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Best neural approximation for 2-quasi inner product spaces

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Abstract

Many researchers studied two-normed spaces for objects of function approximation. Also, neural networks are great instruments for function approximation from function spaces, especially, Lebesque spaces (L_p). In this paper, the aim is to approximate Lebesque integrable functions via two-quasi inner product that is defined here in terms of two-norm. Moreover, there is a definition of a special type of neural network with special weights to estimate the approximation error, which is equivalent to the modulus of smoothness of functions of study. Estimates of infimum and supremum bounds so that the modulus of smoothness is involved in each bound. It is interesting to investigate more work related to what we presented in this study, not only for those fields of neural network function approximation, but also to approximate functions by various types of operators.

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Introduction

Universal Approximation Theorem (UAT) give big role to neural networks as approximation for continuous functions (Cybenko, 1989). Later, many types of functions are approximated by neural networks with degree of approximation that varies from one search to another. Many researches of 2- normed spaces are there to give different expressions and estimates to approximation of functions (Acikgoz, 2014a, 2014b; Almurieb & Bhaya, 2020; Cobzas & Academy, 2017; Elumalai & Vijayaragavan, 2009; Iranmanesh & Soleimany, 2016; Mehmet, 2012). The basic principles of linear 2-normed space was proposed at the first time by S.Gahler in 1965 in his paper(Gahler, 1964). Many other mathematicians' researchers have potentially improved the geometric structure of linear 2-normed space. In his P.hD thesis (Two-Banach spaces), A.White (Albert George White, 1969) that described the linear 2-normed space by defining convergent and Cauchy sequences for the relevant spaces.

Many developments are still introduced by researches. C. Park studied 2-Quasi normed spaces in the paper (Park, 2006) by substituting a condition from the original definition of 2-norm from (Gahler, 1964) so it became suitable for functions from L_p spaces when $0 < p < 1$. The following is the definition of 2-quasi normed space dropping on the space L_p,

Definition 1 (2-Quasi Normed Space) : Let $L_p[0, 1]$ be the Lebesgue space defined as follow

$$\text{“}L_p([0,1]) = \{f : [0,1] \to \mathbb{R}, | \, f \text{ measurable and } \| f_p \| < \infty\}\text{”},$$

Where

$$\text{“}\| f_p \| = \left\{ \int_a^b | f |^p \right\}^{1/p} \text{”},$$

and let $\| .,. \|$ be a real-valued function defined on $L_p[0,1] \times L_p[0,1]$ satisfying the following conditions for all $\varphi_1, \varphi_2, h \in L_p[0,1]$

i. $\| \varphi_1, \varphi_{2p} \| \geq 0$, where $\| \varphi_1, \varphi_{2p} \| = 0$ for any linearly dependent φ_1 and φ_2,

ii. $\| \varphi_1, \varphi_{2p} \| = \| \varphi_2 \ , \varphi_1 \|_p$,

iii. $\| \propto \varphi_1, \varphi_{2p} \| = |\propto| \, \| \varphi_1, \varphi_{2p} \|$, for all $\propto \in R$,

iv. $\| \varphi_1 + \varphi_2, \, h \|_p \leq C(\| \varphi_1, h \|_p + \| \varphi_2, h \|_p)$, for some constant C.

Then $\|.,.\|$ is called 2- norm and the space $(L_{p,2}[0,1], \|.,.\|_p)$ is called 2-normed space.

Note that Definition 1, as well as the later ones, are appropriate for any vector space of dimension greater than 1, but we restrict our study with functions from L_p.

More other related concepts were appeared later, like 2-inner product (Diminnie et al., 1977) and 2-semi inner product (Dragomir, n.d.). We define a new concept relative to them, what we call *2-quasi inner product*. This concept is restricted to $L_p[0, 1]$space in our research but it also true for other spaces as we mentioned above.

Definition 2 (2-quasi inner product space) :

Define the real valued map for functions from $L_p[0, 1] \times L_p[0, 1] \times L_p[0, 1]$ to be 2-quasi-inner product if the following conditions hold for any φ_1, φ_2, h and ψ in $L_p[0, 1]$

i. $\langle \varphi_1, \varphi_2 | \psi \rangle > 0$

ii. $\langle \varphi_1, \varphi_2 | \psi \rangle = \langle \varphi_2, \varphi_1 | \psi \rangle$

iii. $\langle \propto \varphi_1, \varphi_2 | \psi \rangle = \propto \langle \varphi_1, \varphi_2 | \psi \rangle$

iv. $\langle \varphi_1, +\varphi_2, h | \psi \rangle \leq C(\langle \varphi_1, h | \psi \rangle + \langle \varphi_2, h | \psi \rangle)$

We refer $\langle L_{p,2}[0,1] \rangle$ to the 2-quasi inner product space.

Definition 3 : For the space $(L_{p,2}[0,1], \|\cdot,\cdot\|_p)$, we define supremum/ infimum of 2-quasi inner product equipped with $\|\cdot,\cdot\|_p$ by the following

$$\sup \langle \varphi_1, \varphi_2 | \psi \rangle = \lim_{t \to 0^+} \frac{\| \varphi_2(x) + \varphi_1(t), \psi(x) \|_p^p - \| \varphi_2(x), \psi(x) \|_p^p}{pt}$$
$$\inf \langle f_1, f_2 | \psi \rangle = \lim_{t \to 0^-} \frac{\| \varphi_2(x) + \varphi_1(x), \psi(x) \|_p^p - \| \varphi_2(x), \psi(x) \|_p^p}{pt} \quad (1)$$

Properties of SUP/INF IN $\langle L_{p,2}[0,1] \rangle$

It is essential to study the values of (1) in Definition 3, here are some facts about them, they are easily verified by definitions

1. $\sup\langle \alpha\varphi_1, \beta\varphi_2 \mid \psi\rangle = \alpha\beta \sup\langle \varphi_1, \varphi_2 \mid \psi\rangle$,
2. $\sup - \langle \varphi_1, \varphi_2 \mid \psi\rangle = -\sup\langle \varphi_1, \varphi_2 \mid \psi\rangle = \sup\langle \varphi_1, -\varphi_2 \mid \psi\rangle$,
3. $\sup\langle \alpha\varphi_1 + \varphi_2, \varphi_1 \mid \psi\rangle \le C \mid \alpha \mid \sup\langle \varphi_1, \varphi_2 \mid \psi\rangle = \mid \alpha \mid \sup\langle \varphi_2, \varphi_1 \mid \psi\rangle$,

   and the same is true for infimum case for the relations (1, 2 and 3)
4. $\inf\langle \varphi_1, \varphi_2 \mid \psi\rangle \le C \| \varphi_1, \psi \|_p \le \sup\langle \varphi_1, \varphi_2 \mid \psi\rangle$
5. $\varphi_1 \perp_\psi \varphi_2$ iff $\inf\langle \varphi_1, \varphi_2 \mid \psi\rangle \le 0 \le \sup\langle \varphi_1, \varphi_2 \mid \psi\rangle$
6. $\inf\langle \varphi_1, \varphi_2 \mid \psi\rangle = \sup\langle \varphi_1, \varphi_2 \mid \psi\rangle$ iff the norm is Gâteaux differentiable.

On our trip to investigate approximation capabilities by neural networks, we define the following 2- normed for any $\varphi_1, \varphi_2 \in \langle L_{p,2}[0,1]\rangle$ space as follow

$$\| \varphi_1, \varphi \|_{2p} = \left(\int_b^a \left| \varphi_1(x)\varphi_2'(x) - \varphi_1'(x)\varphi_2(x) \right|^p \right)^{\frac{1}{p}} \quad \forall x \in [0,1] \tag{2}$$

We can find the best approximation of any function from $\langle L_{p,2}[0,1]\rangle$ out of the set of neural networks $\mathcal{N}$ of the following form

Definition 4 : Any neural networks N of the space $\mathcal{N}$ is given by

$$N(f, \varnothing) = \sum_{i=1}^{k} c_i \sigma(w_i + b), \tag{3}$$

$$c_i = \Delta_h^k(f, \varnothing) = \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} f\left(x - \frac{kh}{2} + ih \right) \varnothing(x) \tag{4}$$

For any function $\varnothing \in L_p[0,1]$. σ is the ReLU activation function

$$\sigma(x) = \max(0, x)$$

Definition 5 : The best approximation of $f \in \langle L_{p,2}[0,1]\rangle$ out of $\mathcal{N}$ is the network $N_0 \in \mathcal{N}$ that satisfies

$$\| f - N_0, \psi \|_p = \inf\{ \| f - N, \psi \|_p : N \in \mathcal{N} \} \tag{5}$$

We denote the best approximation of *f* by the following

$$B_N^\psi = \{ N_0 \in \mathcal{N}, \inf \| f - N, \psi \|_p : N \in \mathcal{N} \}.$$

The purpose of this paper is to characterize the best approximation in the space $L_{p,2}\langle [0,1]\rangle$ with modulus of smoothness as below

Definition 5 : Modulus of smoothness of order *k* of function $f \in \langle L_{p,2}[0,1]\rangle$ is given by

$$\omega_k(f,\delta)_p = \sup_{0<h\leq\delta} \| \Delta_h^k f(\cdot) \|_p, \delta \geq 0,$$

where the symmetric difference of *f* order *k* is given by

$$\Delta_h^k \ (f,x) := \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f\left(x - \frac{kh}{2} + ih\right), & x \pm \frac{kh}{2} \in I \\ 0, & o.w \end{cases}$$

Auxiliary Lemmas

For our purposes of function approximation by neural networks, we us the 2-normed which is given in (2). The proof of the fisrt two lemmas are easily to be verified by Quasi-triangle inequality and definitions.

Lemma 1 : *The norm*

$$\| \varphi_1, \varphi_2 \|_p = \left(\int_b^a \left| \varphi_1(x)\varphi_2'(x) - \varphi_1'(x)\varphi_2(x) \right|^p dx \right)^{\frac{1}{p}}$$

is 2-normed so the space $(L_{p,} \| .,. \|)$ *is 2-normed space.*

With a little modification to (2) , we get the following lemma

Lemma 2 : *We set*

$$\langle \varphi_1, \varphi_2 \mid h \rangle = \left(\int_b^a | \varphi_1(x)\varphi_2'(x) - \varphi_1'(x)\varphi_2(x) |^p |h(x)|^p \right)^{\frac{1}{p}}$$

as a 2-quasi inner product.

Lemma 3 : *For any* $N \in B_N^{\psi}(f)$, *where* $f \in L_{p,2}\langle[0,1]\rangle$, *we have*

$$\| f - N, \psi \|_p \leq C\omega_k(f,\psi)$$

Proof :

$$\| N, \psi \|_p \leq c(p) \left\| f - \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} f\left(x - \frac{kh}{2} + ih\right) \sigma(w_i + b), \psi \right\|_p$$

$$\le c(p)\sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}\left\| f\left(x-\frac{kh}{2}+ih\right),\psi\right\|_p \| \sigma(w_i+b),\psi \|_p$$

$$\le c(p)\sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}\left\| f\left(x-\frac{kh}{2}+ih\right),\psi\right\|_p \int_0^1 \left|(w_i+b)\psi'(x)\right|^p dx$$

$$\le C\omega_k(f,\psi)$$

□

Main Result

Now, we are ready to state and prove the degree of neural best approximation of functions from $\langle L_{p,2}[0,1]\rangle$ lies between the two bounds, which both are in terms of modulus of smoothness of that function.

Theorem : *For any function* $f\in\langle L_{p,2}[0,1]\rangle$, *there exist a neural network* $N\in B_N^{\psi}(f)$, *that satisfies*

$$\inf\langle f,N\,|\,\psi\rangle \le C\omega_k(f,\psi)\le \sup\langle f,N\,|\,\psi\rangle$$

Proof : Let $N_0\in B_N^{\psi}(f)$, set $g_0=f-N_0$, s.t. $g_0\perp_{\psi} N_0$, then by properties of $\inf\langle\varphi_1,\varphi_2\,|\,\psi\rangle$, we get

$$\inf\langle g_0,N_0\,|\,\psi\rangle \le 0\le \sup\langle g_0,N_0\,|\,\psi\rangle$$

$$\inf\langle f-N_0,N_0\,|\,\psi\rangle \le 0\le \sup\langle f-N_0,N_0\,|\,\psi\rangle$$

Here, we estimate the infimum case, so the supremum one is similar. Again by properties of $\inf\langle f,N\,|\,\psi\rangle$, and lemma3,

$$\begin{aligned}\inf\langle f-N_0,N_0\,|\,\psi\rangle &\le c\inf\langle f,N_0\,|\,\psi\rangle\\ &\le c\,\| N_0,\psi\|_p\\ &\le C\omega_k(f,\psi)\end{aligned}$$

□

Conclusion

Investigating the approximation capabilities of new spaces is important to different fields for the applicable potentials it has. We define a space of functions from $L_{p,2}[0,1]$ associated with 2-quasi inner product space, so we name $\langle L_{p,2}[0,1]\rangle$. The approximation degree for functions above is of *k*th order modulus of smoothness.

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