

Centroids Of Dendriform Algebras

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Abstract

This paper deals with centroids of dendriform algebras. We introduce the notion of the centroid for dendriform algebra and study some of their properties. Subsequently, we determine the centroids of low-dimensional dendriform algebras alongside their classification.

Keywords: Dendriform algebra; Derivation; Central Derivation; Centroid.

1 Introduction

The principal aim of this study is based on centroids of Dendriform algebras. The centroid $\Omega(E)$ of a Lie algebra L is the space of L – module homomorphisms ω on $L : \omega([a, b]) = [a, \omega(b)]$ for all $a, b \in L$.

Classification problems of the dendriform algebras using the algebraic and geometric technique prompted interest in the centroids of algebras. The dendriform algebra introduced by Loday [2] with a motivation to provide dual of dialgebras, and have been further studied with connections to several areas in mathematics and physics, including operads, homology, hopf algebras, Lie and Leibniz algebras, combinatorics, and quantum field.

One of the important results has it that all scalar extension of a simple if and only if the centroid comprises of the scalars in the base field especially the finite-dimensional simple associative algebras. The centroid is crucial in investigation of divisional algebras, Brauer groups.

The centroids of nilpotent are studied by Melville and Benkarta, readers can see [1], [3], [4], [16], [17] and references therein for more details. Our concerns therefore is, How much of these results on centroids can be obtained for Leibniz algebras? In this study, we focus on the centroids of two- dimensional dendriform algebras over complex field. The concept of centroids, An algorithm is found to describe the centroids and this algorithm has been found to be efficient in computing the centroids of other classes of algebras as well. Thereafter, the algorithm is applied to find the centroids of two-dimensional centroids algebras [5].

2 Preliminaries

This section will provide the definitions and results obtained with dendriform algebras to feature the paper as a self-contain piece.

Definition 2.1. *A dendriform algebra is a vector space E composed with maps $\prec: E \times E \rightarrow E$ and $\succ: E \times E \rightarrow E$ satisfying the axioms:*

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c) \tag{1}$$

$$(a \succ b) \prec c = a \succ (b \prec c) \tag{2}$$

$$a \succ (b \succ c) = (a \prec b + a \succ b) \succ c \tag{3}$$

for all a, b and $c \in E$.

Let make E as dendriform algebra and its subset to be A, B . Then we define the following binary operation, denoted \diamond , over subsets of E :

$$A \diamond B := A \prec B + A \succ B,$$

where

$$A \prec B = \text{Span}_{\mathbb{C}}\{a \prec b \mid a \in A, b \in B\} \text{ and } A \succ B = \text{Span}_{\mathbb{C}}\{a \succ b \mid a \in A, b \in B\}.$$

Obviously, if A and B are ideals so is $A \diamond B$. Lets consider the following series: $E^1 = E, E^{k+1} = E^1 \diamond E^k + E^2 \diamond E^{k-1} + \dots + E^k \diamond E^1$.

Let $S \neq \emptyset, S \subseteq E$. The subset

$$Z_E(S) = \{a \in E \mid a * S = S * a = 0\}$$

is called centralizer.

Definition 2.2. *A dendriform algebra derivation E is a linear transfiguration $d : E \rightarrow E$ Satisfying*

$$d(a * b) = d(a) * b + a * d(b)$$

for all $a, b \in E$.

The set of all dendriform algebra derivations E is a subspace of $\text{End}_K(E)$. This subspace is equipped with the bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$, and we denoted by $\text{Der}(E)$. This concept of nilpotent algebra introduced by [6] observations and the advance of the current concept development. And readers can see [10], [11], [12], [13], [14], [15], (also see [8], and [9]).

Definition 2.3. Let (E, \prec, \succ) be a dendriform algebra over a field K . For $z \in E$, define $\text{ad}_z(a) = z * a - a * z$ for all $a \in E$.

Then $\text{ad}_z(a)$ is called inner derivation of E

Definition 2.4. Let E be an arbitrary dendriform algebra over a field K . The left and right centroids $\Omega^\prec(E)$ and $\Omega^\succ(E)$ of E are the spaces of K -linear transformations on E given by

$$\Omega_K^*(E) = \{\omega \in \text{End}_K(E) \mid \omega(a * b) = a * \omega(b) = \omega(a) * b \text{ for all } a, b \in E\},$$

where the $*$ is \prec and \succ , respectively.

We will write $\Omega_K^*(A)$ for $\Omega^*(E)$ if it is important to emphasize the dependence on K . The centroid of the associative dendriform algebra A is defined as $\Omega(A) = \Omega^\prec(E) \cap \Omega^\succ(E)$.

Definition 2.5. Let E be a dendriform algebra and $\omega \in \text{End}(E)$. Then ω is called a central derivation, if $\omega(E) \subseteq Z(E)$ and $\omega(E^2) = 0$.

The set of total central derivations of E is represented by $C(E)$. It is a simple observation to see that $C(E) \subseteq \Omega(E)$. In fact, $C(E)$ is an ideal of $\Omega(E)$.

Definition 2.6. Let consider E as an indecomposable of dendriform algebra. However, the $\Omega(E)$ is small if $\Omega(E)$ is produced by the scalars and the central derivations.

3 Properties of Centroids of dendriform algebras

In this section, we declare the following results on properties of the centroids of dendriform algebras.

Theorem 3.1. Considering (E, \prec, \succ) as a dendriform algebra. Then

i) $\Omega(E)\text{Der}(E) \subseteq \text{Der}(E)$.

ii) $[\Omega(E), \text{Der}(E)] \subseteq \Omega(E)$.

iii) $[\Omega(E), \Omega(E)](E) \subseteq C(E)$ and $[\Omega(E), \Omega(E)](E^2) = 0$

Proof.

The proof of parts i) – iii) is straightforward by using definitions of derivation and centroid. \square

Lemma 3.1. *If the characteristic of K is 0 or not a factor of $n - 1$. Then*

$$C(E) = \Omega(E) \cap Der(E).$$

Proof.

If $\omega \in \Omega(E) \cap Der(E)$ then by definition of $\Omega(E)$ and $Der(E)$, for all $a, b \in E$, we have $\omega(a * b) = \omega(a) * b + a * \omega(b)$ and $\omega(a * b) = (\omega(a)) * b = a * (\omega(b))$, so $\omega(E^2) = 0$ and $\omega(E) \subseteq Z(E)$ where the hypothesis that the characteristic of K is 0 or not a factor of $n - 1$ is used. it easy to show that $\Omega(E) \cap Der(E) \subseteq C(E)$.

To exhibition the inverse inclusion, let $\omega \in C(E)$; then $0 = \omega(a * b) = \omega(a) * b = a * \omega(b)$. Thus $\omega \in \Omega(E) \cap Der(E)$. This implies $C(E) = \Omega(E) \cap Der(E)$ \square

Theorem 3.2. *Let (E, \prec, \succ) be a dendriform algebra. Then for any $d \in Der(E)$ and $\omega \in \Omega(E)$ one has the following.*

- (a) *The composition $d\omega$ is in $\Omega(E)$ if and only if ωd is a central derivation of E ;*
- (b) *The composition $d\omega$ is a derivation of E only if $[E, \omega]$ is a central derivation of E .*

Proof.

Let us prove (a) For any $\omega \in \Omega(E)$, $d \in Der(E), \forall a, b \in E$ by saying $d \circ \omega$ is contained in $\Omega(E)$ is an central derivation of E by i) and ii) in Theorem 3.1 we have

$$\begin{aligned} d \circ \omega(a * b) &= d(\omega(a) * b) = d(\omega(a)) * b + (\omega(a)) * d(b) \\ &= d(\omega(a)) * b + \omega \circ d(a * b) - (\omega \circ d(a)) * b \end{aligned}$$

Therefore, we get $(d \circ \omega - \omega \circ d)(a * b) = ((d \circ \omega - \omega \circ d)(a)) * b$.

(b) If $d \circ \omega \in Der(E)$, using $[d, \omega] \in \Omega(E)$ we get

$$[d, \omega](a * b) = ([d, \omega](a)) * b = a * ([d, \omega](b)). \tag{4}$$

On the other hand, $[d, \omega] = d \circ \omega - \omega \circ d$. Therefore,

$$[d, \omega](a * b) = (d \circ \omega(a)) * b + a * (d \circ \omega(b)) - (\omega \circ d(a)) * b - a * (\omega \circ d(b)). \tag{5}$$

Due to (4) and (5) we get $a * ([d, \omega](b)) = ([d, \omega](a)) * b = 0$ and thus the necessity is proved.

Let now $[d, \omega]$ be a central derivation of E . Then

$$\begin{aligned} (d \circ \omega)(a * b) &= [d \circ \omega](a * b) + (\omega \circ d)(a * b) \\ &= \omega(d(a) * b) + \omega(a * d(b)) \\ &= (\omega \circ d)(a) * b + a * (\omega \circ d)(b), \end{aligned}$$

where $*$ represents the product \succ and \prec . □

3.1 Centroids of short-dimensional dendriform algebras

This section gives the details of the centroids of dendriform algebras in dimension two over the complex field \mathbb{C} . Let $\{r_1, r_2, r_3, \dots, r_n\}$ be a basis of an n -dimensional dendriform algebra E . The product of the basis

$$r_i \prec r_j = \sum_{k=1}^n \gamma_{ij}^k r_k \text{ and } r_s \succ r_t = \sum_{l=1}^n \delta_{st}^l r_l, \quad i, j, s, t, = 1, 2, \dots, n.$$

We have

$$\begin{aligned} \sum_{k=1}^n \gamma_{ij}^t a_{kt} - a_{ti} \gamma_{tj}^k &= 0; & \sum_{k=1}^n \gamma_{ij}^k a_{kt} - a_{tj} \gamma_{it}^k &= 0; \\ \sum_{k=1}^n \delta_{st}^t a_{kt} - a_{ts} \delta_{sl}^k &= 0; & \sum_{k=1}^n \delta_{st}^k a_{kt} - a_{sl} \delta_{st}^k &= 0. \end{aligned}$$

The classification of all two-dimensional dendriform algebras has been given by [5]. Therefore taking into account the classification result on associative algebras.

Theorem 3.3. Any two-dimensional dendriform algebra can be included in one of following classes of algebras:

- $Dend_2^1(\alpha) : r_1 \prec r_1 = r_2, \quad r_1 \succ r_1 = \alpha r_2, \quad \alpha \in \mathbb{C};$
- $Dend_2^2 : r_1 \prec r_1 = r_1, \quad r_1 \succ r_2 = r_2;$
- $Dend_2^3 : r_1 \prec r_1 = r_1, \quad r_2 \prec r_1 = r_2, \quad r_1 \succ r_2 = r_2, \quad r_2 \succ r_1 = -r_2;$
- $Dend_2^4 : r_2 \prec r_1 = r_2, \quad r_1 \succ r_1 = r_1;$
- $Dend_2^5 : r_1 \prec r_2 = -r_2, \quad r_2 \prec r_1 = r_2, \quad r_1 \succ r_1 = r_1, \quad r_1 \succ r_2 = r_2;$
- $Dend_2^6 : r_1 \prec r_1 = r_1, \quad r_2 \succ r_2 = r_2;$
- $Dend_2^7 : r_1 \prec r_2 = -r_2, \quad r_2 \prec r_2 = r_2, \quad r_1 \succ r_1 = r_1, \quad r_1 \succ r_2 = r_2;$

$$Dend_2^8: \quad r_1 < r_1 = r_1 + r_2, \quad r_1 < r_2 = -r_2, \quad r_2 < r_2 = r_2, \quad r_1 > r_1 = -r_2; \\ r_1 > r_2 = r_2;$$

$$Dend_2^9: \quad r_1 < r_1 = r_1, \quad r_2 < r_1 = r_2, \quad r_2 > r_1 = -r_2, \quad r_2 > r_2 = r_2;$$

$$Dend_2^{10}: \quad r_1 < r_1 = -r_2, \quad r_2 < r_1 = r_2, \quad r_1 > r_1 = r_1 + r_2; \quad r_2 > r_1 = -r_2 \\ r_2 > r_2 = r_2;$$

$$Dend_2^{11}: \quad r_2 < r_1 = r_2, \quad r_1 > r_1 = r_1, \quad r_1 > r_2 = r_2;$$

$$Dend_2^{12}: \quad r_1 < r_1 = r_1, \quad r_2 < r_1 = r_2, \quad r_1 > r_2 = r_2.$$

Theorem 3.4. *The centroids of two dimensional complex Dendriform algebras are given as follows:*

Table 1: Centroids of two-dimensional associative algebras

Multiplication table of E	Centroid $\Omega(E)$	Dim	Types of $\Omega(E)$
$Dend_2^1(\alpha)$	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{11} \end{pmatrix}$	2	small
$Dend_2^2$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	1	small
$Dend_2^3$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	1	small
$Dend_2^4$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	1	small
$Dend_2^5$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	1	small
$Dend_2^6$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	1	small
$Dend_2^7$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	1	small
$Dend_2^8$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	1	small
$Dend_2^9$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	1	small
$Dend_2^{10}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	1	small
$Dend_2^{11}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	1	small
$Dend_2^{12}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}$	1	small

Corollary 3.1.

- i) The centroids of 2 –dimensional dendriform algebras are small.
- ii) The centroids of 2 –dimensional dendriform algebras over the complex field \mathbb{C} has dimensions 1 and 2.

References

- [1] Benkarta G., Neherb E., The centroid of extended affine and root graded Lie algebras, *Journal of Pure and Applied Algebra*, 205(1), 2006, 117–145.
- [2] Loday J.-L., Frabetti A., Chapoton F., Goichot F., *Dialgebras and Related Operads*, Lecture Notes in Math., 1763, Springer, Berlin, 2001.
- [3] Melville D.J., Centroids of nilpotent Lie algebras, *Comm. Algebra*, 20(12), 1992, 3649-3682.
- [4] Melville D.J., Centroids and Cartan algebras, *Comm. Algebra*, 21(8), 1993, 2791–2818.
- [5] Rikhsiboev, I. M., Rakhimov, I. S. and Basri, W., The description of dendriform algebra structures on two-dimensional complex space, *Journal of Algebra, Number Theory: Advances and Applications*, 4 (1),2010, 118.
- [6] Jacobson N., A note on automorphisms and derivations of Lie algebras, *Proc. Amer. Math. Soc.*, 6, 281-283, (1955).
- [7] A.Vimal, C.Sivakumar, “A Load Balancing Model Based On Cloud Partitioning”, *International Journal Of Innovations In Scientific And Engineering Research (Ijiser)*, Vol 1 Issue 3 Mar 2014 Pp.196-201.
- [8] Rakhimov I.S., Al-Nashri Al-Hossain, On Derivations of low-dimensional complex Leibniz algebras, *JP Journal of Algebra, Number Theory and Applications*, 21(1), 69-81, (2011).
- [9] R. Karpagam, Dr. S. Suganya , Applications Of Data Mining And Algorithms In Education – A Survey , *International Journal of Innovations in Scientific and Engineering Research (IJISER)*, Vol.3 ,No.4 ,2016.
- [10] Rakhimov I.S. and Al-Hossain N. On derivations of some classes of Leibniz algebras, *Journal Generalized Lie theory and Applications*, 6, Article ID G120501, doi 10:4303/jglta/G120501, (2012).
- [11] Ancochea J.M., Compoamor R., Characteristically nilpotent Lie algebras: a survey, arXiv:1103224v1.[math.RA], (2000).

- [12] Dixmier J. and Lister W.G. Derivations of nilpotent Lie algebras, *Proc. Amer. Math. Soc.*, 8, 155-158, (1957).
- [13] Khakimjanov Yu.B., Characteristically nilpotent Lie algebras, *Math. USSR Sbornik*, 70(1), (1990).
- [14] Khakimjanov Yu.B., On characteristically nilpotent Lie algebras, *Soviet Math. Dokl.*, 40(1), (1990).
- [15] Leger G.F., Togo S., Characteristically nilpotent Lie algebras, *Duke Math. J.*, 26, 623-628, (1959).
- [16] Ravisankar T. S., Characteristically nilpotent algebras, *Canadian J. Math.*, 23, 222- 235, (1971).
- [17] Rakhimov, I. S., and Fiidow, M. A., Centroids of finite dimensional associative dialgebras, *Far East Journal of Mathematical Sciences*, 98(4), 427-443, (2015).
- [18] M. A. Fiidow, I. S. Rakhimov, S. K. Said Hussain, Derivations and Centroids of Associative Algebras. *IEEE proceedings of International Conference on Research and Education in Mathematics (ICREM7)*, 227-232, (2015).

