Sharp Estimates for the Zalcman Conjecture and Second Order Hankel Determinant

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Abstract
In this work, we found sharp estimates for the Zalcman conjecture and second order Hankel determinant for the inverse function when it belongs to the class of starlike functions with respect to symmetric points, denoted by S_S^*. These results are new.

Keyword: Univalent function, regular function; Mobius transformation; close-to-convex; bi-univalent; Hankel Determinant; Zalcman conjecture.

1. Introduction

Hankel matrices are use when we have a sequence of resulting data, and we want to achieve a Markov model. Analyzing the high and low values in the Hankel matrix can help estimate the parameters related to the model. In conclusion, Hankel matrices represent a powerful tool in linear algebra and numerical analysis, and are use in a variety of mathematical and engineering applications.

Let the space of all single-valued analytic maps in $\mathbb{D} = \{z \in C : |z| < 1\}$, stand for H, the subset $\mathcal{A}$ of H be the collection of all functions with Maclaurin's series, namely

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

With $a_1 = 1$, which have been normalized by $f(0)' - 1 = f(0) = 0$, where the coefficients $a_n's$ are complex constants. Geometrically, the consraints $f(0) = 0$ and $f(0)' = 1$ are related in shifting and amounts in spinning with expansion or contraction of the image domain respectively.

The subcolletion $\mathbb{S}$ of $\mathcal{A}$ are univalent it projects $\mathbb{D}$ onto the complete z- plane, minus a crack on the −ve real axis from -0.25 to ∞, because it could be seen by expressing in Koebe function is the significant one namely

$$K(z) = 0.25\left\{\left[\frac{1}{(1-z^2)}\right]^2 - 1\right\} \tag{2}$$

The Area theorem has a significant impact on the approach of schlicht functions, proved first by **[1].** If $t(z) = \frac{1}{z} + \sum_{n\geq 1} d_n z^n$ is univalent function in D taking a pole of first order at the point z = 0 then

$$\sum_{n\geq 1} |d_n|^2 \leq 1 \tag{3}$$

[2].evidence to prove that, $|a_t| \leq 2$ for $f \in \mathbb{S}$ and $|a_t| \leq t$ for $t \geq 2$

Let $\mathfrak{I}$ denotes the collection of holomorphic functions in $\mathbb{D}$ expressed as

$$g(z) = 1 + \sum_{t>1} C_t z^t, \tag{4}$$

With properties $Reg(z) > 0, g(0) = 1$ and $|c_t| \leq 2$ for each $t \geq 1$ is sharp for the Mobius transformation, namely

$$M_0(z) = (1+z)(1-z)^{-1} = 1 + 2z + 2z^2 + 2z^3 + \cdots \tag{5}$$

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Definition 1. A regular function f named bounded turning which derivative has real part, if

$$Ref(z)' > 0, \quad z \in \mathbb{D}, \tag{6}$$

The class of bounded turning functions is represents by $\mathfrak{R}$ **[3, 4].** Conducted a systematic study about the traits of functions in this class and estimated that $|a_n| \leq 2$, $n \geq 2$, the estimate is sharp for the function $-z + log\left(\frac{1+z}{1-z}\right)$ see **[5,6].** Also studied properties of functions in this class.

Definition 2. A holomorphic function $f \in \mathfrak{R}(\alpha), \alpha \in [0,1)$ (according to **[7]**.), if

$$Re\{f(z)'\} > \alpha, \quad z \in \mathbb{D} \tag{7}$$

If $\alpha = 0$, then $\mathfrak{R}(0) = \mathfrak{R}$

Definition 3. A regular function f is named starlike (S^*) with regard to (0, 0). Analytically i.e.

$$Re\left\{\frac{zf(z)'}{f(z)}\right\} > 0, \quad z \in \mathbb{D} \tag{8}$$

Definition 4. A holomorphic function f is termed convex (k), when f in $\mathbb{D}$ onto a convex domain. Analytically

$$Re\left\{\frac{f(z)' + zf(z)''}{f(z)'}\right\} > 0, \quad z \in \mathbb{D} \tag{9}$$

If $f(z) \in K\ then\ |a_t| \leq 1$, for every $t \geq 2$, is sharp for $f(z) = \frac{z}{1-z}$. Further, each function $f \in K$ is starlike and the containment is proper, because the function $K(z) \in S^*$, but not in K.

[3].from Definitions 3. And 4. there is a very close analytic relationship with functions in the families K and S^*such that $f \in K \Leftrightarrow zf' \in S^*$

Definition 5.[8]. A regular function $f \in S^*(\beta)$, with$\beta \in [0,1)$, if

$$Re\left\{\frac{zf(z)'}{f(z)}\right\} - \beta, \quad z \in \mathbb{D} \tag{10}$$

If $\beta = 0$, then $S^*(0) = S^*. where\ S^*(\alpha) = S^*$ when $\alpha \in [0,1)$ and $S^*(\alpha) \subseteq S^*(\beta)$ if $\alpha - \beta \geq 0$

Definition 6. A holomorphic function $f \in K(\beta), \beta \in [0,1)$, if

$$Re\left\{\frac{f(z)' + zf(z)''}{f(z)'}\right\} > \beta, \quad z \in \mathbb{D} \tag{11}$$

[3]. observed that $f \in K(\beta) \Leftrightarrow zf' \in S^*(\beta)$ such that $K \subset S^*\left(\frac{1}{2}\right) \subset S^*$

Definition 7.[9]. A holomorphic transformation f, called close-to-convex, whose collection is denoted by CC, if $\exists$ a function $g(z) \in S^*$ satisfying

$$Re\left\{\frac{zf(z)'}{g(z)}\right\} > 0, \quad z \in \mathbb{D} \tag{12}$$

Where $K \subset S^* \subset CC \subset \mathbb{S}$ if $g(z) = z$ then $CC = \mathfrak{R}$

Definition 8.[10]. Proved the members of $f \in S_s^*$ in CC is univalent if

$$Re\left\{\frac{2zf(z)'}{f(z) - f(-z)}\right\} > 0, \quad z \in \mathbb{D} \tag{13}$$

Definition 9. A holomorphic transformation f is called as convex function with regard to symmetric points, if

$$Re\left(\frac{2zf(z)'}{f(z) - f(-z)}\right)' > 0, \quad z \in \mathbb{D} \tag{14}$$

The collection of all these mappings is represented by K_s, introduced by Das and Singh **[11].**

Remark. Any Hypergeometric function ${}_2F_1(t_1, t_2, t_3; z)$, defined in the unit disc in a series of powers in z ${}_2F_1(t_1, t_2, t_3; z) = \sum_{n=0}^{\infty} \frac{(t_1)_n (t_2)_n z^n}{(t_3)_n} \cdot \frac{z^n}{n!}$ It is not defined if t_3 is a negative integer. Here $(t)_n$ is called (rising) Pochhammer symbol, described as:

$$(t)_n = \begin{cases} 1 & n > 0 \\ t(t+1) \ldots \ldots (t+n-1) & n > 0 \end{cases}$$

An example on Hypergeometric function:

For $p, m, n \in \mathbb{N}$ then

$${}_2F_1\left[1, \frac{p}{m+n}; 1 + \frac{p}{m+n}; z^{m+n}\right] = 1 + \frac{Pz^{m+n}}{m+n+P} + \frac{Pz^{2(m+n)}}{2(m+n)+P} + \cdots + \frac{Pz^{3(m+n)}}{3(m+n)+P}$$

And a function F subordinate to G, both analytic in $\mathbb{D}$, symbolized as $F \prec G$, when there occurs a holomorphic function w(z) in $\mathbb{D}$ fulfilling $w(0) - 1 = 0\ and\ |w(z)| < 1, z \in \mathbb{D}$ named Schwarz's map in this way

Definition 10. A regular function f is bi-univalent, if the couple's f and its analytic extension namely $f^{-1}$are univalent in $\mathbb{D}$, whose class is represented by Ω.

According to Koebe s $\left(\frac{1}{4}\right)^{th}$ - theorem, every holomorphic and univalent function $\omega\ in\ \mathbb{D}$ possesses an inverse

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denoted by ω^{-1}satisfying

$z = \{\omega^{-1}(\omega(z))\}, z \in \mathbb{D}$ and $\omega\{\omega^{-1}(\omega)\} = \omega, \left(|\omega| < p_o(f); p_o(f) \geq \frac{1}{4}\right)$. Where

$$\omega = \left\{\omega + \sum_{n\geq 2} q_n \omega^n\right\} + \sum_{n\geq 2} a_n \left\{\omega + \sum_{n\geq 2} q_n \omega^n\right\}^n \tag{15}$$

Definition 11 let $k \in \mathbb{N}$. A domain ψ is called K-copies of symmetric, if ψ revolves with respect to the origin making an angle $\frac{2\pi}{k}$ takes ψ onto itself in $\mathbb{D}$ if

$$f\left(e^{\frac{2\pi}{k}i} z\right) = e^{\frac{2\pi}{k}i} f(z), z \in \mathbb{D} \tag{16}$$

[12]. [13] and **[14].** Introduce $f \in A_p$ is holomorphic function is a member of $\mathfrak{R}_p$ if

$$Re\left\{\frac{f(z)'}{Pz^{p-1}}\right\} > 0, z \in \mathbb{D} \tag{17}$$

For $p = 1$ in (17), we get $\mathfrak{R}_1(0) = \mathfrak{R}$.

Definition 12. A function f given in (1.7.1) to be in $\mathfrak{R}_p(\alpha)$ with $\alpha \in [0,1)$ if

$$Re\left\{\frac{f(z)'}{Pz^{p-1}}\right\} - \alpha, z \in \mathbb{D} \tag{18}$$

[15, 16].defined the Hankel det of order q for the regular function f specified in (1) as

$$H_{q,t}(f) = \begin{vmatrix} a_t & a_{t+1} \cdots & a_{t+q-1} \\ a_{t+1} & a_{t+2} \cdots & a_{t+q} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{t+q-1} & a_{t+q} & a_{t+2q-2} \end{vmatrix}$$

Here $a_1 = 1$ and q, t are integers, positive in nature. The determinant $H_{q,t}(f)$ has been investigated by many Mathematicians. **[17, 18].** Applied by them in the evaluation of singularities of meromorphic mappings. **[15].** settled the Hankel det for schlit mappings by the inequality $|H_{q,t}(f)| < k_t^{-\left(\frac{1+2\beta}{2}\right)q+\frac{3}{2}}$, with $\beta > 0.00025$ and k dependent on q

[19]. has established a well-built outcome to a really mean univalent functions as $|H_{2,n}(f)| < B.n^{\frac{1}{2}}, where\ B$ is an absolute constant. Given a subfamily of F of $\mathcal{A}$, for specific values of q and t, estimating an upper boundof $H_{q,t}(f)$ is a problem of interest to study, when $f \in F$. A familiar result is that $(a_3 - a_2^2) = H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$, also called Fekete-Szego functional for the functions in $\mathbb{S}$. See **[20, 21].** Supposed that

$$f \in S \Longrightarrow |a_j^2 - a_{2j-1}| \leq (j-1)^2, for\ j = 2.3 \ldots \tag{19}$$

[22]. tested the Zalcman supposition for the functions in the class CC. For $f \in S$

[23]. Put forward a universal Zalcman hypothesis that

$$|a_k a_l - a_{k+l-1}| \leq (k-1)(l-1), \text{ for } k, l = 2,3, \ldots \tag{20}$$

still it is an open problem. Further, he derived (20) for the mappings in the families $\mathbb{S}^*$and $\mathbb{S}_{\mathbb{R}}$

Some Lemma and Result

Lemma 13. [24]. If $g \in G$, then $2 \geq |c_t|$, for every $t \geq 1$, sharp for the mobious transformation, given in (5).

Lemma 14. Let $g \in G$, given in (4) with $c_1 \geq 0$ then

$$c_2 = \frac{1}{2}[c_1^2 + x(4 - c_1^2)]$$

And $c_3 = \frac{1}{4}[c_1^3 + \{2c_1 x - c_1 x^2 + 2(1-|x|^2)y\}(4 - c_1^2)]$, for x and y whose absolute value is at most unity, i.e. $|x| \leq 1$ and $|y| \leq 1$.

Lemma 15.[25]. $g \in J \Longrightarrow$ The bound $2 \geq |c_n - \mu c_k c_{n-k}|$ is sharp, for $n, k \in \mathbb{N}$, where $n > k$ and $0 \leq \mu \leq 1$.

Theorem 16. If $f \in S_s^*$ then $|H_{2,1}(f^{-1})| \leq 1$, the estimate is sharp for

$$f_o = z + z^3 + z^5 + z^7 + \cdots$$

Proof. for $f \in S_s^*$ as per definition 8, a function $g(z)$, is given (4) occurs in such wise

$$\frac{2zf(z)'}{f(z) - f(-z)} = g(z) \tag{21}$$

Employing equivalent representation for $f(-z), f(z) and\ g(z)$ in (21), then

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$$2z\sum_{n\geq 1} na_n z^{n-1} = \left(\sum_{n\geq 1} a_n z^n - \sum_{n\geq 1}(-1)^n a_n z^n\right)\left(1+\sum_{n\geq 1} c_n z^n\right) \tag{22}$$

An easy calculation gives

$$2z(1+2a_2z+3a_3z^2+4a_4z^3+5a_5z^4 \dots) \tag{23}$$

Further simplification gives

$$(4a_2-2c_1)z^2+(4a_3-2c_2)z^3+(-2a_3c_1+8a_4-2c_3)z^4+\cdots=0 \tag{24}$$

After simplifying, the coefficients of $z^i = 2,3,4$, are obtained as

$$a_2=\frac{c_1}{2}; a_3=\frac{c_2}{2}; a_4=\frac{(2c_3+c_2)}{8} \dots \tag{25}$$

Simplifying the expressions (25) in sight of (15), we obtain

$$q_2=\frac{-c_1}{2}; q_3=\frac{-c_2+c_1^2}{2} \, and \; a_4=\frac{-5c_1^3+9c_1c_2-2c_3}{8} \dots \tag{26}$$

Now, based on $H_{2,1}(f)$, we can have

$$H_{2,1}(f^{-1})=\begin{vmatrix} 1 & q_2 \\ q_2 & q_3 \end{vmatrix} = q_3-q_2^2 \tag{27}$$

Placing $q_j, (j=2,3)$ values from (26) in (27), we have

$$H_{2,1}(f^{-1})=\left(\frac{-c_2+c_1^2}{2}\right)-\left(\frac{c_1}{2}\right)^2 \tag{28}$$

After simplifying, we get

$$H_{2,1}(f^{-1})=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right). \tag{29}$$

Taking modulus and employing Lemma 15, we get

$$|H_{2,1}(f^{-1})| \leq \frac{1}{2}\left|c_2-\frac{c_1^2}{2}\right| = 1 \tag{30}$$

From f_o we obtain $q_2 = 0$ and $q_3 = -1$ which follows the result.

Theorem 17. if $f \in S_s^* \Longrightarrow |H_{2,2}(f^{-1})| \leq 1,$
the equality holds for $f_o$specified under Theorem 16

Proof. for $f \in S_s^*$, in view of $H_{2,2}(f)$, we have
Placing $q_j, (j=2,3,4)$ values from (26) in (31), we have

$$H_{2,2}(f^{-1})=\left(-\frac{c_1}{2}\right)\left(\frac{-5c_1^3+9c_1c_2-2c_3}{8}\right)-\left(\frac{-c_2+c_1^2}{2}\right)^2$$

$$H_{2,2}(f^{-1})=\frac{1}{16}(c_1^4-c_1^2c_2-4c_2^2+2c_1c_1^4) \tag{32}$$

Which is equivalent to

$$q_2q_4-q_3^2=\frac{1}{16}[d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4] \tag{33}$$

Here $d_1 = 2; d_2 = -1; d_3 = -4; d_4 = 1$
Employing c2 and c3 values from Lemma 14 on right side of 33, appears as

$$= 4\left[\frac{1}{4}d_1c_1[c_1^3+[2c_1x-c_1x^2+2(1-|x|^2)y](4-c_1^2)]+\frac{1}{2}d_2c_1^2(c_1^2+x(4-c_1^2)) + \frac{1}{4}d_3(c_1^2+x(4-c_1^2))^2+d_4c_1^4\right] \tag{34}$$

$$4[d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4]$$

Taking magnitude on either side, then employing the triangle inequity in (34),
We have

$$4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| \leq [|d_1+2(d_2+2d_4)+d_3|c_1|^4+2|d_1+d_2+d_3||c_1|^2|4 - c_1^2||x|+\{(|d_1|-|d_3|c_1^2-2|d_1||c_1||y|+4|d_3|\}|4-c_1^2||x|^2+2|d_1||c_1||(4-c_1^2||y|] \tag{35}$$

From (33), we can now write

$$|d_1+2d_2+d_3+4d_4|=0; |d_1+d_2+d_3|=3; \{(|d_1|-|d_3|)c_1^2-2|d_1||c_1||y|+4|d_3|\} = (-2c_1^2-4c_1|y|+16). \tag{36}$$

Placing the values from (36), d1 from (32) in (34), it takes the form

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$$4|d_1c_1c_1 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq [(0)c_1^4 + 4c_1(4-c_1^2)|y| + 6c_1^2(4-c_1^2)|x| + (-2c_1^2)|x| + (-2c_1^2 4c_1|y| + 16)(4-c_1^2)|x|^2] \quad (37)$$

Employing triangle inequity, displacing $|x|$ with ρ, besides $1 \geq |y|$, designate $c_1 = c \in [0,2]$, the right side of (37) takes the form

2. Conclusions

In this paper, we studied and presented properties of Zalcman conjecture where, we obtained some theorems and properties associated with a class defined by a second order Hankel determinant.

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