Degree of Best Approximation in terms of Weighted DT Moduli of Smoothness

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Abstract--- It is well known, inverse and direct theorems relate the degree of best trigonometric or algebraic approximation of functions to differential properties of these functions. We define weighted types of moduli of smoothness and prove inverse and direct inequalities for algebraic approximation in terms of theses weighted moduli of smoothness of functions in$L_p[-1,1]$, spaces for$0 < p < 1$. Theses give characterization of some types of classes of smoothness of functions via its degree of algebraic approximation.

Keywords--- Approximation by Polynomials in the L_p −Norm, Degree of Approximation, Direct and Inverse Theorems, L_p Spaces, Moduli of Smoothness.

I. Introduction and Definition

Direct approximation theorems are statements asserting that if$f$ belongs to certain class of smoothness then its degree of best approximation decreases to zero. On classes of continuously differentiable functions, such theorems were first proved in terms of the first-order modulus of continuity by Jackson [10] in 1911. Later, Zygmund[20] and Akhiezerin [1] generalized Jackson's results to the second-order modulus of continuity, and Stechkinin [17] extended these results to the moduli of continuity of an arbitrary integer order k, k ≥ 3.

Inverse approximation theorems are the converse statements that characterize the smoothness properties of a function depending on the speed of convergence to zero of its approximation by some approximating aggregates. These theorems were first obtained by Bernstein [2] in 1912. And already in 1919, direct and inverse approximation theorems, due to Jackson and Bernstein, were given in the book on approximation theory by de la Vall´ee Poussin [19].

Investigations of the connection (direct and inverse) between the smoothness properties of functions and the possible orders of their approximations were carried out by many authors on various classes of functions and for various approximating aggregates. Such results constitute the classics of modern approximation theory and they are described quite fully in the monographs see [4],[5],[9],[18].

In [3] the authors prove direct and inverse inequalities for the degree of best approximation using neural networks. Also in [11] the authors used a direct kind of approximation. They proved a Jackson type theorem for the approximation of functions defined on graphs.

Now we will give some basic definitions.

Let$\|\cdot\|_p := \|\cdot\|_{L_p[-1,1]}$, $0 < p < 1$, and $\varphi(x) = \sqrt{1-x^2}$.

For $k \in \mathbb{N}_0, h \geq 0$,an interval J and$g: J \longrightarrow \mathbb{R}$,(where$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$),

let

$$\Delta_h^k(g,x,J) := \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} g\left(x + \left(i - \frac{k}{2}\right)h\right) & if\ x \mp \frac{kh}{2} \in J, \\ 0 & \text{otherwise} \end{cases} [13]$$

be the "kth symmetric difference", and let$\Delta_h^k(g,x) := \Delta_h^k(g,x,[-1,1])$.

Definition 1.1 Let $0 < p < 1$, $and\ r \in \mathbb{N}_0 . Then, for\ r \geq 1$, let

$$\mathbb{B}_p^r := \left\{ g: \left\| g^{(r)} \varphi^{(r)} \right\|_p < +\infty \right\},$$

Definition 1.2 [13]Let $k \in \mathbb{N}, r \in \mathbb{N}_0$, and$g \in \mathbb{B}_p^r$, $0 < p < 1$, define

$$\omega_{k,r}^{\varphi}\left(g^{(r)}, t\right)_p := \sup_{0 \leq h \leq t} \left\| W_{kh}^r(.) \Delta_{h\varphi(.)}^k \left(g^{(r)}, .\right) \right\|_p,$$

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Where

$$W_\delta(x) := ((1-x-\delta\varphi(x)/2)(1+x-\delta\varphi(x)/2))^{1/2}.$$

For $\delta > 0$, define

$$\mathfrak{D}_\delta := \{x: 1-\delta\varphi(x)/2 \ge |x|\}\backslash\{\mp 1\}$$
$$= \left\{x: |x| \le \frac{4-\delta^2}{4+\delta^2}\right\} = [-1+\mu(\delta), 1-\mu(\delta)]$$

Where

$$\mu(\delta) \coloneqq 2\delta^2/(4+\delta^2).$$

Definition 1.3 [14]Let $k \in \mathbb{N}, r \in \mathbb{N}_0$, and $g \in \mathbb{B}_p^r, 0 < p < 1$.Then, the "averaged modulus of smoothness" is defined as

$$\omega_{k,r}^{*\varphi}\left(g^{(r)}, t\right)_p = \left(\frac{1}{t}\int_0^t \int_{\mathfrak{D}_{k\tau}} \left|W_{k\tau}^r(x)\Delta_{\tau\varphi(x)}^k\left(g^{(r)}, x\right)\right|^p dx d\tau\right)^{1/p}.$$

Definition 1.4 [13]*(K-functional)* For $k \in \mathbb{N}, r \in \mathbb{N}_0, 0 < p < 1, and\ f \in \mathbb{B}_p^{r+k}$, define

$$K_{k,r}^{\varphi}(f^{(r)}, t^k)_p \coloneqq \inf_{g\in\Pi_n}\left(\left\|\left(f^{(r)}-g^{(r)}\right)\varphi^r\right\|_p + t^k\left\|g^{(k+r)}\varphi^{r+k}\right\|_p\right).$$

Definition 1.5 [13] For $f \in \mathbb{B}_p^r$, define

$$\omega_\varphi^k(f,t)_{\varphi^r,p} = \sup_{0<h\le t}\left\|\varphi^r\Delta_{h\varphi}^k f\right\|_{L_p[-1+t^*,1-t_1^*]} + \sup_{0<h\le t^*}\left\|\varphi^r\overrightarrow{\Delta_h^k} f\right\|_{L_p[-1,-1+Bt^*]} + \sup_{0<h\le t^*}\left\|\varphi^r\overleftarrow{\Delta_h^k f}\right\|_{L_p[1-Bt^*,1]},$$

Definition1.6 [13] For $f \in L_p[-1,1], 0 < p < 1$, and let Π_n denote the space of algebraic polynomials of degree not exceeding n, and denote

$$E_n(f)_p = \inf_{P_n\in\Pi_n}\|f-P_n\|_p.$$

The degree of approximation of $f \in L_p[-1,1]$ by element of Π_n.

As usual, we will use the notation $c(v)$ to denote such absolute constants depending on v which are of no important to us and may be different even if they appear in the same line.

II. Notation and Auxiliary Results

In our article we will use the following notations [16]

$$x_i \coloneqq cos\frac{i\pi}{n}, 0 \le i \le n,$$
$$I_i \coloneqq [x_i, x_{i-1}], h_i = |I_i| = x_{i-1} - x_i,$$
$$1 \le i \le n,$$
$$\mathcal{J}_{i:} \coloneqq \begin{cases} I_i \cup I_{i-1}, & if\ 2 \le i \le n, \\ I_1, & if\ i = 1. \end{cases}$$
$$\Delta_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}.$$

It can easily verified that $h_{i\mp 1} < 3h_i$ and $\Delta_n(x) \le h_i \le 5\Delta_n(x)$ for $x \in I_i$. See [16]

To prove our main results, we need the following lemma**s.**

Lemma 2.1 [15]

If $g \in \mathbb{B}_p^{r+1}, r \in \mathbb{N}_0, 0 < p < 1$, and $k \ge 2$,

Then

$$\omega_{k,r}^{\varphi}(g^{(r)}, t)_p \le ct\omega_{k-1,r+1}^{\varphi}\left(g^{(r+1)}, t\right)_p.$$

Lemma 2.2 [15]

let $k \in \mathbb{N}, r \in \mathbb{N}_0, 0 < p < 1$ and $f \in \mathbb{B}_p^r$, then

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \le c(p,k)\left\|f^{(r)}\right\|_p.$$

Lemma 2.3

For a function $f \in \mathbb{B}_p^r, 0 < p < 1, r \in \mathbb{N}_0, and\ , k \in \mathbb{N}$. The following inequality holds.

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$$\sum_{j=1}^{n} \omega_{k,r}^{\varphi}(f^{(r)}, h_j, \mathcal{T}_j)_p^p \leq c(p)\, \omega_{k,r}^{*\varphi}(f^{(r)}, n^{-1})_p^p.$$

Proof

The idea of the proof depend on employment of the inequality

$$\omega_{k,r}^{\varphi}(f^{(r)}, t, [-1,1])_p^p \leq c(p)\frac{1}{t}\int_0^t \int_{-1}^{1} \left|W_{kh}^r(x)\Delta_{h\varphi(.)}^k\big(f^{(r)}, x, [-1,1]\big)\right|^p dx\, dh\,,$$

Which appeared in [6] and using the above inequality we have

$$\omega_{k,r}^{\varphi}(f^{(r)}, h_j, \mathcal{T}_j)_p^p \leq c(p)h_j^{-1}\int_0^{h_j}\int_{\mathcal{T}_j}\left|W_{kh}^r(x)\Delta_{h\varphi(.)}^k\big(f^{(r)}, x, \mathcal{T}_j\big)\right|^p dx\, dh$$

$$\leq c(p)\int_{\mathcal{T}_j}\int_0^{h_j/\varphi(x)} \frac{\varphi(x)}{h_j}\left|W_{kh}^r(x)\Delta_{h\varphi(.)}^k\big(f^{(r)}, x\big)\right|^p dh\, dx\,.$$

Since $\frac{h_j}{\varphi(x)} \sim n^{-1} for\ x \in \mathcal{T}_j,\ j = 3, \cdots, n-1$ we conclude for theses j

$$\omega_{k,r}^{\varphi}(f^{(r)}, h_j, \mathcal{T}_j)_p^p \leq c(p)n\int_{\mathcal{T}_j}\int_0^{cn^{-1}}\left|W_{kh}^r(x)\Delta_{h\varphi(.)}^k\big(f^{(r)}, x\big)\right|^p dh\, dx \quad (2.1).$$

Since$\Delta_{h\varphi(x)}^k\big(f^{(r)}, x\big) = 0\ if \frac{k}{2}\ h\varphi(x) > 1 \mp x$, so it equal zero for$x \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_n \mathrm{h} > 30\mathrm{n}^{-1}$.

Hence, (2.3)also hold for $\mathrm{j} = 1, 2$and n .

Finally, from inequality (2.1)

$$\sum_{j=1}^{n} \omega_{k,r}^{\varphi}(f^{(r)}, h_j, \mathcal{T}_j)_p^p \leq c(p)\, n \sum_{j=1}^{n}\int_{\mathcal{T}_j}\int_0^{cn^{-1}}\left|W_{kh}^r(x)\Delta_{h\varphi(x)}^k\big(f^{(r)}, x\big)\right|^p dh\, dx$$

$$\leq c(p)n\int_{-1}^{1}\int_0^{cn^{-1}}\left|W_{kh}^r(x)\Delta_{h\varphi(x)}^k\big(f^{(r)}, x\big)\right|^p dh\, dx$$

$$\leq c(p)n\int_0^{cn^{-1}}\left\|W_{kh}^r(x)\Delta_{h\varphi(x)}^k\big(f^{(r)}, x\big)\right\|_p^p dh$$

$$\leq c(p)\omega_{k,r}^{*\varphi}(f^{(r)}, n^{-1})_p^p.$$

Lemma 2.4

For $P_n \in \Pi_n, k = 1,2, \cdots and\ 0 < p < 1$, we have

$$\omega_{k,r}^{\varphi}(P_n, t)_p \leq c(nt)^k \|P_n\|_p,$$

Where $c = c(p, k)$

Proof. In view of (Lemma 2.2) applied for $f^{(r)} = P_n$ and using Markov's inequality

$$\omega_{k,r}^{\varphi}(P_n, t)_p \leq t^k\left\|P_n^{(k)}\right\|_p$$

$$\leq t^k n^k \|P_n\|_p.$$

III. Main Results

The following results is our main results.

Theorem 3.1

$$If\ f \in \mathbb{B}_p^r, 0 < p < 1, \text{then}$$

$$E_n(f)_p \leq \frac{c}{n^r}\,\omega_{k,r}^{\varphi}\big(f^{(r)}, n^{-1}\big)_{p,} n \geq k + r.$$

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Proof

It follows from ([7] Theorem1.1)

$$E_n(f)_p \le c(p)\omega^{\varphi}_{k+r}(f, n^{-1})_{p,} \quad n \ge k + r.$$

Since $f \in \mathbb{B}^r_p$, and by apply Lemma 2.1r times we get

$$E_n(f)_p \le c(p)\frac{1}{n^r}\omega^{\varphi}_k\left(f^{(r)}, n^{-1}\right)_{p,}$$
$$\le c(p)\frac{1}{n^r}\omega^{\varphi}_{k,r}\left(f^{(r)}, n^{-1}\right)_{p,}$$

Corollary 3.2

If $f \in \mathbb{B}^r_p, r \in \mathbb{N}_{0,} 0 < p < 1$, and if for some $k \in \mathbb{N}$, $\alpha > r$,

$$\omega^{\varphi}_{k,r}\left(f^{(r)}, n^{-1}\right)_p = O(t^{\alpha - r}), then$$
$$E_n(f)_p \le cn^{-\alpha}, \qquad n \ge k + r.$$

Proof

By the above theorem we have

$$E_n(f)_p \le \frac{c}{n^r}\omega^{\varphi}_{k,r}\left(f^{(r)}, \tfrac{1}{n}\right)_p$$
$$\le \tfrac{c}{n^r} . \tfrac{1}{n^{\alpha - r}} = cn^{-\alpha}.$$

Theorem 3.3

Let $f \in \mathbb{B}^r_p, r \in \mathbb{N}, 0 < p < 1$, and let P_n denotes the polynomial of best approximation of $f\ in\ L_p[-1,1]$, of degree less than *n*. If $\int_0^1 \left(\omega^{\varphi}_{k,r}(f^{(r)}, \tau)/\tau\right) d\tau < \infty$ for some $k \in \mathbb{N}$, then

$$\left\|\left(f^{(r)} - P^{(r)}_n\right)\varphi^r\right\|_p \le c\int_0^{1/n}\left(\omega^{\varphi}_{k,r}\left(f^{(r)}, \tau\right)_p/\tau\right)d\tau,$$

Proof

By using potapov's estimate see [8]

$$\left\|\varphi^u P^{(u)}_n\right\|_p \le c(p, u)n^u\|P_n\|_p,$$

We have

$$\left\|\left(f^{(r)} - P^{(r)}_n\right)\varphi^r\right\|_p \le \sum_{i=1}^{\infty}\left\|\left(P^{(r)}_{2^i n} - P^{(r)}_{2^{i-1}n}\right)\varphi^r\right\|_p$$
$$\le c(p)\sum_{i=1}^{\infty} 2^{ir}n^r\left\|P_{2^i n} - P_{2^{i-1}n}\right\|_p$$
$$\le c(p)\sum_{i=1}^{\infty} 2^{ir}n^r\left(\left\|P_{2^i n} - f\right\|_p + \left\|f - P_{2^{i-1}n}\right\|_p\right)$$
$$\le c(p)\sum_{i=1}^{\infty}\omega^{\varphi}_{k,r}\left(f^{(r)}, 1/(2^i n), I_i\right)_p,$$

Using lemma 2.3 we get

$$\left\|\left(f^{(r)} - P^{(r)}_n\right)\varphi^r\right\|_p \le c(p)\,\omega^{*\varphi}_{k,r}\left(f^{(r)}, 1/n\right)_p$$
$$\le c(p)\int_0^{1/n}\left(\omega^{\varphi}_{k,r}\left(f^{(r)}, \tau\right)_p/\tau\right)d\tau.$$

Corollary 3.4 *Let* $k \in \mathbb{N}, r \in \mathbb{N}_{0,}$ *and* $r < \alpha < r + k$, *and let* $f \in L_p[-1,1], o < p < 1. If$

$$E_n(f)_p \le n^{-\alpha}, \quad n \ge N, \ (3.1)$$

For some $N \ge k + r$, then $f \in \mathbb{B}^r_p$ and

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$$\omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p \le c(\alpha,k,r)t^{\alpha-r} + c(N,k,r)t^k E_{k+r}(f)_{p,} \ t>0.$$

In particular, if $N=k+r$, then (3.1)implies that $f\in \mathbb{B}_p^r$and

$$\omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p \le c(\alpha,k,r)t^{\alpha-r}, t>0.(3.2)$$

Proof

Let P_n be a best approximation to $f^{(r)}$in $L_p, r\ge 1$

$$\omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p \le \omega_{k,r}^{\varphi}\left(f^{(r)}-P_n+P_n,t\right)_p$$
$$\le c(p)\left\|f^{(r)}-P_n\right\|_p + \|P_n\|_p$$
$$\le M_1+M_2.$$

Since P_n best approximation to $f^{(r)}$.

For ℓ *given by* $\ell=\max\{i: 2^i<n\}, 2^\ell=n$ and P_{2^i} be a polynomial of best approximation to $f^{(r)}$, so we may write

$$f^{(r)}-P_n=\sum_{i\in N} P_{2^i n}-P_{2^{i-1}n}$$
$$M_1\le c(p)\sum_{i\in N}\left\|P_{2^i n}-P_{2^{i-1}n}\right\|_p$$
$$\le c(p)\sum_{i\in N} E_{2^i n}(f)_p$$
$$\le \sum_{i\in N}(2^i n)^{-\alpha} \quad r<\alpha<r+k$$
$$\le n^{-\alpha}n^{-\alpha}=n^{-2\alpha}<n^{-\alpha}.$$

To estimate M_2

Let $\bar{P}_n(P_n)$be a best approximation to P_n.

$$M_2=\|P_n\|_p=\|P_n-\bar{P}_n(P_n)+\bar{P}_n(P_n)\|_p$$
$$\le c(p)(\|P_n-\bar{P}_n(P_n)\|_p+\|\bar{P}_n(P_n)\|_p)$$
$$\le E_n(P_n)_p+\|\bar{P}_n(P_n)\|_p.$$
$$\|\bar{P}_n(P_n)\|_p=\left\|\sum_{i=1}^{\ell} P_{2^i}-P_{2^{i-1}}\right\|_p \quad ,$$

Where $i<\ell$, P_{2^i} is best approximation to f. So

$$\|\bar{P}_n(P_n)\|_p\le\sum_{i=1}^{\ell} E_{2^\ell}(f)_p^p$$
$$\le c(p)\, E_n(f)_p\le n^{-\alpha}.$$

Combine M_1 *and* M_2 we get (3.2).

Since $\left\|f^{(r)}\varphi^r\right\|_p\le\infty$. Then we get $f\in\mathbb{B}_p^r$,and the proof is complete.

From Corollaries (3.2) and (3.4), another constructive characterization result was given in the following corollary.

Corollary 3.5

Let $k\in\mathbb{N}, r\in\mathbb{N}_{0,} r<\alpha<r+k$, and let $g\in L_p[-1,1], o<p<1$. Then

$$E_n(f)_p\le cn^{-\alpha}, \text{for all } n\ge k+r, \text{if and only if } f\in\mathbb{B}_p^r \text{and } \omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p\le ct^{\alpha-r}, t>0.$$

Now we present a matching inverse theorem.

Theorem 3.6

Let $k\in\mathbb{N}, r\in\mathbb{N}_{0,}$, $,0<p<1$ *and* $f\in\mathbb{B}_p^r$, we have

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$$\omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p \le ct^k\left(\sum_{0\le n\le 1/t}(n+1)^{kp-1}E_n\left(f^{(r)}\right)_p^p\right)^{1/p} \quad (3.3)$$

Proof

Let $P_n^{(r)} \in \Pi_n$ be a polynomial of best approximation of $f^{(r)}$. For $t > 0$ define $\ell = \ell(t)$ by $2^{-\ell} \le t \le 2^{-(\ell+1)}$.Using Lemma (2.2), we have

$$\begin{aligned}\omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p^p &\le \omega_{k,r}^{\varphi}\left(f^{(r)},2^{-\ell}\right)_p^p\\ &\le \omega_{k,r}^{\varphi}\left(f^{(r)}-P_{2^\ell}^{(r)},2^{-\ell}\right)_p^p + \omega_{k,r}^{\varphi}\left(P_{2^\ell}^{(r)},2^{-\ell}\right)_p^p \quad (3.4)\\ &\le c(p)\left\|f^{(r)}-P_{2^\ell}^{(r)}\right\|_p^p + \omega_{k,r}^{\varphi}\left(P_{2^\ell}^{(r)},2^{-\ell}\right)_p^p\\ &\le c(p)E_{2^\ell}\left(f^{(r)}\right)_p + \omega_{k,r}^{\varphi}\left(P_{2^\ell}^{(r)},2^{-\ell}\right)_p^p.\end{aligned}$$

For $P_{2^{-\ell}}^{(r)} := P_0^{(r)}$ we can use Lemma (2.4) to obtain

$$\begin{aligned}\omega_{k,r}^{\varphi}\left(P_{2^\ell}^{(r)},2^{-\ell}\right)_p^p &\le \sum_{i=0}^{\ell}\omega_{k,r}^{\varphi}\left(P_{2^i}^{(r)}-P_{2^{i-1}}^{(r)},2^{-\ell}\right)_p^p\\ &\le c(p)\sum_{i=0}^{\ell}(2^{i-\ell})^{kp}\left\|P_{2^i}^{(r)}-P_{2^{i-1}}^{(r)}\right\|_p^p \quad (3.5)\\ &\le c(p)2^{-\ell kp}\sum_{i=-1}^{\ell-1}2^{ikp}\,E_{2^i}\left(f^{(r)}\right)_p^p\\ &\left(E_{2^{-1}}\left(f^{(r)}\right)_p := E_0\left(f^{(r)}\right)_p\right.\end{aligned}$$

since $\sum_{i=-1}^{\ell-1}2^{ikp}\,E_{2^i}\left(f^{(r)}\right)_p^p$is equivalentto the right- hand side of (3.3), inequalities (3.4) and (3.5) together with$2^{-\ell kp}\sim t^{kp}$the proof is complete.

As a direct consequence of Theorem 3.1 and Theorem 3.6, we have the following corollary.

Corollary 3.7

Let $k \in \mathbb{N}, r \in \mathbb{N}_{0,}$, $,0 < p < 1$ and $f \in \mathbb{B}_p^r$, we have

$$E_n(f)_p = O(n^{-\alpha}) \Leftrightarrow \omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p = O(t^\alpha)\text{for } 0 < \alpha < k.$$

IV. Conclusion

In field of approximation theory, it is important to study inverse and direct theorems. For function approximation, we can estimate the degree of best approximation by mean of its weighted moduli of smoothness. This implies a characterization for many classes of smoothness of functions via the degree of their weighted algebraic approximation.

References

[1] AkhiezerN.I. Lectures on Approximation Theory. (1965), 2nd ed.Nauka, Moscow, (Russian).

[2] Bernstein.S.N. On the best approximation of continuous functions by polynomials of given degree. *Collected Works, 1,Acad Nauk SSSR*, Moscow,(1912),11-104. (Russian)

[3] Bhaya E.S., Al-sadaa. Z.H. AStechkin-Marchaud Inequality in Terms of Neural Networks Approximation in Lp-Space for 0< p< 1. *In IOP Conference Series: Materials Science and Engineering* Vol. 571(1)(2020),1-5.

[4] Butzer, P.L, Nessel, R.J. Fourier analysis and Approximation. *Pure and Applied Mathematics Series* Volume 7, Birkhauser Verlag,(1971).

[5] DeVore,R. A.,Lorentz,G. G.Constructive Approximation. *Spinger. New york,(*1993).

[6] DeVore, R.A., Popov, V.A. Interpolation of Besov spaces. *Transactions of the American Mathematical*

DOI: 10.5373/JARDCS/V12I4/20201419
ISSN 1943-023X
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Society, 305(1), 1988: 397-414.

[7] Devore, R.A.,Leviatan, D.&Yu,X.M. Polynomial approximation in L_p (0<p<1). *Constructive Approximation*, 8(2),1992, 187-201.

[8] Ditzian,Z.,Totik,V. Moduli of smoothness. Springer Series in Computational Mathematics, Springer, New York(1987).

[9] Dzyadyk, V.K., Shevchuk, I.A. Theory of uniform approximation of functions by polynomials. *Walter de Gruyter*, (2008).

[10] Jackson,D.Uber die Genauigkeit der Annaherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung. Gottingen,(1911).

[11] Khalaf,A.M., Bhaya, E.SL_p,Approximation of Functions on Graphs.Journal of Adv Research in Dynamical&Control Systems,11,10-special issue (2019):767-768.

[12] Kopotun,K. A. Note on Simultaneous Approximation in Lp [-1, 1] (1≤p<∞). Analysis, 15, (1995):151-158.

[13] Kopotun,K. A.,Leviatan,D.&Shevchuk, I. A. New moduli of smoothness. *Publications de l'Institut Mathematique.* 96(110),(2014): 169–180.

[14] Kopotun, K. A., Leviatan,D.&Shevchuk,I. A. New Moduli of Smoothness*:* Weighted DT Moduli Revisited and Applied, Constructive Approximation,42(1),(2015):129-159.

[15] Sharba,Z.A., Bhaya,E.S.New Moduli of Smoothness for Weighted $L_p, 0 < p < 1$ Approximation. Accepted in Conference of the University of Babylon (ISCUB-2019), to be held December 22-4, 2020.

[16] Shevchuk,I. A.Approximation by polynomials and traces of functions continuous on a segment,Kiev: Naukova Dumka,(1992)(Russian).

[17] Stechkin, S.B. On the order of the best approximations of continuous functions (Russian). Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya, 15(3),(1951):219-242.

[18] Timan, M.F. Approximation and properties of periodic functions. Nauk. dumka, Kiev. *properties of periodic functions* (2009).

[19] Vall´ee-Poussin,C.-J. Le¸sons sur l'Approximation des Fonctions d'une Variable R´eelle, Gauthier-Villars (1919).

[20] Zygmund, A. On the continuity module of the sum of the series conjugate to a Fourier series.*Prace Mat. Fiz*, 33,(1924): 25–132.

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