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Why moduli of p–smoothness?

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Abstract

For the increasing importance of discovering new types of moduli of smoothness, more suitable measurements are provided by the moduli of p–smoothness. Measuring fractional smoothness of functions by p–variation is used for many purposes in approximation theory. In this paper, we express $\omega_{k-1/p}(f;\delta)$ in terms of $\omega_{k-1/q}(f;\delta)$ for any $1<p,q<\infty$ to get fractional modulus of smoothness of functions with bounded kth $p-$ varation. Also, embedding of the space $V_{p,\alpha}^{(k)}$ is proved with necessity and sufficient conditions.

Subject Classification: *41A65 , 46S60.*

Keywords: *Fractional smoothness, Bounded* $(p,\alpha)kth-$ *varation, Periodic functions.*

1. Introduction

For several decades, function approximation is well studied by using modulus of smoothness, moduli of p-smoothness, with different versions and porpuses, see for examples [1], [4], [6], [8] and [12]. In addition to the p-continuity, moduli of p-smoothness measures fractional

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smoothness of functions through p-variation. Let g be a real 1- periodic function, $\Pi=[x_0,x_1\dots,x_n]$ is a partition satisfying $x_0<x_1<\cdots<x_n$ where $x_n=x_0+1$, and $p>1$, $p':=p/(p-1)$, and $0\le\alpha\le 1/p'$. Set $\|\Pi\|=\max(x_{j+1}-x_j)$. For any Π, in [9] the authours defined the $v_{p,\alpha}$-variation of a function $f:[a,b]\to\mathbb{R}$, as follow

$$v_{p,\alpha}(g;\,\Pi)=\left(\sum_{k=0}^{n-1}\frac{\left|g(x_{k+1})-g(x_k)\right|^p}{(x_{k+1}-x_k)^{\alpha p}}\right)^{1/p} \tag{1.1}$$

The function $g$is said to be bounded p–varation iff

$$v_{p,\alpha}(g)=\sup_{\Pi} v_{p,\alpha}(g;\Pi)<\infty, \tag{1.2}$$

Define $V_{p,\alpha}$ to be the set of functions of type g in (1.2). When $\alpha=0$, then $V_{p,0}=V_p$. In order to define the bounded second variation, Poussin modified the partition Π, in [5], to be as follow

$$x_0<y_1\le z_1<x_1<y_2\le z_2<x_2<\cdots<y_n\le z_n<x_n=x_0+1. \tag{1.3}$$

With extra modifications by the Riesz in [14], Merentes in [10], the second variation functions f that satisfies the finiteness of the following

$$v^{(2)}_{p,1/p'}(g)^p=\sup_{\Pi}\sum_{k=0}^{n-1}\left|\frac{g(x_{i+1})-g(z_{i+1})}{x_{i+1}-z_{i+1}}-\frac{g(y_{i+1})-g(x_i)}{y_{i+1}-x_i}\right|^p\frac{1}{(x_{i+1}-x_i)^{p/p'}} \tag{1.4}$$

More extensions are made by [9]. Fist, they used the following partition from [3] and [11] for $k\in\mathbb{N}$.

$$\begin{aligned}x_0&=t_{1,1}<t_{1,2}<\cdots<t_{1,i}\le t_{1,i+1}<\cdots<t_{1,2i}\le\cdots\le t_{2,i+1}<\cdots\le\cdots<t_{3,1}<\cdots<t_{j,1}\\&<\cdots<t_{j,i}\le t_{j,i+1}<\cdots<t_{j,2i}\le\cdots\le t_{m,1}<\cdots<t_{m,i}\le t_{m,i+1}<\cdots<t_{m,2i}\\&=x_0+1,\end{aligned} \tag{1.6}$$

The number $v^{(k)}_{p,\alpha}(g)$ is the $(p,\alpha)kth\ p-$variation of g, and is given by

$$v^{(k)}_{p,\alpha}(g;\Pi):=\left(\sum_{j=1}^{m}\left|g[t_{j,k+1},\cdots,t_{j,2k}]-g[t_{j,1},\cdots,t_{j,k}]\right|^p\right)^{\frac{1}{p}}\cdot\frac{1}{(t_{j,2k}-t_{j,1})^{\alpha p}}, \tag{1.6}$$

where

$$g[t_0,t_1,\cdots,t_k]:=\sum_{j=0}^{k}\frac{g(t_j)}{(t_j-t_0)\cdots(t_j-t_{j-1})(t_j-t_{j+1})\cdots(t_j-t_k)}, \tag{1.7}$$

is the *kth* divided difference. Set the space $V_{p,\alpha}^{(k)}$ $(K \in \mathbb{N})$ to be the space of 1- periodic functions g s.t.

$$\nu_{p,\alpha}^{(k)}(g) = \sup_{\Pi} \nu_{p,\alpha}^{(k)}(g,\Pi) < \infty, \tag{1.8}$$

Note that $V_p^{(k)}(g)$ is mentioned to the family of $kth\ p-$variation of g where $\alpha = 0$. Later, Terehin studied the properties of the modulus of continuity of fractional order 1–1⁄*p* [15]. With an important property

$$\omega_{1-1/p}(g;nh) \le n^{1/p'} \omega_{1-1/p}(g;h). \tag{1.9}$$

A generalization of Terehin definition is essential, to get what is called modulus of fractional p-smoothness of order $k-1/p$.

$$\omega_{k-1/p}(g;\,\delta) := \sup_{0\le\delta\le1} \omega_{1-1/p}(\Delta_h^{k-1} g;h) \tag{1.10}$$

where

$$\Delta_h g(x) = g(x+h) - g(x). \qquad \Delta_h^k g(x) = \Delta_h \Delta_h^{k-1} g(x).$$

In [7], Kolyada defined the family of continuous functions Ω_α so to prove that $\Omega_{1/p'}$ is the family of majorants of moduli of p-continuity.

Here, we define the family of *k*-smooth functions Γ_α^k by requiring the following conditions for every $\sigma \in \Gamma_\alpha^k$, we have

i. $\sigma(0) = 0$,

ii. $\sigma(t)$ is nondecreasing, and

iii. $\sigma(t)t^{-\alpha/k}$ is nonincreasing

Unlike Kolyada, condition iii, comes to fit the case of study, (p,α) *k*th p-variation. It is clear that $\Gamma_1^1 = \Omega_1$ is almost like the family of moduli of continuity. $\Gamma_\alpha^1 = \Omega_\alpha$ is the family of Kolyada, whose family is simply $\Omega_{1/p'}$. For $1 < q < \infty$, and for any $\sigma \in \Gamma_{1/p'}$, construct the sequence similar to [7]

$$\sigma_n = \sigma(2^{-n}),\ \bar{\sigma}_n = 2^{-1/kq'} \sigma_n, \tag{1.11}$$

By conditions (ii and iii) above, we get that

$$\begin{cases} \sigma_{n+1} \le \sigma_n \le 2^{-1/kq'} \sigma_{n+1}, \text{and} \\ \quad \bar{\sigma}_{n+1} \le \bar{\sigma}_n \le 2^{-1/kq'} \bar{\sigma}_{n+1} \end{cases} \tag{1.12}$$

But if

$$\lim_{t\to 0+} 2^{-1/kq'}\sigma(t)=\infty, \tag{1.13}$$

then by the constructed sequence of integers $\eta_k=\eta_k(\sigma)$ from [17], [2] and [13] we assume $\eta_1=0,$ and

$$\eta_{k+1}=\min\left\{n\in\aleph:\max\left(\frac{\sigma_n}{\sigma_{n_k}},\frac{\bar{\sigma}_n}{\bar{\sigma}_{n_k}}\right)\le\frac{1}{4}\right\}, \tag{1.14}$$

so that

$$4\sigma_{n_{k+1}}\le\sigma_{n_k}\quad, 4\bar{\sigma}_{n_{k+1}}\le\bar{\sigma}_{n_k}, \tag{1.15}$$

Then

$$4\sigma_{n_{k+1}}-1>\sigma_{n_k}\text{, or}\quad 4\sigma_{n_k}>\bar{\sigma}_{n_{k+1}}-1$$

By (1.11), we get one of the inequalities

$$\sigma_{n_k}<8\sigma_{n_{k+1}},\ \text{or}\ \bar{\sigma}_{n_{k+1}}<8\bar{\sigma}_{n_k}, \tag{1.16}$$

Now, for any $\sigma\in\Gamma^k_{1/p'}$ define the set

$$V^{\sigma}_{q,\alpha}=\{g\in V_{q,\alpha}\ :\omega_{k-1/q'}(g;\delta)=\mathrm{O}(\sigma(\delta))\}$$

In section three, we prove the sufficient and efficient conditions for the embedding $V^{\gamma}_{q,\alpha}\subset V_{q,\alpha}$.

Preliminary Results

If $1<p<\infty,$ then the equality

$$\lim_{\delta\to 0^+}\omega_{1-1/p}(g;\delta)=0, \tag{2.1}$$

holds for non-constant functions g that satisfies (2.1). It is said to be p- continuous, and C_p is the family of all p- continuous functions.

In the following propositions, we introduce some basic properties of the fractional modulus of smoothness of order k–1/p.

Proposition 2.1 : Let f be a real 1- periodic function, let $p'=p/(p-1).$ We have $\omega_{k-1/p}(f;n\delta)\le cn^{k-1/p'}\omega_{k-1/p}(f;\delta),$

where $0\le\delta\le\frac{1}{n}.$

Proof :

$$\omega_{k-1/p}(f;n\delta)=\sup_{0<h\le n\delta}\omega_{1-1/p}\left(\Delta_h^{k-1}f,h\right)=\sup_{0<h\le\delta}\omega_{1-1/p}\left(\Delta_{nh}^{k-1}f,\,h\right),$$

By using property (1.9), we get

$$\omega_{k-1/p}(f;n\delta)\le c\sup_{0<h\le\delta}n^{1/p'}\omega_{1-1/p}(\Delta_{nh}^{k-1}f,h)\le c\sup_{0<h\le\delta}n^{1+1/p'}\omega_{1-1/p}(\Delta_h^{k-1}f,h)$$
$$\le c\sup_{0<h\le\delta}n^{k+1/p'}\omega_{1-1/p}(\Delta_h^{k-1}f,h)\le cn^{k+1/p'}\omega_{k-1/p}(f;n\delta).$$

□

Proposition 2.2 : Let f be a real 1- periodic function, and $p'=p/(p-1)$, we have

$$\mu^{k-\frac{1}{p'}}\omega_{k-1/p}(f,\mu)\le 2^{k-1/p'}\delta^{k-1/p'}\omega_{k-1/p}(f,\delta),$$

where $0<\delta<\mu\le 1$, and $p'=p/(p-1)$.

Proof : By using proposition 2.1 we get

$$\mu^{k-1/p'}\omega_{k-1/p}(f;\mu)$$
$$\le\mu^{k-1/p'}\delta^{k-1/p'}\omega_{k-1/p}\left(f;\frac{1}{\delta^{k-1/p'}}\mu\right)$$
$$\le\delta^{k-1/p'}\mu^{k-\frac{1}{p'}+1}\omega_{k-1/p}\left(f;\frac{1}{\delta^{k-1/p'}}\right)\le 2^{k-1/p'}\delta^{k-1/p'}\omega_{k-1/p}\left(f;\frac{1}{\delta^{k-1/p'}}\right)$$
$$\le 2^{k-1/p'}\delta^{k-\frac{1}{p'}}\omega_{k-1/p}(f;\delta).$$

□

Note 2.3 : For any $f\in C_p^{(k)}$, and let

$$\sigma^*(t)=t^{k-\frac{1}{p'}}\inf_{0<u<t}\frac{\omega_{k-1/p'}(f,u)}{u^{k-1/p'}}$$

Clearly, $\sigma^*\in\Gamma_{1/p'}^k$. Also, by letting $\mu=t,$ and $u=\delta,$ in Proposition 2.2., we get

$$\sigma^*(t)\le\omega_{k-1/p}(f;t)\le 2^{k-1/p'}\delta^{k-\frac{1}{p'}}\sigma^*(t)$$

For the converse, let $\sigma \in \Gamma^k_{1/p'}$ the construction of *f* of Terehin [16], so by (1.10), it is clear that

$$\sigma(t) \le \omega_{k-1/p}(f;t) \le C\sigma(t).$$

We conclude from Note 2.3. that $\Gamma^k_{1/p'}$ is a family of majorants of moduli of p-smoothness.

Proposition 2.4 : For $\sigma \in \Gamma_{1/q'}$ diverges as (1.12), then $\sum_{j=1}^{\infty} 2^{j\vartheta q/k}\sigma_j^q$, converges iff $\sum_{m=1}^{\infty} 2^{\eta_m \vartheta q/k}\sigma_{\eta_m}^q$ converges.

Proof : By means of cases of (1.16), we have

$$\sum_{j=\eta_m}^{\eta_{m+1}-1} 2^{j\vartheta q/k}\sigma_j^q \le 8^q \sigma_{\eta_{m+1}}^q \sum_{j=\eta_m}^{\eta_{m+1}-1} 2^{j\vartheta q/k} \le C2^{\eta_{m+1}\vartheta q/k}\sigma_{\eta_{m+1}}^q$$

or

$$\sum_{j=\eta_m}^{\eta_{m+1}-1} 2^{-j\vartheta q/k}\sigma_j^q = \sum_{j=\eta_m}^{\eta_{m+1}-1} 2^{-jq/(p'-q)k} 8^q \bar{\sigma}_j^q \le 8^q \bar{\sigma}_{\eta_m}^q \sum_{j=\eta_m}^{\infty} 2^{-jq/p'k}$$
$$= C2^{-\eta_m q/p'k}\bar{\sigma}_{\eta_k}^q = C2^{\eta_m \theta q}\sigma_{\eta_m}^q$$

which ends the proof. □

3. The Main Results

Theorem 3.1 : *Let* $1 < p < q < \infty$ *and* $\theta = 1/p - 1/q$. *Let* $g \in V_{q,\alpha}^{(k)}$. *Assume that*

$$\int_0^1 (t^{-\vartheta}\omega_{k-1/q}(g,t))^q \frac{dt}{t} < \infty, \tag{3.1}$$

then $g \in V_{p,\alpha}^{(k)}$ *and*

$$\omega_{k-1/p}(g,\delta) \le 4\left(\int_0^\delta (t^{-\vartheta}\omega_{k-1/q}(g,t))^q \frac{dt}{t}\right)^{\frac{1}{q}}, \tag{3.2}$$

for all $\delta \in [0,1]$.

Proof : Let Π be a partition of the form (1.5), then

$$\nu_{p,\alpha}^{(k)}(g;\Pi)=\left(\frac{\sum_{j=1}^{m}\left|g[t_{j,k+1},\cdots,t_{j,2k}]-g[t_{j,1},\cdots,t_{j,k}]\right|^{p}}{(t_{j,2k}-t_{j,1})^{\alpha\theta p}}\right)^{1/p},$$

$$\le c(q)\left(\frac{\sum_{j=1}^{m}\left|g[t_{j,k+1},\cdots,t_{j,2k}]-g[t_{j,1},\cdots,t_{j,k}]\right|^{q}}{(t_{j,2k}-t_{j,1})^{\alpha\theta q}}\right)^{\frac{1}{q}} \tag{3.3}$$

Now, for a partition Π, define

$$\mathrm{T}_l(\Pi)=\{j:2^{-l-1}<t_{j,2k}-t_{j,1}\le 2^{-l}\},\quad (l=0,1,\cdots).$$

Set also $S_l^{(k)}(\Pi)=(\Sigma_{l\in T_l(\Pi)}\,|\,g[t_{l,k+1},\cdots,t_{l,2k}]-g[t_{l,1},\cdots,t_{l,k}]\,|^{q})^{\frac{1}{q}}$, if $\mathrm{T}_l(\Pi)=\varnothing$ and $S_l^{(k)}=0$ otherwise. By (3.3) we have that

$$\nu_{p,\alpha}^{(k)}(g;\Pi)\le\left(\sum_{l=0}^{\infty}\sum_{j\in T_l(\Pi)}\frac{\left|g[t_{l,k+1},\cdots,t_{l,2k}]-g[t_{l,1},\cdots,t_{l,k}]\right|^{q}}{(t_{j,2k}-t_{j,1})^{\alpha\theta q}}\right)^{\frac{1}{q}}$$

$$\le\left(\sum_{l=0}^{\infty}2^{(l+1)^{\alpha\theta q}}S_l^{(k)}(\Pi)^{q}\right)^{\frac{1}{q}} \tag{3.4}$$

It's clear that

$$S_l^{(k)}(\Pi)\le\omega_{k-1/q}(g,2^{-l}) \tag{3.5}$$

For partition Π, by using (3.4), (3.5) and proposition (2.1), we get

$$\nu_{p,\alpha}^{(k)}(g)\le\left(\sum_{l=0}^{\infty}2^{(l+1)^{\alpha\theta q}}\omega_{k-1/q}(g,2^{-l})^{q}\right)^{1/q}\le 4\left(\int_0^1(t^{-\theta}\omega_{k-1/q}(g,t))^{q}\frac{dt}{t}\right)^{1/q}.$$

Therefore $g\in V_{p,1/p}^{(k)}$. In addition to, let $2^{-s}\le\delta\le 2^{-s+1}$, $s\in N$, and let Π be any partition with $\|\Pi\|\le\delta$, then $\mathrm{T}_l^{(k)}(\Pi)=\varnothing$ and $S_l^{(k)}(\Pi)=0$ for $l<s$.
From (3.4) and (3.5) we get

$$\nu_{p,\alpha}^{(k)}(g;\,\Pi)\le\left(\sum_{l=v}^{\infty}2^{(l+1)^{\alpha\theta q}}\omega_{k-1/q}(g,2^{-l})^{q}\right)^{1/q}\le 4\left(\int_0^{\delta}(t^{-\theta}\omega_{k-1/q}(g,t))^{q}\frac{dt}{t}\right)^{\frac{1}{q}}.$$

□

Theorem 3.2 : *For any* $1 < p < q < \infty, \theta = \frac{1}{p} - \frac{1}{q}$, *and* $\sigma \in \Gamma^{k}_{1/q'}$, *then* $V^{\sigma}_{q} \subseteq V^{(k)}_{p,\alpha}$ *iff* $\int_0^1 (t^{-\theta}\omega(t))^q \frac{dt}{t}$ *is finite*

Proof : The embedding is easily hold by Theorem 3.1. To the contrary, suppose that the necessity doesn't hold, then σ satisfies (1.13). so by (1.14), the sequence $\eta_m = \eta_m(\sigma)$ satisfies $\sum_{m=1}^{\infty} 2^{\varsigma_m \theta q/k} \sigma_{\mathrm{j}}^{q} = \infty$,

So there exists $u_j, j \in \aleph$, s.t. $u_1 = 1$, and $(\sum_{m=u_j}^{u_{j+1}-1} 2^{\eta_m \theta q} \sigma_{\eta_m}^{q})^{1/q} > 2^j$.

Name $\sigma^* = \sum_{j=1}^{\infty} (\sum_{m=u_j}^{u_{j+1}-1} 2^{\eta_m \theta q} \sigma_{\eta_m}^{q})^{1/q}$

To contract the assumption, we prove that $\sigma^* \in V^{\sigma}_{q,\alpha}$, but not $V^{(k)}_{p,\,\alpha}$,

$$\omega_{k-\frac{1}{q'}}(\sigma^*, \delta) = \sup_{\Pi} v^{(k)}_{p,\alpha}(\sigma^*, \Pi) = \sup_{\Pi} \left(\sum_{l=0}^{\infty} 2^{(l+1)\alpha\theta q} \omega_{k-\frac{1}{q'}}(\sigma^*, 2^{-l})^q \right)^{1/q}$$

$$= \sup_{\Pi} \left(\sum_{l=0}^{\infty} 2^{(l+1)\alpha\theta q} \sum_{j=1}^{\infty} \sum_{m=u_j}^{u_{j+1}-1} 2^{\eta_m \theta q} \omega_{k-\frac{1}{q'}}(\sigma_{\eta_m}^{q}, 2^{-l})^q \right)^{1/q}$$

$$= \sup_{\Pi} \left(\sum_{l=0}^{\infty} 2^{((l+1)\alpha + \eta_m)\theta q} \sum_{j=1}^{\infty} \sum_{m=u_j}^{u_{j+1}-1} \omega_{k-\frac{1}{q'}}(\sigma_{\eta_m}^{q}, 2^{-l})^q \right)^{1/q} = C\sigma(2^{-l}).$$

□

Conclusion

The importance of moduli of p-smoothness comes from the need to measure fractional smoothness of functions by using (p,α) *k*th variation. We benefit from replacing the two modulus of p-smoothness and q-smoothness , to prove some relations between the families of functions V^{σ}_{q} and $V^{(k)}_{p,\alpha}$.

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