TRAVELING WAVE SOLUTIONS FOR TIME-FRACTIONAL $B(m,n)$ EQUATIONS

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Received October 10, 2021

Abstract. In this paper, we examine time-fractional Boussinesq-like $B(m,n)$ equations that model some real-life systems, where the Caputo sense of fractional derivative has been used. The examined equations are characterized by a set of initial and boundary conditions. We will apply the homotopy perturbation method for analytical treatment of these equations. A comparison between our solutions and other existing solution in the literature, shows accuracy of the employed analysis. Proper graphs are used to illustrate the obtained results.

Key words: Semi-analytical solution, Homotopy perturbation method, Fractional $B(m,n)$ equation.

1. INTRODUCTION

Semi-analytical methods are powerful tools to solve nonlinear partial differential equations (NLPDEs). Several techniques, such as the variational iteration method (VIM) [1–3], homotopy perturbation method (HPM) [4–6], Adomian decomposition method (ADM) [7, 8], homotopy analysis method (HAM) [9, 10], parameter expansion method (PEM) [11], weighted residual method, Adomian-Laplace method, and other methods have been implemented independently to handle NLPDEs, numerically and analytically.

Many researchers applied different methods to solve fractional problems, for example you can see [12–26]. A substantial amount of research work has been invested for the study of the time-fractional equations in the literature. Recently, there has been considerable attention from the scientific community devoted to the analytical and numerical studies of fractional differential equations (FDEs). This follows from the advantages of using derivatives of arbitrary orders in the modelling of different phenomena in physics, engineering, finance, chemistry, and the life sciences. Fractional calculus (differential and integral operators of non-integer orders) is often used to model real-life systems. Fractional differentiation has been found to be effective to describe many phenomena in biology, fluid flow, chemistry, finance, control theory, psychology, and other areas of science and engineering. That is because of the fact that a reasonable modelling of a physical phenomenon has dependence not only on the time instant, but also on the prior time history, which can be fruitfully achieved by using fractional calculus. This returns to the non-local property of fractional derivatives. Therefore, FDEs are powerful models to describe real world phenomena more accurately than integer-order differential equations. Many real world problems are modelled by FDEs, and finding the solutions of these equations have been recently the subject of many research works. As a matter of fact, in recent two decades, FDEs received much more attention because of their applications in different areas of applied mathematics and physics.

Although the basic mathematical ideas of fractional calculus were developed about three-hundred years ago by Leibniz, Liouville, Riemann, and others, but for two reasons they have not been applied in real-life problems. This is due to the existence of non-equivalent definitions for fractional differentiation such as the Caputo and the Riemann-Liouville definitions. The second reason is that fractional differentiation has no evident geometrical interpretation. However, in spite of these reasons, as different applications can be gracefully modelled by the use of fractional calculus, recently fractional calculus starts too much attention for mathematicians, engineers, and physicists.

We aim in this paper to obtain semi-analytical solutions for the time-fractional $B(m,n)$ equation. The Boussinesq-like $B(m,n)$ or $B(m,n)$ equation is the generalized form of the standard Boussinesq equation with generalized evolution term. We aim also to obtain a variety of solutions of distinct physical structures such as compactons, solitons, and traveling wave solutions. The study of Boussinesq-like equations in variable water depth began to appear in 1967. The Boussinesq equation is a very famous nonlinear evolution equation developed to describe the motion of water with small amplitude and long wave. One may find some applications of Boussinesq-like equations in the study of the dynamics of the thin inviscid layers with free surface, the study of nonlinear string, the shape-memory alloys, the propagation of waves in elastic rods, and in the continuum limit of lattice dynamics or coupled electrical circuits [27–31]. The integer-order Boussinesq-like $B(m,n)$ equation with generalized evolution term has the following form [32]

$$(u^l)_{tt} + a(u^m)_{xx} - b(u^n)_{xxxx} = 0, \tag{1}$$

where the first term in (1) is the generalized evolution term, the second and the third terms, respectively, show the nonlinear and the dispersion terms. The constants a and b have real values, while l, m, and n are integers. Equation (1) changes to Boussinesq equation for $l = m = n = 1$. This equation is not integrable for general values of l, m, and n. In this paper, we focus on the time-fractional case of equation (1) which reads

$$D_t^\alpha u^l + a(u^m)_{xx} - b(u^n)_{xxxx} = 0, \tag{2}$$

where $1 < \alpha \leq 2$, with suitable initial conditions. To obtain approximate solutions for (2) we apply homotopy perturbation method.

The fractional differentiations are considered in the Caputo sense, because it permits boundary and initial conditions to be included in the formulation of the problem (for more details see [33]). The organization of this paper is as follows. In Sec. 2 we give some preliminary definitions of the fractional calculus. A brief review on homotopy perturbation method is presented in Sec. 3. Section 4 applies the method and consists of two subsections devoted to different values of m and n. In that Section, we consider the value of l as $l = 1$. We have similar application for the problem in Sec. 5 for $l = 2$. Finally, some discussions are given in Sec. 6.

2. PRELIMINARIES

In this Section we present some basic concepts, definitions, and properties of the fractional calculus theory that will be useful in the sequel. These basic definitions are presented in [33, 34].

Definition 1. A real function $f(x), x > 0$, is said to be in the space $C_v, v \in \mathbb{R}$ if there exists a real number $q > v$, such that $f(x) = x^q f_1(x)$, where $f_1(x) \in C[0,\infty]$, and it is in the space C_v^m if and only if $f^m \in C_v, m \in \mathbb{N}$.

Definition 2. For $\alpha > 0$, the Caputo fractional derivative of order α, denoted by D^α, is defined by

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} f^m(\tau) d\tau,$$

for $m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0$, and $f \in C_{-1}^m$.

Definition 3. For $\alpha > 0$, the Caputo fractional derivative of order α denoted by D_+^α is defined as:

$${}^cD_+^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^x (x-\tau)^{n-\alpha-1} D^n u(\tau) d\tau.$$

Definition 4. For n to be the smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{(n-\alpha-1)} \frac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, \text{ for } n-1 < \alpha < n, \\ \\ \frac{\partial^n u(x,t)}{\partial t^n} dt, \quad \text{for } \alpha = n \in \mathbb{N}. \end{cases}$$

Definition 5. The Riemann-Liouville fractional integral operator of order $\alpha \geqslant 0$ of a function $u \in C_v, v \geqslant -1$, is defined by

$$J_{0+}^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} u(\tau) d\tau, \ \alpha > 0, \ x > 0, \ J^0 u(x) = u(x).$$

Further, we need the following properties of the operator J:

$$ {}_tJ^{\alpha}_{a+}\left(\frac{t^{\beta}}{\Gamma(\beta+1)}\right)=\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},\ J^{\alpha}J^{\beta}u(x)=J^{\alpha+\beta}u(x),\ J^{\alpha}J^{\beta}u(x)=J^{\beta}J^{\alpha}u(x). $$

3. THE HOMOTOPY PERTURBATION METHOD

In this Section we give a brief review of the homotopy perturbation method (HPM). We refer the interested reader for more details to [35, 36]. For the first time HPM was presented by Ji-Huan He in 1999, which introduced a homotopy parameter p and the value of the parameter p changes from 0 to 1. If $p=0$, the system of equations reduces to a simple form that admits a simple solution, when $p=1$, the equation takes the original form of the equation and gives the desired solution. To have a basic idea of HPM, we consider the following nonlinear fractional differential equation:

$$ D_t^{\alpha}u(x,t)=v(x,t)-Lu(x,t)-Nu(x,t),\ \ m-1<\alpha<m,\ m\in\mathbb{N}, t\geqslant 0, x\in\mathbb{R}, \tag{3} $$

subject to the initial condition $u^{(i)}(0,0)=c_i$ and the boundary condition

$$ B\left(u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x_j}\right)=0,\ \ i=0,1,\cdots,m-1,\ j=0,1,\cdots,n, $$

where v is a known analytic function, D_t^{α} is the fractional Caputo sense derivative, L is a linear operator, N is a nonlinear operator, B is a boundary operator, the ith derivative of u is $u^{(i)}(x,t)$, the specified initial conditions are c_i, $i=0,1,\cdots,m-1$, and we assume that the solution u is a causal function of time, which means that vanishes for $t<0$.

We construct the following homotopy:

$$ (1-p)D_t^{\alpha}u(x,t)+p[D_t^{\alpha}u(x,t)+Lu(x,t)+Nu(x,t)-v(x,t)]=0,\ p\in[0,1], \tag{4} $$

or

$$ D_t^{\alpha}u(x,t)+p[Lu(x,t)+Nu(x,t)-v(x,t)]=0,\ \ p\in[0,1]. \tag{5} $$

The homotopy parameter p changes from zero to unity, if $p=0$, Eq. (4) or (5) becomes $D_t^{\alpha}u(x,t)=0$ and if $p=1$, Eq. (4) or (5) will turn out to the original FDE.

According to HPM, we assume that the solution of Eq. (4) or (5) can be written in the p series as:

$$ u(x,t)=u_0(x,t)+pu_1(x,t)+pu_2(x,t)+\cdots. \tag{6} $$

Now, by setting $Nu(x,t)=M(x,t)$, substituting Eq. (6) into Eq. (5) or (4) and by collecting p's terms of identical powers, we obtain:

$$\begin{aligned}
&p^0 : D_t^\alpha u_0(x,t) = 0,\\
&p^1 : D_t^\alpha u_1(x,t) = -Lu_0(x,t) - M_0(u_0(x,t)) + v(x,t),\\
&p^2 : D_t^\alpha u_2(x,t) = -Lu_1(x,t) - M_1(u_0(x,t), u_1(x,t)),\\
&p^3 : D_t^\alpha u_3(x,t) = -Lu_2(x,t) - M_2(u_0(x,t), u_1(x,t), u_2(x,t)),
\end{aligned} \tag{7}$$

and so on. The functions $M_0, M_1, M_2, \cdots$ satisfy:

$$M(u_0(x,t)+pu_1(x,t)+p^2u_2(x,t)+\cdots) = M_0(u_0(x,t))+pM_1(u_0(x,t),u_1(x,t))+\cdots.$$

By applying J_t^α on both sides of Eq. (7), considering the initial and boundary conditions and setting $p = 1$ we can obtain the approximate solution as:

$$u(x,t) = \Sigma_{i=0}^{\infty} u_i(x,t).$$

In the following Sections, we apply this method on time-fractional $B(m,n)$ equation for two values of l, and some different values of m and n.

4. APPLICATION OF HPM ON TIME-FRACTIONAL $B(m,n)$ EQUATION FOR $l = 1$

In this Section we will apply HPM on time-fractional $B(m,n)$ equation for $l = 1$ in two different cases depending on parameters m and n. We illustrate one example for both cases.

4.1. CASE $B(2,2)$

In this case, for $a = b = 1$, we have the following equation

$$D_t^\alpha u + (u^2)_{xx} - (u^2)_{xxxx} = 0, \qquad 1 < \alpha \leq 2. \tag{8}$$

To follow [12], we consider the initial conditions as:

$$u(x,0) = -\frac{2}{3} + e^{\frac{1}{2}x}, \qquad u_t(x,0) = \frac{1}{2}e^{\frac{1}{2}x}.$$

The standard operator form of the generalized time-fractional $B(2,2)$ is:

$$D_t^\alpha u = -(u^2)_{xx} + (u^2)_{xxxx}. \tag{9}$$

To solve Eq. (8), we construct the following homotopy:

$$D_t^\alpha u - D_t^\alpha u_0 = p\left(-\frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^4(u^2)}{\partial x^4} - D_t^\alpha u_0\right), \tag{10}$$

where $p \in [0,1]$ is the embedding parameter and is considered as a small parameter. Now, suppose that the solution of Eq. (10) has the following form

$$u = u_0 + p^1 u_1 + p^2 u_2 + \cdots, \tag{11}$$

substituting (11) into (10) and collecting p's terms of the same power of p yields:

$$
\begin{aligned}
&p^0: D_t^{\alpha} u_0 - D_t^{\alpha} u_0 = 0,\\
&p^1: D_t^{\alpha} u_1 = -\frac{\partial^2(u_0^2)}{\partial x^2} + \frac{\partial^4(u_0^2)}{\partial x^4} - D_t^{\alpha} u_0,\\
&p^2: D_t^{\alpha} u_2 = -\frac{\partial^2(2u_0u_1)}{\partial x^2} + \frac{\partial^4(2u_0u_1)}{\partial x^4},\\
&p^3: D_t^{\alpha} u_3 = -\frac{\partial^2(2u_0u_1+u_1^2)}{\partial x^2} + \frac{\partial^4(2u_0u_1+u_1^2)}{\partial x^4},
\end{aligned}
$$

and so on.

According to the procedure of HPM we have the following terms:

$$
\begin{aligned}
u_0(x,t) &= -\frac{2}{3} + e^{\frac{1}{2}x} + \frac{1}{2}e^{\frac{1}{2}x}t,\\
u_1(x,t) &= \frac{1}{4}e^{\frac{1}{2}x}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{8}e^{\frac{1}{2}x}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},\\
u_2(x,t) &= \frac{1}{16}e^{\frac{1}{2}x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{1}{32}e^{\frac{1}{2}x}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},
\end{aligned}
$$

and so on.

Therefore, the approximate solution of $u(x,t)$ is:

$$
u(x,t) = u_0 + u_1 + u_2. \tag{12}
$$

When $\alpha = 2$, solution (12) will be

$$
\begin{aligned}
u(x,t) = &-\frac{2}{3} + e^{\frac{1}{2}x} + \frac{1}{2}e^{\frac{1}{2}x}t + \frac{1}{4}e^{\frac{1}{2}x}\frac{t^2}{\Gamma(3)} + \frac{1}{8}e^{\frac{1}{2}x}\frac{t^3}{\Gamma(3)}\\
&+ \frac{1}{16}e^{\frac{1}{2}x}\frac{t^4}{\Gamma(5)} + \frac{1}{32}e^{\frac{1}{2}x}\frac{t^5}{\Gamma(6)}.
\end{aligned}
$$

By continuing the procedure, we can guess the exact solution of Eq. (8) as follows:

$$
\begin{aligned}
u(x,t) &= -\frac{2}{3} + \left[1 + \frac{t}{2} + \frac{t^2}{2^2*2!} + \frac{t^3}{2^3*3!} + \frac{t^4}{2^4*4!} + \cdots\right]e^{\frac{1}{2}x}\\
&= -\frac{2}{3} + e^{\frac{1}{2}x}\left(1 + \sum_{n=0}^{\infty}\frac{t^{n+1}}{2^{n+1}*(n+1)!}\right).
\end{aligned} \tag{13}
$$

It must be noted that this guess is checked by using Maple software. That checking showed that the solution (13) is the exact solution of Eq. (8) for $\alpha = 2$. Thus, we conclude that, the method works as well. Our solution is different from solution (6.11) in Ref. [12], which has obtained by the Adomian decomposition method.

4.2. CASE $B(3,3)$

For $a = b = 1$, the following equation is considered in this case:

$$D_t^{\alpha}u + (u^3)_{xx} - (u^3)_{xxxx} = 0, \tag{14}$$

where $1 < \alpha \leq 2$. Besides, similar to Zhu [37], the initial conditions are considered as:

$$u(x,0) = \frac{\sqrt{6}}{2}\sinh(\frac{x}{3}), \qquad u_t(x,0) = \frac{-1}{\sqrt{6}}\cosh(\frac{x}{3}).$$

In this case, the standard operator form of the generalized time-fractional $B(3,3)$ is:

$$D_t^{\alpha}u = -(u^3)_{xx} + (u^3)_{xxxx}. \tag{15}$$

To solve (14), the following homotopy is constructed:

$$D_t^{\alpha}u - D_t^{\alpha}u_0 = p\left(-\frac{\partial^2(u^3)}{\partial x^2} + \frac{\partial^4(u^3)}{\partial x^4} - D_t^{\alpha}u_0\right), \tag{16}$$

where $p \in [0,1]$ is the embedding small parameter. Now, suppose that the solution of Eq. (16) has the following form

$$u = u_0 + p^1 u_1 + p^2 u_2 + \cdots . \tag{17}$$

Substituting (17) into (16) and collecting p's terms of the same power of p yields:

$$\begin{aligned}
&p^0 : D_t^{\alpha}u_0 - D_t^{\alpha}u_0 = 0,\\
&p^1 : D_t^{\alpha}u_1 = -\frac{\partial^2(u_0^3)}{\partial x^2} + \frac{\partial^4(u_0^3)}{\partial x^4} - D_t^{\alpha}u_0,\\
&p^2 : D_t^{\alpha}u_2 = -\frac{\partial^2(3u_0^2u_1)}{\partial x^2} + \frac{\partial^4(3u_0^2u_1)}{\partial x^4},\\
&p^3 : D_t^{\alpha}u_3 = -\frac{\partial^2(3u_0^2u_2 + 3u_0u_1^2)}{\partial x^2} + \frac{\partial^4(3u_0^2u_2 + 3u_0u_1^2)}{\partial x^4},
\end{aligned}$$

and so on.

According to HPM, we can obtain the followings:

$$\begin{aligned}
u_0(x,t) &= \frac{\sqrt{6}}{2}\sinh(\frac{x}{3}) - \frac{1}{\sqrt{6}}\cosh(\frac{x}{3})t,\\
u_1(x,t) &= \frac{\sqrt{6}}{18}\sinh(\frac{x}{3})\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\sqrt{6}}{54}\cosh(\frac{x}{3})\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\
&\quad - \frac{\sqrt{6}}{81}\sinh(\frac{x}{3})\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{\sqrt{6}}{81}\cosh(\frac{x}{3})\frac{t^{\alpha+3}}{\Gamma(\alpha+4)},
\end{aligned}$$

and so on.
Therefore, we can approximate solution of $u(x,t)$ using two terms

$$u(x,t) = u_0 + u_1. \tag{18}$$

When $\alpha = 2$ solution (18) is as follows:

$$\begin{aligned} u(x,t) = &\frac{\sqrt{6}}{2}\sinh(\frac{x}{3}) - \frac{1}{\sqrt{6}}\cosh(\frac{x}{3})t + \frac{\sqrt{6}}{36}\sinh(\frac{x}{3})t^2 \\ &- \frac{\sqrt{6}}{324}\cosh(\frac{x}{3})t^3 - \frac{\sqrt{6}}{1944}\sinh(\frac{x}{3})t^4 + \frac{\sqrt{6}}{9720}\cosh(\frac{x}{3})t^5 + \cdots. \end{aligned} \tag{19}$$

This solution is as same as solution of Zhu [37] which was obtained by ADM, we can write the closed form of (19) by Taylor series as:

$$u(x,t) = \frac{\sqrt{6}}{2}\sinh(\frac{x-t}{3}).$$

Figure 1 shows the physical behaviour of the numerical solution of $u(x,t)$ of (18) in 3D plots. We have plotted solution (18) for four values of the differentiation order, α. All solutions are compacton ones.

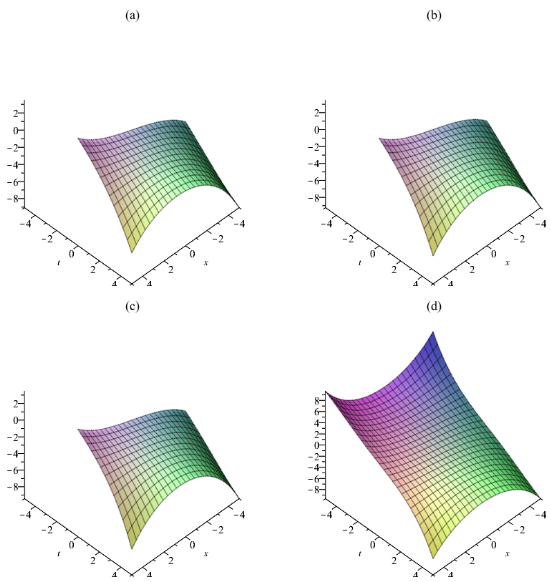

Fig. 1 – Compacton solutions of Eq. (14), (a) plot for $\alpha = 1.1$, (b) plot for $\alpha = 1.4$, (c) plot for $\alpha = 1.7$, (d) plot for $\alpha = 2$.

5. APPLICATION OF HPM ON TIME-FRACTIONAL $B(m,n)$ FOR $l=2$

Here, we apply HPM on time-fractional $B(m,n)$ equation in two different cases depending on values of m and n. For both cases we obtain the approximate solutions and figure out them for some values of the differentiation order, α. Note that in the case of $l=2$, the fractional derivative applies on a nonlinear function, that is, applies on u^2.

5.1. CASE $B(1,2)$

In the first case, we consider $m=1, n=2, a=-1$, and $b=1$. Therefore, we have the following equation:

$$D_t^{\alpha}(u^2)-(u)_{xx}-(u^2)_{xxxx}=0, \tag{20}$$

where $1<\alpha\leq 2$. In this case we consider the following initial conditions:

$$u(x,0)=\frac{-1}{3v^2}\cosh^2\left(\frac{vx}{4}\right), \qquad u_t(x,0)=\frac{1}{6}\cosh\left(\frac{vx}{4}\right)\sinh\left(\frac{vx}{4}\right).$$

By the same procedure in the previous Section, we obtain:

$$\begin{aligned}
u_0(x,t)&=\frac{-1}{3v^2}\cosh^2\left(\frac{vx}{4}\right)+\frac{1}{6}\cosh\left(\frac{vx}{4}\right)\sinh\left(\frac{vx}{4}\right)t,\\
u_1(x,t)&=\left(2\left(\frac{-1}{3v^2}\cosh^2\left(\frac{vx}{4}\right)+\frac{1}{6}\cosh\left(\frac{vx}{4}\right)\sinh\left(\frac{vx}{4}\right)t\right)\right)^{-1}\\
&\quad\left\{\frac{1}{36}\cosh^2(\frac{vx}{4})\left(\cosh^2(\frac{vx}{4})-1\right)v^2\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{v^4}{288}\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right.\\
&\quad\left.-\sinh(\frac{vx}{4})\cosh(\frac{vx}{4})\left(\frac{\cosh^2(\frac{vx}{4})}{9}-\frac{3}{16}\right)v^2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{5}{96}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right\}.
\end{aligned}$$

Therefore, the approximate solution of (20) is presented as:

$$u(x,t)=u_0+u_1. \tag{21}$$

When $\alpha=2$, solution (21) is as follows:

$$
\begin{aligned}
u(x,t) = &\frac{-1}{3v^2}\cosh^2\left(\frac{vx}{4}\right)+\frac{1}{6}\cosh\left(\frac{vx}{4}\right)\sinh\left(\frac{vx}{4}\right)t \\
&+\left(2\left(\frac{-1}{3v^2}\cosh^2\left(\frac{vx}{4}\right)+\frac{1}{6}\cosh\left(\frac{vx}{4}\right)\sinh\left(\frac{vx}{4}\right)t\right)\right)^{-1} \\
&\left\{\frac{1}{36}\cosh^2(\frac{vx}{4})\left(\cosh^2(\frac{vx}{4})-1\right)v^2\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{v^4}{288}\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right. \\
&\left.-\sinh(\frac{vx}{4})\cosh(\frac{vx}{4})\left(\frac{\cosh^2(\frac{vx}{4})}{9}-\frac{3}{16}\right)v^2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{5}{96}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right\}.
\end{aligned} \tag{22}
$$

Figure 2 shows the physical behaviour of approximate solution of $u(x,t)$ (21) in 3D plots. The traveling wave solutions are plotted for four different values of α. Besides, the plots of Fig. 2 shows the continuous dependence of the solution of time-fractional derivatives. Because, as $\alpha \to 2$, the plots of the approximate solution (21) are close to the plots of solution (22), the latter being the approximate solution for the integer order differential equation.

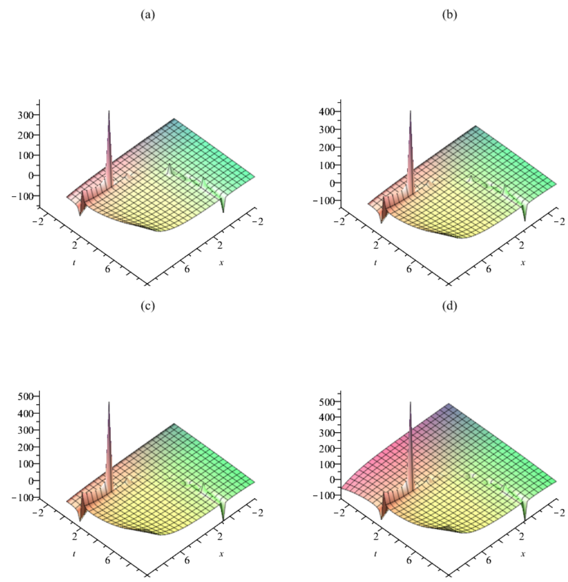

Fig. 2 – Traveling wave solutions of Eq. (20), (a) $\alpha = 1.1$, (b) $\alpha = 1.4$, (c) $\alpha = 1.7$, (d) $\alpha = 2$.

5.2. CASE $B(3,2)$

Here, the last case is considered for $m=3, n=2, a=-1$, and $b=1$. Hence, we must obtain approximate solution for the following equation:

$$D_t^{\alpha}(u^2)-(u^3)_{xx}-(u^2)_{xxxx}=0, \tag{23}$$

where $1<\alpha\leq 2$. In this case we consider the following initial conditions:

$$u_0(x,0)=\frac{-5}{9}v^2\frac{1}{\cosh^2(\frac{vx}{4})}, \qquad u_t(x,0)=-\frac{5}{18}v^4\frac{\sinh(\frac{vx}{4})}{\cosh^3(\frac{vx}{4})}.$$

For this case, the following terms can be obtained by application of HPM:

$$u_0(x,t)=\frac{-5}{9}v^2\frac{1}{\cosh^2(\frac{vx}{4})}-\frac{5}{18}v^4\frac{\sinh(\frac{vx}{4})}{\cosh^3(\frac{vx}{4})}t,$$

$$\begin{aligned}
u_1(x,t)=&\left(2\left(\frac{-5}{9}v^2\frac{1}{\cosh^2(\frac{vx}{4})}-\frac{5}{18}v^4\frac{\sinh(\frac{vx}{4})}{\cosh^3(\frac{vx}{4})}t\right)\right)^{-1}\\
&\left\{\frac{25}{324}\frac{v^{12}}{\cosh^4(\frac{vx}{4})}\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{25}{81}\frac{v^8}{\cosh^4(\frac{vx}{4})}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right.\\
&+\frac{25}{81}v^{10}\frac{\sinh(\frac{vx}{4})}{\cosh^5(\frac{vx}{4})}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{2575}{2592}\frac{v^{12}}{\cosh^6(\frac{vx}{4})}\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\\
&-\frac{2125}{1296}\frac{v^8}{\cosh^6(\frac{vx}{4})}\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{125}{2592}v^{14}\frac{\sinh(\frac{vx}{4})}{\cosh^7(\frac{vx}{4})}\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}\\
&-\frac{2125}{864}v^{10}\frac{\sinh(\frac{vx}{4})}{\cosh^7(\frac{vx}{4})}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{36875}{15552}\frac{v^{12}}{\cosh^8(\frac{vx}{4})}\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\\
&+\frac{11375}{7776}\frac{v^8}{\cosh^8(\frac{vx}{4})}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{625}{3888}v^{14}\frac{\sinh(\frac{vx}{4})}{\cosh^9(\frac{vx}{4})}\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}\\
&+\frac{25}{81}v^{10}\frac{\sinh(\frac{vx}{4})}{\cosh^5(\frac{vx}{4})}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{2575}{2592}\frac{v^{12}}{\cosh^6(\frac{vx}{4})}\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\\
&\left.-\frac{625}{518}v^{14}\frac{\sinh(\frac{vx}{4})}{\cosh^{11}(\frac{vx}{4})}\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}\right\}.
\end{aligned}$$

Therefore, the approximate solution of (23) is

$$u(x,t)=u_0+u_1. \tag{24}$$

We have plotted the approximate solution (24) in 3D cases for some values of α in Fig. 3. As the plots of Fig. 3 show, in all cases we have soliton solutions for (23). The plots $(a),(b)$, and (c) show the continuity dependence on the time-fractional derivatives. Because, they are very similar to plot (d), which shows the integer case $\alpha = 2$.

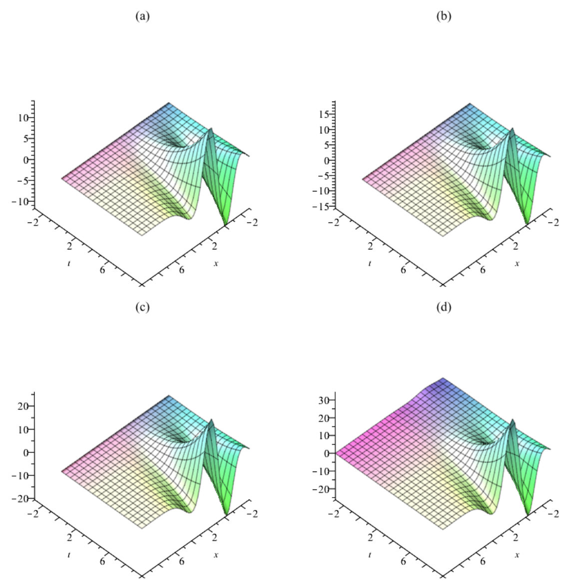

Fig. 3 – Soliton solutions of Eq. (23), (a) $\alpha = 1.1$, (b) $\alpha = 1.4$, (c) $\alpha = 1.7$, (d) $\alpha = 2$.

6. CONCLUSION

In this paper, we have established semi-analytical solutions for some kinds of time-fractional Boussinesq-like $B(m,n)$ equations. Compacton, traveling wave, and soliton solutions are presented for the governing model. The examined equations are characterized by a set of initial and boundary conditions. We used the homotopy perturbation method for analytical treatment of these equations and to illustrate the analysis. Proper graphs were used to illustrate the obtained results. To the best of our knowledge, some of these equations are investigated for the first time in this paper. The obtained results validate the reliability and rapid convergence of the homotopy perturbation method. As a matter of fact, when fractional orders of derivatives approach to an integer value, the graphs of solution for the former cases are close to the graphs of the latter. This shows that, the solution depends continuously on the time-fractional derivative, which confirms the reliability of our solutions.

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