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On $n-$norms of Complex Hilbert Spaces

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Abstract

Researchers have defined and studied many different norms their applications in different areas. We study the $n-$norm over the Complex Vector Space (CVS) and explain the relation between some different inequalities of $n-$norms in a Complex Hilbert Spaces (CHS). Moreover, we discuss bounded $n-$complex linear functional normed spaces and then give some related results and generalizations.

1 Introduction

The norm on a complex space is well known and has been studied extensively (see, for instance, [1], [3], and [4]). A norm is a nonnegative complex-valued

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function on the CVS X that satisfies scaling, triangle inequality and zero only at the origin. For example, if $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n})$ is a vector in C^n, one of the suitable norms is $\|x\| = \sqrt{\sum_{k=1}^{n} |x_k|}$. Note that the coordinates of $\mathbf{x}$ all happen to be complex numbers. Then the above definition agrees with the norm for CVSs. Moreover, a mapping $\|., \dots, .\| : X^n \to C$ is the so called $n-$norm on X, if it satisfies; permutationally invariant, scaling, zero only iff its components are linearly independent, and finally triangle inequality under the first component, whereas $(X, \|., \dots, .\|)$ is called an $n-$normed space. In [2] the complex inner product of two vectors $\mathbf{x}$ and $\mathbf{y}$ in standard form is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x_1}\overline{\mathbf{y_1}} + \cdots + \mathbf{x_n}\overline{\mathbf{y_n}}$ The relation between the norm and the inner product of $\mathbf{x} = (\mathbf{x_1}, \dots, \mathbf{x_n})$ is given by $\|\mathbf{x}\| = \sqrt{|\mathbf{x_1}| + \cdots + |\mathbf{x_n}|} = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ Similarly , we define an inner-product $\langle ., . \rangle$ in the standard $n-$norm on X form through $\|x_1, \dots, x_n\|_{S^*} = \sqrt{|\det(\langle x_i, x_j \rangle)|}$, so under the conditions of the inner product, we get

$$\|\alpha x_1, \dots, x_n\|_{S^{*\prime}} = |\alpha| \|x_1, \dots, x_n\|_{S^*} = |\alpha| \sqrt{|\det(\langle x_i, x_j \rangle)|}$$

Moreover, the above definition denotes the volume of the paralelepiped spanned of dimension n through $x_1, \dots, x_n \in X$. Since X is a CVS, we write

$$\|\alpha x_1, \dots, x_n\|_{G^*} = \sup_{\|g_i\| \leq 1, g_i \in \overline{X^*}} \begin{vmatrix} g_1(\alpha x_1) & \cdots & g_n(\alpha x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_1(\alpha x_n) \end{vmatrix} = |\overline{\alpha}| \sup_{\|g_i\| \leq 1, g_i \in \overline{X^*}} \det[g_i(x_i)]$$

The conjugate space X^* can be specified at the functional $g : X \to C$ of all additive complex-valued, where $g(\alpha x) = \overline{\alpha} g(x)$. In addition, $\overline{X^*}$ is a complex conjugate of the dual space such that $\Theta_{\langle .,. \rangle} : X \to \overline{X^*}$ The $n-$normed space was first studied by Gahler [3] and [4] in the sixties then extensively by Gunawan [2]. In the next section, we give diverse formulas in a CHS. Many researchers gave important results into CVSs (see [5], for example).

2 $n-$norms in CHSs

Let X be a CHS. Using the Riesz Representation Theorem, each $g \in \overline{X^*}$ can be specified through $z \in X$ such that $g(x) = \langle x, z \rangle$, for all $x \in X$. Hence, using generalized Cauchy-Schwarz and Hadamard's inequalities [6], [7], we get

$$\|\alpha x_1, \dots, x_n\|_{G^*} = |\overline{\alpha}| \sup_{z_i \in X, \|z_i\| \leq 1} \det[\langle x_k, z_i \rangle]$$

$$\leq |\overline{\alpha}| \sup_{z_i \in X, \|z_i\| \leq 1} \|x_1, \dots, x_n\|_{S^{*\prime}} \|z_1, \dots, z_n\|_{S^{*\prime}} \leq \|x_1, \dots, x_n\|_{S^{*\prime}} \|z_1\| \dots \|z_n\|$$

We conclude that $\|\alpha x_1, \dots, x_n\|_{G^*} \leq \|\alpha x_1, \dots, x_n\|_{S^{*\prime}}$.
Conversely, suppose $\alpha x_1, \dots, x_n$ are linearly independent. The vectors obtained from $\alpha x_1, \dots, x_n$ equal to $\alpha x_1^*, \dots, x_n^*$ and by the Gram-Schmidt orthogonalization, we get $\|\alpha x_1, \dots, x_n\|_{S^{*\prime}} = |\alpha| \|x_1^*\|, \dots, \|x_n^*\|$.
If $y_i = \frac{1}{\|x_i^*\|}, i = 1, \dots, n$, then we can use determinants as follows:

$$|\overline{\alpha}| \begin{vmatrix} \langle x_1, z_1 \rangle & \cdots & \langle x_1, z_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1 \rangle & \cdots & \langle x_n, z_n \rangle \end{vmatrix} = |\overline{\alpha}| \frac{1}{\|x_1^*\| \dots \|x_n^*\|} \begin{vmatrix} \langle x_1^*, x_1^* \rangle & \cdots & \langle x_1^*, x_n^* \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n^*, x_1^* \rangle & \cdots & \langle x_n^*, x_n^* \rangle \end{vmatrix} = |\overline{\alpha}| \|x_1\|^* \dots \|x_n^*\|$$

Hence,

$$\|\alpha x_1, \dots, x_n\|_{G^*} \geq \|\alpha x_1, \dots, x_n\|_{S^{*\prime}}$$

Suppose that X is a separable space and let $\{e_1, e_2, \dots\}$ be a complete orthonormal subset of X. Subsequently, $\forall x \in X$, we can specify the the sequence $(\langle x, e_k \rangle) \in l^2$. As shown, [5] determines an $n-$norm on X through the following formula:

$$\|\alpha x_1, \dots, x_n\|_2 = \left[|\alpha| \frac{1}{n!} \sum_{k_1} \cdots \sum_{k_n} |\det [\lambda_{ik_j}]|^2 \right]^{\frac{1}{2}},$$

where $\lambda_{ik} = \langle x_1, e_k \rangle$.

Theorem 2.1. *For any separable Hilbert space X, $\|\alpha x_1, \dots, x_n\|_{G^*}$, $\|\alpha x_1, \dots, x_n\|_{S^{*\prime}}$ and $\|\alpha x_1, \dots, x_n\|_2$ are identical.*

In the following, we give other formulas of $n-$norms.

Theorem 2.2. *The function*

$$\|\alpha x_1, \dots, x_n\|_{E^*} = |\alpha| \sup_{z_1, \dots, z_n, \|z_1, \dots, z_n\|_{S^*} \leq 1} \begin{vmatrix} \langle x_1, z_1 \rangle & \cdots & \langle x_1, z_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1 \rangle & \cdots & \langle x_n, z_n \rangle \end{vmatrix}$$

defines an $n-$norm over X.

Proof.
If $\|\alpha x_1, \dots, x_n\|_{E^*} = 0$, then the rows of $\langle x_i, z_i \rangle$ are linearly dependent vectors $\forall z_1, \dots z_n \in X$ with $\|\alpha x_1, \dots, x_n\|_{S^*} \leq 1$. On the contrary, we get

$\|\alpha x_1, \ldots, x_n\|_{E^*} = 0$ for linearly dependent vectors. Using determinants, we get an invariance of $\|\alpha x_1, \ldots, x_n\|_{E^*}$ under permutation. Moreover, we obtain

$$\|\alpha\beta x_1, \ldots, x_n\|_{E^*} = |\alpha\beta| \|x_1, \ldots, x_n\|_{E^*}, \forall \alpha, \beta \in C$$

Finally, for arbitrary elements $x, x^{'}, x_2, x_3, , x_n \in X$, we get

$$\begin{vmatrix} \langle x + x^{'}, z_1 \rangle & \cdots & \langle x + x^{'}, z_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1 \rangle & \cdots & \langle x_n, z_n \rangle \end{vmatrix} = \begin{vmatrix} \langle x, z_1 \rangle & \cdots & \langle x, z_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1 \rangle & \cdots & \langle x_n, z_n \rangle \end{vmatrix} + \begin{vmatrix} \langle x^{'}, z_1 \rangle & \cdots & \langle x^{'}, z_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1 \rangle & \cdots & \langle x_n, z_n \rangle \end{vmatrix}$$

By taking the supermums for the above inequalities, we get

$$\|\alpha(x + x^{'}), \ldots, x_n\|_{E^*} \leq \|\alpha x, \ldots, x_n\|_{E^*} + \|\alpha x^{'}, \ldots, x_n\|_{E^*} \square$$

As a result, we get

Theorem 2.3. *The formulas $\|\alpha x_1, \ldots, x_n\|_{E^*}$ and $\|\alpha x_1, \ldots, x_n\|_{G^*}$ are identical.*

Proof.

Suppose $z_1, \ldots, z_n$, where $\|z_1, \ldots, z_n\|_{S^{*\prime}} \leq \|z_1\|, \ldots, \|z_n\|$, we obtain , $\|z_1, \ldots, z_n\|_{S^{*\prime}} \leq 1, j = 1, \ldots, n$. Subsequently $\|x_1, \ldots, x_n\|_{G^*} \leq \|x_1, \ldots, x_n\|_{E^*}$.
To prove the second part, if $\|z_1, \ldots, z_n\|_{S^{*\prime}} \leq 1$, then, by the generalized Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\alpha| |\langle x_k, z_i \rangle| &\leq \sqrt{|\langle x_k, x_i \rangle|} \sqrt{|\langle z_k, z_i \rangle|} = |\alpha| \|x_1, \ldots, x_n\|_{S^{*\prime}} \|z_1, \ldots, z_n\|_{S^{*\prime}} \\ &\leq |\alpha| \|x_1, \ldots, x_n\|_{S^{*\prime}} = |\alpha| \|x_1, \ldots, x_n\|_{G^*} \end{aligned}$$

So, $\|\alpha x_1, \ldots, x_n\|_{E^*} \leq \|\alpha x_1, \ldots, x_n\|_{G^*} \square$

Corollary 2.4. *When X is a separable CHS, the $n-$normed spaces $(X, \|., \ldots, .\|_{G^*}), (X, \|., \ldots$ and $(X, \|., \ldots, .\|_{E^*})$ are identical.*

In $(X, \|.\|)$, we define a norm on the complex conjugate of the dual space X^* using $\|g\| = \sup_{\|x\| \leq 1} |g(x)|, g \in \overline{X^*}$.

Theorem 2.5. *For an $n-$normed space $(X, \|., \ldots, .\|^{'})$, $\|., \ldots, .\|^{'} : \overline{(X^*)^n} \rightarrow C$ presented by*

$$\|\alpha g_1, \ldots, g_n\|^{'} = \sup_{x_i \in X, \|x_1, .., x_n\| \leq 1} \begin{vmatrix} g_1(\alpha x_1) & \cdots & g_n(\alpha x_n) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n) \end{vmatrix}$$

shown an $n-$norm on $\overline{X^*}$

3 Bounded $n-$linear Functionals

Consider a CVS X with an $n-$norm $\|.,\ldots,.\|$ on X. The functional G: $X^n \to C$ is called an n-linear functional on a CVS X. G is bounded if $|\mathrm{G}(\alpha x_1,\ldots,x_n)| \leq h\|\alpha x_1,\ldots,x_n\|$, for a constant h, $(x_1,\ldots,x_n) \in X^n$, $\alpha \in C$. If G is bounded , then

$$\|G\| = \sup_{\|x_1,\ldots,x_n\|\neq 0} \frac{|G(\alpha x_1,\ldots,x_n)|}{\|\alpha x_1,\ldots,x_n\|} \text{ or } \|G\| = \sup_{\|x_1,\ldots,x_n\|=1} |G(\alpha x_1,\ldots,x_n)|$$

Let Γ denote the set of all bounded $n-$linear functionals. In what follows, we show two alternate formulas for $\|G\|$:

Theorem 3.1. *Assume that $G \in \Gamma$ on C. Then*

$$\|G\| = \inf_h\{|G(\alpha x_1,....,x_n)|\} \leq h\|\alpha x_1,...,x_n\| = \sup_{\|x_1,...,x_n\|\leq 1} |G(\alpha x_1,....,x_n)|,$$

where$(x_1,\ldots,x_n) \in X^n, \alpha \in C$

Proof. Suppose that $H = \{h : |G(\alpha x_1,....,x_n)| \leq h\|\alpha x_1,...,x_n\|, (x_1,....,x_n) \in X^n, \alpha \in C\}$. It is clear that $\|G\| \in H$ and so $\inf H \leq \|G\|$.
On the other hand, $\forall h \in H$, we have $\frac{|G(\alpha x_1,...,x_n)|}{\|\alpha x_1,...,x_n\|} \leq |\alpha|h$ when $\|x_1,...,x_n\| \neq 0$; therefore, $\|G\| \leq |\alpha|h$ and this is true $\forall h \in H$. We have $\|G\| \leq \inf H$. So $\|G\| = \inf H$. Next, if $\|x_1,...,x_n\| \leq 1$, then $|G(\alpha x_1,....,x_n)| \leq |\alpha|\|G\|\|x_1,...,x_n\| \leq |\alpha|\|G\|$. So $\sup_{\|x_1,...,x_n\|\leq 1} |G(\alpha x_1,...,x_n)| \leq \|G\|$.
Conversely, we get $\|G\| = \sup_{\|x_1,...,x_n\|=1} |G(\alpha x_1,...,x_n)| \leq sup_{\|x_1,...,x_n\|\leq 1}|G(\alpha x_1,...,x_n)|$.
Consequently, $\|G\| = sup_{\|x_1,...,x_n\|\leq 1}|G(\alpha x_1,...,x_n)|\Box$.
As an example, consider the $n-$normed space $(C^n, \|.,...,.\|_{S^{*\prime}})$ with basis $\{e_1,...,e_n\}$. Define G by $G(\alpha x_1,...,x_n) = |\alpha| \begin{vmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{vmatrix} = |\alpha| \det[\lambda_{ij}]$, where $x_k = \sum_{i=1}^n e_i, k = 1,...,n$. Then, $G \in \Gamma$ is on C^n, $\|G\| = |\alpha|$.

Theorem 3.2. *Suppose that $(X, \langle .,.\rangle)$ be a CVS, with an $n-$norm $\|.,\ldots,.\|_{S^{*\prime}}$ in standard form. For installed elements $z_1,...,z_n \in X$, we define G on X^n by $G(\alpha x_1,\ldots,x_n) = \det[\langle \alpha x_k, z_i\rangle]$. Then $G \in \Gamma$ on X, and $\|G\| = |\alpha|\|z_1,\ldots,z_n\|_{S^{*\prime}}$*

Proof. *Using Theorem 3.1, we get*

$$\|G\| = \sup_{\|x_1,...,x_n\|_{S^{*\prime}}\leq 1} |\det[\langle \alpha x_k, z_i\rangle]|$$

By the generalized Cauchy-Schwarz inequality, we have

$$\|G\| \le \sup_{\|x_1,...,x_n\|_{S^{*\prime}} \le 1} \|\alpha x_1, ..., x_n\|_{S^{*\prime}} \|z_1, ..., z_n\|_{S^{*\prime}} \le |\alpha| \|z_1, ..., z_n\|_{S^{*\prime}}$$

Assume $x_i = \frac{z_i}{\|z_1,...,z_n\|_{S^{*\prime}}^{\frac{1}{n}}}$. *Then we obtain* $\|G\| = |\alpha| \|z_1, ..., z_n\|_{S^{*\prime}} \square$.

Any member in Γ on the space $L_p, 1 \le p \le \infty$,

$$\|\alpha x_1, ..., x_n\|_p = \left[|\alpha| \frac{1}{n!} \sum_{k_1} \cdots \sum_{k_n} \left| \det \left[x_{ik_j} \right] \right|^p \right]^{\frac{1}{p}}.$$

4 Conclusions

In this paper, we studied CHS with some types of $n-$norms there and the relationships among them. Also, we presented the concept of a bounded $n-$linear functional with some related results.

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