European Journal of Scientific Research
ISSN 1450-216X Vol.57 No.4 (2011), pp.577-582

http://www.eurojournals.com/ejsr.htm

Characterizations of Continuity and Compactness with Respect to Weak Forms of ω-Open Sets

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Abstract

In this paper we use the weak ω - open sets defined by T. Noiri, A. Al-Omari, M. S. M. Noorani in [5], to define new weak types of continuity and compactness and prove some theorems about them.

Keywords: Weak open set, weak continuity, weak compactness, ω-open set.

1. Introduction and Preliminaries

Through out this paper, (X,T) stands for topological space. Let (X,T) be a topological space and A a subset of X. A point x in X is called ***condensation point*** of A if for each U in T with x in U, the set $U \bigcap A$ is uncountable [3]. In 1982 the ω-closed set was first introduced by H. Z. Hdeib in [3], and he defined it as: A is ***ω-closed*** if it contains all its condensation points and the ***ω-open*** set is the complement of the ω-closed set. Equivalently. A subset W of a space (X,T), is ω-open if and only if for each $x \in W$, there exists $U \in T$ such that $x \in U$ and $U \setminus W$ is countable. The collection of all ω-open sets of (X,T) denoted T_ω form topology on *X* and it is finer than T. Several characterizations of ω-closed sets were provided in [1, 3, 4, 6]. For a subset *A* of *X*, the closure of *A* and the ω -interior of *A* will be denoted by $cl(A)$ and $\operatorname{int}_\omega(A)$ respectively. The ***ω- interior*** of the set *A* defined as the union of all ω- open sets contained in *A* .

In 2009 in [5] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced and investigated new notions called $\alpha-\omega-$open, $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open sets which are weaker than ω-open set. Let us introduce these notions in the following definition:

Definition 1.1. [5] A subset *A* of a space *X* is called**:**

1. ***$\alpha-\omega-$open*** if $A \subseteq \operatorname{int}_\omega(cl(\operatorname{int}_\omega(A)))$, the complement is called ***$\alpha-\omega-$ closed*** set.
2. ***$pre-\omega-$open*** if $A \subseteq \operatorname{int}_\omega(cl(A))$, the complement is called ***$pre-\omega-$closed*** set.
3. ***$b-\omega-$open*** if $A \subseteq \operatorname{int}_\omega(cl(A)) \bigcup cl(\operatorname{int}_\omega(A))$, the complement is called ***$b-\omega-$closed*** set.
4. ***$\beta-\omega-$open*** if $A \subseteq cl(\operatorname{int}_\omega(cl(A)))$, the complement is called ***$\beta-\omega-$closed*** set.

In [5] T. Noiri, A. Al-Omari, M. S. M. Noorani introduced relationships among the weak open sets above by the lemma below:

Lemma 1.2. [5] In any topological space:

1. Any open set is ω-open.
2. Any ω-open set is $\alpha-\omega-$open.
3. Any $\alpha-\omega-$open set is $pre-\omega-$open.
4. Any $pre-\omega-$open set is $b-\omega-$open.
5. Any $b-\omega-$open set is $\beta-\omega-$open.

The converse is not true [5].

Remark 1.3. [5] The intersection of two $pre-\omega-$open, (resp. $b-\omega-$open and $\beta-\omega-$open) sets need not be $pre-\omega-$open, (resp. $b-\omega-$ open and $\beta-\omega-$open) sets.

Lemma 1.4. [5] The intersection of an $\alpha-\omega-$open (resp. $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) subset of any topological space and an open subset is $\alpha-\omega-$open (resp. $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) set.

Remark 1.5. The union of an $\alpha-\omega-$closed (resp. $pre-\omega-$closed, $b-\omega-$ closed and $\beta-\omega-$closed) subset of any topological space and a closed subset is $\alpha-\omega-$closed (resp. $pre-\omega-$ closed, $b-\omega-$closed and $\beta-\omega-$ closed) set.

Theorem 1.6. [5] If $\{A_\alpha : \alpha\in\Delta\}$ is a collection of $\alpha-\omega-$open (resp. $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) subsets of the topological space (X,T) then $\bigcup_{\alpha\in\Delta} A_\alpha$ is $\alpha-\omega-$open (resp. $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) set.

Corollary 1.7. If $\{A_\alpha : \alpha\in\nabla\}$ is a collection of $\alpha-\omega-$closed (resp. $pre-\omega-$closed, $b-\omega-$closed and $\beta-\omega-$closed) subsets of the topological space (X,T), then $\bigcap_{\alpha\in\nabla} A_\alpha$ is $\alpha-\omega-$closed (resp. $pre-\omega-$closed, $b-\omega-$closed and $\beta-\omega-$closed) set.

Definition 1.8. [5] A space (X,T) is called a ***door space*** if every subset of *X* is either open or closed.

Example 1.9. The space (X,T) for $X=\{a,b\}$, and $T=\{X,\phi,\{a\}\}$, is a door space.

Lemma 1.10. [5] If (X,T) is a door space, then every $pre-\omega-$open set is $\omega-$open.

Theorem 1.11. Let *A* be a $\beta-\omega-$open set in the topological space (X,T), then A is $b-\omega-$open , whenever *X* is door space.

Proof:

Let *A* be a $\beta-\omega-$open subset of *X*. If *A* is open then by Lemma 1.4 it is $b-\omega-$open. Then if *A* is closed we get $A\subseteq cl(\text{int}_\omega(A)\subseteq(\text{int}_\omega(cl(A))\bigcup cl(\text{int}_\omega(A))$. Thus *A* is $b-\omega-$open set in *X* X

Definition 1.12. [5] A subset *A* of a space *X* is called**:**

1. An $\omega-t-$***set***, if $\text{int}(A)=\text{int}_\omega(cl(A))$.
2. An $\omega-B-$ ***set*** ,if $A=U\bigcap V$, where *U* is an open set and *V* is an $\omega-t-$ set.
3. An $\omega-t_\alpha-$***set***, if $\text{int}(A)=\text{int}_\omega(cl(\text{int}_\omega(A)))$.
4. An $\omega-B_\alpha-$***set*** if $A=U\bigcap V$, where U is an open set and *V* is an $\omega-t_\alpha-$set.
5. An $\omega-$***set*** if $A=U\bigcap V$, where *U* is an open set and $\text{int}(V)=\text{int}_\omega(V)$.

Definition 1.13. Let (X,T) be topological space. It said to be satisfy

1. The $\omega-$ ***condition*** if every $\omega-$ open set is $\omega-$set.
2. The $\omega-B_{\alpha}$ ***condition*** if every $\alpha-\omega-$open set is $\omega-B_{\alpha}-$set.
3. The $\omega-B-$***condition*** if every $pre-\omega-$open is $\omega-B-$set.

Now let us introduce the following lemma from [5].

Lemma 1.14. [5] For any subset *A* of a space *X*, We have

1. *A* is open if and only if *A* is $\omega-$open and $\omega-$set.
2. *A* is open If and only if *A* is $\alpha-\omega-$open and $\omega-B_{\alpha}$-set.
3. *A* is open if and only if *A* is $pre-\omega-$open and $\omega-B-$set.

2. Decomposition of Continuity

Let us now use the weak $\omega-$open sets to define a decomposition of continuity. Also we introduce some theorems about this notion.

Definition 2.1. A function $f:(X,\sigma)\to(Y,\tau)$ is called $\omega-$***continuous*** (resp. $\alpha-\omega-$***continuous*** , $pre-\omega-$***continuous***, $b-\omega-$***continuous and*** $\beta-\omega-$***continuous***), if for each $x\in X$, and each $\omega-$-open (resp. $\alpha-\omega-$open, $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) set *V* containing f(x), there exists an $\omega-$open (resp. $\alpha-\omega-$open, $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open,) set *U* containing *x*, such that $f(U)\subset V$.

Proposition 2.2. A function $f:(X,\sigma)\to(Y,\tau)$ is ω-continuous (resp. $\alpha-\omega-$continuous, $pre-\omega-$continuous, $b-\omega-$continuous and $\beta-\omega-$continuous) if and only if for each ω-open (resp. $\alpha-\omega-$open, $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) set *V* in *Y*, $f^{-1}(V)$ is ω-open (resp. $\alpha-\omega-$open, $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) set in *X* .

Proof:

Let *f* be an ω-continuous map from *X* to *Y*, and let $x\in X$,and *V* be an $\omega-$open subset of *Y* containingf(x). We must show that $f^{-1}(V)$ is $\omega-$open subset of *X* containing *x*, let $x\in f^{-1}(V)$, then by the $\omega-$continuity of *f* we can find an $\omega-$open set *U* in *X* and containing *x*, such that $f(U)\subset V$, then $U\subset f^{-1}(V)$, which is true for any $x\in f^{-1}(V)$. This implies $f^{-1}(V)$ is $\omega-$ open subset of *X*. For the opposite side, let us assume that the inverse image of any $\omega-$open set is also an $\omega-$open to prove *f* is ω-continuous map. Let $x\in X$ and let *V* be an ω-open subset of *Y* containing f(x) , by the hypothesis $f^{-1}(V)$ is $\omega-$open subset of *X*, so for any $x\in f^{-1}(V)$, $f(f^{-1}(V))\subset V$, and *f* is $\omega-$continuous. By the same way we can prove the other cases X

Theorem 2.3. Let (X,σ) and (Y,τ) be two topological spaces such that *X* satisfies the $\omega-B_{\alpha}$**-condition** , and $f:(X,\sigma)\to(Y,\tau)$ be a map. If *f* is $\alpha-\omega-$continuous then it is ω-continuous.

Proof:

Let $f:(X,\sigma)\to(Y,\tau)$ be an $\alpha-\omega-$continuous, to prove it is $\omega-$continuous, let $x\in X$ and *V* be an $\omega-$open (so it is $\alpha-\omega-$open) set containing f(x). Since *f* is $\alpha-\omega-$continuous there exists an $\alpha-\omega-$open subset *U* of *X* containing *x* such that $f(U)\subset V$. Then since *X* satisfies the $\omega-B_{\alpha}$-condition we have *U* is an ω-open of *X* containing *x* such that $f(U)\subset V$. This implies *f* is ω-continuous X

Theorem 2.4. Let (X,σ) and (Y,τ) be two topological spaces such that *X* is door space, and $f:(X,\sigma)\to(Y,\tau)$ be a map.

1. If *f* is $pre-\omega-$continuous then it is ω-continuous.

2. If f is $\beta-\omega-$continuous then it is $b-\omega-$continuous.

Proof:

By the same way as the proof of Theorem 2.3, using Lemma 1.2, Lemma 1.10 and Theorem 1.11, we can prove this theorem X

Theorem 2.5. Let (X,σ) and (Y,τ) be two topological spaces that satisfy the $\omega-$condition then the map $f:(X,\sigma)\to(Y,\tau)$ is continuous if and only if it is $\omega-$continuous.

Proof:

Let be $f:(X,\sigma)\to(Y,\tau)$ a continuous map , $x\in X$ and V be an $\omega-$ open set in Y and containing f(x). Since X satisfy $\omega-$condition, so V is also open in Y. And by the continuity of f there is an open set U (also it is $\omega-$ open) with $f(U)\subset V$. For the converse let f be an $\omega-$continuous map and V be an open set in Y and containing f(x), so it is also $\omega-$open and by the $\omega-$ continuity of f, there is an $\omega-$open set U in X containing x with $f(U)\subset V$, and since X satisfies the $\omega-$condition U is an open set therefore f is continuous X

Remark 2.6. Theorem 2.5. is not true in general. It mean if $f:(X,\sigma)\to(Y,\tau)$ is $\omega-$continuous, then it is not necessarily continuous. As we see in the following example.

Example 2.7. Let $X=\{a,b,c\}$, $\sigma=\{\phi,X,\{c\}\}$, $Y=\{d,e,f\}$, $\tau=\{\phi,Y,\{d\}\}$, and let $f:(X,\sigma)\to(Y,\tau)$ be a map defined by $f(a)=f(b)=d, f(c)=e.f$ is $\omega-$continuous but not continuous.

Note that since X and Y are countable, so any subset of them is $\omega-$open. If $x=a$, we have $f(x)=d$. $V_1=\{d\}, V_2=\{d,e\}, V_3=\{d,f\}$, and $V_4=Y$ are $\omega-$open sets in Y containing $f(x)$, so there exist $U_1=\{a,b\}, U_2=\{a,c\}, U_3=\{a\}$ and $U_4=X$ such that $f(U_1)=V_1$, $f(U_2)=V_2$, $f(U_3)=V_3$ and $f(U_4)=V_4$. Similarly for $x=b$, and $x=c$, Therefore f is $\omega-$continuous map.

Next f is not continuous. Let $x=b, f(x)=d$ if $V=\{d\}$, then when $U=X$, we have $f(U)=\{d,e\}\not\subset\{d\}=V$. Hence f is not continuous map .

Theorem 2.8. Let (X,σ) and (Y,τ) be two topological spaces that satisfy the $\omega-B_\alpha$ condition then the map $f:(X,\sigma)\to(Y,\tau)$ is continuous if and only if it is $\alpha-\omega-$continuous.

Theorem 2.9. Let (X,σ) and (Y,τ) be two topological spaces that satisfy the $\omega-$B-condition then the map $f:(X,\sigma)\to(Y,\tau)$ is continuous if and only if it is $pre-\omega-$continuous.

Theorem 2.10. Let (X,σ) and (Y,τ) be two door topological spaces and $f:(X,\sigma)\to(Y,\tau)$ be a map. Then

1. f is pre - $\omega-$continuous if and only if it is $\omega-$continuous.
2. f is $\beta-\omega-$continuous if and only if it is b- $\omega-$continuous.

Proof of (1):

Let f be a pre-$\omega-$continuous, and let V be an $\omega-$open set in Y and containing $f(x)$, therefore it is $pre-\omega-$open and since f is $pre-\omega-$continuous, there is a pre-$\omega-$open set U in X containing x and $f(U)\subset V$. Since X is a door space U is also an $\omega-$open set. For the converse let f be an $\omega-$continuous map and V be a pre - $\omega-$open set in Y. Then since Y is door space we get V is $\omega-$open , and by the $\omega-$continuity of f there exists an $\omega-$open set U in X containing x (also pre-$\omega-$open) with $f(U)\subset V$. And so f is a $\omega-$continuous. Similarly we can prove (2) X

3. Weak $\omega-$Compactness

In this article we shall introduce weak $\omega-$compactness. It is defined that every cover by such weak open sets contains a finite subcover. So let us state new definitions for the weak new types of $\omega-$compact sets, and prove several rather simple theorems about it.

Definition 3.1. Let *X* be a topological space. We say that a subset *A* of *X* is $\omega-$***compact*** [2] (resp. $\alpha-\omega-$***compact*** , $pre-\omega-$***compact***, $b-\omega-$***compact*** and $\beta-\omega-$***compact*** **)** if for each cover of $\omega-$open (resp. $\alpha-\omega-$open , $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open) sets from *X* contains a finite sub cover for *A*.

Theorem 3. 2. In any topological space, every $\beta-\omega-$compact set is compact.

Proof :

Let *X* be a topological space , and let *A* be a $\beta-\omega-$compact subset of *X* , to prove *A* is compact, let *C* be an open cover for *A*. Since we can consider *C* as a cover of $\beta-\omega-$open sets by lemma 1.2 and *A* is $\beta-\omega-$compact subset of *X*. Then *C* contains a finite sub cover , Thus *X* is compact set.

Theorem 3.4. Let (X,T) be a topological space

1. If (X,T) is door space, then any α-compact set is pre-ω-compact.
2. If (X,T) is door space, then any b-ω-compact set is $\beta-\omega-$compact.
3. If (X,T) satisfies the ω-condition, then any compact set is ω-compact.
4. If (X,T) satisfies the $\omega-B_\alpha$ condition, then any compact set is $\omega-\alpha-$compact.
5. If (X,T) satisfies the ω**-B-**condition, then any compact set is pre-ω- compact.

Proof:

1. Let *X* be a topological door space, and let *A* be an ω-compact subset of *X* , and *C* be a cover of pre-ω- open subsets of *X* . Since *X* is a door space so we can consider *C* as a cover of ω-open sets. And by the ω-compactness of *X*, *C* contains a finite sub cover of pre-α-open sets. Hence *A* is pre-ω-compact.

Similarly we can prove (2).

3. Let *X* be a topological space satisfies the ω-condition, and *A* be a compact subset of *X*, to prove *A* is ω- compact, let *C* be a cover of ω-open sets for *A*. Since *X* satisfies the $\omega-$condition, we can consider *C* as a cover of open sets and by the compactness of *A* ,*C* contains a finite subcover of open(also ω-open) sets for *A*. This implies *X* is ω-compact. Similarly we can prove (4) and (5) X

Theorem 3.5. An $\omega-$closed (resp. $\alpha-\omega-$closed, $pre-\omega-$closed, $b-\omega-$closed and $\beta-\omega-$closed) sub set of ω- compact (resp. $\alpha-\omega-$compact , $pre-\omega-$compact, $b-\omega-$compact and $\beta-\omega-$compact) subspace is cω-ompact (resp. $\alpha-\omega-$compact, $pre-\omega-$compact, $b-\omega-$compact and $\beta-\omega-$compact).

Proof:

Let *Y* be an $\omega-$compact subspace of the topological space *X*, and let *F* be an ω- closed subset of *Y*. Let $C=\{G_\lambda,\lambda\in\Lambda\}$ be a cover of oω-open sets for *F*. Then $C\bigcup(Y\setminus F=D$ is a cover of $\omega-$open sets for *Y*. Since *Y* is ω-compact there is a finite subcover D' of *D* for *Y*, and hence without $Y\setminus F$, a cover for *F* (because *F* and $Y\setminus F$ are disjoint). So we have shown that a finite sub collection of *C* cover *F*. Thus *F* is ω-compact. Similarly we can prove the other cases X

Theorem 3.6. Let $f:X\rightarrow Y$ *b*e an $\omega-$continuous (resp. $\alpha-\omega-$continuous, pre-ω-continuous, b-ω-continuous, and $\beta-\omega-$ continuous) map from the $\omega-$compact (resp. $\alpha-\omega-$compact, $pre-\omega-$ compact, $b-\omega-$ compact, and $\beta-\omega-$ compact) space *X* onto a topological space *Y*. Then *Y* is ω-compact (resp. $\alpha-\omega-$compact, $pre-\omega-$ compact, $b-\omega-$ compact and $\beta-\omega-$ compact) space.

Proof:

Let $f: X \to Y$ be an ω-continuous map from the $\omega-$compact space *X* on to *Y*. Let $\{Y_\lambda, \lambda \in \Lambda\}$ be a cover of ω- open sets for *Y*, then since f is ω-continuous map so $\left\{f^{-1}(Y_\lambda), \lambda \in \Lambda\right\}$ is a cover of ω-open sets for *X*. Since *X* is ω-compact so it has a finite sub cover $\{f^{-1}(Y_{\lambda_i}) : i = 1,2,.......n\}$. Then by the surjection of f we get $\{Y_{\lambda_i} : i = 1,2,.......n\}$ is an ω-open cover for *Y*. Hence Y is ω-compact. With a simple modification to that prove one can prove the other cases X

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