Separation Axioms Via Kernel Set in Topological Spaces

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Abstract

In this paper deals with the relation between the separation axioms T_i-space , i= 0,1,…,4 and R_i-space i= 0,1,2,3 throughout kernel set associated with the closed set . Then we prove some theorems related to them.

Keywords: separation axioms, Kernel set and weak separation axioms.

1. INTRODUCTION AND PRELIMINARIES

In1943, N.A.Shainin [4] offered a new weak separation axiom called R_0 to the world of the general topology. In 1961, A.S.Davis [1] rediscovered this axiom and he gave several interesting characterizations of it. He defined R_0, R_1 and R_2 entirely. He did not submit clear definition of R_3-space but stated it throughout this note: (But the usual definition of "normality" must be modified slightly if R_3 is to be the axiom for normal spaces.)

The present study presents the definition of R_3-spaces as follows:(A topological space is called an R_3-space iff it is normal space and R_1-space). This definition of R_3-space satisfied with: Every R_3 is an R_2-spaces. On the other hand (X, T) is aT_4-space if and only if it is an R_3 -space and T_{k-1}-space, $k = 0, 1, 2, 3,4$.

We proved R_i-spaces, $i =$ 0,1,2,3, by using kernel set[2,5] associated with the closed set. We prove the topological space is aT_0-space if and only if either $y \notin ker\{x\}$ or $x \notin \mathrm{ker}\{y\}$ for each $x \neq y \in X$ .and a topological space (X,T) is a T_1-space if and only if for each $x \neq y \in X$,then $x \notin \mathrm{ker}\{y\}$ and $y \notin \mathrm{ker}\{x\}$, also (X,T) is a T_1-spacce iff $ker\{x\} = \{x\}$, and by using kernel set, we states the relation between T_i-spaces $i = 0,1,2,3,4$ and R_i-spaces $i = 0,1,2,3$.

Definition 1.1.[2]

The intersection of all open subset of (X,T) containing A is called the kernel of A (briefly ker(A)), this means ker(A)=⋂{G∈T:A⊆G}

Definition 1.2.[1,2]

A topological space (X,T) is called an R_0-space if for each open set U and $x \in U\ then\ cl\{x\} \subseteq X$.

Definition 1.3.[1,2]

A topological space (X,T) is called an R_1- space if for each two distinct point x , y of X with $cl\{c\} \neq cl\{y\}$, there exist disjoint open sets U,V such that $cl\{c\} \subseteq U and\ cl\{c\} \subseteq V$..

Corollary 1.4.[2]

Let (X, T)be a topological space. Then (X,T) is R_0-space if and only if, $cl\{x\} = \mathrm{ker}\{x\}$, for each $x \in X$.

Definition 1.5 [1]

A topological space (X, T) is called an R_2-space are those which are property regular space.

Remark 1.6 [1]

The usual definition of “normality” must be modified slightly if R_3 is to be the axiom for normal spaces

Remark 1.7 [1]

Each separation axiom is defined as the conjunction of two weaker axioms: T_k-space $=R_{k-1}$-space and T_{k-1}-space = R_{k-1}-space and T_0-space

Remark 1.8 [1]

Every R_k-space is an R_{k-1}-space.

Theorem 1.9 [3]

Every compact Hausdorf space is aT_3-space (and consequently regular).

Theorem 1.10 [3]

Every compact Hausdorf space is a normal space (T_4-space).

***Lemma* 1.11** [2]

Let (X, T) be topological space then $x \in cl\{y\}\ iff\ y \in \ker\{x\}$. for each $x \neq y \in X$

2. R_i-Spaces, $i = 0, 1, 2, 3$

Theorem 2.1

A topological space (X, T) is an R_0-space if and only if for each F closed set and $x \in F$ then $ker\ \{x\} \subseteq F$.

Proof

Let a topological space (X,T) be a R_0-space and F be a closed set and $x \in F$.Then for each $y \notin F$ implies $y \in F^c$ is open set ,then $cl\{y\} \subseteq F^c$ [since (X,T) is R_0-space], so $x \notin cl\{y\}$. Hence by theorem 1.11, $y \notin ker\{x\}$. Thus $\ker\{x\} \subseteq F$

Conversely

Let for each F closed set and $x \in F$ then $ker\ \{x\} \subseteq F$and let $U \in T$, $x \in U$ then for each $y \notin U$ implies $y \in U^c$ is a closed set implies $\ker\{y\} \subseteq U^c$. Therefore $x \notin \ker\{y\}$ and $y \notin cl\{x\}$ [By theorem 1.11]. So $cl\{x\} \subseteq U$. Thus (X, T) is an R_0-space.

Corollary 2.2

A topological space (X, T) is an R_0-space if and only if for each U open set and $x \in U$ then $cl(ker\{x\}) \subseteq U$.

Theorem 2.3

A topological space (X, T) is an R_1-space if and only if for each$x \neq y \in X$ with $ker\{x\} \neq ker\{y\}$ then there exist closed sets F_1, F_2 such that $\ker\{x\} \subseteq F_1, \ker\{x\} \cap F_2 = \emptyset and \ker\{y\} \subseteq F_2, \ker\{y\} \cap F_1 = \emptyset and F_1 \cup F_2 = X$

Proof

Let a topological space (X, T) be an R_1-space. Then for each $x \neq y \in X$ with $ker\{x\} \neq ker\{y\}$. Since every $R_1 - space\ is\ an\ R_o\_space$ [by remark 1.8, hence by theorem 1.4, $cl\{x\} \neq cl\{y\}$, then there exist open sets G_1, G_2 such that $cl\{x\} \subseteq G_1\ and\ cl\{y\} \subseteq G_2 and\ G_1 \cap G_2 = \emptyset$ [Since (X, T) is R_1-space], then $G_1^c\ and\ G_2^c$ are closed sets such that$G_1^c \cup G_2^c = X$. Put $F_1 = G_1^c and F_2 = G_2^c$ Thus $x \in G_1 \subseteq F_2\ and\ y \in G_2 \subseteq F_1$ so that $\ker\{x\} \subseteq G_1 \subseteq F_2\ and\ ker\{y\} \subseteq G_2 \subseteq F_1$.

Conversely

Let for each $x \neq y \in X$ with $ker\{x\} \neq ker\{y\}$, there exist closed sets F_1, F_2 such that $\ker\{x\} \subseteq F_1, \ker\{x\} \cap F_2 = \emptyset and \ker\{y\} \subseteq F_2, \ker\{y\} \cap F_1 = \emptyset and F_1 \cup F_2 = X$, then $F_1^c\ and\ F_2^c$ are open sets such that $F_1^c \cap F_2^c = \emptyset$. Put $F_1^c = G_2$ *and* $F_2^c = G_1$. Thus $\ker\{x\} \subseteq G_1$ *and* $ker\{y\} \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$, so that $x \in G_1$ *and* $y \in G_2$ implies $x \notin cl\{y\}$ *and* $y \notin cl\{x\}$, then $cl\{x\} \subseteq G_1$ *and* $cl\{y\} \subseteq G_2$. Thus (X,T) is an R_1-space.

Corollary 2.4

A topological space (X, T) is an R_1-space if and only if for each $x \neq y \in X$ with $cl\{x\} \neq cl\{y\}$ then there exist disjoint open sets U, V such that $cl(ker\{x\}) \subseteq U$ *and* $cl(ker\{y\}) \subseteq V$

Proof

Let (X, T) be an R_1-space and let $x \neq y \in X$ with $cl\{x\} \neq cl\{y\}$, then there exist disjoint open sets U, V such that $cl\{x\} \subseteq U$ *and* $cl\{y\} \subseteq V$.

Also (X,T) is R_0-spece [by remark 1.8] implies for each $x \in X$, then $cl\{x\} = \ker\{x\}$ [By theorem 1.4], but $cl\{x\} = cl(cl\{x\}) = cl(\ker\{x\})$. Thus $cl(ker\{x\}) \subseteq U$ *and* $cl(ker\{y\}) \subseteq V$

Conversely

Let for each $x \neq y \in X$ with $cl\{x\} \neq cl\{y\}$ then there exist disjoint open sets U, V such that $cl(ker\{x\}) \subseteq U$ *and* $cl(ker\{y\}) \subseteq V$. Since $\{x\} \subseteq \ker\{x\}$ *then* $cl\{x\} \subseteq cl(\ker\{x\})$ for each $x \in X$ So we get $cl\{x\} \subseteq U$ *and* $cl\{$

Theorem 2.5

A topological space (X,T) is a regular space(R_2-space) if and only if for each closed subset G of X and $x \notin G$ with $\ker(G) \neq ker\{x\}$ then there exist closed sets F_1 , F_2 such that $\ker(G) \subseteq F_1, \ker(G) \cap F_2 = \emptyset$ *and* $\ker\{x\} \subseteq F_2, \ker\{x\} \cap F_1 = \emptyset$ *and* $F_1 \cup F_2 = X$

Proof

Let a topological space (X,T) be a regular space(R_2-space) and let G be a closed set, $x \notin G$,then there exist open sets U , V such that $G \subseteq U$, $x \in V$ *and* $U \cap V = \emptyset$, then U^c *and* V^c are closed sets such that $U^c \cup V^c = X$ .Put $F_2 = U^c$ *and* $F_1 = V^c$, so we get $\ker(G) \subseteq U \subseteq F_1, \ker(G) \cap F_2 = \emptyset$ *and* $\ker\{x\} \subseteq V \subseteq F_2, \ker\{x\} \cap F_1 = \emptyset$ *and* $F_1 \cup F_2 = X$.

Conversely

Let for each closed subset G of X and $x \notin G$ with $\ker(G) \neq ker\{x\}$ then there exist closed sets F_1 , F_2 such that $\ker(G) \subseteq F_1, \ker(G) \cap F_2 = \emptyset$ *and* $\ker\{x\} \subseteq F_2, \ker\{x\} \cap F_1 = \emptyset$ *and* $F_1 \cup F_2 = X$ .Then F_1^c *and* F_2^c are open sets such that $F_1^c \cap F_2^c = \emptyset$ and $\ker(G) \cap F_1^c = \emptyset$, $\ker\{x\} \cap F_2^c = \emptyset$,so that $G \subseteq F_2^c$ *and* $x \in F_1^c$. Thus (X, T) is a regular space (R_2-space).

***Lemma* 2. 6**

Let (X,T) be a regular space and F be a closed set. Then $\ker(F) = cl(F) = F$.

Proof

Let (X,T) be a regular space and F be a closed set. Then for each $x \notin F$,there exist disjoint open sets U, V such that $F \subseteq U$ and $x \in V$. Since $\ker(F) \subseteq U$,implies $\ker(F) \cap V = \emptyset$,thus $x \notin cl(\ker(F))$. We showing that if $x \notin F$ implies $x \notin cl(\ker(F))$, therefore $cl(\ker(F) \subseteq cl(F) = F$. As $cl(F) = F \subseteq \ker(F)$[By definition 1.1]. Thus $\ker(F) = cl(F) = F$.

Theorem 2.7

A topological space (X,T) is a regular space (R_2-space) if and only if for each closed subset F of X and $x \notin F$ with $cl(\ker(F)) \neq cl(ker\{x\})$ then there exist disjoint open sets U, V such that $cl(\ker(F)) \subseteq U$ and $cl(ker\{x\}) \subseteq V$.

Proof

Let a topological space (X,T) be a regular space (R_2-space)and let F be a closed set, $x \notin F$. Then there exist disjoint open set U, V such that $F \subseteq U$ and $x \in V$. By lemma 2.6, $cl(\ker(F)) = cl(F) = F$ .in the other hand (X,T) is an R_0-space [By remark 1.8]. Hence, by theorem 1.4, $cl\{x\} = \ker\{x\}$ for each $x \in X$. Thus $cl(\ker(F)) \subseteq U$ and $cl(ker\{x\}) \subseteq V$.

Conversely

Let for each closed set F and $x \notin F$ with cl(ker(F)) $\neq cl(ker\{x\})$ then there exist disjoint open sets U, V such that $cl(\ker(F)) \subseteq U$ and $cl(ker\{x\}) \subseteq V$ .Then $F \subseteq U$ and $x \in V$. Thus (X, T) a regular space(R_2-space).

Definition 2.8

A topological space (X,T) is an R_3-space if and only if (X, T) is a normal and R_1-space.

Theorem 2.9

Every R_3-space is a regular space(R_2-space).

Proof

Let F be a closed and $x \notin F$. Then $x \in F^c$ is an open set implies for each $y \in F$, $y \notin \ker\{x\}$, therefore $\ker\{x\} \neq \ker\{y\}$. Then there exist closed sets G_y , H_y such that $ker\{y\} \subseteq Gy$, $ker\{y\} \cap Hy = \emptyset$ and $ker\{x\} \subseteq Hy$, $Ker\{x\} \cap Gy = \emptyset$[Since(X,T) is R_1-space by assumption and by theorem 2.3], let $\beta = \cap\{H_y : x \in H_y\}$, is a closed set such that $\beta \cap F = \emptyset$. Hence (X, T) is a normal space then there exist disjoint open sets U, V such that F⊆U and $\beta \subseteq V$, so that $x \in V$. Thus (X, T) is a regular space.

3. T_i-Spaces, $i = 0, 1, \dots, 4$

Theorem 3.1

A topological space (X,T) is a T_0-space if and only if either $y \notin ker\{x\}$ or $x \notin ker\{y\}$, for each $x \neq y \in X$.

Proof

Let (X, T) is a T_0-space then for each $x \neq y \in X$, there exists an open set G such that $x \in G, y \notin G$ or $x \notin G, y \in G$. Thus either $x \in G, y \notin G$ implies $y \notin ker\{x\}$ or $x \notin G, y \in G$ implies $x \notin ker\{y\}$.

Conversely

Let either $y \notin ker\{x\}$ or $x \notin ker\{y\}$, for each $x \neq y \in X$. Then there exists an open set G such that x $\in G$, $y \notin G$ or $x \notin G, y \in G$. Thus (X, T) is a T_0 space.

Theorem 3.2

A topological space (X, T) is a T_1-space if and only if for each $x \neq y \in X$. $y \notin ker\{x\}$ and $x \notin ker\{y\}$

Proof

Let (X, T) is a T_1-space then for each $x \neq y \in X$, there exists an open sets U, V such that $x \in U, y \notin U$ or $y \in V, x \notin V$. Implies $y \notin ker\{x\}$ and $x \notin ker\{y\}$.

Conversely

Let $y \notin ker\{x\}$ and $x \notin ker\{y\}$, for each $x \neq y \in X$. Then there exists an open sets U, V such that $x \in U, y \notin U$ and $y \in V, x \notin V$. Thus (X, T) is a T_1- space.

Theorem 3.3

A topological space (X, T) is a T_1-space if and only if for each $x \in X$ then $ker\{x\} = \{x\}$.

Proof

Let (X, T) is aT_1-space and let $ker\{x\} \neq \{x\}$, then $ker\{x\}$ contains anther point distinct from x say y. So $y \in ker\{x\}$. Hence by theorem 3.2, (X, T) is not a T_1-space this is contradiction. Thus $ker\{x\} = \{x\}$

Conversely

Let $ker\{x\} = \{x\}$, for each $x \in X$ and let (X, T) is not a T_1-space. Then $y \in ker\{x\}$ (say)[By theorem 3.2], implies $ker\{x\} \neq \{x\}$, this is contradiction . Thus (X, T) is a T_1-space.

Theorem 3.4

A topological space (X, T) is a T_1-space if and only if for each $x \neq y \in X$ implies ker{x} ∩ker$\{y\} = \emptyset$.

Proof

Let a topological space (X, T) be a T_1-space.Then $ker\{x\} = \{x\}$ and $ker\{y\} = \{y\}$[By theorem 3.3]. Thus $\ker\{x\} \cap \ker\{y\} = \emptyset$.

Conversely

Let for each $x \neq y \in X$ implies $\ker\{x\} \cap \ker\{y\} = \emptyset$and let (X,T) is not T_1-space Then for each $x \neq y \in X$ implies $y \in ker\{x\}$ or $x \in ker\{y\}$, The $\ker\{x\} \cap \ker\{y\} \neq \emptyset$. This is contradiction. Thus (X, T) is a T_1-space.

Corollary 3.5

A topological an T_0-space is a T_2-space if and only if for each $x \neq y \in X$ with $ker\{x\} \neq ker\{y\}$ then there exist closed sets F_1, F_2 such that $\ker\{x\} \subseteq F_1, \ker\{x\} \cap F_2 = \emptyset and \ker\{y\} \subseteq F_2, \ker\{y\} \cap F_1 = \emptyset and F_1 \cup F_2 = X$

Proof

By theorem 2.3 and remark 1.7.

***Corollary* 3.6**

A topological T_1-space is a T_2-space if and only if one of the following conditions holds:

1) For each $x \neq y \in X$ with $cl\{x\} \neq cl\{y\}$ then there exist open sets U, V such that $cl(ker\{x\}) \subseteq U$ *and* $cl(ker\{y\}) \subseteq V$

2) for each $x \neq y \in X$ with $ker\{x\} \neq ker\{y\}$ then there exist closed sets F_1, F_2 such that $\ker\{x\} \subseteq F_1, \ker\{x\} \cap F_2 = \emptyset and \ker\{y\} \subseteq F_2, \ker\{y\} \cap F_1 = \emptyset and F_1 \cup F_2 = X$.

***Proof* (1)**

By corollary 2.4 and remark 1.7.

Proof (2)

By theorem 2.3 and remark 1.7.

Theorem 3.7

A topological R_1-space is a T_2-space if and only if one of the following conditions holds:

1) *For each* $x \in X, ker\{x\} = \{x\}$.

2) For each $x \neq y \in X$,$\ker\{x\} \neq \ker\{y\}$implies $\ker\{x\} \cap ker\{y\} = \emptyset$.

3) For each for each $x \neq y \in X$, *either* $x \notin \ker\{y\}$ *or* $y \notin \ker\{x\}$

4) For each for each $x \neq y \in X$ then $x \notin \ker\{y\}$ *and* $y \notin \ker\{x\}$.

Proof (1)

Let (X, T) be a T_2-space .Then (X, T) is a T_1-space and R_1-space [By remark 1.7]. Hence by theorem 3.3, ker{x} ={x} for each $x \in X$.

Conversely

Let for each $x \in X$, ker{x} ={x}, then by theorem 3.3, (X, T) is a T_1-space. Also (X, T) is an R_1-space by assumption. Hence by remark1.7, (X, T) is aT_2 space.

***Proof* (2)**

Let (X, T) be a T_2-space .Then (X, T) is a T_1-space. Hence by theorem 3.4, $\ker\{x\} \cap ker\{y\} = \emptyset$ for each $x \neq y \in X$.

Conversely

Assume that for each $x \neq y \in X$, $\ker\{x\} \neq \ker\{y\}$ implies $\ker\{x\} \cap ker\{y\} = \emptyset$, so by theorem 3.4, the topological space (X, T) is a T_1-space, also (X, T) is an R_1-space by assumption. Hence by remark 1.7, (X, T) is a T_2-space.

Proof (3)

Let (X, T) be a T_2-space .Then (X, T) is a T_0-space. Hence by theorem 23.1, *either* $x \notin \ker\{y\}$ *or* $y \notin \ker\{x\}$ for each $x \neq y \in X$.

Conversely

Assume that for each $x \neq y \in X$, $either\ x \notin \ker\{y\}\ or\ y \notin \ker\{x\}$for each $x \neq y \in X$.

, so by theorem 3.1, (X, T) is a T_0-space also (X, T) is an R_1-space by assumption. Thus (X, T) is a T_2-space [By remark 1.7].

Proof (4)

Let (X, T) be a T_2-space.Then (X,T) is a T_1-space and an R_1-space [By remark 1.7]. Hence by theorem 3.2, $x \notin \ker\{y\}\ and\ y \notin \ker\{x\}$.

Conversely

Let for each $x \neq y \in X$ then $x \notin \ker\{y\}\ and\ y \notin \ker\{x\}$. Then by theorem 3.2, (X, T) is a T_1-space. Also (X, T) is an R_1-space by assumption. Hence by remark 1.7, (X, T) is a T_2-space.

Theorem 3.8

A topological space (X,T) is a normal space if and only if for each disjoint closed sets G , H with $ker(G) \neq ker(H)$ then there exist closed sets F_1 , F_2 such that $\ker(G) \subseteq F_1, \ker(G) \cap F_2 = \emptyset and\ \ker(H) \subseteq F_2, \ker(H) \cap F\_1 = \emptyset and\ F\_1 \cup F\_2 = X$.

Proof

Let (X,T) be a normal topological space and let for each disjoint closed sets G , H with $cl(G) \neq cl(H)$ then there exist disjoint open sets U , V such that$G \subseteq U$, $H \subseteq V\ and\ U \cap V = \emptyset$, then $U^c\ and\ V^c$ are closed sets such that $U^c \cup V^c = X$ and $\ker(G) \cap U^c = \emptyset$, $ker(H) \cap V^c = \emptyset$. Put $F_2 = U^c\ and\ F_{1=}V^c$ .Thus $\ker(G) \subseteq U \subseteq F_1, \ker(G) \cap F_2 = \emptyset\ and\ \ker(H) \subseteq V \subseteq F_2, \ker(H) \cap F_1 = \emptyset\ and\ F_1 \cup F_2 = X$.

Conversely

Let for each disjoint closed sets G, H with $ker(G) \neq ker(H)$, there exist closed sets F_1 , F_2 such that $\ker(G) \subseteq F_1, \ker(G) \cap F_2 = \emptyset\ and\ \ker(H) \subseteq F_2, \ker(H) \cap F_1 = \emptyset\ and\ F_1 \cup F_2 = X$, implies $F_1^c\ and\ F_2^c$ are open sets such that $F_1^c \cap F_2^c = \emptyset$ and $\ker(G) \cap F_1^c = \emptyset$, $\ker(H) \cap F_2^c = \emptyset$,so that $G \subseteq F_2^c\ \ and\ H \subseteq F_1^c$. Thus (X,T) is a normal space.

Theorem 3.9

A topological compact an R_1-space is a T_3-space if and only if one of the following conditions holds:

1)$for\ each\ x \in X, \ker\{x\} = \{x\}$

2)$for\ each\ x \neq y \in X, \ker\{x\} \cap ker\{y\} = \emptyset$.

3)$for\ each\ x \neq y \in X\ then\ x \notin \ker\{y\}\ and\ y \notin \ker\{x\}$

4)$for\ each\ x \neq y \in X\ either\ x \notin \ker\{y\}\ or\ y \notin \ker\{x\}$

Proof (1)

Let (X, T) be a T_3-space .Then (X,T) is a T_1-space , by theorem 3.3, $for\ each\ x \in X, \ker\{x\} = \{x\}$

Conversely

Let for each $x \in X, ker\{x\} = \{x\}$, then by theorem 3.3. (X, T) is a T_1-space. Also (X, T) is a compact R_1-space by assumption. So by remark1.7, we get (X, T) is a compact T_2- space .Hence by theorem 1.9, (X, T) is a T_3-space

Proof (2)

Let (X, T) be a T_3-space .Then (X, T) is a T_1-space. Hence by theorem 3.4, $for\ each\ x \neq y \in X, \ker\{x\} \cap ker\{y\} = \emptyset$

Conversely

Assume that $for\ each\ x \neq y \in X, \ker\{x\} \cap ker\{y\} = \emptyset$, so by theorem 3.4, the topological space (X, T) is a T_1- space, also (X,T) is a compact R_1-space by assumption. So by remark1.7, (X,T) is a compact T_2-space. Hence by theorem 1.9, (X, T) is a T_3- space.

Proof (3)

Let (X, T) be a T_3-space .Then (X, T) is a T_1-space. Hence by theorem 3.3, then $for\ each\ x \neq y \in X\ then\ x \notin \ker\{y\}\ and\ y \notin \ker\{x\}$

Conversely

Assume that $for\ each\ x \neq y \in X\ then\ x \notin \ker\{y\}\ and\ y \notin \ker\{x\}$, then by theorem 3.3, (X, T) is a T_1-space also(X, T) is a Compact R_1-space by assumption. Thus (X, T) is a compact T_2-space [By remark 1.7]. Hence by theorem 1.9, (X, T) is a T_3-space.

Proof (4)

Let (X, T) be a T_3-space.Then (X,T) is a T_0-space. So by theorem 3.1, $either\ x \notin \ker\{y\}\ or\ y \notin \ker\{x\}$, for each $x \neq y \in X$

Conversely

Let $for\ each\ x \neq y \in X\ either\ x \notin \ker\{y\}\ or\ y \notin \ker\{x\}$. Then by theorem 3.1, (X, T) is a T_1-space. Also (X, T) is a compact R_1-space by assumption . Hence by remark 1.7, (X, T) is a compact T_2-space. Henceby theorem1.9, (X,T) is a T_3-space.

Theorem 3.11

A topological compact an R_1-space is a T_4-space if and only if one of the following conditions holds:

a) $for\ each\ x \in X, \ker\{x\} = \{x\}$

b) $for\ each\ x \neq y \in X, \ker\{x\} \cap ker\{y\} = \emptyset$

c) $for\ each\ x \neq y \in X\ then\ x \notin \ker\{y\}\ and\ y \notin \ker\{x\}$

d) $for\ each\ x \neq y \in X\ either\ x \notin \ker\{y\}\ or\ y \notin \ker\{x\}$

Proof (1)

Let (X, T) be a T_4-space .Then (X, T) is a T_1-space, by theorem 3.3, $ker\{x\} = \{x\}$ for each $x \in X$.

Conversely

Let for each $x \in X$, $ker\{x\} = \{x\}$, then by theorem 3.3. (X, T) is a T_1-space. Also (X, T) is a compact R_1-space by assumption. So by remark1.7, we get (X, T) is a compact T_2- space .Hence by theorem1.10, (X, T) is a T_4-space

Proof (2)

Let (X, T) be a T_4-space .Then (X,T) is a T_1-space. Hence by theorem 3.4, $for\ each\ x \neq y \in X, \ker\{x\} \cap ker\{y\} = \emptyset$

Conversely

Assume that $for\ each\ x \neq y \in X, \ker\{x\} \cap ker\{y\} = \emptyset$ so by theorem 3.4, the topological space (X, T) is a T_1-space, also (X, T) is a compact R_1-space by assumption. So by remark1.7, (X, T) is a compact T_2-space. Hence by theorem 1.10, (X, T) is a T_4-space .

Proof (3)

Let (X, T) be a T_4-space .Then (X, T) is a T_1-space. Hence by theorem 3.2, then $for\ each\ x \neq y \in X\ then\ x \notin \ker\{y\}\ and\ y \notin \ker\{x\}$

Conversely

Assume that $for\ each\ x \neq y \in X\ then\ x \notin \ker\{y\}\ and\ y \notin \ker\{x\}$, then by theorem 3.2, (X, T) is a T_1-space also(X, T) is a compact R_1-space by assumption. Thus (X,T) is a compact T_2-space [By remark 1.7]. Hence by theorem 1.10, (X,T) is a T_4-space

Proof (4)

Let (X, T) be a T_4-space.Then (X, T) is a T_0-space. So by theorem 3.1, $for\ each\ x \neq y \in X\ either\ x \notin \ker\{y\}\ or\ y \notin \ker\{x\}$

Conversely

Let $for\ each\ x \neq y \in X\ either\ x \notin \ker\{y\}\ or\ y \notin \ker\{x\}$ Then by Theorem 3.1, (X, T) is a T_0-space. Also (X, T) is a compact R_1-space by assumption. Hence by remark 1.7, (X, T) is a compact T_2-space. Hence by theorem1.9,(X,T) is a T_4-space.

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