

Solution of Heat Equation

We want to solve the heat equation,

$$u_t = cu_{xx}, \quad t > 0, \quad 0 \leq x \leq L$$

where $c > 0$ is a constant (the thermal conductivity of the material).

subject to the boundary: $u(0, t) = 0$, $u(L, t) = 0$

and the initial condition $u(x, 0) = f(x)$.

We applied separation of variables to solve the heat equation $u_t = cu_{xx}$, $t > 0$
 $0 < x < L$; $c > 0$ with conductions $u(0, t) = u(L, t) = 0$ and $u(x, 0) = f(x)$.

The following examples explain how to do it.

Example 1: Apply the method of separation of variables to solve the heat equation

$u_t - 2u_{xx} = 0$ over $0 < x < 3, t > 0$ for the boundary conditions
 $u(0, t) = u(3, t) = 0$ and the initial condition $u(x, 0) = 5 \sin 4\pi x$

Solution: Assume the solution is $u(x, t) = F(x)G(t)$

$$u_t = FG' \quad \text{and} \quad u_{xx} = F''G$$

Substituting in the given equation, we have

$$FG' = 2F''G \Leftrightarrow \frac{F''}{F} = \frac{G'}{2G} = -\lambda^2$$

$$\frac{F''}{F} = -\lambda^2 \quad \text{and} \quad \frac{G'}{2G} = -\lambda^2$$

$$F'' + \lambda^2 F = 0 \quad \text{and} \quad G' + 2G\lambda^2 = 0$$

$$F = A \sin \lambda x + B \cos \lambda x \quad \text{and} \quad G = Ce^{-2\lambda^2 t}$$

$$\text{So } u(x, t) = (A \sin \lambda x + B \cos \lambda x) \times Ce^{-2\lambda^2 t}$$

$$\text{Or } u(x, t) = e^{-2\lambda^2 t}(D \sin \lambda x + E \cos \lambda x); \quad AC = D, \quad BC = E$$

$$u(0, t) = 0 \quad \text{since} \quad \sin 0 = 0 \quad \text{and} \quad \cos 0 = 1 \quad \text{this must imply that} \quad E = 0$$

$$u(3, t) = 0 \Leftrightarrow De^{-2\lambda^2 t} \sin 3\lambda = 0 \Leftrightarrow e^{-2\lambda^2 t} \neq 0$$

$$D = 0 \Leftrightarrow u = 0 \quad (\text{trivial solution})$$

The only sensible deduction is that $\sin 3\lambda = 0 \Leftrightarrow 3\lambda = n\pi \Leftrightarrow \lambda = n\pi/3$

$$\text{Then } u(x, t) = D_n e^{-\frac{2n^2\pi^2}{9}t} \sin \frac{n\pi}{3} x$$

$$u(x, 0) = 5 \sin 4\pi x \Leftrightarrow 5 \sin 4\pi x = D_n \sin \frac{n\pi}{3} x$$

$$4\pi = \frac{n\pi}{3} \Leftrightarrow n = 12 \text{ so } D_{12} = 5$$

$$\text{Then } u(x, t) = 5e^{-\frac{288\pi^2}{9}t} \sin(4\pi x) = 5e^{-32\pi^2 t} \sin(4\pi x)$$

Example 2: Solve by the method of separation of variables the heat equation

$$u_t = u_{xx} ; \quad 0 < x < 1 , t > 0 \text{ with } u_x(0, t) = u_x(1, t) = 0$$

and the initial condition $u(x, 0) = 3 \cos 2\pi x$

Solution: Assume the solution is $u(x, t) = F(x)G(t)$

$$u_t = FG' \text{ and } u_{xx} = F''G$$

$$FG' = F''G \Leftrightarrow \frac{F''}{F} = \frac{G'}{G} = -\lambda^2$$

$$\frac{F''}{F} = -\lambda^2 \text{ and } \frac{G'}{G} = -\lambda^2$$

$$F'' + \lambda^2 F = 0 \text{ and } G' + G\lambda^2 = 0$$

$$F = A \sin \lambda x + B \cos \lambda x \quad \text{and} \quad G = Ce^{-\lambda^2 t}$$

$$\text{So } u(x, t) = (A \sin \lambda x + B \cos \lambda x) \times Ce^{-\lambda^2 t}$$

$$\text{Or } u(x, t) = e^{-\lambda^2 t}(D \sin \lambda x + E \cos \lambda x) ; \quad AC = D, \quad BC = E$$

$$u_x(x, t) = e^{-\lambda^2 t}(D\lambda \cos \lambda x - E\lambda \sin \lambda x)$$

$$u_x(0, t) = 0 \Leftrightarrow D = 0$$

$$u_x(1, t) = 0 \Leftrightarrow -E\lambda e^{-\lambda^2 t} \sin \lambda = 0 \Leftrightarrow \lambda e^{-\lambda^2 t} \neq 0$$

$$E = 0 \Leftrightarrow u = 0 \quad (\text{trivial solution})$$

$$\sin \lambda = 0 \Leftrightarrow \lambda = n\pi, \quad n = 1, 2, 3, \dots$$

$$\text{Then } u(x, t) = E_n e^{-n^2 \pi^2 t} \cos n\pi x$$

$$u(x, 0) = 3 \cos 2\pi x \Leftrightarrow 3 \cos 2\pi x = E_n \cos n\pi x$$

$$2\pi = n\pi \Leftrightarrow n = 2 \text{ so } E_2 = 3$$

$$\text{Then } u(x, t) = 3e^{-4\pi^2 t} \cos(2\pi x)$$

Example 3: Apply the method of separation of variables to solve the heat equation

$u_t = 3u_{xx}$ over $0 < x < \pi, t > 0$ for the boundary conditions

$u(0, t) = u(\pi, t) = 0$ and the initial condition $u(x, 0) = 3 \sin 2x - 6 \sin 5x$

Solution: Assume the solution is $u(x, y) = F(x)Y(y)$ then

$$u_t = FG' \text{ and } u_{xx} = F''G$$

$$FG' = 3F''G \Rightarrow \frac{F''}{F} = \frac{G'}{3G} = -\lambda^2$$

$$\frac{F''}{F} = -\lambda^2 \text{ and } \frac{G'}{3G} = -\lambda^2$$

$$F'' + \lambda^2 F = 0 \text{ and } G' + 3G\lambda^2 = 0$$

$$F = A \sin \lambda x + B \cos \lambda x \quad \text{and} \quad G = Ce^{-3\lambda^2 t}$$

$$\text{So } u(x, t) = (A \sin \lambda x + B \cos \lambda x) \times Ce^{-3\lambda^2 t}$$

$$\text{Or } u(x, t) = e^{-3\lambda^2 t}(D \sin \lambda x + E \cos \lambda x) ; \quad AC = D, \quad BC = E$$

$u(0, t) = 0$ since $\sin 0 = 0$ and $\cos 0 = 1$ this must imply that $E = 0$

$$u(\pi, t) = 0 \Rightarrow De^{-3\lambda^2 t} \sin \pi \lambda = 0 ; \quad e^{-3\lambda^2 t} \neq 0$$

$$D = 0 \Rightarrow u = 0 \quad (\text{trivial solution})$$

$$\therefore \sin(\lambda \pi) = 0 \Rightarrow \lambda \pi = n\pi \Rightarrow \lambda = n, \quad n = 1, 2, 3, \dots$$

$$\text{Then } u(x, t) = D_n e^{-3n^2 t} \sin nx$$

$$u(x, 0) = 3 \sin 2x - 6 \sin 5x$$

$$3 \sin 2x - 6 \sin 5x = D_2 \sin 2x + D_5 \sin 5x$$

$$D_2 = 3, D_5 = -6$$

$$\text{Then } u(x, t) = 3e^{-12t} \sin 2x - 6e^{-75t} \sin 5x$$

H.W: Apply the method of separation of variables to solve the heat equation

$$1. \quad u_t = u_{xx}; \quad 0 < x < 1, t > 0 \quad \text{with} \quad u(0, t) = u_x(1, t) = 0$$

and the initial condition $u(x, 0) = 5 \sin \frac{3\pi}{2} x$

$$2. \quad u_t = u_{xx}; \quad 0 < x < 1, t > 0 \quad \text{with} \quad u_x(0, t) = u(1, t) = 0$$

and the initial condition $u(x, 0) = 4 \cos \frac{5\pi}{2} x$