Derivatives

The derivative of $y = f(x)$, denoted by y', $f'(x)$ or $\frac{dy}{dx}$, is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Rules for finding derivatives

1. Constant Rule

The derivative of a constant is always zero. That is if $f(x) = c$ then $f'(x) = 0$.

2. Power Rule The derivative of a power function, $f(x) = x^n$. Here n is a number of any kind: integer, rational, positive, negative, even irrational, if $f(x) = x^n$,

then the derivative is $f'(x) = nx^{n-1}$

Examples:

No	y	$\frac{dy}{dx}$
1.	$y = x^4$	$\frac{dy}{dx} = 4x^3$
2.	$y = x^{-4}$	$\frac{dy}{dx} = -4x^{-5} = -\frac{4}{x^5}$
3.	$y = \frac{1}{x^2} = x^{-2}$	$\frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$
4.	$y = x^{\frac{3}{5}}$	$\frac{dy}{dx} = \frac{3}{5}x^{\frac{3}{5}-1} = \frac{3}{5}x^{-\frac{2}{5}}$

3. Multiplication by constant: If $g(x) = cf(x)$ then $g'(x) = cf'(x)$.

4. Sum Rule: If $h(x) = f(x) + g(x)$, then $h'(x) = f'(x) + g'(x)$.

5. Difference Rule: If $h(x) = f(x) - g(x)$, then $h'(x) = f'(x) - g'(x)$.

6. Product Rule: The derivative of the product of two functions is not the product of the functions' derivatives; rather, it is described by the equation below:

$$\frac{d}{dx}\big(f(x) \times g(x)\big) = f(x) \times g'(x) + f'(x) \times g(x)$$

Example 6: Find derivative of the function $y = (3x^2 + 5)(2x^3 - 5x - 4)$

$$\frac{dy}{dx} = (3x^2 + 5) \times (6x^2 - 5) + (2x^3 - 5x - 4) \times 6x$$

7. Quotient Rule: The derivative of the quotient of two functions is not the quotient of the functions' derivatives; rather, it is described by the equation below:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \times f'(x) - f(x) \times g'(x)}{\left(g(x)\right)^2}$$

Example 7: Find derivative of the function

$$y = \frac{x^2 - 1}{x^2 + 1}$$

$$\frac{dy}{dx} = \frac{(x^2 + 1) \times 2x - (x^2 - 1) \times 2x}{(x^2 + 1)^2} = \frac{2x^3 + 2x - (2x^3 - 2x)}{(x^2 + 1)^2}$$

$$= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}$$

8. Chain Rule:

Suppose that we have two functions $f(x)$ and $g(x)$ and they are both differentiable.

1.If we define $F(x) = (fog)(x)$ then the derivative of $F(x)$ is,

$$F'(x) = f'\big(g(x)\big)g'(x)$$

2.If we have $y = f(u)$ and $u = g(x)$ then the derivative of y is,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example 8: Find derivatives of the functions

1. $y = (2x + 3)^4 \quad ⇨ \quad \frac{dy}{dx} = 4(2x + 3)^3 \times 2 = 8(2x + 3)^3$

2. $y = \sqrt{x^2 + 3x} \quad ⇨ \quad y = (x^2 + 3x)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 + 3x)^{-1/2}(2x + 3) = \frac{(2x + 3)}{2\sqrt{x^2 + 3x}}$$

3. $y = \dfrac{x}{\sqrt{x^2+1}} \Rightarrow \dfrac{dy}{dx} = \dfrac{\sqrt{x^2+1} \times 1 - x \times (1/2)(x^2+1)^{-1/2} \times 2x}{x^2+1}$

$$= \frac{\sqrt{x^2+1} - \dfrac{x^2}{\sqrt{x^2+1}}}{x^2+1} = \frac{\dfrac{x^2+1-x^2}{\sqrt{x^2+1}}}{\sqrt{x^2+1}\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$$

Example 9: If $y = u^2 - 2u$ and $u = \sqrt{3x+1}$, find $\dfrac{dy}{dx}$

$\dfrac{dy}{du} = 2u - 2 = 2(u-1)$ and $\dfrac{du}{dx} = \dfrac{3}{2\sqrt{3x+1}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2(u-1) \times \frac{3}{2\sqrt{3x+1}} = \frac{3(u-1)}{\sqrt{3x+1}} = \frac{3\left(\sqrt{3x+1}-1\right)}{\sqrt{3x+1}}$$

Example 10: If $y = t + \dfrac{1}{t}$ and $x = t - \dfrac{1}{t}$, find $\dfrac{dy}{dx}$

$\dfrac{dy}{dt} = 1 - \dfrac{1}{t^2} = \dfrac{t^2-1}{t^2}$ and $\dfrac{dx}{dt} = 1 + \dfrac{1}{t^2} = \dfrac{t^2+1}{t^2}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{t^2-1}{t^2} \times \frac{t^2}{t^2+1} = \frac{t^2-1}{t^2+1}$$

Higher Derivatives

If the derivative $f'(x)$ of a function $f(x)$ exists in the domain of $f(x)$, then we have a new function. Now that we have agreed that the derivative of a function is a function, we can repeat the process and try to differentiate the derivative. The result, if it exists, is called the **second derivative**. It is denoted $f''(x)$. The derivative of the second derivative is called the third derivative, written $f'''(x)$, and so on.

The nth derivative of $f(x)$ is denoted $f^{(n)}(x)$. Thus
Leibniz' notation for the nth derivative of $y = f(x)$ is $\dfrac{d^n y}{dx^n}$.

Be careful to distinguish the second derivative from the square of the first derivative. Usually

$$\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$$

Example 11: Find $f'(x)$, $f''(x)$, $f^{(3)}(x)$ and $f^{(4)}(x)$ for $f(x) = 2x^3 + 3x^2 - 4x + 5$

$$f'(x) = 6x^2 + 6x - 4$$

$$f''(x) = 12x + 6$$

$$f^{(3)}(x) = 12$$

$$f^{(4)}(x) = 0$$

Example 12: Compute the first, second and third derivatives of $y = \sqrt{x+2}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4(x+2)^{3/2}}$$

$$\frac{d^3y}{dx^3} = \frac{3}{8(x+2)^{5/2}}$$

Exercises

Find the derivative in each of the following problems $(1-4)$

1. $y = (x^2 - 1)^4$
2. $y = x^2\sqrt{2x^2 + 3}$
3. $y = \dfrac{1+\sqrt{x}}{1-\sqrt{x}}$
4. $y = \sqrt{\dfrac{1-x}{x^2+1}}$

Compute the first, second and third derivatives in the following problems $(5-8)$

5. $y = 2x^3 - 5x^2 + 3x$
6. $y = \sqrt{2x+3}$ at $x = 3$
7. $y = x\sqrt{x}$ at $x = 4$
8. $y = (x^2+4)^{5/2}$ at $x = 0$
9. If $y = u\sqrt{2u+5}$ and $x = (4u)^{\frac{1}{3}}$ find $\dfrac{dy}{dx}$
10. If $u = s + \sqrt{s}$ and $v = s - \sqrt{s}$ find $\dfrac{du}{dv}$