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A thermodynamic analysis of non-equilibrium heat conduction in a semi-infinite medium subjected to a step change in temperature

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Abstract

The problem of non-equilibrium heat conduction in a semi-infinite medium subjected to a step change in temperature is analyzed thermodynamically using the extended irreversible thermodynamic approach. The results show clearly the wave nature of the dimensionless temperature distribution, Stanton number and the dimensionless entropy change profiles. The non-equilibrium profiles approach the equilibrium profiles as the speed of wave propagation is increased. The results also show that the non-equilibrium temperature is higher than the equilibrium temperature but the difference decreases as the wave propagation speed increases.

Keywords: Heat conduction; Thermodynamics; Temperature; Non-equilibrium

1. Introduction

A great deal of research has been devoted to developing theories dealing with non-equilibrium thermodynamics and non-Fourier heat conduction. These researches help to understand the connection between the two different subjects. Three different approaches are developed, which are the classical irreversible thermodynamics (CIT), the rational thermodynamics (RT) and the extended irreversible thermodynamics (EIT) approaches.

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Nomenclature

c	thermal wave speed (m/s)
C_p	specific heat at constant pressure (J/kg K)
erf( )	error function
erfc( )	complementary error function
I_n	modified Bessel function of first kind of order n
ierfc( )	imaginary complementary error function
J^s	entropy flux (W/m^3 K)
k	thermal conductivity (W/m K)
q	heat flux (W/m^3)
St	conduction Stanton number
s	specific entropy per unit volume (J/kg K m^3)
T	temperature (K)
t	time (s)
u	specific internal energy per unit volume (J/kg m^3)
u( )	unit step function
v	specific volume (m^3/kg)
X	generalized thermodynamic force (K/m)
x	distance (m)

Greek symbols

α	thermal diffusivity (m^2/s)
β	dimensionless time
δ	dimensionless distance
θ	dimensionless temperature distribution
ρ	density (kg/m^3)
σ^s	rate of entropy production per unit volume (W/m^3 K)
Δs	entropy change (J/kg K)
τ	phase lag or relaxation time of heat flux (s)

Subscripts

eq	equilibrium
f	Fourier
nf	non-Fourier
0	initial
w	wall

Superscripts

$*$	dimensionless
$\cdot$	time derivative

The most popular approach is the CIT approach, which depends on the hypothesis of local equilibrium [1]. Local equilibrium implies:

1. Equilibrium is stable.
2. Variables used in the local equilibrium case are the same as those used in classical thermodynamics, such as entropy and temperature.
3. The relationships between the state variables used in classical thermodynamics remain valid in the case of local equilibrium.

The basic thermodynamic equations used in this approach are the energy balance equation and Gibbs equation [1].

$$\rho \dot{u} = -\nabla \cdot q \tag{1}$$

$$\mathrm{d}s = T^{-1}\,\mathrm{d}u \tag{2}$$

$$\rho \dot{s} = T^{-1}\rho \dot{u} \tag{3}$$

$$\therefore\ \rho \dot{s} = -T^{-1}\nabla \cdot q \tag{4}$$

$$T^{-1}\nabla \cdot q = \nabla \cdot (T^{-1}q) + T^{-2}(q \cdot \nabla T) \tag{5}$$

$$\therefore\ \rho \dot{s} = -\nabla \cdot (T^{-1}q) - T^{-2}(q \cdot \nabla T) \tag{6}$$

The following entropy balance equation is also used [2].

$$\rho \dot{s} = -\nabla \cdot J^s + \sigma^s \geqslant 0 \tag{7}$$

where J^s is the entropy flux vector and σ^s is the entropy production per unit volume per unit time, which is a positive definite quantity ($\sigma^s \geqslant 0$).

By comparing Eqs. (6) and (7), it is seen that

$$J^s = T^{-1}q$$

and

$$\sigma^s = -T^{-2}(q \cdot \nabla T) = -q \cdot (T^{-2}\nabla T)$$

According to Onsagers law, q is the thermodynamic flux and $(T^{-2}\nabla T)$ is the thermodynamic force [3]. A linear relationship between the flux and the force is assumed [4].

$$q = -\varepsilon T^{-2}\nabla T \tag{8}$$

Comparing Eq. (8) with Fourier's law ($q = -K\nabla T$) yields that

$$\varepsilon = KT^2 \tag{9}$$

The second approach is the RT [3] approach, which is based on the assumption that the internal energy depends on the physical fluxes in addition to the to the classical variables and that the constitutive equations are time functional [5]. Also, the approach depends on the Clausius–Duhem inequality of the second law of thermodynamics [2].

$$\rho \dot{s} \geqslant \nabla \cdot (T^{-1}q) \tag{10}$$

The third approach is the EIT approach, which is based on the following assumptions [1]:

1. The entropy is a function of the classical variables and the physical fluxes.
2. The entropy flux is equal to the heat flux divided by temperature.
3. The phenomenological laws of EIT are unsteady and non-linear.
4. In EIT, a signal propagates at a finite speed.

This approach depends on the Cattaneos equation and a generalized Gibbs equation, Eqs. (11) and (12), respectively [6].

$$\tau q^{\cdot} + q = -K\nabla T \tag{11}$$

$$s^{\cdot} = T^{-1}u^{\cdot} + \tau\nabla T^{-1}q^{\cdot} \tag{12}$$

2. The present work

The present work aims at examining the entropy change during non-Fourier heat conduction in a semi-infinite medium subjected to a step change in temperature using the EIT approach. It also aims at predicting the temperature distribution and the heat flux.

The non-Fourier heat conduction in an isotropic solid is described in the EIT approach in terms of the heat flux q and the classical variable u or T [2,6]. It is defined by the Cattaneos equation:

$$\tau q^{\cdot} + q = -K\nabla T \tag{11}$$

$$q^{\cdot} = -\frac{1}{\tau}[q + K\nabla T] \tag{13}$$

The entropy variation during non-Fourier heat conduction is described by the following state equation:

$$s = s(u, q) \tag{14}$$

$$\mathrm{d}s = \left(\frac{\partial s}{\partial u}\right)_q \mathrm{d}u + \left(\frac{\partial s}{\partial q}\right)_u \mathrm{d}q \tag{15}$$

In analogy with CIT, the first partial derivative of Eq. (15) can be written as

$$\left(\frac{\partial s}{\partial u}\right)_q = T^{-1}(u, q) \tag{16}$$

where $T^{-1}(u, q)$ is the non-equilibrium temperature.

The second partial derivative is defined as [6]

$$\left(\frac{\partial s}{\partial q}\right)_u = \frac{\gamma}{\rho}T^{-1}(u, q) \tag{17}$$

The parameter γ is given as [6]

$$\gamma = \gamma^{-}q \tag{18}$$

where γ^- is a function of the specific internal energy or the local equilibrium temperature as will be indicated later.

$$\therefore \mathrm{d}s = T^{-1}\,\mathrm{d}u + \frac{\gamma^-}{\rho}T^{-1}q\,\mathrm{d}q \tag{19}$$

$$\rho\mathrm{d}s = \rho T^{-1}\,\mathrm{d}u + \gamma^- T^{-1}q\,\mathrm{d}q \tag{20}$$

By integrating Eq. (20) and taking the time derivative of the resulting equation, the following equation is obtained.

$$\rho s^{\cdot} = \rho T^{-1}u^{\cdot} + \gamma^- T^{-1}qq^{\cdot} \tag{21}$$

Since $\rho u^{\cdot} = -\nabla \cdot q$, hence,

$$\therefore \rho s^{\cdot} = -T^{-1}\nabla \cdot q + \gamma^- T^{-1}qq^{\cdot} \tag{22}$$

Substituting Eq. (5) in Eq. (22) gives

$$\rho s^{\cdot} = -\nabla \cdot (T^{-1}q) - T^{-2}(q \cdot \nabla T) + \gamma^- T^{-1}qq^{\cdot} \tag{23}$$

By comparing Eqs. (7) and (23), it is seen that

$$\sigma^s = -T^{-2}(q \cdot \nabla T) + \gamma^- T^{-1}qq^{\cdot} \geqslant 0 \tag{24}$$

$$\sigma^s = q(\gamma^- T^{-1}q^{\cdot} - T^{-2}\nabla T) \geqslant 0 \tag{25}$$

According to Onsager, the term q in Eq. (25) is the thermodynamic flux, and the term $(\gamma^- T^{-1}q^{\cdot} - T^{-2}\nabla T)$ is the thermodynamic force (X) [3].

$$\sigma^s = X \cdot q \geqslant 0 \tag{26}$$

A linear relationship between flux and force is assumed [4].

$$X = \mu \cdot q \tag{27}$$

where μ is a phenomenological coefficient dependent only on u or T.

$$\sigma^s = \mu q^2 \geqslant 0 \tag{28}$$

Equating Eqs. (25) and (28) and re-arranging the resulting equation gives

$$q^{\cdot} = T(\gamma^-)^{-1}\{\mu q + T^{-2}\nabla T\} \tag{29}$$

For steady state heat conduction ($q^{\cdot} = 0$), Eq. (29) becomes

$$q = -\mu^{-1}T^{-2}\nabla T \tag{30}$$

By comparing Eq. (30) with Fourier's law, it is seen that

$$\mu = (KT^{-2})^{-1} \tag{31}$$

Substituting Eq. (31) in Eq. (29) and re-arranging the resulting equation gives

$$q^{\cdot} = \frac{1}{KT\gamma^-}[q + K\nabla T] \tag{32}$$

Comparison of Eq. (32) with Eq. (13) shows that

$$\gamma^{-} = -\tau(KT)^{-1} \tag{33}$$

Substituting Eq. (33) in Eq. (19) leads to

$$\mathrm{d}s = T^{-1}\,\mathrm{d}u - \frac{\tau}{\rho K T^2} q\,\mathrm{d}q \tag{34}$$

Depending on the local equilibrium assumption, Eq. (34) can be written in terms of the local equilibrium temperature as

$$\mathrm{d}s = T_{\mathrm{eq}}^{-1}\,\mathrm{d}u - \frac{\tau}{\rho K T_{\mathrm{eq}}^2} q\,\mathrm{d}q \tag{35}$$

The integration of Eq. (35) leads to the general non-equilibrium entropy equation for non-Fourier heat conduction.

$$s(T,q) = s_{\mathrm{eq}}(T) - \frac{\tau}{2\rho K T_{\mathrm{eq}}^2} q^2 \tag{36}$$

For equilibrium conditions, (τ) equals zero, and Eq. (36) reduces to the equilibrium entropy equation.

Let,

$$\Delta s = s_{\mathrm{eq}}(T) - s(T,q) \tag{37}$$

Therefore,

$$\Delta s = \frac{\tau}{2\rho K T_{\mathrm{eq}}^2} q^2 \tag{38}$$

Lebon and Casas-Vazquez [7] indicate that the following stability criteria can be used if the physical properties are independent of temperature.

$\Delta s > 0$ for spontaneous change
$\Delta s = 0$ for equilibrium
$\Delta s < 0$ criteria of stability

The solution of Eq. (38) requires knowledge of the heat flux q and the equilibrium temperature distribution. In order to obtain the temperature distribution, the damped wave equation of the temperature distribution (Eq. (39)) must be solved with the following assumptions:

1. The system is one dimensional.
2. Non-equilibrium convection and radiation are negligible.
3. The physical properties are constant and independent of temperature.
4. The heat pulse is uniformly distributed.

$$\frac{1}{c^2}\frac{\partial^2 T}{\partial^2 t} + \frac{1}{\alpha}\frac{\partial T}{\partial t} = \nabla^2 T \tag{39}$$

The solution was developed by Baumeister and Hamill [8] using a Laplace transform technique for the following initial and boundary conditions.

$$
\begin{array}{ll}
T = T_0 \text{ and } \frac{\partial T}{\partial t} = 0 & \text{when } t = 0,\ x > 0 \\
T = T_w & \text{when } t > 0,\ x = 0 \\
T \to T_0 & \text{when } t > 0,\ x \to \infty
\end{array}
$$

The solution is

$$
\frac{T(x,t) - T_0}{T_w - T_0} = u(ct - x)\left\{ \exp\left(-\frac{cx}{2\alpha}\right) + \int_{x/c}^{t} \left[\left(\frac{cx}{2\alpha}\right) \exp\left(-\frac{c^2\phi}{2\alpha}\right) \left[\frac{I_1\left(\sqrt{\frac{c^4\phi^2}{4\alpha^2} - \frac{c^2\alpha^2}{4\alpha^2}}\right)}{\left(\frac{2\alpha}{c^2}\right)\sqrt{\frac{c^4\phi^2}{4\alpha^2} - \frac{c^2\alpha^2}{4\alpha^2}}} \right] \right] \mathrm{d}\phi \right\} \tag{40}
$$

where ϕ is a dummy variable and $u(ct - x)$ is the unit step function defined as [8]

$$
u(ct - x) = \begin{cases} 1 & \text{for } ct \geqslant tx \\ 0 & \text{for } ct < x \end{cases}
$$

and I_1 represents the solution of the modified Bessel function [8].

To simplify the analysis, Eq. (39) and its solution are written in dimensionless form using the following dimensionless parameters:

$$
\theta = \frac{T(x,t) - T_0}{T_w - T_0} \tag{41}
$$

$$
\beta = \frac{c^2 t}{2\alpha} \tag{42}
$$

$$
\delta = \frac{cx}{2\alpha} \tag{43}
$$

The dimensionless form of Eq. (39) is

$$
\frac{\partial^2\theta}{\partial\beta^2} + 2\frac{\partial\theta}{\partial\beta} = \frac{\partial^2\theta}{\partial\delta^2} \tag{44}
$$

The dimensionless initial and boundary conditions are

$$
\begin{array}{ll}
\theta = 0 \text{ and } \frac{\partial\theta}{\partial\beta} = 0 & \text{when } \beta = 0,\ \delta > 0 \\
\theta = 1.0 & \text{when } \beta > 0,\ \delta = 0 \\
\theta \to 0 & \text{when } \beta > 0,\ \delta \to \infty
\end{array}
$$

The dimensionless form of Eq. (40) is

$$
\theta = u(\beta - \delta)\left\{ \exp[-\delta] + \delta \int_0^{\sqrt{\beta^2 - \delta^2}} \frac{\exp\left(-\sqrt{\eta^2 + \delta^2}\right)}{\sqrt{\eta^2 + \delta^2}} I_1(\eta)\,\mathrm{d}\eta \right\} \tag{45}
$$

$$\eta = \sqrt{\beta^2 - \delta^2}$$
$$u(\beta - \delta) = \begin{cases} 1 & \text{for } \beta \geqslant \delta \\ 0 & \text{for } \beta < \delta \end{cases} \tag{46}$$

The non-Fourier heat flux is obtained by solving Cattaneos equation (Eq. (11)) using the exponential integrating factor method [9], and the solution becomes

$$q(x,t) = -\frac{c^2K}{\alpha}\exp\left(-\frac{c^2t}{\alpha}\right)\int_0^t\left(\frac{\partial T}{\partial x}\right)\exp\left(\frac{c^2\varsigma}{\alpha}\right)\mathrm{d}\varsigma \tag{47}$$

where $\varsigma = t/\tau$.

The solution is made dimensionless using the same dimensionless parameters (δ and β).

$$q(\delta,\beta) = -\frac{cK}{\alpha}(T_\mathrm{w} - T_0)\exp[-2\beta]\int_0^\beta\left(\frac{\partial\theta}{\partial\delta}\right)\exp(2\eta)\,\mathrm{d}\eta \tag{48}$$

The term $(\partial\theta/\partial\delta)$ is obtained by differentiating the temperature distribution equation (Eq. (45)).

At $\delta = 0$, the heat flux becomes

$$q(0,\beta) = \frac{cK}{\alpha}(T_\mathrm{w} - T_0)\exp(-\beta)I_0(\beta) \tag{49}$$

If the conduction Stanton number is defined as

$$St = \frac{q}{\rho C_p c(T_\mathrm{w} - T_0)} \tag{50}$$

Then, the non-Fourier–Stanton number when $\delta = 0$ becomes

$$St_\mathrm{nf} = \exp(-\beta)I_0(\beta) \tag{51}$$

Similarly, for $\delta > 0$, the non-Fourier–Stanton number becomes

$$St_\mathrm{nf} = -\exp(2\beta)\int_0^\beta\left(\frac{\partial\theta}{\partial\delta}\right)\exp(2\eta)\,\mathrm{d}\eta \tag{52}$$

The Stanton number based on Fourier's law (equilibrium case) is

$$St_\mathrm{f} = \frac{1}{\sqrt{2\pi\beta}} \tag{53}$$

Having known the heat flux, the entropy change during the non-Fourier heat conduction is calculated from Eq. (38).

Let $\Delta s^* = \Delta s/C_p$,

$$\Delta s^* = \frac{\tau}{2\rho C_p K T_\mathrm{eq}^2}q^2 \tag{54}$$

Since $\tau = \alpha/c^2$, $\rho C_p = K/\alpha$ and hence, $\tau = K/\rho C_p c^2$

$$\therefore \Delta s^* = \frac{q^2}{2(\rho C_p c)^2 T_\mathrm{eq}^2} \tag{55}$$

or,

$$\Delta s^* = \frac{1}{2}\left[\frac{q}{\rho C_p c(T_w - T_0)}\right]^2 \frac{(T_w - T)^2}{T_{eq}^2} \tag{56}$$

Let $\theta_{eq} = T_{eq}/(T_w - T_0)$,

$$\therefore \Delta s^* = 0.5\left[\frac{St_{nf}}{\theta_{eq}}\right]^2 \tag{57}$$

3. Results and discussion

The unsteady conduction heat transfer in a semi-infinite medium subjected to a step change in temperature is analyzed thermodynamically using the EIT theory. A relationship to calculate the entropy change during the unsteady heat conduction is derived.

Because of the wave nature of the non-Fourier temperature distribution, the medium is divided into two regions, namely the disturbed or thermal wave region and the undisturbed region, as shown in Fig. 1. The line where $\delta = \beta$ separates the two regions. The effect of a temperature change at the boundary ($x = 0$) is felt within the disturbed or thermal wave region only. The thermal wave region is characterized by short distances from the boundary ($x = 0$) and long times, while the undisturbed region is characterized by long distance and short times.

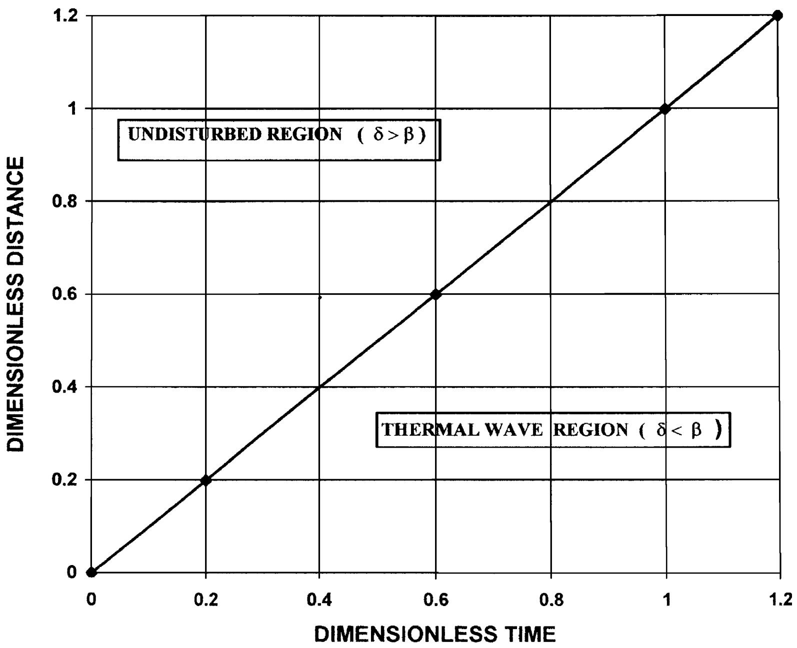

Fig. 1. Map of undisturbed and thermal wave regions.

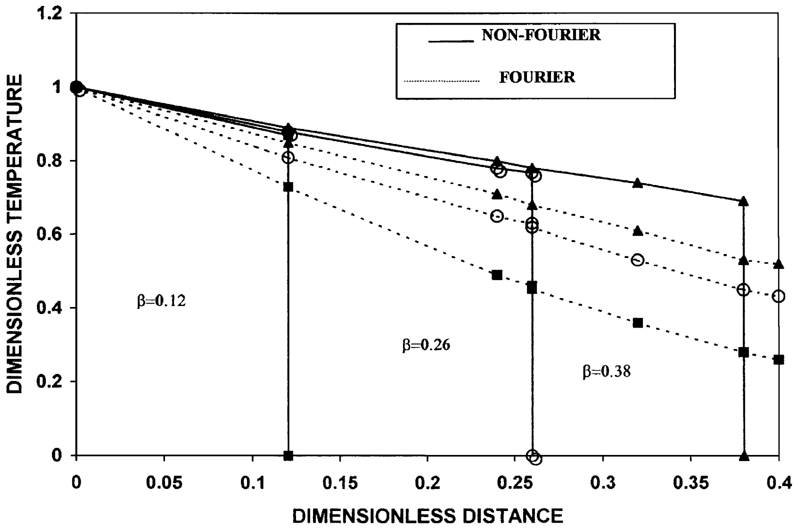

Fig. 2. Temperature distribution in a semi-infinite solid with a step change in temperature (short time behavior).

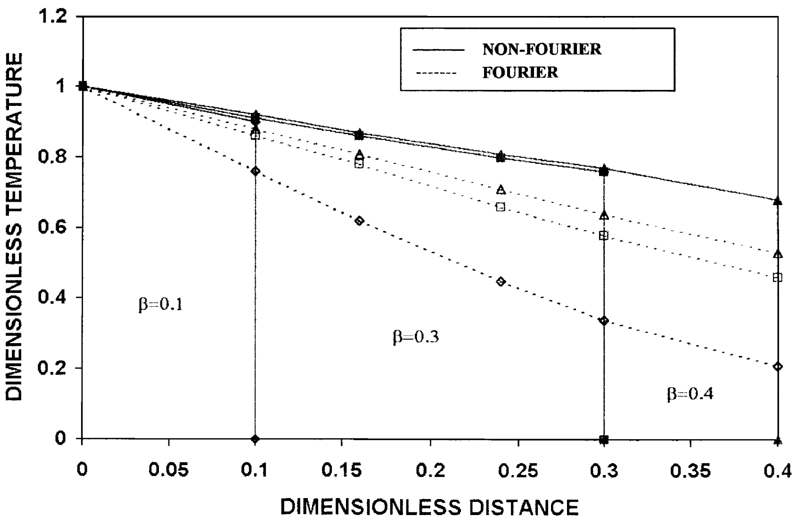

Fig. 3. Temperature distribution in a semi-infinite solid with a step change in temperature (short time behavior).

Figs. 2 and 3 show a comparison between the dimensionless temperature distribution predicted by the Fourier (diffusion) theory and that predicted by the non-Fourier (wave) theory. The wavefront is clearly shown and occurs at $\delta = \beta$ where the distribution drops sharply to zero. The wave propagates at a finite speed through the medium. The diffusion theory predicts a continuous profile where the dimensionless temperature decreases gradually along the medium. The maximum difference between the two profiles occurs at the wavefront. It is clear that the wave theory gives a higher temperature than the diffusion theory due to the irreversibility associated with the actual unsteady heat conduction.

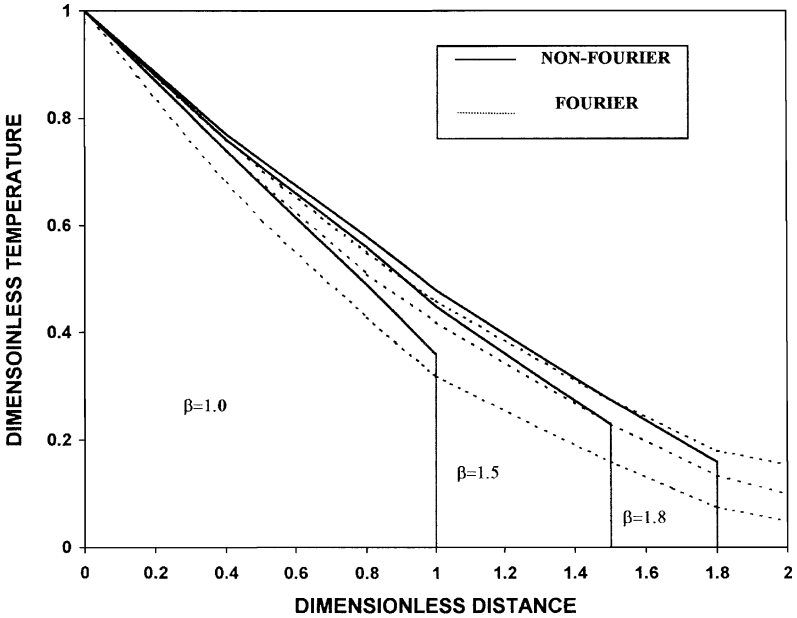

Fig. 4. Temperature distribution in a semi-infinite solid with a step change in temperature (long time behavior).

Fig. 4 shows the same comparison for large values of β, i.e. long times after the temperature change is imposed. It is seen that the temperature distribution predicted by the wave theory approaches that predicted by the diffusion theory as the value of β is increased. Large values of β mean greater thermal wave speed. It is also seen that as β increases, the wavefront travels further through the medium while its amplitude diminishes greatly.

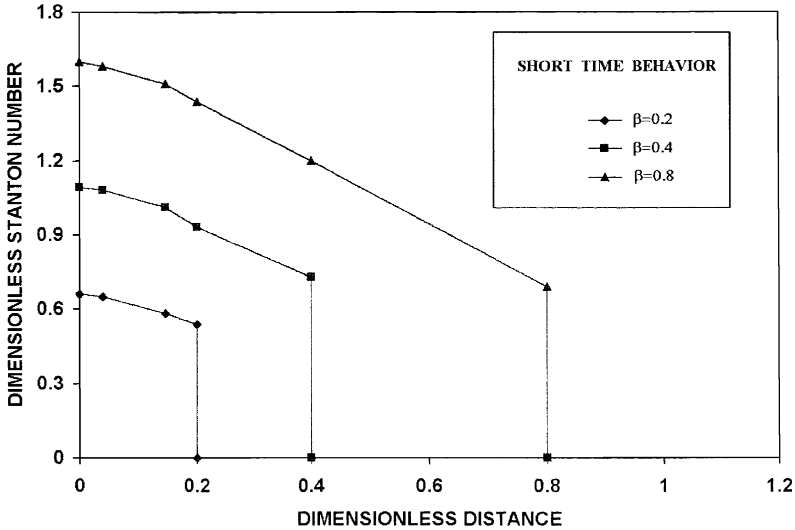

Fig. 5. Variation of dimensionless Stanton number with dimensionless distance for different values of β (short time behavior).

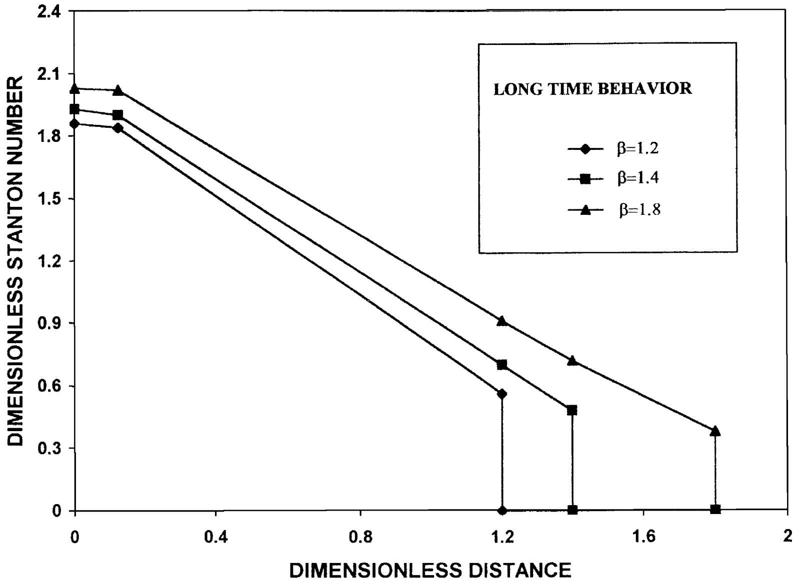

Fig. 6. Variation of dimensionless Stanton number with dimensionless distance for different values of β (long time behavior).

Figs. 5 and 6 show the variation of Stanton number, or in other words, the heat flux, with dimensionless distance for short and long time behaviors, respectively. These figures show clearly the disturbed and the undisturbed regions and the wavefront which occurs at $\beta = \delta$ where the

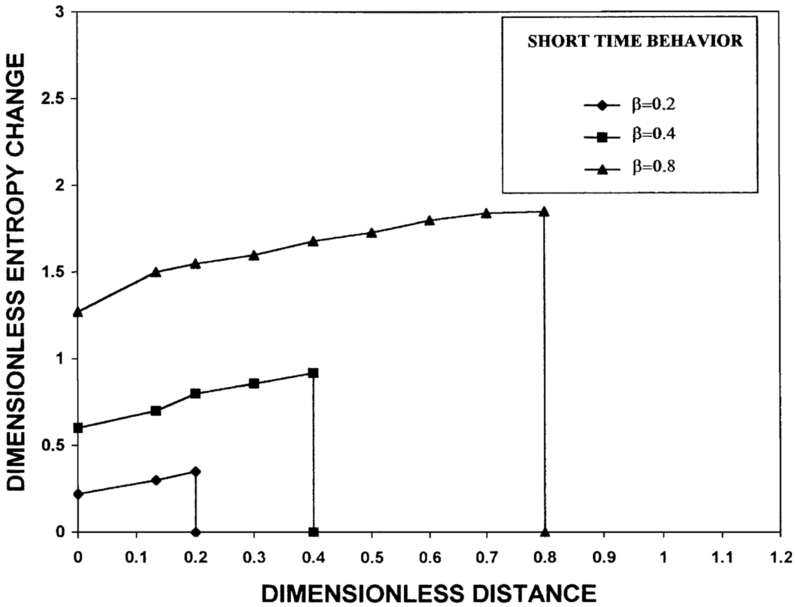

Fig. 7. Variation of dimensionless entropy change with dimensionless distance for different values of β (short time behavior).

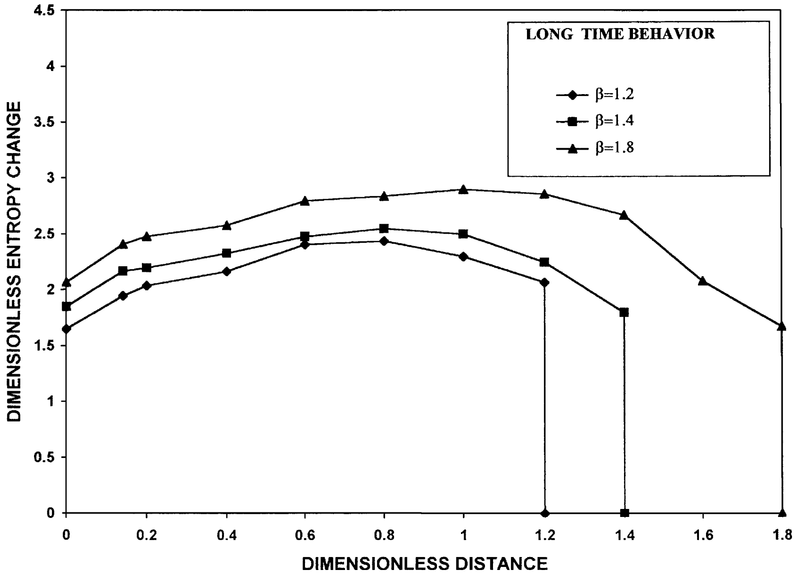

Fig. 8. Variation of dimensionless entropy change with dimensionless distance for different values of β (long time behavior).

Stanton number drops sharply to zero. Fig. 6 shows that the Stanton number profiles become closer as the dimensionless time is increased and finally approach the equilibrium profiles. This finding agrees with the results of Vazquez et al. [10] for the same conditions.

The same wavefront phenomenon is noticed when plotting the dimensionless entropy change against the dimensionless distance for different values of β, as shown in Figs. 7 and 8. The entropy change increases as is β increased and finally approaches the equilibrium profile as β approaches infinity, since the heat flux profile approaches that of equilibrium.

4. Conclusions

1. The local temperature predicted by the non-Fourier theory is higher than that predicted by the Fourier theory.
2. The difference between the non-Fourier and Fourier temperature decreases as the wave propagation speed is increased.
3. The dimensionless entropy change exhibits a wave-like behavior.
4. The profiles of the dimensionless temperature, Stanton number and dimensionless entropy change relax to equilibrium profiles as the speed of wave propagation approaches infinity.

References

[1] Lebon G. From classical irreversible thermodynamics to extended thermodynamics. Acta Physica Hungarica 1989;66(1–4):241–9.

[2] Lebon G, Jou D, Casas-Vazquez J. In: Extended irreversible thermodynamics: standards and prospects, entropy and entropy generation: fundamentals and applications. Kluwer Academic Publishers; 1969. p. 37–54.
[3] Luikov AV. Application of irreversible thermodynamics methods to investigation of heat and mass transfer. Int J Heat Mass Transfer 1966;9(2):139–52.
[4] Lambermont J, Lebon G. On a generalization of the Gibbs equation for heat conduction. Phys Lett 1973;42A(7):499–500.
[5] Truesdell G. Rational thermodynamics. New York: McGraw-Hill Book Company; 1969.
[6] Jou D, Casas-Vazquez J. Extended irreversible thermodynamics of heat conduction. Eur J Phys 1988;9:329–33.
[7] Lebon G, Casas-Vazquez J. On the stability conditions for heat conduction with finite wave speed. Phys Lett 1976;55A(7):393–4.
[8] Baumeister KJ, Hamill YD. Hyperbolic heat conduction equation—a solution for the semi-infinite body problem. ASME J Heat Transfer 1971;93(1):126–7.
[9] Wylie CR, Barrett LC. Advanced engineering mathematics. 5th ed. New York: McGraw-Hill Book Company; 1987.
[10] Vazquez JA, Del Rio JA, Aguirre AA. Non-linear heat wave in extended irreversible thermodynamics. J Non-Equil Thermodyn 1995;20(3):252–62.