap Y$, there exist B sub set of X such that ${}_YB^{\theta}_{y_2} = {}_XA^{\theta}_{y_2} \cap Y$ and ${}_YB^{\theta}_{y_1} = {}_XA^{\theta}_{y_1} \cap Y$ with ${}_YB^{\theta}_{y_2} \cap {}_YB^{\theta}_{y_1} = ({}_XA^{\theta}_{y_2} \cap Y) \cap ({}_XA^{\theta}_{y_1} \cap Y)$ $= ({}_XA^{\theta}_{y_2} \cap {}_XA^{\theta}_{y_1}) \cap Y) = \emptyset \cap Y = \emptyset$. Hence Y is S_2-space**.**

Theorem 3.5: For topological space (X,T), then the following properties hold:

1. Every $M_\circ$_space is I^*_$T_\circ$_space.
2. Every M_1_space is I^*_T_1_space.
3. Every M_2_space is I^* _T_2_space.

Proof (1): Let x, y∈ X with $x \neq y$ and X be $M_\circ$_space then there exist a sub set A of X such that y∈ A^{θ}_x or x∈ A^{θ}_y assume x∈ A^{θ}_y .By theorem 1.26 then A^{θ}_y=$((A^c)^{*y})^c$ so x∉ $(A^c)^{*y}$,it follows x∉ $((A^c)^{*y})^{*x}$=∅. Put $(A^c)^{*y}$=B then B^{*x} =∅ and y ∉ B^{*x}. Hence X is I^*_$T_\circ$_space.

Proof (2): Let x, y∈ X with $x \neq y$ and X be M_1_space then there exist a sub set A of X such that y∈ A^{θ}_x and x∈ A^{θ}_y` .By theorem 1.26 A^{θ}_y =$((A^c)^{*y})^c$ and A^{θ}_x=$((A^c)^{*x})^c$so x∉ $(A^c)^{*y}$ and y∉ $(A^c)^{*x}$it follows x∉ $((A^c)^{*y})^{*x}$=∅ and y∉ $((A^c)^{*x})^{*y}$.Put $(A^c)^{*y}$=B and $(A^c)^{*x}$= C .Then(B^{*x} =∅ and $C^{*y} = \emptyset$), (y ∉ B^{*x} and x∉ C^{*y}). Hence X is I^*_T_1_space.

Proof (3): Let x, y∈ X with $x \neq y$ and X be M_2_space then there exist a sub set A of X such that y∈ A^{θ}_x and x∈ $A^{\theta}_y$with $A^{\theta}{}_{x} \cap A^{\theta}_y = \emptyset$. By theorem 1.26 then A^{θ}_y =$((A^c)^{*y})^c$ and A^{θ}_x =$((A^c)^{*x})^c$ so x∉ $(A^c)^{*y}$ and y∉ $(A^c)^{*x}$, it follows x∉ $((A^c)^{*y})^{*x}$=Ø and y∉ $((A^c)^{*x})^{*y}$. Put $(A^c)^{*y}$=B and $(A^c)^{*x}$= C. Then(B^{*x} =Ø and $C^{*y} = \emptyset$), (y ∉ B^{*x} and x∉ C^{*y}) with $B^{*x} \cap C^{*y} = \emptyset \cap \emptyset$=Ø. Hence X is I^*_T_2_space.

Theorem 3.6: For topological space (X,T), then the following properties hold:

1. Every $M_\circ$_space is I^{**}_$T_\circ$_space.
2. Every M_1_space is I^{**}_T_1_space.

Proof (1): Let x, y∈ X with $x \neq y$ and X be $M_\circ$_space then there exist a sub set A of X such that y∈ A_x^θ or x∈ A_y^θ.Assume x∈ A_y^θ.By theorem 1.26 then A_y^θ=$((A^c)^{*y})^c$ so x∉ $(A^c)^{*y}$ .Hence X is I^{**}-$T_\circ$-space.

proof (2) : Let x, y∈ X with $x \neq y$ and X be M_1_space then there exist a sub set A of X such that y∈ A_x^θ and x∈ A_y^θ. By theorem 1.26 then A_y^θ=$((A^c)^{*y})^c$ and A_x^θ=$((A^c)^{*x})^c$ so x∉ $(A^c)^{*y}$ and y∉ $(A^c)^{*x}$. Hence X is I^{**}-T_1-space.

Proposition 3.7: If f :(X,T)⟶ (Y,ρ) is open and bijection map, then f (A_x^θ)=($f(A)_{f(x)}^\theta$.

Proof : Let A ⊆ X and x∈X. Then by definition 1.26 we have A_x^θ = { z∈X: there exist $U_Z \in T(z)$such that U_z- A∈ I_x}. Since f is open, it follows that $f(U_z)$ is open subsets of Y and containing f (z). Also f is injective by Remark 1.22, part (2) we have $f(I_x)$= $I_{f(x)}$. $f(A_x^\theta)$= {f (z) ∈ f (X) there exist f (U_z) ∈ neighborhood of (f(z)) such that $f(U_z)$ –f (A) ∈ $f(I_x)$= {f (z)∈Y there exist f (U_z) ∈ neighborhood of f (z) such that $f(U_z)$ –f (A) ∈ $I_{f(x)}$}=$(f(A))_{f(x)}^\theta$.

Theorem 3.8: Let f be bijection map from M_i_space (X,T) into (Y,ρ), then (Y,ρ) is M_i_space if f is open map. for i=0,1,2.

Proof : Assume i=2 let $y_1 \neq y_2$ ∈Y. Since f is bijection so there exists $x_1 \neq x_2$ ∈X such that y_1 =$f(x_1)$ and y_2 =$f(x_2)$. But X is M_2_space then there exist a sub set A of X such that $x_1 \in A_{x_2}^\theta$ and $x_2 \in A_{x_1}^\theta$ with $A_{x_2}^\theta \cap A_{x_1}^\theta = \emptyset$. It follows f (A) is a sub set of Y and $y_1 = f(x_1) \in f(A)_{f(x_2)=y_2}^\theta$ and y_2=$f(x_2) \in f(A)_{f(x_1)=y_1}^\theta$with $f(A)_{f(x_2)}^\theta \cap f(A)_{f(x_1)}^\theta = f(A)_{y_2}^\theta \cap f(A)_{y_1}^\theta$=$f$ (Ø)=Ø .Then Y is M_2_space.

Proposition 3.9: If f :(X,T) ⟶ (Y,ρ) is continuous and bijection map then f^{-1} (A_y^θ)= f^{-1} $(A)_{f^{-1}(y)}^\theta$.

Proof : Let A ⊆ Y and y∈Y. Then by definition 1.25 we have A_y^θ = { z∈Y: there exist $U_Z \in T(z)$such that U- A∈ I_y}. Since f is continuous ,it follows that $f^{-1}(U_z)$ is open subsets of X and containing f^{-1}(z). Also f is bijection by Remark 1. 22, part (1) we have $f^{-1}(I_y)$= $I_{f^{-1}(y)}$. f^{-1} (A_y^θ)= {f^{-1} (z)∈ f^{-1} (Y) there exist f^{-1} (U_z) ∈ neighborhood of(f^{-1}(z)) such that f^{-1} (U_z) –f^{-1} (A)

$\in f^{-1}(I_y)\}$= $\{f^{-1}$ (z)∈X there exist f^{-1} $(U_z) \in$ neighborhood of $(f^{-1}$ (z) such that f^{-1} (U_z) –f^{-1} (A) ∈ $I_{f^{-1}(y)}\}$=$(f(A))^{\theta}_{f^{-1}(y)}$.

Theorem 3.10: Let f be injection and continuous map from (X,T) space onto M_i- space (Y,ρ),then (X,T) is M_i- space for each i=0,1,2.

Proof : Assume i=2 let x_1, x_2 ∈X. Since f is injection $f(x_1) \neq f(x_2)$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$ so that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(x_2)$.Then $y_1 \neq y_2 \in$ Y. Since Y is M_2_space then there exist a sub set B of Y such that $y_1 \in B^{\theta}_{y_2}$ and $y_2 \in B^{\theta}_{y_1}$ with $B^{\theta}_{y_2} \cap B^{\theta}_{y_1} = \emptyset$, it follows f^{-1}(B) is a sub set of X and $x_1 = f^{-1}(y_1) \in f^{-1}(B)^{\theta}_{x_2 = f^{-1}(y_2)}$ and $x_2 \in f^{-1}(B)^{\theta}_{x_1 = f^{-1}(y_1)}$with$f^{-1}(B^{\theta}_{y_2} \cap B^{\theta}_{y_1}) = f^{-1}(B)^{\theta}_{x_2} \cap f^{-1}(B)^{\theta}_{x_1} = f^{-1}(\emptyset) = \emptyset$. Then X is M_2_space.

References

[1] F. Arenas,J. Dontchev , and L. Puertas ,"idealization some weak separation axioms *"Acts Math .Hunger* , 2,(2008) 47-53.

[2] J.N. Sharma, "general Topology" Krishna prakashan , Meerut (U.P)(1997).

[3] J. Kelley and John L. " *General Topology"*. Springer. ISBN ,(1991).

[4] K. Kuratowski, Topological I, Warszawa, 1933.

[5] L.A AL-Swidi and B.AL-sadaa "Turing Point of Proper Ideal *"in Archive Des Science*, 7(2013) (213-220).

[6] L.A AL-Swidi and A.AL-Nafee, A, "New separation axioms using the ideal "Gem set" in topology space :*Mathematical Theory and Modeling*, 3(2013)1-10.

[7] L. A AL-Swidi and M. AL-Rubaye ,"On separation axioms with respect to Gem"" *international journal of Engineering Research and Technology*, 3(2014)2789 – 2792.

[8] R. Vaidyanathaswamy, The localization theory in set topology, *Proc. Indian Acad. Sci*. 20 (1945), 51 – 61.

[9] R. Balaji and N. Rajesh, " Some New Separation Axioms In Ideal Topological Spaces" *international journal of Engineering Research and Technology*,.2(2013) 38-48.

[10] N. Shanin, On separation in topological spaces, *Dokl. Akad. Nauk. SSSR*.,

[11] S. Davis, Indexed systems of neighborhoods for general topological spaces, *Amer. Math. Monthly* 68(1961), 886-893

[12] T.Noiri and A.Keskin ,"ΛI-sets and Some weak Axioms "*Int .Journal of Math. Analysis*,5(2011),539-548.

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