Method (4)

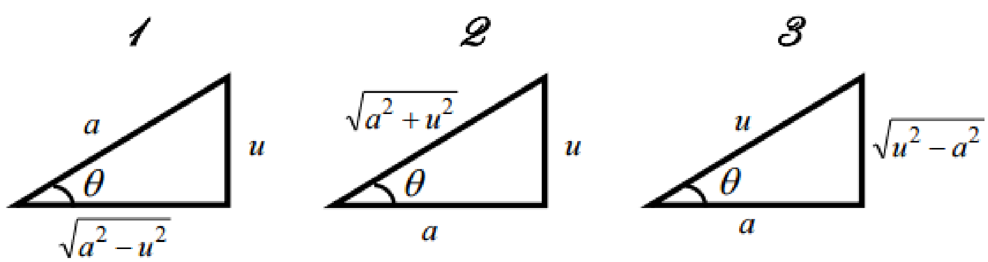
Trigonometric Substitutions

If the integral involve one of the forms $\left(a^2+u^2, \sqrt{a^2-u^2}, \sqrt{a^2+u^2}, \sqrt{u^2-a^2}\right)$ then the substitutions as follows :

1 - If $\sqrt{a^2-u^2}$ Let $u = a\sin(\theta) \Rightarrow a^2-u^2 = a^2\cos^2(\theta)$

2 - If $\sqrt{a^2+u^2}$, a^2+u^2 Let $u = a\tan(\theta) \Rightarrow a^2+u^2 = a^2\sec^2(\theta)$

3 - If $\sqrt{u^2-a^2}$ Let $u = a\sec(\theta) \Rightarrow u^2-a^2 = a^2\tan^2(\theta)$



Trig Identities

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

Example: Find $\boldsymbol{I = \int \frac{x^3}{\sqrt{9-x^2}}\, dx}$

<u>Sol.</u> Let $x = a\sin\theta = 3\sin\theta$

$$dx = 3\cos\theta\ d\theta \quad \Rightarrow \quad d\theta = \frac{dx}{3\cos\theta}$$

$$\Rightarrow \int \frac{x^3}{\sqrt{9-x^2}}\, dx = \int \frac{(3\sin\theta)^3}{\sqrt{9-(3\sin\theta)^2}} \,.3\cos\theta\ d\theta$$

$$= 27\int \frac{3\sin^3\theta\, cos\,\theta}{\sqrt{9-(3\,sin\,\theta)^2}}\quad d\theta$$

$$= 27\int \frac{3\ \sin^3\theta\, cos\,\theta}{\sqrt{9-9\,sin^2\theta}}\quad d\theta$$

$$= 27\int \frac{3\sin^3\theta\, cos\,\theta}{\sqrt{9(1-sin^2\theta)}}\quad d\theta$$

$$\because\quad cos^2\theta + sin^2\theta = 1 \quad\Rightarrow\quad cos^2\theta = 1 - sin^2\theta$$

$$\Rightarrow\qquad = 27\int \frac{3\sin^3\theta\, cos\,\theta}{\sqrt{9(\,cos^2\theta)}}\quad d\theta$$

$$\because\quad \sqrt{9(\,cos^2\theta)} = 3\cos\theta$$

$$= 27\int \frac{3\ \sin^3\theta\, cos\,\theta}{3\cos\theta}\quad d\theta$$

$$= 27\int \sin^3\theta \quad d\theta$$

$$= 27\int \sin^2\theta \ \ \sin\theta\ \ d\theta$$

$$= 27\int (1-\cos^2\theta)\ \ \sin\theta\ \ d\theta$$

$$= 27\int \sin\theta\ \ d\theta - 27\int \cos^2\theta\ \ \sin\theta\ \ d\theta$$

$\int \cos^2\theta\ \ \sin\theta\ d\theta$

Let u= cos θ du= -sin θ $d\theta$

$$\therefore \int u^2\ \ -du \ = \frac{-u^3}{3} + c$$

$$= \frac{-\cos^3\theta}{3} + c$$

$$= 27\,(-\cos\theta\,) + 27\ \frac{\cos^3\theta}{3} + c$$

$$= 27\left(-\sqrt{1-\sin^2\theta}\,\right) + \frac{27}{3}\ (1-\sin^2\theta)^{\frac{3}{2}} + c$$

$$\because\ \boldsymbol{x = 3\,sin\,\theta} \quad\Rightarrow\quad \boldsymbol{sin\,\theta = \frac{x}{3}}$$

$$= 27\left(-\sqrt{1-\left(\frac{x}{3}\right)^2}\,\right) + \frac{27}{3}\left(1-\left(\frac{x}{3}\right)^2\right)^{\frac{3}{2}} + c$$

$$= 27\left(-\sqrt{1-\frac{x^2}{9}}\,\right) + 9\ \left(1-\frac{x^2}{9}\right)^{\frac{3}{2}} + c$$

$$= 27\left(-\sqrt{\frac{9-x^2}{9}}\,\right) + 9\ \left(\frac{9-x^2}{9}\right)^{\frac{3}{2}} + c$$

Lect. Huda A. Hadi

$$= 27\left(-\sqrt{\frac{\mathbf{9-x^2}}{\mathbf{9}}}\right) + 9\left(\frac{\mathbf{9-x^2}}{\mathbf{9}}\right)^{\frac{3}{2}} + c$$

$$= -\frac{27}{3}\sqrt{9-x^2} + \frac{9}{27}(9-x^2)^{\frac{3}{2}} + c$$

$$= -9\sqrt{9-x^2} + \frac{1}{3}(9-x^2)^{\frac{3}{2}} + c$$

Example: Evaluate $\boldsymbol{I = \int \frac{dx}{4+x^2}}$

<u>Sol.</u> Let $x = 2\tan\theta \quad \Rightarrow \quad \tan\theta = \frac{x}{2} \quad \Rightarrow \quad \theta = tan^{-1}\left(\frac{x}{2}\right)$

$$dx = 2\,sec^2\theta\ \ d\theta \qquad \Rightarrow \quad d\theta = \frac{dx}{2\,sec^2\theta}$$

$$\int \frac{2\,sec^2\theta\ \ d\theta}{\mathbf{2^2}+2\,tan^2\theta}$$

$$\because a^2 + x^2 = a^2\ sec^2\ \theta$$

$$\int \frac{2\,sec^2\theta\ \ d\theta}{\mathbf{2^2}+2\,tan^2\theta} = \int \frac{2\,sec^2\theta\ \ d\theta}{\mathbf{2^2}\ sec^2\theta} = \int \frac{1}{2}\,d\theta = \frac{1}{2}\int d\theta = \frac{1}{2}\,\theta + c$$

$$= \frac{1}{2}\ tan^{-1}\left(\frac{x}{2}\right) \quad + c$$

Example: Find the integral of $\int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}\ dx$

Sol. Let $x = \sin\theta \quad \Rightarrow \quad dx = \cos\theta\ d\theta$

at $\quad x = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \frac{\sqrt{3}}{2} = \sin\theta \quad \Rightarrow \quad \theta = sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \ \Rightarrow \ \theta = \frac{\pi}{3}$

at $\quad x = -\frac{1}{2} \quad \Rightarrow \quad -\frac{1}{2} = \sin\theta \quad \Rightarrow \quad \theta = sin^{-1}\left(-\frac{1}{2}\right) \ \Rightarrow \ \theta = \frac{-\pi}{6}$

$$\int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}\ dx = \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1-sin^2\theta}\ \cos\theta \quad d\theta$$

$$= \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{cos^2\theta}\ \cos\theta \quad d\theta$$

$$= \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} cos\theta\ \cos\theta \quad d\theta$$

$$= \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} cos^2\theta \quad d\theta$$

$$= \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1+\cos(2\theta)}{2} \quad d\theta$$

$$= \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{2}\ d\theta + \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\cos(2\theta)}{2} \quad d\theta$$

$$= \frac{1}{2}\left(\int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \quad d\theta + \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \cos(2\theta)\ d\theta\right)$$

$$= \frac{1}{2}\left(\int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \quad d\theta + \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \cos(2\theta) \ \ d\theta\ \right)$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} + c$$

$$= \frac{1}{2}\left(\frac{\pi}{3} + \frac{1}{2}\sin\left(2\frac{\pi}{3}\right)\right) - \frac{1}{2}\left(-\frac{\pi}{6} + \frac{1}{2}\sin\left(-\frac{\pi}{3}\right)\right)$$

$$= \frac{\pi+\sqrt{3}}{4}$$

Ex. **Evaluate**

$$I = \int \frac{\sqrt{x^2 - 7}}{x} dx$$

Sol.

$$x = \sqrt{7}\sec(\theta) \Rightarrow \sec(\theta) = \frac{x}{\sqrt{7}} \Rightarrow \theta = \sec^{-1}(\frac{x}{\sqrt{7}}) \Rightarrow dx = \sqrt{7}\sec(\theta)\tan(\theta)d\theta$$

$$\Rightarrow I = \int \frac{\sqrt{7\sec^2(\theta) - 7}}{\sqrt{7}\sec(\theta)} . \sqrt{7}\sec(\theta)\tan(\theta)d\theta = \int \sqrt{7}\tan^2(\theta)d\theta = \sqrt{7}\int (\sec^2(\theta) - 1)d\theta$$

$$\Rightarrow \quad = \sqrt{7}[\tan(\theta) - \theta] + c = \sqrt{7}\left(\tan[\sec^{-1}(\frac{x}{\sqrt{7}})] - \sec^{-1}(\frac{x}{\sqrt{7}}) \right) + c$$

Method (5)

Integral Involving Quadratic Function

Quadratic functions are functions in the form $ax^2 + bx + c = 0$. Integrating functions that include a quadratic can sometimes be a little difficult. Therefore, we can find the solution by **Method (1)**, **Partial fractions**, or we **can reduce it** to the form

$$ax^2 + bx + c = \left[ax^2 + bx + \left(\frac{1}{2}b\right)^2 + c - \left(\frac{1}{2}b\right)^2\right]$$

$$\Rightarrow \quad = \left[ax^2 + bx + \frac{b^2}{4} + c - \frac{b^2}{4}\right]$$

$$= \left[ax + \frac{b}{2}\right]^2 + \left[c - \frac{b^2}{4}\right]$$

$$= u^2 + B$$

where, $u = ax + \frac{b}{2}$ and $B = c - \frac{b^2}{4}$, then the solution can be found by **Method (4).**

Ex. 1: Evaluate the integral.

$$\boldsymbol{I = \int \frac{6x}{3x^2 - 1} dx}$$

<u>Sol.</u> Let $u = 3x^2 - 1 \quad \Rightarrow \quad du = 6x\, dx$

$\int \frac{6x}{3x^2-1} dx = \int \frac{1}{u} du = ln\,|u| + c$

$$= ln\,|3x^2 - 1| + c$$

Ex. 2: Evaluate the integral.

$$\boldsymbol{I = \int \frac{dx}{2x - x^2}}$$

Sol.

$$\Rightarrow \ I = \int \frac{dx}{\sqrt{-(x^2 - 2x + 1 - 1)}} = \int \frac{dx}{\sqrt{-[(x-1)^2 - 1]}} = \int \frac{dx}{\sqrt{[1 - (x-1)^2]}}$$

Let $u = x - 1 \ \Rightarrow \ du = dx$

$$I = \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1}(u) + c = \sin^{-1}(x-1) + c$$

Ex. 3: Evaluate the integral.

$$\boldsymbol{I = \int \frac{dx}{x^2 + 3x + 5}}$$

<u>Sol.</u>

$$\Rightarrow x^2 + 3x + 5 = \left[x^2 + 3x + \left(\frac{1}{2}(3)\right)^2 + 5 - \left(\frac{1}{2}(3)\right)^2\right]$$

$$= \left[x^2 + 3x + \frac{9}{4} + 5 - \frac{9}{4}\right]$$

$$= \left(x + \frac{3}{2}\right)^2 + \frac{11}{4}$$

$\therefore \quad \int \frac{\boldsymbol{dx}}{\left(\boldsymbol{x}+\frac{3}{2}\right)^2 + \frac{11}{4}}$

Let $u = x + \frac{3}{2} \quad \Rightarrow \quad du = dx$

$$\therefore \quad \int \frac{\boldsymbol{dx}}{\left(\boldsymbol{x}+\frac{3}{2}\right)^2 + \frac{11}{4}} = \int \frac{\boldsymbol{du}}{\boldsymbol{u}^2 + \frac{11}{4}}$$

We have $\quad u^2 + a^2 \quad \Rightarrow \quad u = x + \frac{3}{2}$ and $\quad a^2 = \frac{11}{4} \quad \Rightarrow \quad a = \sqrt{\frac{11}{4}} = \frac{\sqrt{11}}{2}$

$\int \frac{du}{u^2 + \frac{11}{4}} = \frac{1}{a}\, tan^{-1}\left(\frac{u}{a}\right) + c$

Depend on table of Integral Formula (Standard Form) p.88

$$= \frac{1}{\frac{\sqrt{11}}{2}}\, tan^{-1}\left(\frac{u}{\frac{\sqrt{11}}{2}}\right) + c$$

$$= \frac{2}{\sqrt{11}}\, tan^{-1}\left(\frac{2u}{\sqrt{11}}\right) + c$$

$$= \frac{2}{\sqrt{11}}\, tan^{-1}\left(\frac{2(x+\frac{3}{2})}{\sqrt{11}}\right) + c$$

Ex. 4: Evaluate the integral.

$$I = \int \frac{9x + 8}{3x^2 + 10x - 8} \; dx$$

Sol.

$$\int \frac{9x+8}{3x^2+10x-8} \; dx$$

We notice that the quadratic function in the denominator of the fraction can be factored.

$$= \int \frac{9x+8}{(x+4)(3x-2)} dx$$

Here we'll use a partial fractions decomposition to split the integral in two.

$$= \int \frac{A}{(x+4)} dx + \int \frac{B}{(3x-2)} dx$$

$$(x + 4) = 0 \quad \Rightarrow \quad x = -4$$

$$(3x - 2) = 0 \quad \Rightarrow \quad 3x = 2 \quad \Rightarrow \quad \frac{2}{3}$$

$$A = |_{x=-4} \; \frac{9(-4)+8}{(3(-4)-2)} = \frac{-36+8}{-12-2} = \frac{-28}{-14} = 2$$

$$B = |_{x=\frac{2}{3}} \; \frac{9(\frac{2}{3})+8}{\left(\frac{2}{3}\right)+4} = \frac{6+8}{\frac{2+12}{3}} = \frac{14}{\frac{14}{3}} = 3$$

$$\Rightarrow \; \int \frac{2}{(x+4)} dx + \int \frac{3}{(3x-2)} dx$$

$$= \; 2 \, ln \, |x + 4| + ln \, |3x - 2| + c$$