On Neutrosophic Generalized Semi Generalized Closed Sets

Qays Hatem Imran[1]*, Ali H. M. Al-Obaidi[2], Florentin Smarandache[3] and Said Broumi[4]

[1]Department of Mathematics, College of Education for Pure Science, Al-Muthanna University, Samawah, Iraq.

[2]Department of Mathematics, College of Education for Pure Science, University of Babylon, Hillah, Iraq.

[3]Department of Mathematics, University of New Mexico 705 Gurley Ave. Gallup, NM 87301, USA.

[4]Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Morocco.

E-mails: qays.imran@mu.edu.iq, aalobaidi@uobabylon.edu.iq, smarand@unm.edu, broumisaid78@gmail.com

*Correspondence: qays.imran@mu.edu.iq

Abstract

The article considers a new generalization of closed sets in neutrosophic topological space. This generalization is called neutrosophic $\mathcal{gsg}$-closed set. Moreover, we discuss its essential features in neutrosophic topological spaces. Furthermore, we extend the research by displaying new related definitions such as neutrosophic $\mathcal{gsg}$-closure and neutrosophic $\mathcal{gsg}$-interior and debating their powerful characterizations and relationships.

Mathematics Subject Classification (2010): 54A05, 54B05.

Keywords: $Neu^{\mathcal{gsg}}CS$, $Neu^{\mathcal{gsg}}OS$, $Neu^{\mathcal{gsg}}$-closure and $Neu^{\mathcal{gsg}}$-interior.

1. Introduction

The neutrosophic set theory was contributed by Smarandache in [1,2]. The neutrosophic topological space (simply Neu^{TS}) was offered by Salama et al. in [3]. The definition of semi-α-open sets in neutrosophic topological spaces was displayed by Imran et al. in [4]. The neutrosophic generalized homeomorphism was submitted by PAGE et al. in [5]. The class of generalized neutrosophic closed sets was given by Dhavaseelan et al. in [6]. The concepts of neutrosophic generalized $\alpha\mathcal{g}$-closed sets and neutrosophic generalized $\alpha\mathcal{g}$-continuous functions were provided by Imran et al. in [7]. The objective of this article is to show the sense of neutrosophic $\mathcal{gsg}$-closed set (briefly $Neu^{\mathcal{gsg}}CS$) and investigate their main characteristics in Neu^{TS}. Moreover, we argue neutrosophic $\mathcal{gsg}$-closure (in word $Neu^{\mathcal{gsg}}$-closure) and neutrosophic $\mathcal{gsg}$-interior (fleetingly $Neu^{\mathcal{gsg}}$-interior) with revealing several of their vital spots.

DOI: https://doi.org/10.54216/IJNS.18030101
Received: March 02, 2022 Accepted: May 30, 2022

2. Preliminaries

In this work, $(\mathfrak{U},\zeta)$ (or simply $\mathfrak{U}$) always mean Neu^{TS}. Let $\mathfrak{P}$ be a neutrosophic set in a Neu^{TS} $(\mathfrak{U},\zeta)$, we denote the neutrosophic closure, the neutrosophic interior, and the neutrosophic complement of $\mathfrak{P}$ by $NeuCl(\mathfrak{P})$, $NeuInt(\mathfrak{P})$ and $\mathfrak{P}^c = 1_{Neu} - \mathfrak{P}$, respectively.

Definition 2.1: [3]
The family ζ of neutrosophic subsets of a non-empty neutrosophic set $\mathfrak{U} \neq \emptyset$ is called a neutrosophic topology (in short, Neu^{T}) on $\mathfrak{U}$ if it satisfies the below axioms:
(i) $0_{Neu}, 1_{Neu} \in \zeta,$
(ii) $\mathfrak{P}_1 \sqcap \mathfrak{P}_2 \in \zeta$ being $\mathfrak{P}_1, \mathfrak{P}_2 \in \zeta,$
(iii) $\sqcup \mathfrak{P}_i \in \zeta$ for arbitrary family $\{\mathfrak{P}_i | i \in \Lambda\} \sqsubseteq \zeta.$
In this case, we signified Neu^{TS} by $(\mathfrak{U},\zeta)$ or $\mathfrak{U}$. Moreover, the neutrosophic set in ζ is named neutrosophic open (in short, $NeuOS$). Furthermore, for any $NeuOS$ $\mathfrak{P}$, then $\mathfrak{P}^c$ is titled neutrosophic closed set (briefly, $NeuCS$) in $\mathfrak{U}$.

Definition 2.2:
Let $\mathfrak{P}$ be a neutrosophic subset of a $Neu^{TS}(\mathfrak{U},\zeta)$, then it is called to be:
(i) a neutrosophic semi-open set and denoted by $Neu^{s}OS$ if $\mathfrak{P} \sqsubseteq NeuCl(NeuInt(\mathfrak{P}))$. [8]
(ii) a neutrosophic semi-closed set and denoted by $Neu^{s}CS$ if $NeuInt(NeuCl(\mathfrak{P})) \sqsubseteq \mathfrak{P}$. The intersection of entire $Neu^{s}CSs$, including $\mathfrak{P}$ is named a neutrosophic semi-closure, and it is symbolized by $Neu^{s}Cl(\mathfrak{P})$.[8]
(iii) a neutrosophic α-open set and denoted by $Neu^{\alpha}OS$ if $\mathfrak{P} \sqsubseteq NeuInt(NeuCl(NeuInt(\mathfrak{P})))$. [9]
(iv) a neutrosophic α-closed set and denoted by $Neu^{\alpha}CS$ if $NeucCl(NeuInt(NeuCl(\mathfrak{P}))) \sqsubseteq \mathfrak{P}$. The intersection of the whole $Neu^{\alpha}CSs$ including $\mathfrak{P}$ is named neutrosophic α-closure, and it is symbolized by $Neu^{\alpha}Cl(\mathfrak{P})$.[9]

Definition 2.3:
Let $\mathfrak{P}$ be a neutrosophic subset of a $Neu^{TS}(\mathfrak{U},\zeta)$, and let $\mathfrak{M}$ be a a $NeuOS$ in $(\mathfrak{U},\zeta)$ such that $\mathfrak{P} \sqsubseteq \mathfrak{M}$ then $\mathfrak{P}$ is called to be:
(i) a neutrosophic generalized closed set, and it is denoted by $Neu^{g}CS$ if $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The complement of a $Neu^{g}CS$ is a $Neu^{g}OS$ in $(\mathfrak{U},\zeta)$. [10]
(ii) a neutrosophic αg-closed set, and it is denoted by $Neu^{\alpha g}CS$ if $Neu^{\alpha}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The complement of a $Neu^{\alpha g}CS$ is a $Neu^{\alpha g}OS$ in $(\mathfrak{U},\zeta)$. [11]
(iii) a neutrosophic $g\alpha$-closed set, and it is denoted by $Neu^{g\alpha}CS$ if $Neu^{\alpha}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The complement of a $Neu^{g\alpha}CS$ is a $Neu^{g\alpha}OS$ in $(\mathfrak{U},\zeta)$. [12]
(iv) a neutrosophic sg-closed set, and it is denoted by $Neu^{sg}CS$ if $Neu^{s}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The complement of a $Neu^{sg}CS$ is a $Neu^{sg}OS$ in $(\mathfrak{U},\zeta)$. [13]
(v) a neutrosophic gs-closed set, and it is denoted by $Neu^{gs}CS$ if $Neu^{g}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The complement of a $Neu^{gs}CS$ is a $Neu^{gs}OS$ in $(\mathfrak{U},\zeta)$. [14]

Proposition 2.4:[9,10]
In a $Neu^{TS}(\mathfrak{U},\zeta)$, then the next arguments stand, and the opposite of every argument is not valid:
(i) Each $NeuOS$ (resp. $NeuCS$) is a $Neu^{\alpha}OS$ (resp. $Neu^{\alpha}CS$).
(ii) Each $NeuOS$ (resp. $NeuCS$) is a $Neu^{g}OS$ (resp. $Neu^{g}CS$).

DOI: https://doi.org/10.54216/IJNS.18030101
Received: March 02, 2022 Accepted: May 30, 2022

(iii) Each $Neu^{\alpha}OS$ (resp. $Neu^{\alpha}CS$) is a $Neu^{s}OS$ (resp. $Neu^{s}CS$).

Proposition 2.5:[11,12]

In a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$, then the next arguments stand, and the opposite of every argument is not valid:

(i) Each $Neu^{g}OS$ (resp. $Neu^{g}CS$) is a $Neu^{\alpha g}OS$ (resp. $Neu^{\alpha g}CS$).

(ii) Each $Neu^{\alpha}OS$ (resp. $Neu^{\alpha}CS$) is a $Neu^{g\alpha}OS$ (resp. $Neu^{g\alpha}CS$).

(iii) Each $Neu^{g\alpha}OS$ (resp. $Neu^{g\alpha}CS$) is a $Neu^{\alpha g}OS$ (resp. $Neu^{\alpha g}CS$).

Proposition 2.6:[13-15]

In a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$, then the next arguments stand, and the opposite of every argument is not valid:

(i) Each $Neu^{g}OS$ (resp. $Neu^{g}CS$) is a $Neu^{gs}OS$ (resp. $Neu^{gs}CS$).

(ii) Each $Neu^{s}OS$ (resp. $Neu^{s}CS$) is a $Neu^{sg}OS$ (resp. $Neu^{sg}CS$).

(iii) Each $Neu^{sg}OS$ (resp. $Neu^{sg}CS$) is a $Neu^{gs}OS$ (resp. $Neu^{gs}CS$).

(iv) Each $Neu^{g\alpha}OS$ (resp. $Neu^{g\alpha}CS$) is a $Neu^{gs}OS$ (resp. $Neu^{gs}CS$).

3. Neutrosophic Generalized sg-Closed Sets

In this sector, we present and analyse the neutrosophic generalized sg-closed sets and some of their features.

Definition 3.1:

Suppose that $\mathfrak{P}$ is a neutrosophic set in a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ and assume that $\mathfrak{M}$ is a $Neu^{sg}OS$ in $(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. The set $\mathfrak{P}$ is termed as a neutrosophic generalized sg-closed set, and it is signified by $Neu^{gsg}CS$ if $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The collection of all $Neu^{gsg}CSs$ in a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ is signified by $Neu^{gsg}C(\mathfrak{U})$.

Theorem 3.2:

In a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$, the subsequent arguments are valid:

(i) Each $NeuCS$ is a $Neu^{gsg}CS$.

(ii) Each $Neu^{gsg}CS$ is a $Neu^{g}CS$.

Proof:

(i) Let $NeuCS$ $\mathfrak{P}$ and $Neu^{sg}OS$ $\mathfrak{M}$ be in a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Then $NeuCl(\mathfrak{P}) = \mathfrak{P} \sqsubseteq \mathfrak{M}$. Therefore $\mathfrak{P}$ is a $Neu^{gsg}CS$.

(ii) Let $Neu^{gsg}CS$ $\mathfrak{P}$ and $NeuOS$ $\mathfrak{M}$ be in a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $NeuOS$ is a $Neu^{sg}OS$, we get $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, $\mathfrak{P}$ is a $Neu^{g}CS$. ▪

The reverse of the above theorem is inaccurate, as displayed in the subsequent instances.

Example 3.3:

Suppose that $\mathfrak{U} = \{u_1, u_2\}$ is a set and assume that $\zeta = \{0_{Neu}, \mathfrak{P}_1, \mathfrak{P}_2, 1_{Neu}\}$ is a $Neu^{\mathcal{T}}$ defined on $\mathfrak{U}$. Suppose that we have the sets $\mathfrak{P}_1 = \langle u, (0.6,0.7), (0.1,0.1), (0.4,0.2)\rangle$ and $\mathfrak{P}_2 = \langle u, (0.1,0.2), (0.1,0.1), (0.8,0.8)\rangle$ are given. Then the neutrosophic set $\mathfrak{P}_3 = \langle u, (0.2,0.2), (0.1,0.1), (0.6,0.7)\rangle$ is a $Neu^{gsg}CS$. However, this latter set is not a $NeuCS$.

Example 3.4:

Suppose that $\mathfrak{U} = \{u_1, u_2, u_3\}$ is a set and assume that $\zeta = \{0_{Neu}, \mathfrak{P}_1, \mathfrak{P}_2, 1_{Neu}\}$ is a $Neu^{\mathcal{T}}$ defined on $\mathfrak{U}$. Suppose that we have the following sets $\mathfrak{P}_1 = \langle u, (0.5,0.5,0.4), (0.7,0.5,0.5), (0.4,0.5,0.5)\rangle$ and $\mathfrak{P}_2 =$

DOI: https://doi.org/10.54216/IJNS.18030101
Received: March 02, 2022 Accepted: May 30, 2022

$\langle u, (0.3,0.4,0.4), (0.4,0.5,0.5), (0.3,0.4,0.6)\rangle$ are given. Then the neutrosophic set $\mathfrak{P}_3 = \langle u, (0.4,0.6,0.5), (0.4,0.3,0.5), (0.5,0.6,0.4)\rangle$ is a $Neu^{g}CS$. However, this latter set is not a $Neu^{gsg}CS$.

Theorem 3.5:
In a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$, the subsequent arguments are valid:
(i) Each $Neu^{gsg}CS$ is a $Neu^{\alpha g}CS$.
(ii) Each $Neu^{gsg}CS$ is a $Neu^{g\alpha}CS$.
(iii) Each $Neu^{gsg}CS$ is a $Neu^{sg}CS$.
(iv) Each $Neu^{gsg}CS$ is a $Neu^{gs}CS$.
Proof:
(i) Let $Neu^{gsg}CS$ $\mathfrak{P}$ and $NeuOS$ $\mathfrak{M}$ be in a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $NeuOS$ is a $Neu^{\alpha g}CS$, we get $Neu^{\alpha}Cl(\mathfrak{P}) \sqsubseteq NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The latter implies $Neu^{\alpha}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, $\mathfrak{P}$ is a $Neu^{\alpha g}CS$.
(ii) Let $Neu^{gsg}CS$ $\mathfrak{P}$ and $Neu^{\alpha}OS$ $\mathfrak{M}$ be in a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $Neu^{\alpha}OS$ is a $Neu^{s}OS$, which is a $Neu^{sg}OS$, we get $Neu^{\alpha}Cl(\mathfrak{P}) \sqsubseteq NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The latter implies $Neu^{\alpha}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, $\mathfrak{P}$ is a $Neu^{g\alpha}CS$.
(iii) Let $Neu^{gsg}CS$ $\mathfrak{P}$ and $Neu^{s}OS$ $\mathfrak{M}$ be in a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $Neu^{s}OS$ is a $Neu^{sg}OS$, we get $Neu^{s}Cl(\mathfrak{P}) \sqsubseteq NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The latter implies $Neu^{s}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, $\mathfrak{P}$ is a $Neu^{sg}CS$.
(iv) Let $Neu^{gsg}CS$ $\mathfrak{P}$ and $NeuOS$ $\mathfrak{M}$ be in a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $NeuOS$ is a $Neu^{sg}OS$, we get $Neu^{s}Cl(\mathfrak{P}) \sqsubseteq NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. That implies $Neu^{s}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, $\mathfrak{P}$ is a $Neu^{gs}CS$. ▪
The reverse of the above theorem is inaccurate, as displayed in the subsequent instances.

Example 3.6:
Let $\mathfrak{U} = \{u_1, u_2\}$ be a set and assume that $\zeta = \{0_{Neu}, \mathfrak{P}_1, \mathfrak{P}_2, 1_{Neu}\}$ is a $Neu^{\mathcal{T}}$ defined on $\mathfrak{U}$. Suppose that we have the following sets $\mathfrak{P}_1 = \langle u, (0.5,0.6), (0.3,0.2), (0.4,0.1)\rangle$ and $\mathfrak{P}_2 = \langle u, (0.4,0.4), (0.4,0.3), (0.5,0.4)\rangle$ are given. Then the neutrosophic set $\mathfrak{P}_3 = \langle u, (0.5,0.4), (0.4,0.4), (0.4,0.5)\rangle$ is a $Neu^{\alpha g}CS$ and hence $Neu^{g\alpha}CS$ but not a $Neu^{gsg}CS$.

Example 3.7:
Let $\mathfrak{U} = \{u_1, u_2\}$ and let $\zeta = \{0_{Neu}, \mathfrak{P}_1, 1_{Neu}\}$ be a $Neu^{\mathcal{T}}$ on $\mathfrak{U}$. Take $\mathfrak{P}_1 = \langle u, (0.3,0.4,0.6), (0.6,0.6,0.4)\rangle$. Then the neutrosophic set $\mathfrak{P}_2 = \langle u, (0.3,0.2,0.5), (0.6,0.6,0.8)\rangle$ is a $Neu^{sg}CS$ but not a $Neu^{gsg}CS$.

Example 3.8:
Let $\mathfrak{U} = \{u_1, u_2\}$ and let $\zeta = \{0_{Neu}, \mathfrak{P}_1, 1_{Neu}\}$ be a $Neu^{\mathcal{T}}$ on $\mathfrak{U}$. Where $\mathfrak{P}_1 = \langle u, (0.3,0.2,0.3), (0.8,0.6,0.7)\rangle$. Then the neutrosophic set $\mathfrak{P}_2 = \langle u, (0.3,0.2,0.6), (0.8,0.9,0.8)\rangle$ is a $Neu^{gs}CS$. However, this latter set is not a $Neu^{gsg}CS$.

Remark 3.9:
The $Neu^{gsg}CS$ are independent of $Neu^{\alpha}CS$ and $Neu^{s}CS$.

Definition 3.10:
A neutrosophic subset $\mathfrak{P}$ of a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ is called a neutrosophic generalized sg-open set (in short, $Neu^{gsg}OS$) iff $1_{Neu} - \mathfrak{P}$ is a $Neu^{gsg}CS$. The collection of entire $Neu^{gsg}OS$s of a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ is signified by $Neu^{gsg}O(\mathfrak{U})$.

Proposition 3.11:
Let $\mathfrak{P}$ be a $NeuOS$ in $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$, then this set $\mathfrak{P}$ is $Neu^{gsg}OS$ in the space $(\mathfrak{U},\zeta)$.

DOI: https://doi.org/10.54216/IJNS.18030101
Received: March 02, 2022 Accepted: May 30, 2022

Proof:

Let $\mathfrak{P}$ be a $NeuOS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $1_{Neu} - \mathfrak{P}$ is a $NeuCS$ in $(\mathfrak{U},\zeta)$. According to theorem (3.2), point (i), $1_{Neu} - \mathfrak{P}$ is a $Neu^{gsg}CS$. Therefore, $\mathfrak{P}$ is a $Neu^{gsg}OS$ in $(\mathfrak{U},\zeta)$. ▪

Proposition 3.12:

Let $\mathfrak{P}$ be a $Neu^{gsg}OS$ in $Neu^{TS}(\mathfrak{U},\zeta)$, then this set $\mathfrak{P}$ is $Neu^{g}OS$ in the space $(\mathfrak{U},\zeta)$.

Proof:

Let $\mathfrak{P}$ be a $Neu^{gsg}OS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $1_{Neu} - \mathfrak{P}$ is a $Neu^{gsg}CS$ in $(\mathfrak{U},\zeta)$. According to theorem (3.2), point (ii), $1_{Neu} - \mathfrak{P}$ is a $Neu^{g}CS$. Therefore, $\mathfrak{P}$ is a $Neu^{g}OS$ in $(\mathfrak{U},\zeta)$. ▪

Theorem 3.13:

In a $Neu^{TS}(\mathfrak{U},\zeta)$, the subsequent arguments are valid:

(i) Each $Neu^{gsg}OS$ is a $Neu^{\alpha g}OS$ and $Neu^{g\alpha}OS$.

(ii) Each $Neu^{gsg}OS$ is a $Neu^{sg}OS$ and $Neu^{gs}OS$.

Proof:

Similar to above proposition. ▪

Proposition 3.14:

If $\mathfrak{P}$ and $\mathfrak{Q}$ are $Neu^{gsg}CSs$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $\mathfrak{P}\sqcup\mathfrak{Q}$ is a $Neu^{gsg}CS$.

Proof:

Let $\mathfrak{P}$ and $\mathfrak{Q}$ be two $Neu^{gsg}CSs$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ and let $\mathfrak{M}$ be any $Neu^{sg}OS$ in $\mathfrak{U}$ such that $\mathfrak{P} \sqsubseteq \mathfrak{M}$ and $\mathfrak{Q} \sqsubseteq \mathfrak{M}$. Then we have $\mathfrak{P}\sqcup\mathfrak{Q} \sqsubseteq \mathfrak{M}$. Since $\mathfrak{P}$ and $\mathfrak{Q}$ are $Neu^{gsg}CSs$ in $\mathfrak{U}$, $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$ and $NeuCl(\mathfrak{Q}) \sqsubseteq \mathfrak{M}$. Now, $NeuCl(\mathfrak{P}\sqcup\mathfrak{Q}) = NeuCl(\mathfrak{P})\sqcup NeuCl(\mathfrak{Q}) \sqsubseteq \mathfrak{M}$ and so $NeuCl(\mathfrak{P}\sqcup\mathfrak{Q}) \sqsubseteq \mathfrak{M}$. Hence $\mathfrak{P}\sqcup\mathfrak{Q}$ is a $Neu^{gsg}CS$ in $\mathfrak{U}$. ▪

Proposition 3.15:

If $\mathfrak{P}$ is a $Neu^{gsg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $NeuCl(\mathfrak{P}) - \mathfrak{P}$ does not include non-empty $NeuCS$ in $(\mathfrak{U},\zeta)$.

Proof:

Let $\mathfrak{P}$ be a $Neu^{gsg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ and let $\mathfrak{F}$ be any $NeuCS$ in $(\mathfrak{U},\zeta)$ such that $\mathfrak{F} \sqsubseteq NeuCl(\mathfrak{P}) - \mathfrak{P}$. Since $\mathfrak{P}$ is a $Neu^{gsg}CS$, we have $NeuCl(\mathfrak{P}) \sqsubseteq 1_{Neu} - \mathfrak{F}$. This implies $\mathfrak{F} \sqsubseteq 1_{Neu} - NeuCl(\mathfrak{P})$. Then $\mathfrak{F} \sqsubseteq NeuCl(\mathfrak{P})\sqcap(1_{Neu} - NeuCl(\mathfrak{P})) = 0_{Neu}$. Thus, $\mathfrak{F} = 0_{Neu}$. Hence $NeuCl(\mathfrak{P}) - \mathfrak{P}$ does not include non-empty $NeuCS$ in $(\mathfrak{U},\zeta)$. ▪

Proposition 3.16:

A neutrosophic set $\mathfrak{P}$ is $Neu^{gsg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ iff $NeuCl(\mathfrak{P}) - \mathfrak{P}$ does not include non-empty $Neu^{sg}CS$ in $(\mathfrak{U},\zeta)$.

Proof:

Let $\mathfrak{P}$ be a $Neu^{gsg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ and let $\mathfrak{R}$ be any $Neu^{sg}CS$ in $(\mathfrak{U},\zeta)$ such that $\mathfrak{R} \sqsubseteq NeuCl(\mathfrak{P}) - \mathfrak{P}$. Since $\mathfrak{P}$ is a $Neu^{gsg}CS$, we have $NeuCl(\mathfrak{P}) \sqsubseteq 1_{Neu} - \mathfrak{R}$. This implies $\mathfrak{R} \sqsubseteq 1_{Neu} - NeuCl(\mathfrak{P})$. Then $\mathfrak{R} \sqsubseteq NeuCl(\mathfrak{P})\sqcap(1_{Neu} - NeuCl(\mathfrak{P})) = 0_{Neu}$. Thus, $\mathfrak{R}$ is empty.

Conversely, suppose that $NeuCl(\mathfrak{P}) - \mathfrak{P}$ does not include non-empty $Neu^{sg}CS$ in $(\mathfrak{U},\zeta)$. Let $\mathfrak{P} \sqsubseteq \mathfrak{M}$ and $\mathfrak{M}$ is $Neu^{sg}OS$. If $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$ then $NeuCl(\mathfrak{P})\sqcap(1_{Neu} - \mathfrak{M})$ is non-empty. Since $NeuCl(\mathfrak{P})$ is $NeuCS$ and $1_{Neu} - \mathfrak{M}$ is $Neu^{sg}CS$, we have $NeuCl(\mathfrak{P})\sqcap(1_{Neu} - \mathfrak{M})$ is not empty $Neu^{sg}CS$ of $NeuCl(\mathfrak{P}) - \mathfrak{P}$, which is a contradiction. Therefore $NeuCl(\mathfrak{P}) \not\sqsubseteq \mathfrak{M}$. Hence $\mathfrak{P}$ is a $Neu^{gsg}CS$. ▪

DOI: https://doi.org/10.54216/IJNS.18030101
Received: March 02, 2022 Accepted: May 30, 2022

Proposition 3.17:
If $\mathfrak{P}$ is a $Neu^{gs\not g}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ and $\mathfrak{P} \sqsubseteq \mathfrak{Q} \sqsubseteq NeuCl(\mathfrak{P})$, then $\mathfrak{Q}$ is a $Neu^{gs\not g}CS$ in $(\mathfrak{U},\zeta)$.
Proof:
Assume the set $\mathfrak{P}$ is a $Neu^{gs\not g}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$. Suppose the set $\mathfrak{M}$ is a $Neu^{s\not g}OS$ in $(\mathfrak{U},\zeta)$ where $\mathfrak{Q} \sqsubseteq \mathfrak{M}$. So, $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because $\mathfrak{P}$ is a $Neu^{gs\not g}CS$, it observes that $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Currently, $\mathfrak{Q} \sqsubseteq NeuCl(\mathfrak{P})$ suggests $NeuCl(\mathfrak{Q}) \sqsubseteq NeuCl(NeuCl(\mathfrak{P})) = NeuCl(\mathfrak{P})$. Thus, $NeuCl(\mathfrak{Q}) \sqsubseteq \mathfrak{M}$. Hence $\mathfrak{Q}$ is a $Neu^{gs\not g}CS$. ▪

Proposition 3.18:
Let $\mathfrak{P} \sqsubseteq \mathfrak{D} \sqsubseteq \mathfrak{U}$ and if $\mathfrak{P}$ is a $Neu^{gs\not g}CS$ in $\mathfrak{U}$ then $\mathfrak{P}$ is a $Neu^{gs\not g}CS$ relative to $\mathfrak{D}$.
Proof:
$\mathfrak{P} \sqsubseteq \mathfrak{D} \sqcap \mathfrak{M}$ where $\mathfrak{M}$ is a $Neu^{s\not g}OS$ in $\mathfrak{U}$. Then $\mathfrak{P} \sqsubseteq \mathfrak{M}$ and hence $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$.
This implies that $\mathfrak{D} \sqcap NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{D} \sqcap \mathfrak{M}$. Thus $\mathfrak{P}$ is a $Neu^{gs\not g}CS$ relative to $\mathfrak{D}$. ▪

Proposition 3.19:
If $\mathfrak{P}$ is a $Neu^{s\not g}OS$ and a $Neu^{gs\not g}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $\mathfrak{P}$ is a $NeuCS$ in $(\mathfrak{U},\zeta)$.
Proof:
Suppose that $\mathfrak{P}$ is a $Neu^{s\not g}OS$ and a $Neu^{gs\not g}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{P}$ and since $\mathfrak{P} \sqsubseteq NeuCl(\mathfrak{P})$. Thus, $NeuCl(\mathfrak{P}) = \mathfrak{P}$. Hence $\mathfrak{P}$ is a $NeuCS$. ▪

Theorem 3.20:
For each $u \in \mathfrak{U}$ either $\langle u,(0.1,0.1)\rangle$ is a $Neu^{s\not g}CS$ or $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is a $Neu^{gs\not g}CS$ in $\mathfrak{U}$.
Proof:
If $\langle u,(0.1,0.1)\rangle$ is not a $Neu^{s\not g}CS$ in $\mathfrak{U}$ then $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is not a $Neu^{s\not g}OS$ and the only $Neu^{s\not g}OS$ containing $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is the space $\mathfrak{U}$ itself. Therefore $NeuCl(1_{Neu} - \langle u,(0.1,0.1)\rangle) \sqsubseteq 1_{Neu}$ and so $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is a $Neu^{gs\not g}CS$ in $\mathfrak{U}$. ▪

Proposition 3.21:
If $\mathfrak{P}$ and $\mathfrak{Q}$ are $Neu^{gs\not g}OSs$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $\mathfrak{P} \sqcap \mathfrak{Q}$ is a $Neu^{gs\not g}OS$.
Proof:
Let $\mathfrak{P}$ and $\mathfrak{Q}$ be $Neu^{gs\not g}OSs$ in a $Neu^{TS}(\mathfrak{U},\zeta)$. Then $1_{Neu} - \mathfrak{P}$ and $1_{Neu} - \mathfrak{Q}$ are $Neu^{gs\not g}CSs$. By proposition (3.14), $(1_{Neu} - \mathfrak{P}) \sqcup (1_{Neu} - \mathfrak{Q})$ is a $Neu^{gs\not g}CS$. Since $(1_{Neu} - \mathfrak{P}) \sqcup (1_{Neu} - \mathfrak{Q}) = 1_{Neu} - (\mathfrak{P} \sqcap \mathfrak{Q})$. Hence $\mathfrak{P} \sqcap \mathfrak{Q}$ is a $Neu^{gs\not g}OS$. ▪

Theorem 3.22:
A neutrosophic set $\mathfrak{P}$ is $Neu^{gs\not g}OS$ iff $\mathfrak{S} \sqsubseteq NeuInt(\mathfrak{P})$ where $\mathfrak{S}$ is a $Neu^{gs\not g}CS$ and $\mathfrak{S} \sqsubseteq \mathfrak{P}$.
Proof:
Suppose that $\mathfrak{S} \sqsubseteq NeuInt(\mathfrak{P})$ where $\mathfrak{S}$ is a $Neu^{gs\not g}CS$ and $\mathfrak{S} \sqsubseteq \mathfrak{P}$. Then $1_{Neu} - \mathfrak{P} \sqsubseteq 1_{Neu} - \mathfrak{S}$ and $1_{Neu} - \mathfrak{S}$ is a $Neu^{s\not g}OS$ by theorem (3.13) part (ii). Now, $NeuCl(1_{Neu} - \mathfrak{P}) = 1_{Neu} - NeuInt(\mathfrak{P}) \sqsubseteq 1_{Neu} - \mathfrak{S}$. Then $1_{Neu} - \mathfrak{P}$ is a $Neu^{gs\not g}CS$. Hence $\mathfrak{P}$ is a $Neu^{gs\not g}OS$.
Conversely, let $\mathfrak{P}$ be a $Neu^{gs\not g}OS$ and $\mathfrak{S}$ be a $Neu^{gs\not g}CS$ and $\mathfrak{S} \sqsubseteq \mathfrak{P}$. Then $1_{Neu} - \mathfrak{P} \sqsubseteq 1_{Neu} - \mathfrak{S}$. Since $1_{Neu} - \mathfrak{P}$ is a $Neu^{gs\not g}CS$ and $1_{Neu} - \mathfrak{S}$ is a $Neu^{s\not g}OS$, we have $NeuCl(1_{Neu} - \mathfrak{P}) \sqsubseteq 1_{Neu} - \mathfrak{S}$. Then $\mathfrak{S} \sqsubseteq NeuInt(\mathfrak{P})$. ▪

Theorem 3.23:
If $\mathfrak{P} \sqsubseteq \mathfrak{Q} \sqsubseteq \mathfrak{U}$ where $\mathfrak{P}$ is a $Neu^{gs\not g}OS$ relative to $\mathfrak{Q}$ and $\mathfrak{Q}$ is a $Neu^{gs\not g}OS$ in $\mathfrak{U}$, then $\mathfrak{P}$ is a $Neu^{gs\not g}OS$ in $\mathfrak{U}$.

DOI: https://doi.org/10.54216/IJNS.18030101
Received: March 02, 2022 Accepted: May 30, 2022

Proof:

Let $\mathfrak{F}$ be a $Neu^{sg}CS$ in $\mathfrak{U}$ and suppose that $\mathfrak{F} \sqsubseteq \mathfrak{P}$. Then $\mathfrak{F} = \mathfrak{F} \sqcap \mathfrak{Q}$ is a $Neu^{sg}CS$ in $\mathfrak{Q}$. But $\mathfrak{P}$ is a $Neu^{gsg}OS$ relative to $\mathfrak{Q}$. Therefore $\mathfrak{F} \sqsubseteq NeuInt_{\mathfrak{Q}}(\mathfrak{P})$. Since $NeuInt_{\mathfrak{Q}}(\mathfrak{P})$ is a $NeuOS$ relative to $\mathfrak{Q}$. We have $\mathfrak{F} \sqsubseteq \mathfrak{M} \sqcap \mathfrak{Q} \sqsubseteq \mathfrak{P}$, for some $NeuOS$ $\mathfrak{M}$ in $\mathfrak{U}$. Since $\mathfrak{Q}$ is a $Neu^{gsg}OS$ in $\mathfrak{U}$, we have $\mathfrak{F} \sqsubseteq NeuInt(\mathfrak{Q}) \sqsubseteq \mathfrak{Q}$. Therefore $\mathfrak{F} \sqsubseteq NeuInt(\mathfrak{Q}) \sqcap \mathfrak{M} \sqsubseteq \mathfrak{Q} \sqcap \mathfrak{M} \sqsubseteq \mathfrak{P}$. It follows that $\mathfrak{F} \sqsubseteq NeuInt(\mathfrak{P})$. Thus, $\mathfrak{P}$ is a $Neu^{gsg}OS$ in $\mathfrak{U}$. ▪

Theorem 3.24:

If $\mathfrak{P}$ is a $Neu^{gsg}OS$ in a $Neu^{TS}(\mathfrak{U}, \zeta)$ and $NeuInt(\mathfrak{P}) \sqsubseteq \mathfrak{Q} \sqsubseteq \mathfrak{P}$, then $\mathfrak{Q}$ is a $Neu^{gsg}OS$ in $(\mathfrak{U}, \zeta)$.

Proof:

Suppose that $\mathfrak{P}$ is a $Neu^{gsg}OS$ in a $Neu^{TS}(\mathfrak{U}, \zeta)$ and $NeuInt(\mathfrak{P}) \sqsubseteq \mathfrak{Q} \sqsubseteq \mathfrak{P}$. Then $1_{Neu} - \mathfrak{P}$ is a $Neu^{gsg}CS$ and $1_{Neu} - \mathfrak{P} \sqsubseteq 1_{Neu} - \mathfrak{Q} \sqsubseteq NeuCl(1_{Neu} - \mathfrak{P})$. Then $1_{Neu} - \mathfrak{Q}$ is a $Neu^{gsg}CS$ by proposition (3.17). Hence, $\mathfrak{Q}$ is a $Neu^{gsg}OS$. ▪

Theorem 3.25:

For a neutrosophic subset $\mathfrak{P}$ of a $Neu^{TS}(\mathfrak{U}, \zeta)$, the following statements are equivalent:

(i) $\mathfrak{P}$ is a $Neu^{gsg}CS$.

(ii) $NeuCl(\mathfrak{P}) - \mathfrak{P}$ contains no non-empty $Neu^{sg}CS$.

(iii) $NeuCl(\mathfrak{P}) - \mathfrak{P}$ is a $Neu^{gsg}OS$.

Proof:

Follows from proposition (3.16) and proposition (3.18). ▪

Remark 3.26:

The subsequent illustration reveals the relative among the diverse kinds of $NeuCS$:

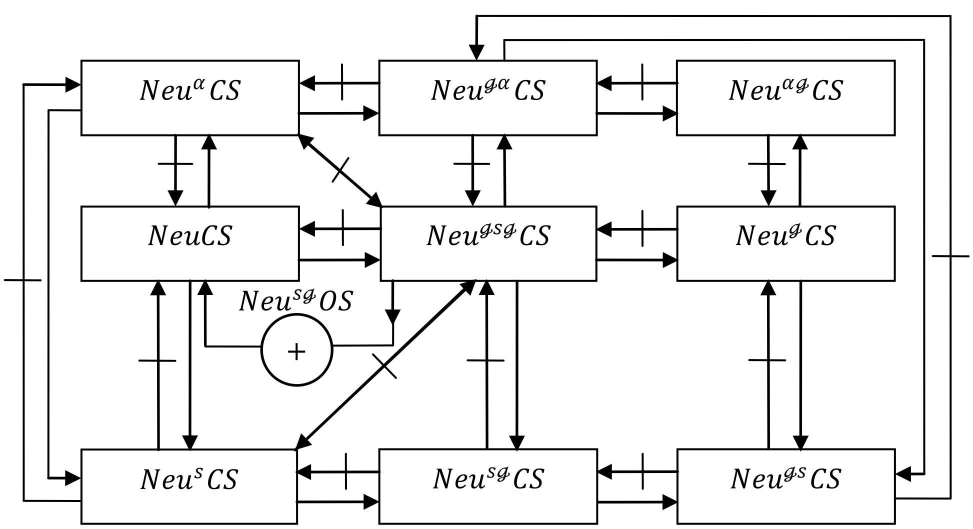

Fig. 3.1

4. Neutrosophic gsg-Closure and Neutrosophic gsg-Interior

We present neutrosophic gsg-closure and neutrosophic gsg-interior and obtain some of its properties in this section.

DOI: https://doi.org/10.54216/IJNS.18030101
Received: March 02, 2022 Accepted: May 30, 2022

Definition 4.1:
The intersection of all $Neu^{gsg}CSs$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ containing $\mathfrak{P}$ is called neutrosophic gsg-closure of $\mathfrak{P}$ and is denoted by $Neu^{gsg}Cl(\mathfrak{P})$.

Definition 4.2:
The union of all $Neu^{gsg}OSs$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ contained in $\mathfrak{P}$ is called neutrosophic gsg-interior of $\mathfrak{P}$ and is denoted by $Neu^{gsg}Int(\mathfrak{P})$.

Proposition 4.3:
Let $\mathfrak{P}$ be any neutrosophic set in a $Neu^{TS}(\mathfrak{U},\zeta)$. Then the following properties hold:
(i) $Neu^{gsg}Int(\mathfrak{P}) = \mathfrak{P}$ iff $\mathfrak{P}$ is a $Neu^{gsg}OS$.
(ii) $Neu^{gsg}Cl(\mathfrak{P}) = \mathfrak{P}$ iff $\mathfrak{P}$ is a $Neu^{gsg}CS$.
(iii) $Neu^{gsg}Int(\mathfrak{P})$ is the largest $Neu^{gsg}OS$ contained in $\mathfrak{P}$.
(iv) $Neu^{gsg}Cl(\mathfrak{P})$ is the smallest $Neu^{gsg}CS$ containing $\mathfrak{P}$.
Proof:
(i), (ii), (iii) and (iv) are obvious. ▪

Proposition 4.4:
Let $\mathfrak{P}$ be any neutrosophic set in a $Neu^{TS}(\mathfrak{U},\zeta)$. Then the following properties hold:
(i) $Neu^{gsg}Int(1_{Neu} - \mathfrak{P}) = 1_{Neu} - (Neu^{gsg}Cl(\mathfrak{P}))$,
(ii) $Neu^{gsg}Cl(1_{Neu} - \mathfrak{P}) = 1_{Neu} - (Neu^{gsg}Int(\mathfrak{P}))$.
Proof:
(i) By definition, $Neu^{gsg}Cl(\mathfrak{P}) = \sqcap\{\mathfrak{Q}: \mathfrak{P} \sqsubseteq \mathfrak{Q}, \mathfrak{Q} \text{ is a } Neu^{gsg}CS\}$

$$\begin{aligned} 1_{Neu} - (Neu^{gsg}Cl(\mathfrak{P})) &= 1_{Neu} - \sqcap\{\mathfrak{Q}: \mathfrak{P} \sqsubseteq \mathfrak{Q}, \mathfrak{Q} \text{ is a } Neu^{gsg}CS\} \\ &= \sqcup\{1_{Neu} - \mathfrak{Q}: \mathfrak{P} \sqsubseteq \mathfrak{Q}, \mathfrak{Q} \text{ is a } Neu^{gsg}CS\} \\ &= \sqcup\{\mathfrak{M}: \mathfrak{M} \sqsubseteq 1_{Neu} - \mathfrak{P}, \mathfrak{M} \text{ is a } Neu^{gsg}OS\} \\ &= Neu^{gsg}Int(1_{Neu} - \mathfrak{P}). \end{aligned}$$

(ii) The evidence is analogous to (i). ▪

Theorem 4.5:
Let $\mathfrak{P}$ and $\mathfrak{Q}$ be two neutrosophic sets in a $Neu^{TS}(\mathfrak{U},\zeta)$. Then the following properties hold:
(i) $Neu^{gsg}Cl(0_{Neu}) = 0_{Neu}$, $Neu^{gsg}Cl(1_{Neu}) = 1_{Neu}$.
(ii) $\mathfrak{P} \sqsubseteq Neu^{gsg}Cl(\mathfrak{P})$.
(iii) $\mathfrak{P} \sqsubseteq \mathfrak{Q} \Longrightarrow Neu^{gsg}Cl(\mathfrak{P}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{Q})$.
(iv) $Neu^{gsg}Cl(\mathfrak{P}\sqcap\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P})\sqcap Neu^{gsg}Cl(\mathfrak{Q})$.
(v) $Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q}) = Neu^{gsg}Cl(\mathfrak{P})\sqcup Neu^{gsg}Cl(\mathfrak{Q})$.
(vi) $Neu^{gsg}Cl(Neu^{gsg}Cl(\mathfrak{P})) = Neu^{gsg}Cl(\mathfrak{P})$.
Proof:
(i) and (ii) are obvious.
(iii) By part (ii), $\mathfrak{Q} \sqsubseteq Neu^{gsg}Cl(\mathfrak{Q})$. Since $\mathfrak{P} \sqsubseteq \mathfrak{Q}$, we have $\mathfrak{P} \sqsubseteq Neu^{gsg}Cl(\mathfrak{Q})$. But $Neu^{gsg}Cl(\mathfrak{Q})$ is a $Neu^{gsg}CS$. Thus $Neu^{gsg}Cl(\mathfrak{Q})$ is a $Neu^{gsg}CS$ containing $\mathfrak{P}$. Since $Neu^{gsg}Cl(\mathfrak{P})$ is the smallest $Neu^{gsg}CS$ containing $\mathfrak{P}$, we have $Neu^{gsg}Cl(\mathfrak{P}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{Q})$.
(iv) We know that $\mathfrak{P}\sqcap\mathfrak{Q} \sqsubseteq \mathfrak{P}$ and $\mathfrak{P}\sqcap\mathfrak{Q} \sqsubseteq \mathfrak{Q}$. Therefore, by part (iii), $Neu^{gsg}Cl(\mathfrak{P}\sqcap\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P})$ and $Neu^{gsg}Cl(\mathfrak{P}\sqcap\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{Q})$. Hence $Neu^{gsg}Cl(\mathfrak{P}\sqcap\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P})\sqcap Neu^{gsg}Cl(\mathfrak{Q})$.
(v) Since $\mathfrak{P} \sqsubseteq \mathfrak{P}\sqcup\mathfrak{Q}$ and $\mathfrak{Q} \sqsubseteq \mathfrak{P}\sqcup\mathfrak{Q}$, it follows from part (iii) that $Neu^{gsg}Cl(\mathfrak{P}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q})$ and $Neu^{gsg}Cl(\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q})$. Hence $Neu^{gsg}Cl(\mathfrak{P})\sqcup Neu^{gsg}Cl(\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q})$………… (1)

DOI: https://doi.org/10.54216/IJNS.18030101
Received: March 02, 2022 Accepted: May 30, 2022

Since $Neu^{gsg}Cl(\mathfrak{P})$ and $Neu^{gsg}Cl(\mathfrak{Q})$ are $Neu^{gsg}CSs$, $Neu^{gsg}Cl(\mathfrak{P})\sqcup Neu^{gsg}Cl(\mathfrak{Q})$ is also $Neu^{gsg}CS$ by proposition (3.14). Also $\mathfrak{P} \sqsubseteq Neu^{gsg}Cl(\mathfrak{P})$ and $\mathfrak{Q} \sqsubseteq Neu^{gsg}Cl(\mathfrak{Q})$ implies that $\mathfrak{P}\sqcup\mathfrak{Q} \sqsubseteq Neu^{gsg}Cl(\mathfrak{P})\sqcup Neu^{gsg}Cl(\mathfrak{Q})$. Thus $Neu^{gsg}Cl(\mathfrak{P})\sqcup Neu^{gsg}Cl(\mathfrak{Q})$ is a $Neu^{gsg}CS$ containing $\mathfrak{P}\sqcup\mathfrak{Q}$. Since $Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q})$ is the smallest $Neu^{gsg}CS$ containing $\mathfrak{P}\sqcup\mathfrak{Q}$, we have $Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P})\sqcup Neu^{gsg}Cl(\mathfrak{Q})$………… (2)

From (1) and (2), we have $Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q}) = Neu^{gsg}Cl(\mathfrak{P})\sqcup Neu^{gsg}Cl(\mathfrak{Q})$.

(vi) Since $Neu^{gsg}Cl(\mathfrak{P})$ is a $Neu^{gsg}CS$, we have by proposition (4.3) part (ii), $Neu^{gsg}Cl(Neu^{gsg}Cl(\mathfrak{P})) = Neu^{gsg}Cl(\mathfrak{P})$. ▪

Theorem 4.6:

Let $\mathfrak{P}$ and $\mathfrak{Q}$ be two neutrosophic sets in a $Neu^{TS}(\mathfrak{U},\zeta)$. Then the following properties hold:

(i) $Neu^{gsg}Int(0_{Neu}) = 0_{Neu}$, $Neu^{gsg}Int(1_{Neu}) = 1_{Neu}$.

(ii) $Neu^{gsg}Int(\mathfrak{P}) \sqsubseteq \mathfrak{P}$.

(iii) $\mathfrak{P} \sqsubseteq \mathfrak{Q} \Longrightarrow Neu^{gsg}Int(\mathfrak{P}) \sqsubseteq Neu^{gsg}Int(\mathfrak{Q})$.

(iv) $Neu^{gsg}Int(\mathfrak{P}\sqcap\mathfrak{Q}) = Neu^{gsg}Int(\mathfrak{P})\sqcap Neu^{gsg}Int(\mathfrak{Q})$.

(v) $Neu^{gsg}Int(\mathfrak{P}\sqcup\mathfrak{Q}) \sqsupseteq Neu^{gsg}Int(\mathfrak{P})\sqcup Neu^{gsg}Int(\mathfrak{Q})$.

(vi) $Neu^{gsg}Int(Neu^{gsg}Int(\mathfrak{P})) = Neu^{gsg}Int(\mathfrak{P})$.

Proof:

(i), (ii), (iii), (iv), (v) and (vi) are obvious. ▪

Definition 4.7:

A $Neu^{TS}(\mathfrak{U},\zeta)$ is called a neutrosophic $T_{\frac{1}{2}}$-space (in short, $NeuT_{\frac{1}{2}}$-space) if each $Neu^{g}CS$ in this space is a $NeuCS$.

Definition 4.8:

A $Neu^{TS}(\mathfrak{U},\zeta)$ is called a neutrosophic T_{gsg}-space (in short, $NeuT_{gsg}$-space) if each $Neu^{gsg}CS$ in this space is a $NeuCS$.

Proposition 4.9:

Every $NeuT_{\frac{1}{2}}$-space is a $NeuT_{gsg}$-space.

Proof:

Let $(\mathfrak{U},\zeta)$ be a $NeuT_{\frac{1}{2}}$-space and let $\mathfrak{P}$ be a $Neu^{gsg}CS$ in $\mathfrak{U}$. Then $\mathfrak{P}$ is a $Neu^{g}CS$, by theorem (3.2) part (ii). Since $(\mathfrak{U},\zeta)$ is a $NeuT_{\frac{1}{2}}$-space, then $\mathfrak{P}$ is a $NeuCS$ in $\mathfrak{U}$. Hence $(\mathfrak{U},\zeta)$ is a $NeuT_{gsg}$-space. ▪

Theorem 4.10:

For a $Neu^{TS}(\mathfrak{U},\zeta)$, the following statements are equivalent:

(i) $(\mathfrak{U},\zeta)$ is a $NeuT_{gsg}$-space.

(ii) Every singleton of a $Neu^{TS}(\mathfrak{U},\zeta)$ is either $Neu^{sg}CS$ or $NeuOS$.

Proof:

(i) $\Longrightarrow$(ii) Assume that for some $u \in \mathfrak{U}$ the neutrosophic set $\langle u,(0.1,0.1)\rangle$ is not a $Neu^{sg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$. Then the only $Neu^{sg}OS$ containing $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is the space $\mathfrak{U}$ itself and $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is a $Neu^{gsg}CS$ in $(\mathfrak{U},\zeta)$. By assumption $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is a $NeuCS$ in $(\mathfrak{U},\zeta)$ or equivalently $\langle u,(0.1,0.1)\rangle$ is a $NeuOS$.

DOI: https://doi.org/10.54216/IJNS.18030101
Received: March 02, 2022 Accepted: May 30, 2022

(ii) ⟹(i) Let $\mathfrak{P}$ be a $Neu^{gsg}CS$ in $(\mathfrak{U},\zeta)$ and let $u \in NeuCl(\mathfrak{P})$. By assumption $\langle u,(0.1,0.1)\rangle$ is either $Neu^{sg}CS$ or $NeuOS$.

Case(1). Suppose $\langle u,(0.1,0.1)\rangle$ is a $Neu^{sg}CS$. If $u \notin \mathfrak{P}$ then $NeuCl(\mathfrak{P}) - \mathfrak{P}$ contains a non-empty $Neu^{sg}CS$ $\langle u,(0.1,0.1)\rangle$ which is a contradiction to proposition (3.18). Therefore $u \in \mathfrak{P}$.

Case(2). Suppose $\langle u,(0.1,0.1)\rangle$ is a $NeuOS$. Since $u \in NeuCl(\mathfrak{P})$, $\langle u,(0.1,0.1)\rangle \sqcap \mathfrak{P} \neq 0_{Neu}$ and therefore $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{P}$ or equivalently $\mathfrak{P}$ is a $NeuCS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$. ▪

5. Conclusion

The concept of $Neu^{gsg}CS$ identified utilizing $Neu^{sg}CS$ constructs a neutrosophic topology and sits between the concept of $NeuCS$ and the concept of $Neu^{g}CS$. The $Neu^{gsg}CS$ can be used to derive a new decomposition of Neu^{gsg}-continuity and new Neu^{gsg}-separation axioms.

Funding: This research received no external funding.

Acknowledgements: The authors are highly grateful to the Referees for their constructive suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. F. Smarandache, A unifying field in logics: neutrosophic logic, neutrosophy, neutrosophic set, neutrosophic probability. American Research Press, Rehoboth, NM, (1999).
2. F. Smarandache, Neutrosophy and neutrosophic logic, first international conference on neutrosophy, neutrosophic logic, set, probability, and statistics, University of New Mexico, Gallup, NM 87301, USA (2002).
3. A. Salama and S. A. Alblowi, Neutrosophic set and neutrosophic topological spaces. IOSR Journal of Mathematics, 3(2012), 31-35.
4. Q. H. Imran, F. Smarandache, R. K. Al-Hamido and R. Dhavaseelan, On neutrosophic semi-α-open sets. Neutrosophic Sets and Systems, 18(2017), 37-42.
5. Md. Hanif PAGE and Q. H. Imran, Neutrosophic generalized homeomorphism. Neutrosophic Sets and Systems, 35(2020), 340-346.
6. R. Dhavaseelan and S. Jafari, Generalized Neutrosophic closed sets. New trends in Neutrosophic theory and applications, 2(2018), 261-273.
7. Q. H. Imran, R. Dhavaseelan, A. H. M. Al-Obaidi and M. H. Page, On neutrosophic generalized alpha generalized continuity. Neutrosophic Sets and Systems, 35(2020), 511-521.
8. P. Iswarya and Dr. K. Bageerathi, On neutrosophic semi-open sets in neutrosophic topological spaces. International Journal of Mathematics Trends and Technology (IJMTT), Vol.37, No.3 (2016), 24-33.
9. I. Arokiarani, R. Dhavaseelan, S. Jafari and M. Parimala, On Some New Notions and Functions in Neutrosophic Topological Spaces. Neutrosophic Sets and Systems, 16(2017), 16-19.
10. A. Pushpalatha and T. Nandhini, Generalized closed sets via neutrosophic topological spaces. Malaya Journal of Matematik, 7(1), (2019), 50-54.
11. D. Jayanthi, α Generalized Closed Sets in Neutrosophic Topological Spaces. International Journal of Mathematics Trends and Technology (IJMTT)- Special Issue ICRMIT March (2018), 88-91.

DOI: https://doi.org/10.54216/IJNS.18030101
Received: March 02, 2022 Accepted: May 30, 2022

12. D. Sreeja and T. Sarankumar, Generalized Alpha Closed sets in Neutrosophic topological spaces. Journal of Applied Science and Computations, 5(11), (2018), 1816-1823.
13. P. Iswarya and Dr. K. Bageerathi, A study on neutrosophic semi generalized closed sets in neutrosophic topological spaces. Acta Ciencia Indica, Vol. XLIII M, No.4, (2017), 239-245.
14. V. K. Shanthi, S. Chandrasekar and K. Safina Begam, Neutrosophic generalized semi closed sets in neutrosophic topological spaces. International Journal of Research in Advent Technology, Vol.6, No.7, (2018), 1739-1743.
15. P. Iswarya and Dr. K. Bageerathi, A study on neutrosophic generalized semi closed sets in neutrosophic topological spaces. Journal of Emerging Technologies and Innovative Research (JETIR), Volume 6, Issue 2, (2019), 452-457.

DOI: https://doi.org/10.54216/IJNS.18030101
Received: March 02, 2022 Accepted: May 30, 2022