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Coconvex multi approximation in the uniform norm

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Abstract

In this research, we study the results about coconvex multi approximation. Recently this subject received a lot of attention. The first result in the topic is a direct theorem for the error bound of the convex approximation for functions of two continuous derivatives that were introduced by Kopotun. In 2006 Kopotun, Leviatan and Shevchuk gave definitive answers to open problems of Jackson's estimates by the moduli of smoothness of Ditzian-Totiktype, also in 2019 they stated many open problems in the coconvex case and mentioned recent developments on this subject. In convex multi approximation, we have a function $f \in C[-1,1]^d$ change its convexity at a finite number of times in $[-1, 1]^d$ and we take care of error bound of approximation of the function *f* using multi algebraic polynomials, in other words, multi algebraic polynomials change its convex particularly at the same points where *f* is. We can get some of Jackson's estimates in which the intended constants depend on the place of the convexity change points. In order to clarify the complexities, we discuss in this research invers inequality for the coconvex approximation of the function *f* using multi algebraic polynomials.

Subject Classification: *41A55; 30E20.*

Keywords: *Shape preserving approximation, Coconvex approximation, Moduli of smoothness.*

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1. Introduction

Let $f \in C[-1,1]^d$ change its convexity at a finite number of times, say $r \geq 0$ times, in $[-1, 1]^d$ and denote by C the space of all continuous multi variate functions defined on $[-1, 1]^d$ and C^s be the space of all functions of $s-$ times continuous derivatives defined on $[-1, 1]^d$. We estimate the error bound of approximation of the function f using multi polynomials which are convex with it, in other words, multi-polynomials that specifically change their convexity at the same points where f is. The entry into this topic began in the early sixties through the work of Lorentz, Shisha and Zeller on monotone approximation. It took great attention in the seventies and early eighties and this is what we find in DeVore's work on monotone approximation, and works of Newman, Beatson, Shvedov and Leviatan on comonotone approximation. In [1] Kopotun, Leviatan and Shevchuk proved that if a function $f \in C[-1,1]$ changes convexity at Y_s then

$$E_n^{(2)}(f, Y_s) \leq c\omega_3^{\varphi}\left(f, \frac{1}{n}\right) \leq c\omega_3\left(f, \frac{1}{n}\right), n \geq N \tag{1.1}$$

In [5] Wu and Zhou proved that for $k \geq 4$, the error bound in (1.1) can't be had with ω_3 takes his place ω_k. In 2014 M.A. Kareem [4] proved that provided n is large enough and this depends on the place of the points Y_s, the approximation rate is estimated by the moduli of smoothness of the multi third Ditzian –Totik type multiplied by a constant C. In 2019 Kopotun, Leviatan and Shevchuk[2] discussed uniform recent estimates in co-monotone approximation, stated many open problems in coconvex approximation and mentioned recent developments on this subject. Thus, in this work we have direct and inverse inequality on coconvex multi approximation on $[-1,1]^d$ in the uniform norm by multi polynomials.

2. The Main Results

Given $f \in C$, and $k \in N$, let

$$\Delta_h^k f((x_1, \dots, x_d)) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f\left(\left(x_1 - \frac{k}{2}h_1 + ih_1, \dots, x_d - \frac{k}{2}h_d + ih_d\right)\right),$$

be the symmetric multi kth difference, defined for all $x = (x_1, \dots, x_d)$ and $h_{\mathcal{J}} \geq 0, \mathcal{J} = 1, \dots, d \ni x_{\mathcal{J}} \pm \frac{k}{2}h_{\mathcal{J}} \in [-1,1]$, $k \in N$. We define $\|f\| = \sup_{\substack{x_{\mathcal{J}} \in [-1,1] \\ \mathcal{J}=1,\dots,d}} |f((x_1, \dots, x_d))|$. To prove our main results, we need to define the moduli of smoothness $\omega_k^{\varphi}(f,t)$ of the multi *kth* order of Ditzian –Totik type , we set

$$\omega_k^{\varphi}(f,t)=\sup_{0\le h_{\mathcal{J}}\le t_{\mathcal{J}}}\ \sup_{\substack{x_{\mathcal{J}}\\ \mathcal{J}=1,\dots,d}}|\Delta^k_{h\varphi(x)}f((x_1,\dots,x_d))|,t_{\mathcal{J}}\ge 0,$$

$h\varphi(x)=(h_1\varphi(x_1),\dots,h_d\varphi(x_d)),h=(h_1,\ \dots,\ h_d)$ and $t=(t_1,\ \dots,\ t_d)$, where $\varphi(x_{\mathcal{J}})=\sqrt{1-x_{\mathcal{J}}^2}$, we take the inner supremum on all $x=(x_1,\dots,x_d)$ $\ni x_{\mathcal{J}}\pm\frac{k}{2}h_{\mathcal{J}}\varphi(x_{\mathcal{J}})\in[-1,1]$. Also, We need to define the following ordinary moduli of smoothness $\omega_k(f,t)=\sup_{0\le h_{\mathcal{J}}\le t_{\mathcal{J}}}\ \sup_{\substack{x_{\mathcal{J}}\\ \mathcal{J}=1,\dots,d}}|\Delta^k_h f((x_1,\dots,x_d))|,\ t_{\mathcal{J}}\ge 0$, we take the inner supremum on all $x=(x_1,\dots,x_d)\ni x_{\mathcal{J}}\pm\frac{k}{2}h_{\mathcal{J}}\in[-1,1]$.

Let $\mathbb{A}_r, r\in N$ be the set of all collections $A_r=\{(y_{11},\dots,y_{1d}),\dots,(y_{r1},\dots,y_{rd})\}$, such that $-1<y_{r1}<\dots<y_{11}<1,\dots,-1<y_{rd}<\dots<y_{1d}<1$. For later reference set $y_{0\mathcal{J}}=1$ and $y_{r\mathcal{J}+1}=-1$. Finally , let $\Delta^2(A_r)$ be the set of $f\in C[-1,1]^d$ that are convex variable at the set A_r and are convex in $[y_{11},1]\times[y_{12},1]\times\dots\times[y_{1d},1]$. Let $n\in N,n>1$, we denote $x_{e\mathcal{J}}=x_{e\mathcal{J},n}$ $=\cos(e\mathcal{J}\pi/n)$ where $e=0,\dots,n$ and $\mathcal{J}=1,\dots,d$, the partition of chebyshev on $[-1,1]^d$ also assume $J_e=J_{e,n}=[x_{e1},x_{e1-1}]\times\dots\times[x_{ed},x_{ed-1}]$, $e=1,\dots,n$. Let $\Omega_{k,n}$ denote the set of each continuous piecewise multipolynomials of degree k - 1, on the partition of chebyshev , also let $\Omega^1_{k,n}\subseteq\Omega_{k,n}$, denote the subset of all functions of continuously differentiable. i.e. , when $B\in\Omega_{k,n}$, we get $B|_{J_e}=q_e$, $e=1,\ \dots,\ n$, where $q_e\in\Pi_{k-1}$, the set of multipolynomials of degree less than or equal $k-1$, and $q_e((x_{e1},\dots,x_{ed}))=q_{e+1}((x_{e1},\dots,x_{ed}))$, $e=1,\dots,n-1$, also when $B\in\Omega^1_{k,n}$, we have $q'_e((x_{e1},\dots,x_{ed}))=q'_{e+1}((x_{e1},\dots,x_{ed}))$, $e=1,\dots,n-1$. Given $A_r\in\mathbb{A}_r$, let $D_i=D_{i,n}(A_r)=(x_{e1+1},x_{e1-2})\times\dots\times(x_{ed+1},x_{ed-2})$, if $y_i\in[x_{e1},x_{e1-1})\times\dots\times[x_{ed},x_{ed-1})$, where $x_{nj+1}=-1,x_{j-1}=1$, and denote $D=D(n,A_r)=\bigcup_{i=1}^r D_i$, $D(n,\varnothing)=\varnothing$. Finally, we write $e\in H=H(n,A_r)$, if $J_e\cap D=\varnothing$. Denote by $\Omega_{k,n}(A_r)\subseteq\Omega_{k,n}$ and $\Omega^1_{k,n}(A_r)\subseteq\Omega^1_{k,n}$, the subsets of those piecewise multi polynomials for which $q_e\equiv q_{e+1}$, whenever both $e,e+1\notin H$. We want to find the approximation of $f\in\Delta^2(A_r)$, using multipolynomials which are coconvex with *f*, and belong to $\Delta^2(A_r)$. Let $E_n^{(2)}(f,A_r)=\inf\|f-q_n\|$ $:q_n\in\Pi_n\cap\Delta^2(A_r)$ be the error bounded of the coconvex approximation for the function *f* using multi algebraic polynomials , where q_n defined as $q_n(x)=(a_{01}+a_{02}+\dots+a_{0d})+a_1(x_1+x_2+\dots+x_d)+a_2(x_1^2+x_2^2+\dots+x_d^2)+\dots+$

$a_n(x_1^n + x_2^n + \ldots + x_d^n)$, and Π_n be the set of multipolynomials of degree less than or equal n. Now let us introduce our main results:

Theorem 1 : *For all $f \in C^s \cap \Delta^2(A_r)$, for no $k \geq 1$, $s = 0,1,2,3$ and $r \geq 2$, is it possible to have constants $c = (k, s, r, d)$ and $N = (k, s, r, d)$, just depend on k, s, r and d, such that the inequality*

$$E_n^{(2)}(f, A_r) \leq \frac{c(d)}{n^s}\omega_k(f^{(s)}, \eta), \eta = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right), \tag{2.1}$$

holds for all $n \geq N$.

Theorem 2 : *For each $k, n \in N$ and $r \in N_0$ there exists constants c and $\hat{c}$, such that if $B \in \Omega_{k,n}(A_r) \cap \Delta^2(A_r)$, then there exists a multi polynomial $Q_n \in \Delta^2(A_r)$ of degree less than or equal $\hat{c}\, n$, satisfying*

$$\left\| B - Q_n \right\| \leq c(d)\omega_k^{\varphi}(B, \eta), \eta = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right). \tag{2.2}$$

Theorem 3 : *For each $k, n \in N$ and $r \in N_0$ there exists constants c and $\hat{c}$, such that if $B \in \Omega_{k,n}^1(A_r) \cap \Delta^2(A_r)$, then there exists a multi polynomial $Q_n \in \Delta^2(A_r)$ of degree less than or equal $\hat{c}\, n$, satisfying* (2.2).

We observe by proving Lemma 1 , the proof of theorem 2 is achieved, and thus sufficient for proof theorem 3 .

Lemma 1 : *Let $k \geq 3$. Then for every $B \in \Omega_{k,n}(A_r) \cap \Delta^2(A_r)$, there is an $\tilde{B} \in \Omega_{k,n}^1(A_r) \cap \Delta^2(A_r)$, such that*

$$\left\| B - \tilde{B} \right\| \leq c\,(d)\,\omega_k^{\varphi}(B, \eta), \eta = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right). \tag{2.3}$$

***Proof*:** for every $2 \leq e \leq n$, set

$$a_e(x_1, \ldots, x_d) = \frac{1}{2}\frac{(x_{e1-1} - x_{e1-2}).(x_{e2-1} - x_{e2-2})\ldots(x_{ed-1} - x_{ed-2})}{(x_{e1-1} - x_{e1}).(x_{e2-1} - x_{e2})\ldots(x_{ed-1} - x_{ed})}\cdot$$
$$\frac{q_{e-1}'((x_{e1-1}, \ldots, x_{ed-1})) - q_e'((x_{e1-1}, \ldots, x_{ed-1}))}{(x_{e1} - x_{e1-2})\ldots(x_{ed} - x_{ed-2})}\cdot$$
$$(x_1 - x_{e1})^2 \ldots (x_d - x_{ed})^2,$$

$$\text{if } e,(e-1)\in H, a_e(x_1,\dots,x_d)=\frac{1}{2}\frac{q'_{e-1}((x_{e1-1},\dots,x_{ed-1}))-q'_e((x_{e1-1},\dots,x_{ed-1}))}{(x_{e1-1}-x_{e1})\dots(x_{ed-1}-x_{ed})}\cdot$$
$$(x_1-x_{e1})^2\dots(x_d-x_{ed})^2$$

If $e,(e-1)\notin H$, and $a_e(x_1,\dots,x_d)=0$, if $e\notin H$. For each $1\le e\le n-1$,

$$b_e(x_1,\dots,x_d)=\frac{1}{2}\frac{(x_{e1}-x_{e1+1}).(x_{e2}-x_{e2+1})\dots(x_{ed}-x_{ed+1})}{(x_{e1}-x_{e1-1}).(x_{e2}-x_{e2-1})\dots(x_{ed}-x_{ed-1})}\cdot$$
$$\frac{q'_e((x_{e1},\dots,x_{ed}))-q'_{e+1}((x_{e1},\dots,x_{ed}))}{(x_{e1+1}-x_{e1-1})\dots(x_{ed+1}-x_{ed-1})}\cdot$$
$$(x_1-x_{e1-1})^2\dots(x_d-x_{ed-1})^2,$$

$$\text{if } e,(e+1)\in H, b_e(x_1,\dots,x_d)=\frac{1}{2}\frac{q'_e((x_{e1},\dots,x_{ed}))-q'_{e+1}((x_{e1},\dots,x_{ed}))}{(x_{e1-1}-x_{e1})\dots(x_{ed-1}-x_{ed})}\cdot$$
$$(x_1-x_{e1-1})^2\dots(x_d-x_{ed-1})^2,$$

if $e\in H,(e+1)\notin H$, and $b_e(x_1,\dots,x_d)=0$, if $e\notin H$. Finally set $a_1(x_1,\dots,x_d)=0$ and $b_n(x_1,\dots,x_d)=0$. Then

$$\tilde{B}=q_e((x_1,\dots,x_d))+a_e((x_1,\dots,x_d))+b_e((x_1,\dots,x_d))+I((x_1,\dots,x_d)),$$

$x\in J_e$ be the desired function , I is a constant piecewise function has jumps in at most the $2r$ points $x_e=(x_{e1},\dots,x_{ed})$ near the y_i's, $y_i=(y_{i1},\dots,y_{id})$, honestly, the jumps for these points are

$$I((x_{e1}+,\dots,x_{ed}+))-I((x_{e1}-,\dots,x_{ed}-))$$
$$=\begin{cases}\frac{1}{2}[q'_e((x_{e1},\dots,x_{ed}))-q'_{e+1}((x_{e1},\dots,x_{ed}))]\,(x_{e1}-x_{e1+1})\cdots(x_{ed}-x_{ed+1}),\\ \text{if } e\notin H,\ (e+1)\in H\\ \frac{1}{2}[q'_e((x_{e1},\dots,x_{ed}))-q'_{e+1}((x_{e1},\dots,x_{ed}))](x_{e1}-x_{e1-1})\cdots(x_{ed}-x_{ed-1}),\\ \text{if } e\in H,\ (e+1)\notin H.\end{cases}$$

In fact, the direct calculations show that $\tilde{B}\in\Omega^1_{k,n}(A_r)\cap\Delta^2(A_r)$, and by Markov's inequality we get

$$q'_e((x_{e1},\dots,x_{ed}))-q'_{e+1}((x_{e1},\dots,x_{ed}))$$
$$\le\frac{(2k^2)^d}{(x_{e1-1}-x_{e1})\dots(x_{ed-1}-x_{ed})}\left\|q_e-q_{e+1}\right\|_{J_e}.$$

Thus (2.3) easily track by the inequality $\left\|q_e-q_{e+1}\right\|_{J_e}\le c(d)\,\omega_k^{\varphi}(B,\eta)$.

3. Negative Result

By [3], there exists $0 < b < 1$, set $g_b''(x) = \begin{cases} -b^{-4}(x^2-b^2)^2, |x|<b, \\ 0, \quad \text{otherwise,} \end{cases}$

and let $g_b(x) = \int_0^x (x-u)\, g_b''(u)\, du$. Thus let $G_b(x) = g_b(x_1) + \ldots + g_b(x_d)$, where $g_b(x_{\mathcal{J}}) = \int_0^{x_{\mathcal{J}}} (x_{\mathcal{J}} - u_{\mathcal{J}})\, g_b''(u_{\mathcal{J}})\, du_{\mathcal{J}}$, $\mathcal{J} = 1,\ldots,d$. It is clear $G_b \in C^3$ and $\| G_b \| = \| g_b + \ldots + g_b \| \le \| g_b \| + \ldots + \| g_b \|$

$$\le \frac{2b}{3} + \ldots + \frac{2b}{3} = d\left(\frac{2b}{3}\right), \tag{3.1}$$

$$\| G_b' \| = \| g_b' + \ldots + g_b' \| \le \| g_b' \| + \ldots + \| g_b' \| = d\left(\frac{8b}{15}\right),$$

$$\| G_b'' \| = \| g_b'' + \ldots + g_b'' \| \le \| g_b' \| + \ldots + \| g_b' \| = 1 + \ldots + 1 = d, \text{ and } G_b^{(3)} \le d(2b^{-1}).$$

Lemma 2 : *Let* $n \ge 1$, *for each multi polynomial* Q_n *of degree not exceeding n, and satisfying* $(x_1^2 - b^2)\, q''(x_1) + (x_2^2 - b^2)\, q''(x_2) + \ldots + (x_d^2 - b^2)\, q''(x_d) \ge 0, x \in \left[\frac{-1}{2}, \frac{1}{2}\right]^d$, *with* $b = \frac{1}{2} n^{\frac{-4}{3}}$, *we have* $\| G_b - Q_n \| > \frac{db}{40}$, *where* $Q_n = q_n(x_1) + q_n(x_2) + \ldots + q_n(x_d)$ *and* $q_n(x_1) = a_{01} + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n$

$$q_n(x_2) = a_{02} + a_1 x_2 + a_2 x_2^2 + \ldots + a_n x_2^n$$

$$\vdots$$

$$q_n(x_d) = a_{0d} + a_1 x_d + a_2 x_d^2 + \ldots + a_n x_d^n.$$

Proof : We note that $Q_n''((\pm b,\ldots,\pm b)) = 0$ and that $Q_n''((x_1,\ldots,x_d)) \le 0$, for $-b < x_{\mathcal{J}} < b$, $\mathcal{J} = 1,\ldots,d$. Suppose that for some $-b < x_{0\mathcal{J}} < b$, $Q_n''((x_{01},\ldots,x_{0d})) < \frac{-d}{4}$. Then

$$|[Q_n''; -b, (x_{01},\ldots,x_{0d}), b]| = \frac{|q_n''(x_{01})|}{(b-x_{01})(b+x_{01})} + \ldots + \frac{|q_n''(x_{0d})|}{(b-x_{0d})(b+x_{0d})} > \frac{d}{4b^2}.$$

Because $[Q_n''; -b, (x_{01},\ldots,x_{0d}), b] = \frac{d}{2} Q_n^{(4)}((\theta_1,\ldots,\theta_d))$, for some $-b < \theta_{\mathcal{J}} < b$, and by Bernstein's inequality we get $n^4 \| Q_n \| \ge \frac{d}{2}$ $| Q_n^{(4)}((\theta_1,\ldots,\theta_d)) | > \frac{d}{4b^2}$. Now by (3.1) and the assessed value of b,

$$\| G_b - Q_n \| \geq \| Q_n \| - \| G_b \| > d\left(\frac{1}{4n^4 b^2} - \frac{2b}{3} \right) = d\left(\frac{4b}{3} \right). \tag{3.2}$$

If on the other hand, $Q_n''((x_1,\dots,x_d)) \geq \frac{-d}{4}$ for every $-b < x_{\mathcal{J}} < b$, $\mathcal{J} = 1,..,d$, then we put Q_n in the formula

$$\begin{aligned} Q_n((x_1,\dots,x_d)) &= Q_n((0,\dots,0)) + (x_1 \dots x_d) Q_n'((0,\dots,0)) \\ &\quad + (\int_0^{x_1} (x_1 - u_1) q_n''(u_1)\, du_1 + \dots \\ &\quad + \int_0^{x_d} (x_d - u_d) q_n''(u_d)\, du_d). \end{aligned}$$

Because $Q_n''((x_1,\dots,x_d)) \geq 0$ for $-b \leq |x_{\mathcal{J}}| \leq \frac{1}{2}$, we get

$$\begin{aligned} & Q_n\left(\left(\frac{-1}{2},\dots,\frac{-1}{2}\right)\right) - 2Q_n((0,\dots,0)) + Q_n\left(\left(\frac{1}{2},\dots,\frac{1}{2}\right)\right) \\ &= \left(\int_0^{\frac{1}{2}} \left(\frac{1}{2} - u_1\right) q_n''(u_1)\, du_1 + \int_0^{\frac{-1}{2}} \left(\frac{-1}{2} - u_1\right) q_n''(u_1)\, du_1 \right) \\ &+ \dots \\ &+ \left(\int_0^{\frac{1}{2}} \left(\frac{1}{2} - u_d\right) q_n''(u_d)\, du_d + \int_0^{\frac{-1}{2}} \left(\frac{-1}{2} - u_d\right) q_n''(u_d)\, du_d \right) \\ &\geq \left(\int_0^{b} \left(\frac{1}{2} - u_1\right) q_n''(u_1)\, du_1 + \int_0^{b} \left(\frac{1}{2} - u_1\right) q_n''(-u_1)\, du_1 \right) \\ &+ \dots \\ &+ \left(\int_0^{b} \left(\frac{1}{2} - u_d\right) q_n''(u_d)\, du_d + \int_0^{b} \left(\frac{1}{2} - u_d\right) q_n''(-u_d)\, du_d \right) \\ &\geq \frac{-db}{4}. \end{aligned}$$

Also the same, $G_b\left(\left(\frac{-1}{2},\dots,\frac{-1}{2}\right)\right) - 2G_b((0,\dots,0)) + G_b\left(\left(\frac{1}{2},\dots,\frac{1}{2}\right)\right)$

$$= 2\left(\int_0^b \left(\frac{1}{2} - u_1\right) g''(u_1)\, du_1 + \dots + \int_0^b \left(\frac{1}{2} - u_d\right) g''(u_d)\, du_d \right) = \frac{-8db}{15} + \frac{db^2}{3}.$$

Hence $4\left\|G_b - Q_n\right\| \geq \left(Q_n\left(\left(\frac{-1}{2},\ldots,\frac{-1}{2}\right)\right) - G_b\left(\left(\frac{-1}{2},\ldots,\frac{-1}{2}\right)\right)\right) - 2(Q_n((0,\ldots,0))$ $- G_b((0,\ldots,0))) + \left(Q_n\left(\left(\frac{1}{2},\ldots,\frac{1}{2}\right)\right) - G_b\left(\left(\frac{1}{2},\ldots,\frac{1}{2}\right)\right)\right) \geq \frac{-db}{4} + \frac{8db}{15} - \frac{db^2}{3} \geq \frac{db}{10}$. Thus using (3.2), to prove Lemma 2. As a direct result we obtain

Corollary 1 : *For all $T > 1$, where T is constant, there is a large enough $N(T)$ such that if $n > N(T)$, then for any $r \geq 2$, there exists a function $G = G_n \in C^3[-1,1]^d$, which changes the convex r of times in $[-1,1]^d$, and any multi polynomial Q_n of degree not exceeding n which is convex with it, such that*

$$\left\|G - Q_n\right\| > \frac{T\left\|G^{(3)}\right\|}{n^3},\ \left\|G - Q_n\right\| > \frac{T\left\|G''\right\|}{n^2},\ \text{and}\ \left\|G - Q_n\right\| > \frac{T\left\|G'\right\|}{n}.$$

***Proof*:** Let $N(T) = (80\,T)^3$ and let $r \geq 2$. We put $b = b_n$, $n > N(T)$ as in Lemma 2, and suppose that the function $G = G_b$ changes convexity at $y_{2\mathcal{J}} = -b$ and $y_{1\mathcal{J}} = b, \mathcal{J} = 1,..,d,$ it is convex in $[y_{11},1]\times[y_{12},1]\times\ldots\times[y_{1d},1]$, and if $r > 2$, then we take points arbitrarily r - 2 satisfying $-1 < y_{r1} < \ldots < y_{31} < \frac{-1}{2}$, $\ldots, -1 < y_{rd} < \ldots < y_{3d} < \frac{-1}{2}$, as for G it has the same convex change at these points as well , therefore $G \in \Delta^2(A_r)$. The multi polynomial Q_n is co convex with G, implies that it satisfies the conditions of Lemma 2 . Hence, using Lemma 2 we get

$$\left\|G - Q_n\right\| > \frac{db}{40} \geq \frac{\left\|G^{(3)}\right\|b^2}{80} > \frac{T\left\|G^{(3)}\right\|}{n^3},\ \left\|G - Q_n\right\| > \frac{db}{40} \geq \frac{T\left\|G''\right\|}{n^2},$$

and $\left\|G - Q_n\right\| > \frac{db}{40} = \frac{3n\left\|G'\right\|}{64n} > \frac{T\left\|G'\right\|}{n}$.

Remark : Note that G_b above is a function independent of T.

Proof of Theorem 1 : Our proof comes easily through the information that indicates, for each $k \geq 1, \omega_k(f,t) \leq (2^{k-1})^d\,\omega_k(f,t) \leq (2^{k-1})^d\, t\left\|f'\right\|$, which by Corollary 1 does not apply to case $s = 0$ in (2.1) and $\omega_k(f,t) \leq (2^k)^d\left\|f\right\|$, which is interested in the other cases.

4. Conclusion

Multi direct coconvex theorem for approximation in terms of the k -*th* order ordinary modulus of smoothness for functions of 3-continuous derivatives. Then to strength this result we prove a negative theorem for it.

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