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A Study on Certain Classes of Harmonic Univalent function, Bi-Univalent on which the convolution operator

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Abstract— *In this work. We use the convolution operator associated with generalized distribution series, inclusion relations between various subclasses k_H^0 , $S_H^{*,0}$ of Harmonic univalent functions are established. Precisely, such inclusions with Harmonic structure and harmonic convex mappings.*

Keywords— Univalent, normalized conditions, BI-univalent, Convolution, harmonic convex function, locally univalent.

1. Introduction and Primilinaries:

Let $\mathbb{C}$ be the complex plane and ϑ be a domain in $\mathbb{C}$. Let f be an analytic function in ϑ, if f does not map onto the same value for different points in ϑ then we call f as univalent. If ϑ not the whole, complex plane is a simply connected region and let a function f under which ϑ is set onto the open unit disc $\mathcal{U} = \{s: s \in \mathbb{C}\ and\ |s| < 1\}$. Thus, it is sufficient to consider analytic univalent functions in Ω which satisfies the normalization conditions $f(0) = f'(0) - 1 = 0$. Geometrically $f(0) = 0$ translates the image of the domain whereas by the condition $f'(0) = 1$.

The class $\mathcal{A}$ formed by the analytic functions in $\mathcal{U}$ that satisfy normalization conditions. These functions in $\mathcal{A}$ are represented as

$$f(s) = s + \sum_{k=2}^{\infty} a_k s^k \quad (s \in \mathcal{U}). \qquad (1.1)$$

The class δ is formed by the functions that are analytic and univalent which satisfy the normalized conditions for example $k(s) = \frac{s}{(1-s)^2}$

Is a member in δ. This function k maps $\mathcal{U}$ onto the complex plane except on the points from $-\infty$ to $-1/4$.

It is obvious f^{-1}, the inverse function of f in δ exists and is written as

$$f^{-1}\big(f(s)\big) = s \qquad (s \in \mathcal{U}).$$

If both f and $f^{-1}$are univalent in $\mathcal{U}$, f is known as bi-univalent in $\mathcal{U}$. Such functions form a class denoted as φ

Let a domain $\mathcal{D}$ be simply connected and $f = u + vi$ is in $\mathcal{D}$ which continuous complex is valued. For the real harmonic u and v, f is called harmonic in $\mathcal{D}$.

$$h(s) = s + \sum_{k\geq 2} A_k s^k, \qquad g(s) = \sum_{k\geq 1} B_k s^k, \qquad |B_1| < 1 \qquad (1.2)$$

Let H denotes the collection of harmonic functions such that $f = h + \bar{g}$. A subclass S_H of H was established **[1]**. Using univalent and sense preserving functions in $\mathcal{U}$ that are complex valued and harmonic and their properties were studied. A function f in S_H is represented as $f = h + \bar{g}$ such that

$$h(s) = s + \sum_{k\geq 2} h_k s^k \qquad g(s) = \sum_{k\geq 1} g_k s^k \qquad |g_1| < 1 \qquad (1.3)$$

That is prerequisite and satisfactory for $f = h + \bar{g} \in S_H$ to be locally univalent and sense preserving in $\mathcal{U}$ and is given by $|f(s)'| > |g(s)'||$, for all s $\in \mathcal{U}$. **[2]**. proved several basic results on this class S_H in their works. When $g(s)$ given as in (1.3) satisfies the condition that $g(s) \equiv 0$, for every δ in $\mathcal{U}$, then S_H is same that of δ with analytic functions.

[3]. introduced and studied the classes S_H and S_H^0. $f(s)$ in S_H expressed as in (1.3) is named harmonic starlike of order α for $0 \leq \alpha < 1$ is givenly

$$\frac{\partial}{\partial\theta}(arg f(s)) > \alpha, \qquad s \in \mathcal{U}$$

The functions that satisfy the above condition form a class, which is represented as $S_H^*(\alpha)$. The function f is called harmonic convex function of order α for $0 \leq \alpha < 1$ is givenly

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta}f(s)\right)\right) > \alpha, \qquad s \in \mathcal{U}$$

$k_H(\alpha)$ is the class, which is formed by functions that satisfy the above condition. **[4]**. Pioneered the study on the classes of $S_H^*(\alpha)$. In addition $k_H(\alpha)$. **[5]**. in their study, they showed that, the classes $S_H^*(\alpha)$ and $k_H(\alpha)$ become S_H^* and k_H respectively, when α takes the value $\alpha = 0$. Further they also proved that $\alpha = B_1 = 0$, the above said classes become $S_H^{*,0}$ H and k_H^0.

The subclasses of harmonic function class S_H^0 in $\mathcal{U}$, namely convex subclass, starlike subclass and close-to-convex subclass are denoted as k_H^0, $S_H^{*,0}$ and C_H^0 respectively.These subclasses are studied extensively see **[1]**. And **[2]**.

The generalized distribution was launched recently see **[6]**. With interesting applications on functions that are univalent.

$I(\theta_1,\theta_2): H \to H$ is Called integral operator given by

$$I(\theta_1,\theta_2)\theta(s) = \hbar(s) + \overline{g(s)} \qquad (1.4)$$

Where

$\hbar(s) = h(s) * K_{\theta_1}(s)$ and $g(s) = g(s) * K_{\theta_2}(s)$

Or in other word

$$\hbar(s) = s + \sum_{k\geq 2}\frac{A_k t_{k-1}}{s_1}s^k \ and \ g(s) = s + \sum_{k\geq 2}\frac{B_k r_{k-1}}{s_2}s^k \qquad (1.4)$$

$$\theta_1(s) = \sum_{k\geq 0} t_k\, s^k \quad and \quad \theta_2(s) = \sum_{k\geq 0} r_k\, s^k$$

And

$$s_1 = \theta_1(1) \ and \ s_2 = \theta_2(1)$$

Recently there are relations between several subclasses of univalent functions that are analytic and harmonic, which are accessible, using the convolution operator $I(\theta_1,\theta_2)$.

We establish the relations amongst the classes k_H^0, $S_H^{*,0}$ and C_H^0

2. **Set of Lemma**

Lemma 2.1. [2]. If h and g are represented as in (5.1) and $f \in k_H^0$, expressed as $f = h + \bar{g}$ and $B_1 = 0$ we have

$$|A_k| \leq \frac{k+1}{2} \qquad |B_k| \leq \frac{k-1}{2}$$

Lemma 2.2. [4]. If h and g are described as in (5.1) and f is written as $f = h + \bar{g}$. If for any α where $0 \leq \alpha < 1$ and if

$$\sum_{k\geq 2}(k-\alpha)\,|A_k| + \sum_{k\geq 1}(k+\alpha)\,|B_k| \leq 1-\alpha \qquad (2.1)$$

Then f in $\mathcal{U}$ fulfills the criteria for the function to be harmonic which is sense preserving and univalent also $f \in S_H^*(\alpha)$

Now suppose

$$h(s) = s - \sum_{k\geq 2}|A_k|s^k \quad and \quad g(s) = \sum_{k\geq 2}|B_k|s^k \qquad (|B_1| < 1)$$

Such that $f = h + \bar{g}$ it is necessary that the condition (2.1) be satisfied. Furthermore for f in $TS_H^*(\alpha)$

$$|A_k| < \frac{1-\alpha}{k-\alpha} \geq 2 \quad and \quad |B_k| < \frac{1-\alpha}{k+\alpha} \geq 1$$

Lemma 2.3. [4]. Let h and g be expressed as in (1.2) and $f = h + \bar{g}$. For a given α where $0 \leq \alpha < 1$ we have,

$$\sum_{k\geq 2}k(k-\alpha)|A_k| + \sum_{k\geq 1}k(k+\alpha)|B_k| \leq 1-\alpha \qquad (2.3)$$

This implies $f \in k_H(\alpha)$ is in $\mathcal{U}$ which is harmonic and sense-preserving univalent function

Lemma 2.4. [2]. Let h and g be described as in (1.2) and $f \in S_H^{*,0}$, or C_H^0 is expressed as $f = h + \bar{g}$ then $B_1 = 0$, we get

$$|A_k| \leq \frac{2k^2+3k+1}{6} \quad and \quad |B_k| \leq \frac{2k^2-3k-1}{6}$$

The next theorem gives the condition which is sufficient for the operator $I(\theta_1,\theta_2)$ to be harmonic starlike in $\mathcal{U}$

Theorem 2.1. If $f = h + \bar{g} \in H$ is expressed as in (1.2) with $B_1 = 0$ and the inequality

$$\frac{1}{S_1}\big((\theta_1'')(1) + (4-\alpha)(\theta_1')(1)\big) + \frac{1}{S_2}\big((\theta_2'')(1) + (2+\alpha)(\theta_2')(1)\big) \leq \frac{2(1-\alpha)}{S_1}\theta_1(0)$$

Is satisfied then $I(\theta_1,\theta_2)\, k_H^0 \subseteq S_H^{*,0}(\alpha)$

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Proof. Let $f \in k_H^0$ be represented as $f = h + \bar{g}$ and is defined as in (1.2) with $B_1 = 0$. for $\hbar$ and g analytic in $\mathcal{U}$ as described in (1.5) we establish that $I(\theta_1, \theta_2)(f) = \hbar + \bar{g} \in S_H^{*,0}(\alpha)$.

From Lemma 2.2, it sufficies the condition given below is true

$$\sum_{k\geq 2}(k-\alpha)\frac{|A_k|t_{k-1}}{s_1} + \sum_{k\geq 2}(k+\alpha)\frac{|B_k|r_{k-1}}{s_2} \leq 1-\alpha \qquad (2.4)$$

Using Lemma 2.1, we arrive

$$\sum_{k\geq 2}(k-\alpha)\frac{|A_k|t_{k-1}}{s_1} + \sum_{k\geq 2}(k+\alpha)\frac{|B_k|r_{k-1}}{s_2}$$
$$= \sum_{k\geq 2}(k-\alpha)\frac{(k+1)t_{k-1}}{2s_1} + \sum_{k\geq 2}(k+\alpha)\frac{(k+1)r_{k-1}}{2s_2}$$
$$\sum_{k\geq 2}(k-\alpha)\frac{(k+1)}{2}\frac{t_{k-1}}{s_1} + \sum_{k\geq 2}(k+\alpha)\frac{(k+1)}{2}\frac{r_{k-1}}{s_2}$$
$$= \frac{1}{2}\left[\frac{1}{s_1}\sum_{k\geq 2}(k-\alpha)(k+1)t_{k-1} + \frac{1}{s_2}\sum_{k\geq 2}(k+1)(k-1)r_{k-1}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s_1}\sum_{k\geq 2}\{(k-1)(k-2)+(4-\alpha)(k-1)+2(1-\alpha)\}\,t_{k-1} + \frac{1}{s_2}\sum_{k\geq 2}\{(k-1)(k-2)+(2+\alpha)(k-1)+2(1-\alpha)\}r_{k-1}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s_1}\sum_{k\geq 1}\{k(k-1)+(4-\alpha)k+2(1-\alpha)\}t_k + \frac{1}{s_2}\sum_{k\geq 1}\{k(k-1)+(2+\alpha)k\}\,r_k\right]$$
$$= \frac{1}{2}\left[\frac{1}{s_1}(\theta_1''(1)+(4-\alpha)\theta_1'(1)+2(1-\alpha)(\theta_1'(1)-\theta_1(0)) + \frac{1}{s_2}(\theta_2''(1)+(2+\alpha)\theta_2'(1)\right] \leq 1-\alpha$$

The prove of theorem is complete

Theorem 2.2. if $f \in H$ such that $f = h + \bar{g}$ as expressed in (1.2) with $B_1 = 0$ and if

$$\frac{1}{s_1}\left(\theta_1'''(1)+(7-\alpha)\theta_1''(1)+2(5-2\alpha)\theta_1'(1)\right) + \frac{1}{s_2}(\theta_2'''(1)+(5+\alpha)\theta_2''(1)+(4+2\alpha)\theta_2'(1) \leq \frac{2(1-\alpha)}{s_1}\theta_1(0)$$

Is true, we have $I(\theta_1, \theta_2)k_H^0 \subset K_H^0(\alpha)$

Proof. If h, g are defined as in (1.2) and let $\in K_H^0$, and is expressed as $f = h + \bar{g}$ such that $B_1 = 0$. It is required to derived $I(\theta_1, \theta_2)f \in K_H(\alpha)$, the function $\hbar$, g are analytic in $\mathcal{U}$ are expressed as in (1.5) with $B_1 = 0$. Using lemma 2.3 We can show

$$\sum_{k\geq 2}k(k-\alpha)\frac{|A_k|t_{k-1}}{s_1} + \sum_{k\geq 2}k(k+\alpha)\frac{|B_k|r_{k-1}}{s_2} \leq 1-\alpha \qquad (2.5)$$

From lemma 2.1 we have

$$\sum_{k\geq 2}k(k-\alpha)\frac{|A_k|t_{k-1}}{s_1} + \sum_{k\geq 2}k(k+\alpha)\frac{|B_k|r_{k-1}}{s_2}$$
$$= \sum_{k\geq 2}k(k-\alpha)\frac{(k+1)}{2}\frac{t_{k-1}}{s_1} + \sum_{k\geq 2}k(k+\alpha)\frac{(k-1)}{2}\frac{r_{k-1}}{s_2}$$
$$= \frac{1}{2}\left[\frac{1}{s_1}\sum_{k\geq 2}k(k-\alpha)(k+1)t_{k-1} + \frac{1}{s_2}\sum_{k\geq 2}k(k+\alpha)(k-1)r_{k-1}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s_1}\sum_{k\geq 2}[(k^2-3k+2)(k-3)+(7-\alpha)(k^2-3k+2)+2(5-2\alpha)(k-1)+2(1-\alpha)]t_{k-1} + \frac{1}{s_2}\sum_{k\geq 2}[(k^2-3k+2)(k-3)\right.$$
$$\left. + (5+\alpha)(k^2-3k+2)+(4+2\alpha)(k-1)]r_{k-1}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s_1}\sum_{k\geq 1}[k(k^2-3k+2)(7-\alpha)k(k-1)+2(5-2\alpha)k+2(1-\alpha)]t_k + \frac{1}{s_2}\sum_{k\geq 1}[k(k^2-3k+2)+(5+\alpha)k(k-1)\right.$$
$$\left. + (4+2\alpha)k]r_n\right]$$

$$=\frac{1}{2}\left[\frac{1}{s_1}\left(\theta_1'''(1)+(7-\alpha)\theta_1''(1)+2(5-2\alpha)\theta_1'(1)+2(1-\alpha)\big(\theta_1(1)-\theta_1(0)\big)+\frac{1}{s_2}(\theta_2'''(1)+(5+\alpha)\theta_2''(1)+(4+2\alpha)\theta_2'(1)\right]\right.$$
$$\leq 1-\alpha$$

the prove of theorem is complete.

Theorem 2.3 if h, g are stated as in (1.2) and if $f \in H$ is expressed as $f = h + \bar{g}$ with $B_1 = 0$, where the inequality

$$\frac{1}{s_1}(2\theta_1'''(1)+(15-2\alpha)\theta_1''(1)+(24-9\alpha)\theta_1'(1)+\frac{1}{s_2}\big(2\theta_2'''(1)+(9+2\alpha)\theta_2''(1)+3(2+\alpha)\theta_2'(1)\big)\leq\frac{6(1-\alpha)}{s_1}\theta_1'(0),$$

Is satisfied then $I(\theta_1,\theta_2)(s_H^{*,0}) \subset s_H^*(\alpha)$ and $I(\theta_1,\theta_2)(C_H^0) \subset s_H^*(\alpha)$

Proof. Let f be stated as in theorem. We want to show that

$$\sum_{k\geq 2}(k-\alpha)\frac{|A_k|t_{k-1}}{s_1}+\sum_{k\geq 2}(k+\alpha)\frac{|B_k|r_{k-1}}{s_2}\leq 1-\alpha$$

.

By using lemma 2.4, we have

$$\sum_{k\geq 2}(k-\alpha)\frac{|A_k|t_{k-1}}{s_1}+\sum_{k\geq 2}(k+\alpha)\frac{|B_k|r_{k-1}}{s_2}$$
$$=\frac{1}{6}\left[\frac{1}{s_1}\sum_{k\geq 2}\{2(k^2-3k+2)(k-3)+(15-2\alpha)(k^2-3k+2)+(24-9\alpha)(k-1)+6(1-\alpha)\}t_{k-1}+\frac{1}{s_2}\sum_{k\geq 2}\{2(k^2-3k\right.$$
$$\left.+2)(k-3)+(9+2\alpha)(k^2-3k+2)+(6+3\alpha)(k-1)\}r_{k-1}\right]$$
$$=\frac{1}{6}\left[\frac{1}{s_1}\sum_{k\geq 1}\{2k(k^2-3k+2)+(15-2\alpha)k(k-1)+(24-9\alpha)k+6(1-\alpha)\}t_k\right.$$
$$\left.+\frac{1}{s_2}\sum_{k\geq 1}\{2k(k^2-3k+2)+(9+2\alpha)k(k-1)+(6+3\alpha)k\}r_n\right]$$
$$=\frac{1}{6}\left[\frac{1}{s_1}(2\theta_1'''(1)+(15-2\alpha)\theta_1''(1)+(24-9\alpha)\theta_1'(1)+6(1-\alpha)\big(\theta_1'(1)-\theta_1(0)\big)+\frac{1}{s_2}(2\theta_2'''(1)+(9+2\alpha)\theta_2''(1)+3(2\right.$$
$$\left.+\alpha)\theta_2'(1)\right]\leq 1-\alpha$$

By the given condition. The prove is complete.

Theorem 2.4 the function f in H such that $f = h + \bar{g}$ is mentioned as in (1.2) with $B_1 = 0$ for which

$$\frac{1}{s_1}\big(2\theta_1^{(iv)}(1)+(23-2\alpha)\theta_1'''(1)+(69-15\alpha)\theta_1''(1)+(54-2\alpha)\theta_1'\big)+\frac{1}{s_2}(2\theta_2^{(iv)}(1)+(17+2\alpha)\theta_2'''(1)+(33+9\alpha)\theta_2''(1)$$
$$+(12+6\alpha)\theta_2'(1))\leq\frac{6(1-\alpha)}{s_1}\theta_1(0)$$

Is satisfied, then $I(\theta_1,\theta_2)(S_H^{*,0}) \subset k_H(\alpha)$ and $I(\theta_1,\theta_2)(C_H^0) \subset k_H(\alpha)$

Proof. Let f be stated as in theorem. It suffices to show that

$$\sum_{k\geq 2}k(k-\alpha)\frac{|A_k|t_{k-1}}{s_1}+\sum_{k\geq 2}k(k+\alpha)\frac{|B_k|r_{r-1}}{s_2}\leq 1-\alpha$$

From lemma 2.4

$$\sum_{k\geq 2}k(k-\alpha)\frac{|A_k|t_{k-1}}{s_1}+\sum_{k\geq 2}k(k+\alpha)\frac{|B_k|r_{k-1}}{s_2}$$
$$=\frac{1}{6}\left[\frac{1}{s_1}\sum_{k\geq 2}(k^2-k\alpha)(2k+1)(k+1)t_{k-1}+\frac{1}{s_2}\sum_{k\geq 2}k(k+\alpha)(2k-1)(k-1)r_{k-1}\right]$$
$$=\frac{1}{6}\left[\frac{1}{s_1}\sum_{k\geq 2}\{2k(k^2-3k+2)(k-3)+(23-2\alpha)k(k^2-3k+2)+(69-15\alpha)k(k-1)+(54-24\alpha)k+6(1-\alpha)\}t_k\right.$$
$$\left.+\frac{1}{s_2}\sum_{k\geq 2}\{2k(k^2-3k+2)(k-3)+(17+2\alpha)k(k-1)(k-2)+(33+9\alpha)(k^2-k)+(12+6\alpha)k\}r_k\right]$$

$$=\frac{1}{6}\left[\frac{1}{s_1}\left(2\theta_1^{(iv)}(1)+(23-2\alpha)\theta_1'''(1)+(69-15\alpha)\theta_1''(1)+(54-24\alpha)\theta_1'(1)+6(1-\alpha)(\theta_1(1)-\theta_1(0)\right)+\frac{1}{s_2}(2\theta_2^{(iv)}(1)\right.$$
$$\left.+(17+2\alpha)\theta_2'''(1)+(33+9\alpha)\theta_2''(1)+(12+6\alpha)\theta_2'(1))\right]\leq 1-\alpha$$

The prove is complete

Theorem 2.5. Let h and g be described as in (5.6) and $f=h+g\ \in H$ such that $B_1=0$. Then the prerequisite and satisfactory condition for $I(\theta_1,\theta_2)(JS_H^{*,0}(\alpha))\subset JS_H^{*,0}(\alpha)$ is

$$\frac{\theta_1(0)}{s_1}+\frac{\theta_2(0)}{s_2}\geq 1$$

Proof. Consider $f=h+\bar{g}\in\ JS_H^{*,0}(\alpha)$ which is expressed as mentioned in (2.3) to establish the desired result $(\theta_1,\theta_2)f\in JS_H^{*,0}(\alpha)$, it is enough if we prove the following

$$\sum_{k\geq 2}(k-\alpha)\frac{|A_k|t_{k-1}}{s_1}+\sum_{k\geq 2}(k+\alpha)\frac{|B_k|r_{k-1}}{s_2}\leq 1-\alpha$$

By using remark.2.1, we have

$$\sum_{k\geq 2}(k-\alpha)\frac{|A_k|t_{k-1}}{s_1}+\sum_{k\geq 2}(k+\alpha)\frac{|B_k|r_{k-1}}{s_2}=(1-\alpha)\left[\sum_{k\geq 2}\frac{t_{k-1}}{s_1}+\sum_{k\geq 2}\frac{r_{k-1}}{s_2}\right]$$
$$=(1-\alpha)\left[\sum_{k\geq 2}\frac{t_k}{s_1}+\sum_{k\geq 2}\frac{r_k}{s_2}\right]$$
$$=(1-\alpha)\left[\frac{\theta_1(1)-\theta_1(0)}{s_1}+\frac{\theta_2(1)-\theta_2(0)}{s_2}\right]\leq 1-\alpha$$

By prove is complete

Theorem 2.6. Consider $f=h+\bar{g}\ \in H$ is expressed as in (2.3) with $B_1=0$. Then the necessary and sufficient criterion for $I(\theta_1,\theta_2)(JS_H^{*,0}(\alpha))\subset Jk_H^0(\alpha)$ is

$$\frac{\theta_1{}'(1)}{s_1}+\frac{\theta_2{}'(1)}{s_2}\leq\frac{\theta_1(0)}{s_1}+\frac{\theta_2(0)}{s_2}-1$$

Proof. Let h and g are defined as in (2.3) and let $f\ \in\ JS_H^{*,0}(\alpha)$ and is expressed as $f=h+\bar{g}$ such that $B_1=0$. For $I(\theta_1,\theta_2)f$ to be contained in $Jk_H^0(\alpha)$, we have to show

$$\sum_{k\geq 2}k(k-\alpha)\frac{|A_k|t_{k-1}}{s_1}+\sum_{k\geq 2}k(k+\alpha)\frac{|B_k|r_{k-1}}{s_2}\leq 1-\alpha$$

2. By using remark 2.1, we obtain

$$\sum_{k\geq 2}k(k-\alpha)\frac{|A_k|t_{k-1}}{s_1}+\sum_{k\geq 2}k(k+\alpha)\frac{|B_k|r_{k-1}}{s_2}$$
$$=(1-\alpha)\left[\sum_{k\geq 2}\frac{kt_{k-1}}{s_1}+\sum_{k\geq 2}\frac{kr_{k-1}}{s_2}\right]$$
$$=(1-\alpha)\left[\sum_{k\geq 2}\frac{(k+1)t_k}{s_1}+\sum_{k\geq 2}\frac{(k+1)r_k}{s_2}\right]$$
$$=(1-\alpha)\left[\frac{\theta_1{}'(1)-\theta_1(1)-\theta_1(0)}{s_1}+\frac{\theta_2{}'(1)+\theta_2(1)-\theta_2(0)}{s_2}\right]\leq 1-\alpha$$

The proof is complete.

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