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Abstract

Through the concept of local function, we were able to define new boundaries for the set that we called **-frontier. There is a closed and important relationship between this concept and ψ-operator, as has been highlighted in this paper.

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1. Introduction

In every research or paper, we try to find definitions, relationships and mathematical concepts that make it easier for us to solve some mathematical problems or be an introduction to building important mathematical models with a clear impact, through the use of some mathematical concepts with multiple uses especially in the subject of topology. Among among these concepts is the local function, and the exploitation of its highly flexible properties in modification and maneuvering. This drew our attention as well as of many other researches, as we enriched this function with a number of researches, whether by studying it in traditional, ambiguous or flexible spaces. For more information, it is possible to use the papers [1,2,3].

2. The Main Ideas

Definition 2.1 : [4] Let $(\dot{K}, \boldsymbol{t}, I)$ be a ideal

1. For any subset K, $\psi^*(k) = \dot{K} - (\dot{K} - k)^{**}$.
2. $k \subseteq \dot{K}$ is called I^{ϖ} -dense iff $k^{**} = \dot{K}$.

Theorem 2.2 : [5,6] Let $(\dot{K}, \boldsymbol{t}, I)$ is ideal, the statement, are equivalents:

1. $\boldsymbol{t} \cap I = \varphi$.
2. If $\mathcal{J} \in I$, so int $(\mathcal{J}) = \varphi$.
3. $H \subset H^*, \forall H \subseteq \mathcal{J}$.
4. $\dot{K} = \dot{K}^*$.

So, Hayashi-Samuel space, for any $H \subset \boldsymbol{t}$, $H^* = H^{**}$. If we have any ideal topological spaces $(\dot{K}, \boldsymbol{t}, I)$, we have three functions of any subset $\mathfrak{C}$ of $\dot{K}$ as follows: $Fr^*(C) \subseteq \boldsymbol{t}^*$ $-Fr(C) \subseteq \boldsymbol{t}$ $-Fr(C)$ where $Fr(C) =$ $\text{cl}_t(\dot{K} - C)$ $\cap \text{cl}_t(C)$.$\boldsymbol{t}^*$ $-Fr(C) =$ $\text{cl}_{t^*}(C) \cap$ $\text{cl}_{t^*}(\dot{K} - C)$ and $Fr^*r(C) = C^* \cap (\dot{K} - C)^*$.

Definition 2.3 : Let $(\dot{K}, \boldsymbol{t}, I)$ be an ideal, then operator $Fr^{**}(k) : \mathcal{P}(\dot{K}) \to \mho(\dot{K})$, defined by $Fr^{**}(k) = k^{**} \cap (\dot{K} - K)^{**}$, is called *-boundary of *k*.

Example 2.4 : Let $\dot{K} = \{\hat{a_1}, \hat{a_2}, \hat{a_3}\}$, $\boldsymbol{t} = \{\varnothing, \dot{K}, \hat{a_1}\}, \{\hat{a_3}, \hat{a_2}\}\}$, $I = \{\varnothing, \{\hat{a_2}\}, \{\hat{a_3}\}, \{\hat{a_3}, \hat{a_2}\}\}$. Then $Fr^{**}(\{\hat{a_1}, \hat{a_2}\},) = \{\hat{a_1}\}$, so, we see that $Fr^{**}(k) \subseteq Fr^*(k)$. For any $a \in \dot{K}$, $I_{\mathfrak{x}} = \{\mathfrak{C} \subseteq \dot{K},\ a \in \dot{K} - \mathfrak{C}\}$ is an ideal on $\dot{K}$, so, $Fr^{**}(k) = \varphi$. For any $k \subseteq \dot{K}$, because $k^{**} = \varphi$ if $a \notin k$, if $a \in k$, then $(\dot{K} - k)^{**} = \varphi$. Now for,

$I=\{\varnothing\}$, we get that all frontiers of any set are equal. But if $I=\mathcal{P}(\dot{\mathrm{K}})$ then $Fr^{**}(a)= Fr^{*}(\mathfrak{C})=\varphi$.

Proposition 2.5: Let $(\dot{\mathrm{K}},t,I)$ is ideal space. For any subset $¥, f$ of $\dot{\mathrm{K}}$, the following are carried:

1. $k\in Fr^{**}(¥)$ iff $\in k^{**}-\psi^{*}(¥)$.
2. $Fr(¥)=\varphi$ iff $¥^{**}\subseteq\psi^{*}(¥)$.
3. $Fr^{**}(¥)=(\dot{\mathrm{K}}-¥)^{**}$ iff $\dot{\mathrm{K}}-¥^{**}\subseteq\psi^{*}(¥)$.
4. If $¥$ is I^{ϖ}-dense, then $Fr^{**}(¥)=(\dot{\mathrm{K}}-¥)^{**}$.

Proof of (1) : $k\in Fr^{**}(¥)$ iff $k\in(¥)^{**}$ and iff $k\in(\dot{\mathrm{K}}-¥)^{**}$ iff $k\in(¥)^{**}$ and $k\notin\psi^{*}(¥)$ iff $k\in ¥^{**}-\psi^{*}(¥)$.

Proof of (4) : $¥$ is I^{ϖ}-dense, then $¥^{**}=\dot{\mathrm{K}}$, so, $Fr^{**}(¥)= ¥^{**}\bigcap(\dot{\mathrm{K}}-¥)^{**}$ $=(\dot{\mathrm{K}}-¥)^{**}$. By Proposition 2.5 part (1)[5], $\psi^{*}(¥)\subseteq ¥^{**}$ for any subset $¥$ in the Hayashi-Samuel space. There are some properties that connect $\psi^{*}-$ operator with **-Frontier in the Hayashi -Samuel space.

Proposition 2.6 : Let $(\dot{\mathrm{K}},t,I)$ is Hayashi-Samuel space, the following statements are correct.

1. $Fr^{**}(k)=\varphi$ iff $k^{**}=\psi^{*}(k)$.
2. For each closed set k $Fr^{**}(k)=k^{**}\subseteq \mathrm{int}(k)$.

Proof **:** Let k be closed subset of $\dot{\mathrm{K}}$, $Fr^{**}(k)=k^{**}\bigcap(\dot{\mathrm{K}}-k)^{**}=k^{**}\bigcap$cl $(\dot{\mathrm{K}}-k)$, by using Theorem 2.2. , $\mathrm{cl}(\dot{\mathrm{K}}-k)=(\dot{\mathrm{K}}-k)^{**}=(\dot{\mathrm{K}}-k)^{*}$, so, $Fr^{**}(k)=k^{**}\bigcap(\dot{\mathrm{K}}-\mathrm{int}(k))=k^{**}\subseteq\mathrm{int}(k)$. There are multiple properties and characteristics to **-boundary set, as in the following proposition.

Proposition 2.7 : For any subset k_1, k_2 in ideal $(\dot{\mathrm{K}},t,I)$, the properties are true.

1. $Fr^{**}(\varphi)=Fr^{**}(\dot{\mathrm{K}})=\varphi$.
2. For any $J\in I, Fr^{**}(J)=\varphi$.
3. $Fr^{**}(k_1\bigcup k_2)= Fr^{**}(k_1)\bigcup Fr^{**}(k_2)$
4. $Fr^{**}(k_1)=k_1^{**}-\psi^{*}(k_1)$.
5. $Fr^{**}(\ddot{F}^{**}r(k_1))\subseteq Fr^{**}(k_1)$.
6. $Fr^{**}(k_1)=(\dot{\mathrm{K}}-k_1)^{**}-\psi^{*}(\dot{\mathrm{K}}-k_1)$.

7. $Fr^{**}(\dot{K}-k_1)=Fr^{**}(k_1)$.
8. $\dot{K}-Fr^{**}(k)=\psi^*(\dot{K}-k_1)\cup\psi^*(k_1)$.
9. $\dot{K}=\psi^*(\dot{K}-k_1)\cup\psi^*(k_1)\cup\ddot{F}^{**}r(k_1)$.

Proof of (6) : $Fr^{**}(k_1) = k_1^{**}\cap(\dot{K}-k_1)^{**} = k_1^{**}\cap\dot{K}- (\dot{K}-(\dot{K}-k_1)^{**})=$ $k_1^{**}\cap\dot{K}-\psi^*(k)$.

Proof of (8) : $\dot{K}-Fr^{**}(k_1) = \dot{K}-(k_1^{**}\cap(\dot{K}-k_1)^{**}) = (\dot{K}-k_1^{**})\cup(\dot{K}-(\dot{K}-k_1)^{**})=\dot{K}-(\dot{K}-(\dot{K}-k_1)^{**}\cup\psi^*(k_1)=\psi^*(\dot{K}-k_1)\cup\psi^*(k_1)$.

Proof of (9) : $\psi^*(\dot{K}-k_1)\cup\psi^*(k_1)\cup Fr^{**}(k_1) = \dot{K}-(\dot{K}-(\dot{K}-k_1)^{**})\cup(\dot{K}-(\dot{K}-k_1)^{**} Fr^{**}(k_1) = (\dot{K}-k_1)^{**}\cup(\dot{K}- (\dot{K}-k_1)^{**} Fr^{**}(k_1) = \dot{K}\cup Fr^{**}(k_1)=$ $\dot{K}$. For the Definition 2.1. it is simple to show that for any open set $\ddot{\mathcal{U}}$, $Fr^{**}(\ddot{\mathcal{U}})\subseteq\ddot{\mathcal{U}}^{**}-\ddot{\mathcal{U}}$.

Proposition 2.8 : For any subset L_1, L_2 in ideal $(\dot{K},t,I)$, the following are correct.

1. $Fr^{**}(L_1\cap L_2)\subseteq Fr^{**}(L_1)\cup Fr^{**}(L_2)$.
2. $Fr^{**}(L_1-L_2)\subseteq Fr^{**}(L_1)\cup Fr^{**}(L_2)$.
3. $Fr^{**}(L_1)\cup Fr^{**}(L_2)=Fr^{**}(L_1-L_2)\cup Fr^{**}(L_2-L_1)\cup Fr^{**}(L_1\cap L_2)$.

Proof of (1) : By Proposition 2.7 part (3,7) we get that $Fr^{**}(L_1\cap L_2)=$ $Fr^{**}(\dot{K}-L_1\cap L_2) = Fr^{**}(\dot{K}-L_1\cup\dot{K}-L_2)\subseteq Fr^{**}(\dot{K}-L_1)\cup Fr^{**}(\dot{K}-L_2) =$ $Fr^{**}r(L_1)\cup Fr^{**}(L_2)$.

Proof of (2) : By part (1) and Proposition 2.7 part (3,7) we get that $Fr^{**}(L_1-L_2) = Fr^{**}(L_1\cap(\dot{K}-L_2))\subseteq Fr^{**}(L_1)\cup Fr^{**}(\dot{K}-L_2) = Fr^{**}(L_1)$ $\cup Fr^{**}(L_2)$.

The concept of symmetric difference of L_1, L_2 usually denoted by $L_1\Delta L_2$ and equal to L_1-L_2 union to L_2-L_1, also equal to $(L_1\cup L_2)-(L_1\cap L_2)$ and the important property is $H\cap(L_1\Delta L_2)=(H\cap L_1)\,\Delta(H\cap L_2)$. Through these observations, there are important properties that relate symmetric difference and $**-$boundary, as shown by the following properties.

Proposition 2.9 : For any subset L_1, L_2 in ideal $(\dot{K},t,I)$, the properties are true.

1. $Fr^{**}(L_1)\cup Fr^{**}(L_2)=Fr^{**}(L_1\cap L_2)\cup Fr^{**}(L_1-L_2)\cup Fr^{**}(L_1\cup L_2)$

2. $Fr^{**}(L_1) \cup Fr^{**}(L_2) = Fr^{**}(L_1 \cup L_2) \cup Fr^{**}(L_2 - L_1) \cup Fr^{**}(L_1 \cap L_2)$.
3. $Fr^{**}(L_1) \cup Fr^{**}(L_2) = Fr^{**}(L_1 - L_2) \cup Fr^{**}(L_2 - L_1) \cup Fr^{**}(L_1 \cap L_2)$.
4. $Fr^{**}(L_1) \cup Fr^{**}(L_1 \Delta L_2) = Fr^{**}(L_1 - L_2) \cup Fr^{**}(L_1 \cap L_2) \cup Fr^{**}(L_2 - L_1)$.
5. $Fr^{**}(L_2) \cup Fr^{**}(L_1 \Delta L_2) = Fr^{**}(L_1 - L_2) \cup Fr^{**}(L_1 \cap L_2) \cup Fr^{**}(L_2 - L_1)$.

Proof of (1) : By using Proposition 2.5 part (2, 5) we get that

$$Fr^{**}(L_1) \cup Fr^{**}(L_2) = Fr^{**}(L_1) \cup Fr^{**}(\dot{K} - L_2) = Fr^{**}(L_1 - (\dot{K} - L_2))$$

$$\cup Fr^{**}(L_1 \cap (\dot{K} - L_2)) \cup Fr^{**}((\dot{K} - L_2) - L_1)$$

$$= Fr^{**}(L_1 \cap L_2) \cup Fr^{**}(L_1 - L_2) \cup Fr^{**}(\dot{\mathbb{X}} - (L_1 \cup L_2)) =$$

$$Fr^{**}(L_1 \cap L_2) \cup Fr^{**}(L_1 - L_2) \cup Fr^{**}(L_1 \cup L_2).$$

Proof of (2) : By using proposition 2.5 part (2,6) we have:

$$Fr^{**}(L_1) \cup Fr^{**}(L_2) = Fr^{**}(L_2) \cup Fr^{**}(\dot{K} - L_1) = Fr^{**}((\dot{K} - L_1) - L_2) \cup Fr^{**}$$
$$((\dot{K} - L_1) \cap L_2) \cup Fr^{**}(L_2 - (\dot{K} - L_1)) = Fr^{**}(\dot{K} - (L_1 \cup L_2)) \cup Fr^{**}(L_1 - L_2) \cup$$
$$Fr^{**}(L_1 \cap L_2) = Fr^{**}(L_1 \cup L_2)) \cup Fr^{**}(L_2 - L_1) \cup Fr^{**}(L_2 \cap L_1).$$

Proof of (4) : By using Proposition (2.8) we get $Fr^{**}(L_1) \cup Fr^{**}(L_1 \Delta L_2) = Fr^{**}(L_1 - (L_1 \Delta L_2)) \cup Fr^{**}(L_1 \cap (L_1 \Delta L_2)) \cup Fr^{**}((L_1 \Delta L_2) - L_2) \ldots(1)$.

Since $L_1 - (L_1 \Delta L_2) = L_1 \cap [(L_1 \cap L_2^c) \cup (L_1^c \cap L_2)]^c = L_1 \cap [(L_1^c \cup L_2) \cap (L_1 \cap L_2^c)]^c = L_1 \cap (L_1^c \cap L_2) = L_1 \cap L_2$. Then $Fr^{**}(L_1 - (L_1 \Delta L_2)) = Fr^{**}(L_1 \cap L_2)$. But $L_1 - (L_1 \Delta L_2) = L_1 \cap L_1 \, \Delta L_2 \cap L_2 = L_1 \Delta (L_1 \cap L_2) = [L_1 \cap (\dot{K} - L_1 \cup \dot{K} - L_2)] \cup [(L_1 \cap L_2 \cap (\dot{K} - L_1)] = L_1 \cup (\dot{K} - L_2) = \dot{K} - (L_1 - L_2)$. Then, $Fr^{**}(L_1 \cap (L_1 \Delta L_2)) = Fr^{**}(\dot{K} - (L_1 - L_2)) = Fr^{**}(L_1 - L_2)$. Finally, since $(L_1 \Delta L_2) - L_2 = (L_1 \Delta L_2) \cap (\dot{K} - L_2) = [L_1 \cap (\dot{K} - L_2)] \, \Delta [L_2 \cap (\dot{K} - L_1)] = (L_1 - L_2) \Delta \varphi = L_1 - L_2$, so $Fr^{**}((L_1 \Delta L_2) - L_2)) = Fr^{**}(L_1 - L_2)$.

3. Discussion and Conclusion

1. By using Proposition 2.7 part (9) we see that space is divided into three parts: $\psi^*(k), \psi^*(\dot{K} - k)$ and $Fr^{**}(k)$, for any subset k in $\dot{K}$.
2. We any modify the **-frontier set by change open set to $\mathfrak{w}$-open, through it, the concepts are modified by papers [1,6,7].

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